"Sur les caractéristiques des équations aux dérivées partielles," Bull. Soc. Math. France **25** (1897), 108-120.

On the characteristics of partial differential equations

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In a note that was presented to l'Académie des Sciences ([1]), I rapidly indicated an extension of the notion of characteristic to partial differential equations of order higher than one and to more than two independent variables. Here, I propose to give the proof of the facts that were announced; however, before I do that, I will point out a terminology that will be of great utility ([2]).

If z is an analytic function of n variables $x_1, \ldots, x_n$ then one will have relations of the form:

$$(a) \qquad d\frac{\partial^k z}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} = \sum_{i=1}^{n}\frac{\partial^{k+1} z}{\partial x_1^{\alpha_1}\cdots\partial x_i^{\alpha_{i+1}}\cdots\partial x_n^{\alpha_n}}\,dx_i \qquad (\alpha_1 + \ldots + \alpha_n = k).$$

One can consider the quantities $x_1, \ldots, x_n, z, \dfrac{\partial^k z}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$ (k varies from 0 to p) to be independent variables. One calls any system of values that are attributed to those symbols an *element of order p in n* + 1*-dimensional space.*

Two infinitely-close elements that verify those relations are said to be *united.*

Any system of equations in the coordinates of an element that verifies equations (a) defines a multiplicity M^p. In each of those systems, there are some relations between just $x_1, \ldots, x_n, z$ that define a point-like multiplicity that one calls the *support* of M^p.

There are multiplicities M^p such that each point of the support corresponds to just one element of order p. If q is the number of dimensions of the support then we shall call them *multiplicities* M_q^p.

1. – First consider a partial differential equation that is linear with respect to the second-order derivatives; let:

$$(1) \qquad \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}\,p_{ik} + \varphi = 0 = \Phi, \qquad p_i = \frac{\partial z}{\partial x_i}, \qquad p_{ik} = \frac{\partial^2 z}{\partial x_i\,\partial x_k},$$

([1]) J. BEUDON, "Sur les singularités des équations aux dérivées partielles," Comptes rendus des séances **124** (29 March 1897).

([2]) For more details, see J. BEUDON, "Sur les systèmes d'équations aux dérivées partielles dont les caracteréristiques dépendent d'un nombre fini de paramètres," Ann. de l'E. N. S. Supp., 1896.

in which the A_{ik} and φ are given functions of the $x_1, \ldots, x_n, z, p_1, \ldots, p_n$ for such an equation.

From the general theorems of Cauchy on the existence of integrals of differential systems, one knows that any analytic integral of that equation can be defined by giving a multiplicity M_{n-1}^1 that it must contain. The following equations:

$$\frac{\partial z}{\partial x_i} = p_i + p_n \frac{\partial x_n}{\partial x_i} \qquad (i = 1, 2, \ldots, n-1), \tag{2}$$

in which z, x_n , and p_n are arbitrary functions of $x_1, \ldots, x_{n-1}$, represent a multiplicity M_{n-1}^1.

In order to calculate the second derivatives as functions of $x_1, \ldots, x_{n-1}$, one mst make use of the formulas:

$$\frac{\partial p_\rho}{\partial x_i} = p_{\rho i} + p_{\rho n} \frac{\partial x_n}{\partial x_i} \qquad \begin{pmatrix} \rho = 1, 2, \ldots, n \\ i = 1, 2, \ldots, n-1 \end{pmatrix}, \tag{3}$$

from which, one will deduce that:

$$p_{\rho n} = \frac{\partial p_n}{\partial x_\rho} - p_{nn} \frac{\partial x_n}{\partial x_\rho}, \tag{4}$$

$$p_{\rho i} = \frac{\partial p_\rho}{\partial x_i} - \frac{\partial x_n}{\partial x_i}\frac{\partial p_n}{\partial x_\rho} + p_{nn} \frac{\partial x_n}{\partial x_i}\frac{\partial x_n}{\partial x_\rho}. \tag{5}$$

Upon substituting this into the proposed equation (1), one will finally get:

$$\left(\sum_{\rho=1}^{n-1}\sum_{i=1}^{n-1} A_{\rho i}\frac{\partial x_n}{\partial x_i}\frac{\partial x_n}{\partial x_\rho} - \sum_{\rho=1}^{n-1} A_{\rho n}\frac{\partial x_n}{\partial x_\rho} + A_{nn}\right) p_{nn}$$

$$+ \sum_{\rho=1}^{n-1}\sum_{i=1}^{n-1} A_{\rho i}\left(\frac{\partial p_\rho}{\partial x_i} - \frac{\partial x_n}{\partial x_i}\frac{\partial p_n}{\partial x_\rho}\right) - \sum_{\rho=1}^{n-1} A_{\rho n}\frac{\partial p_n}{\partial x_\rho} + \varphi = 0.$$

In order for indeterminacy to exist, one must have:

$$\sum_{\rho=1}^{n-1}\sum_{i=1}^{n-1} A_{\rho i}\frac{\partial x_n}{\partial x_i}\frac{\partial x_n}{\partial x_\rho} - \sum_{\rho=1}^{n-1} A_{\rho n}\frac{\partial x_n}{\partial x_\rho} + A_{nn} = 0, \tag{6}$$

$$\sum_{\rho=1}^{n-1}\sum_{i=1}^{n-1} A_{\rho i}\left(\frac{\partial p_\rho}{\partial x_i} - \frac{\partial x_n}{\partial x_i}\frac{\partial p_n}{\partial x_\rho}\right) - \sum_{\rho=1}^{n-1} A_{\rho n}\frac{\partial p_n}{\partial x_\rho} + \varphi = 0. \tag{7}$$

I shall call the multiplicities M_{n-1}^1 that are defined by equations (2), (6), and (7) *singular multiplicities.* It is easy to see their degree of generality, while remaining quite general. Indeed, if one chooses x_n arbitrarily as a function of $x_1, \ldots, x_{n-1}$ then equation (6) will permit one to obtain p_n, and as a result, $p_1, \ldots, p_{n-1}$. Upon replacing those symbols with their values in equation (7), one will have a linear, second-order partial differential equation that defines z.

If one knows an integral surface of equation (1) then the supports of the singular multiplicities that are placed on that integral surface will be defined by the first-order partial differential equation (6), and the orientations of the first-order elements of those singular multiplicities will be defined by equation (7). As one can easily see, those two equations have the same characteristics.

We shall next see that those characteristics have great importance.

2. – I shall now perform a change of variables that leaves $x_1, \ldots, x_{n-1}$ unaltered, but is such that x_n will become a function of $x_1, \ldots, x_{n-1}$, and the new variable y. I will get formulas (2), (3), and:

(9) $$\frac{\partial z}{\partial y} = \frac{\partial p_{\rho n}}{\partial x_i}\frac{\partial x_n}{\partial y} - \frac{\partial p_{\rho n}}{\partial y}\frac{\partial x_n}{\partial x_i} \qquad \left(\begin{matrix} \rho = 1, 2, \ldots, n \\ i = 1, 2, \ldots, n-1 \end{matrix}\right).$$

I deduce the relations:

(10) $$\frac{\partial p_{\rho i}}{\partial y} = \frac{\partial p_{\rho n}}{\partial x_i}\frac{\partial x_n}{\partial y} - \frac{\partial p_{nn}}{\partial x_\rho}\frac{\partial x_n}{\partial x_i}\frac{\partial x_n}{\partial y} + \frac{\partial p_{nn}}{\partial y}\frac{\partial x_n}{\partial y}\frac{\partial x_n}{\partial x_\rho}$$

from that.

Under those conditions, $z, x_n, p_1, \ldots, p_n$ will be functions of $x_1, x_2, \ldots, x_{n-1}$, and y. I shall determine the change of variables in such a way that those functions will represent a singular multiplicity M_{n-1}^1 for any value that is attributed to y. Suppose that one has a singular multiplicity for $y = y_0$; equation (1) will then be verified. I would like to write down that the derivative of its left-hand side with respect to y is zero.

To simplify notations, I will set:

$$\frac{d}{dx_n} = \frac{\partial}{\partial x_n} + \frac{\partial}{\partial z}p_n + \sum_{i=1}^{n}\frac{\partial}{\partial p_i}p_{in}.$$

Upon taking equations (9) and (10) into account, I will have:

$$0 = \frac{\partial p_{nn}}{\partial y}\left(\sum_{i=1}^{n-1}\sum_{\rho=1}^{n-1}A_{\rho i}\frac{\partial x_n}{\partial x_i}\frac{\partial x_n}{\partial x_\rho} - \sum_{\rho=1}^{n-1}A_{\rho n}\frac{\partial x_n}{\partial x_i} + A_{nn}\right)$$

$$+ \frac{\partial x_n}{\partial y}\left[\sum_{i=1}^{n-1}\sum_{\rho=1}^{n-1}A_{\rho i}\left(\frac{\partial p_{\rho n}}{\partial x_i} - \frac{\partial x_n}{\partial x_i}\frac{\partial p_{nn}}{\partial x_\rho}\right) + \sum_{\rho=1}^{n-1}A_{\rho n}\frac{\partial p_{\rho n}}{\partial x_i} + \sum_{i=1}^{n-1}\sum_{k=1}^{n-1}p_{ik}\frac{dA_{ik}}{dx_n} + \frac{d\varphi}{dx_n}\right].$$

The coefficient of $\partial p_{nn} / \partial y$ is zero by hypothesis, and since $\partial x_n / \partial y$ is not identically zero, one will have:

$$0 = \sum_{i=1}^{n-1}\sum_{\rho=1}^{n-1} A_{\rho i}\left(\frac{\partial p_{nn}}{\partial x_i} - \frac{\partial x_n}{\partial x_i}\frac{\partial p_{nn}}{\partial x_\rho}\right) + \sum_{\rho=1}^{n-1} A_{\rho n}\frac{\partial p_{nn}}{\partial x_\rho} + \sum_{i=1}^{n}\sum_{k=1}^{n} p_{ik}\frac{\partial A_{ik}}{\partial x_n} + \frac{\partial \varphi}{\partial x_n}. \tag{11}$$

3. – I say that equation (11) represents the condition that must be verified by the orientation of the second-order elements of any integral that contains the multiplicity M_{n-1}^1 on that singular multiplicity. In order to prove that fact, I shall seek to calculate the values of the third-order derivatives, but first, I shall introduce the notation:

$$\frac{d}{dx_j} = \frac{\partial}{\partial x_j} + \frac{\partial}{\partial z} p_j + \sum_{i=1}^{n} \frac{\partial}{\partial p_i} p_{ij} \qquad (j = 1, 2, \ldots, n-1).$$

When the proposed equation is differentiated with respect to x_j that will give:

$$\sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}\, p_{ikj} + \sum_{i=1}^{n}\sum_{k=1}^{n} p_{ik}\frac{\partial A_{ik}}{\partial x_j} + \frac{\partial \varphi}{\partial x_j} = 0. \tag{12}$$

We remark that one will have:

$$\frac{\partial p_{ik}}{\partial x_j} = p_{ikj} + p_{ikn}\frac{\partial x_n}{\partial x_j} \qquad \begin{pmatrix} i, k = 0, 1, 2, \ldots, n \\ j = 1, 2, \ldots, n-1 \end{pmatrix} \tag{13}$$

on the multiplicity M_{n-1}^1, so those relations will permit one to calculate all of the p_{ikj} as functions of p_{nnn} ; upon substituting that into equation (12), one will get:

$$\sum_{i=1}^{n-1}\sum_{k=1}^{n-1} A_{ik}\left\{\frac{\partial p_{ik}}{\partial x_j} - \frac{\partial x_n}{\partial x_j}\left[\frac{\partial p_{nn}}{\partial x_i} - \frac{\partial x_n}{\partial x_j}\left(\frac{\partial p_{nn}}{\partial x_i} - p_{nnn}\frac{\partial x_n}{\partial x_k}\right)\right]\right\}$$

$$+ \sum_{i=1}^{n-1} A_{in}\left[\frac{\partial p_{in}}{\partial x_j} - \frac{\partial x_n}{\partial x_j}\left(\frac{\partial p_{nn}}{\partial x_i} - p_{nnn}\frac{\partial x_n}{\partial x_i}\right)\right] + A_{nn}\left(\frac{\partial p_{nn}}{\partial x_j} - p_{nnn}\frac{\partial x_n}{\partial x_j}\right) + \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial A_{ik}}{\partial x_j} p_{ik} + \frac{d\varphi}{dx_j} = 0.$$

The coefficient of p_{nnn} in this equation is precisely the left-hand side of equation (6). It is zero, since we are on a singular multiplicity M_{n-1}^1. The coefficient of $\partial x_n / \partial x_j$ is the left-hand side of equation (11), which we assume to be verified. What will finally remain is:

$$\sum_{i=1}^{n-1}\sum_{k=1}^{n-1} A_{ik}\frac{\partial p_{ik}}{\partial x_j} + \sum_{i=1}^{n-1} A_{in}\frac{\partial p_{in}}{\partial x_j} + \sum_{i=1}^{n}\sum_{k=1}^{n} p_{ik}\frac{\partial A_{ik}}{\partial x_j} + \frac{d\varphi}{dx_j} = 0.$$

That equation is verified, since the second-order elements that one infers from formulas (2) and (3) will satisfy the proposed equation identically when one places oneself upon a singular multiplicity. We have then established that if the orientations of the chosen second-order elements on the singular multiplicity verify equation (11) then the calculation of the third-order derivatives will be indeterminate.

4. – Equations (2), (3), (6), (7), and (11) define a family of multiplicities M_{n-1}^2 that I shall call *singular multiplicities.* If one includes formulas (13) then one will get an infinitude of multiplicities M_{n-1}^3 that verify the proposed equations and its derived ones. However, they are not all located on an n-dimensional integral multiplicity. In order to find the conditions under which that situation will present itself, I shall make a change of variables that is analogous to the one that was employed in no. **2**; i.e., such that one will be dealing with singular multiplicities M_{n-1}^2 for any y, since x_n, z, p_i, p_{ik}, p_{ikj} are functions of $x_1, \ldots, x_{n-1}$, and the new variable y.

I must append the formulas:

$$\frac{\partial p_{\rho i}}{\partial x_k} = p_{\rho ik} + p_{\rho in}\frac{\partial x_n}{\partial x_k} \quad \text{and} \qquad \frac{\partial p_{\rho i}}{\partial y} = p_{\rho in}\frac{\partial x_n}{\partial y}\,.$$

Under those conditions, one will have:

$$\frac{\partial \Phi}{\partial x_j} + \frac{\partial \Phi}{\partial x_n}\frac{\partial x_n}{\partial x_j} = 0,$$

because the multiplicities M_{n-1}^3 verify equations (12) identically. In order for that to be true for any y, one must have:

$$\frac{\partial^2 \Phi}{\partial y\,\partial x_j} + \frac{\partial x_n}{\partial x_j}\frac{\partial^2 \Phi}{\partial x_n\,\partial y} + \frac{\partial \Phi}{\partial x_n}\frac{\partial^2 x_n}{\partial x_j\,\partial y} = 0.$$

Now, $\partial\Phi / \partial x_n$ is verified identically, so it will then suffice that one should have:

$$\frac{\partial^2 \Phi}{\partial y\,\partial x_j} = 0.$$

This is what one obtains when one takes the integrability conditions into account:

$$(14)\qquad \sum_{i=1}^{n-1}\sum_{k=1}^{n-1} A_{ik}\left(\frac{\partial p_{knn}}{\partial x_i} - \frac{\partial x_k}{\partial x_i}\frac{\partial p_{nnn}}{\partial x_k}\right) + \sum_{i=1}^{n-1} A_{in}\frac{\partial p_{nnn}}{\partial x_i} + \sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\frac{dA_{ik}}{dx_n}p_{ikn} + \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{d^2 A_{ik}}{dx_n^2}p_{ik} + \frac{d^2\varphi}{dx_n^2} = 0,$$

in which one sets:

$$\frac{df}{dx_n} = \frac{\partial f}{\partial x_n} + \frac{\partial f}{\partial z} p_n + \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} p_{in} + \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial f}{\partial p_{ij}} p_{ijn} .$$

When equation (14) is combined with equations (13), that will express the condition for the multiplicities M_{n-1}^3 to be contained in n-dimensional integral multiplicities. One can verify that by noting that the fourth-order elements are indeterminate. We thus obtain the *singular multiplicities* M_{n-1}^3.

Upon proceeding in that way, step-by-step, one will prove the existence of *singular multiplicities* M_{n-1}^p for all values of the order p. One will also see that once a singular multiplicity M_{n-1}^p has been chosen, the set of multiplicities M_{n-1}^{p+1} that correspond to it will depend upon an arbitrary function of n – 2 arguments.

We shall prove a little later that those singular multiplicities indeed contain integral multiplicities; i.e., that the preceding conditions that we have recognized to be necessary will be, in fact, sufficient, as well.

5. – I shall now envision a nonlinear second-order partial differential equation. For the sake of simplicity, I shall suppose that it is rational in the symbols that enter into it; let that equation be:

(15) $$f(x_1, \ldots, x_n, z, p_i, p_{ik}) = 0.$$

The degree of generality is defined in the same manner as it is for linear equations – i.e., by formulas (2). In order to calculate the second-order derivatives, we make use of formulas (3). In order for there to be indeterminacy, it is necessary that:

(16) $$\sum_{\rho=1}^{n-1} \sum_{i=1}^{n-1} \frac{\partial f}{\partial p_{\rho i}} \frac{\partial x_n}{\partial x_i} \frac{\partial x_n}{\partial x_\rho} - \sum_{\rho=1}^{n-1} \frac{\partial f}{\partial p_{\rho n}} \frac{\partial x_n}{\partial x_i} + \frac{\partial f}{\partial p_{nn}} = 0.$$

However, that condition, which is necessary, is no longer sufficient. If equation (16) no longer includes p_{nn} then the discussion will proceed as it did for linear equations; we then suppose that this is not true.

Equations (2), (3), (15), and (16) define a family of multiplicities M_{n-1}^2. However, those multiplicities are not all placed on the n-dimensional integral multiplicities of the proposed equation. In order to recognize the cases in which that will be true, we again make use of the change of variables that has proved useful to us, and after some calculations that I shall omit, we will find that one must have:

(17) $$\sum \frac{\partial f}{\partial p_\rho} p_{\rho n} + \frac{\partial f}{\partial x_n} + \frac{\partial f}{\partial z} p_n + \sum_{\rho=1}^{n-1} \sum_{i=1}^{n-1} \frac{\partial f}{\partial p_{\rho i}} \left(\frac{\partial p_{\rho n}}{\partial x_i} - \frac{\partial x_n}{\partial x_i} \frac{\partial p_{nn}}{\partial x_\rho} \right) + \sum_{\rho=1}^{n} \frac{\partial f}{\partial p_{\rho n}} \frac{\partial p_{nn}}{\partial x_\rho} = 0.$$

Equations (2), (3), (15), (16), and (17) define a family of multiplicities that I shall call the *singular multiplicities* M_{n-1}^2. One verifies indeterminacy in entirely the same manner as above in the calculation of the third-order derivatives, and one will prove the existence of *singular multiplicities* M_{n-1}^{p} for all values of p.

If a singular multiplicity M_{n-1}^{p} is given then the singular multiplicities that correspond to it will depend upon an arbitrary function of $n - 2$ arguments.

6. – It now remains for me to establish that if one is given a singular multiplicity M_{n-1}^{2} then there will indeed be an infinitude of n-dimensional integral multiplicities that contain it ([1]). I can always perform a change of variables such that the support of that singular multiplicity has the equations:

$$z = 0, \qquad x_n = 0.$$

One will then have, in turn:

$$p_1 = 0, \ \ldots, \qquad p_{n-1} = 0, \qquad p_n = f(x_1, \ldots, x_{n-1}),$$

$$p_{\rho i} = 0 \qquad (\rho, i = 1, 2, \ldots, n - 1), \qquad p_{ni} = \frac{\partial f}{\partial x_i}, \qquad p_{nn} = \varphi(x_1, \ldots, x_{n-1}),$$

and if one sets:

$$z = z' + x_n f + \frac{x_n^2 \varphi}{z}$$

then one will come down to:

$$z = 0, \qquad x_n = 0, \qquad p_i = 0, \qquad p_{ik} = 0 \qquad (i, k = 1, 2, \ldots, n).$$

Since one is dealing with a singular multiplicity, equations (16) and (17) must be verified, as well as the equations that are obtained by differentiating the proposed equations with respect to $x_1, \ldots, x_{n-1}$. One will then have:

$$\frac{\partial f}{\partial p_{nn}} = 0, \qquad \frac{\partial f}{\partial x_i} = 0$$

for the preceding values of the arguments. One can then put the proposed equation into the form:

$$(18) \qquad p_{n\,n-1} = \alpha_1 p_1 + \ldots + \alpha_n p_n + \sum_{i=1}^{n-2}\sum_{k=1}^{n-2} \alpha_{ik}\, p_{ik} + \sum_{i=1}^{n-2} \beta_i\, p_{ni} + \sum_{i=1}^{n-1} \gamma_i\, p_{n-1\,i} + \ldots;$$

the unwritten terms have higher degree. Upon replacing x_{n-1} with $x_{n-1} + \lambda x_n$, one can make the term in $p_{n-1\ n-1}$ disappear. Before going further, I would like to prove the following theorem:

([1]) For that proof, I was inspired by the method that was indicated by Goursat. *Leçons sur les équations aux dérivées partielles du second ordre*, pp. 188.

If one is given the equation:

$$p_{nn-1} = F\,(x_1, \ldots, x_n, p_1, \ldots, p_1, p_{n1}, \ldots, p_{nn-2}, p_{ik}, p_{nn}),$$

such that $\partial F / \partial p_{nn}$ *and* $\partial F / \partial p_{n-1\,n-1}$ *are zero for the following values:*

$$x_1 = x_1^0, \qquad \ldots., \qquad x_n = x_1^0,$$

$$z_0 = \psi_1(x_1^0, \ldots, x_{n-2}^0), \quad p_i^0 = \left(\frac{\partial \psi_1}{\partial x_i}\right)_0 \qquad (i = 1, 2, \ldots, n-2),$$

$$p_{n-1}^0 = \psi_2^0, \qquad p_n^0 = \varphi_2^0,$$

$$p_{ik}^0 = \left(\frac{\partial^2 \psi_1}{\partial x_i\,\partial x_k}\right)_0 \qquad (i, k = 1, 2, \ldots, n-2),$$

$$p_{n-1i}^0 = \left(\frac{\partial \psi_2}{\partial x_i}\right)_0 \qquad (i = 1, 2, \ldots, n-2), \quad p_{ni}^0 = \left(\frac{\partial \varphi_2}{\partial x_i}\right)_0,$$

$$p_{n-1n-1}^0 = \psi_3^0, \quad p_{nn}^0 = \varphi_3^0$$

then it will admit an integral that is holomorphic in a neighborhood of $x_1 = x_1^0, \ldots, x_n = x_n^0$ *that reduces to:*

$$\Psi = \psi_1\,(x_1, \ldots, x_{n-2}) + (x_{n-1} - x_{n-1}^0)\,\psi_2\,(x_1, \ldots, x_{n-2}) + \ldots \qquad \text{for } x_n = x_n^0,$$

$$\Phi = \psi_1\,(x_1, \ldots, x_{n-2}) + (x_n - x_n^0)\,\varphi_2\,(x_1, \ldots, x_{n-2}) + \ldots \qquad \text{for } x_{n-1} = x_{n-1}^0,$$

One can, without inconvenience, suppose that all of the initial givens are zero. In order to that, it will suffice to set:

$$x_i = x_i^0 + x_i',$$

$$z = z' + \psi_1\,(x_1, \ldots, x_{n-2}) + (x_{n-1} - x_{n-1}^0)\,\psi_2 + \ldots + (x_n - x_n^0)\,\varphi_2 + \ldots$$

It results from this that if one calculates the values of the successive derivatives for $x_1 = 0, \ldots, x_n = 0$, step-by-step, then one will have, by hypothesis:

$$p_{ijk}^0 = 0, \qquad p_{nnn,\ldots,nijk,\ldots}^0 = 0, \qquad p_{n-1n-1,\ldots,n-1ijk,\ldots}^0 = 0 \qquad (i, j, k, \ldots, \leq n-2).$$

The proposed equation then gives all of the $p_{nn-1,ijk,\dots}$ by differentiating with respect to x_1 , …, x_{n-2} . The absence of terms of degree one in p_{nn} and p_{n-1n-1} permits one to calculate the terms in p_{nnn-1}, $p_{nn-1n-1}$ unambiguously, and as a result, the terms in $p_{nnn-1,ijk,\dots}$, $p_{nn-1\,n-1,ijk,\dots}$, as well, and so on.

In order to prove the convergence of the development thus-obtained, I shall employ the method of majorizing functions and consider the following auxiliary equation:

(19) $$p_{nn-1} = \frac{M}{\left(1-\dfrac{x_1+\cdots+x_n+z+p_1+\cdots+p_n}{\rho}\right)\left(1-\dfrac{\sum_{i=1}^{n-2} p_{ni}}{R}\right)\left(1-\dfrac{\sum_{i=1}^{n-2} p_{n-1i}}{R}\right)\left(1-\dfrac{p_{nn}+p_{n-1n-1}}{R}\right)} - M\left(1+\frac{p_{nn}+p_{n-1n-1}}{R}\right).$$

ρ and R are the radii of the circles of convergence for F for the corresponding symbols, and M is the maximum modulus of F. The right-hand side is obviously a majorizing function for F. It will then suffice to establish the convergence for that equation; for that reason, I shall set:

$$x_1 + \dots + x_{n-2} = u, \qquad x_{n-1} + x_n = v.$$

Equations (18) will become:

$$\frac{\partial^2 z}{\partial v^2} = \frac{M}{\left[1-\dfrac{(n-2)u+2v+z+(n-2)\dfrac{\partial z}{\partial u}+2\dfrac{\partial z}{\partial v}}{\rho}\right]\left[1-\dfrac{(n-1)n\dfrac{\partial^2 z}{\partial u^2}}{R}\right]\left[1-\dfrac{(n-2)}{R}\dfrac{\partial^2 z}{\partial u\,\partial v}\right]\left[1-\dfrac{2}{R}\dfrac{\partial^2 z}{\partial v^2}\right]} - M\left(1+\frac{2}{R}\frac{\partial^2 z}{\partial v^2}\right),$$

or rather:

$$\frac{\partial^2 z}{\partial v^2} = M\left[\frac{(n-1)\,n}{2R}\frac{\partial^2 z}{\partial u^2}+\frac{2(n-2)}{R}\frac{\partial^2 z}{\partial u\,\partial v}\right]+\left(\frac{4M}{R^2}+\frac{2}{R}\right)\left(\frac{\partial^2 z}{\partial v^2}\right)^2+\dots,$$

in which the unwritten terms are either of order less than 2 or of degree greater than 1 if they are of order equal to or greater than 2, resp. That equation admits a holomorphic integral such that z and $\partial z / \partial v$ reduce to 0 for $v = 0$. One will have:

$$\left.\begin{aligned} \frac{\partial^k z}{\partial u^k} &= 0 \\ \frac{\partial^{k+1} z}{\partial u^k \partial v} &= 0 \end{aligned}\right\} \qquad \text{for} \qquad u = 0, \quad v = 0 \quad \text{for any } k,$$

moreover, so one can unambiguously calculate the values of $\frac{\partial^{k1} z}{\partial u^{k-2} \partial v^2}$, then the value of $\frac{\partial^3 z}{\partial v^3}$, and so on, which is easy to recognize, *and all of the coefficients that one obtains will be positive.* It results immediately that the development that is given by equation (19) is convergent, along with the one that is provided by the proposed equation.

I shall now return to equation (18), which is the principal object of that paragraph; the application of the preceding theorem is immediate. There is a holomorphic integral that reduces to zero for $x_n = 0$ and to:

$$x_n^3 \varphi_3 + x_n^4 \varphi_4 + \ldots$$

for $x_{n-1} = 0$, in which $\varphi_3, \varphi_4, \ldots$ are arbitrary functions of $x_1, x_2, \ldots, x_{n-2}$. One sees that:

$$p^0_{i,j.k,\ldots} = 0 \qquad \text{for} \qquad x_1 = 0, \ldots, x_n = 0,$$

$$p^0_{n-1n-1,\ldots,n-1,ijk,\ldots} = 0,$$

and since one has, on the other hand:

$$\frac{\partial f}{\partial x_i} = 0 \qquad \text{for} \qquad z = 0,\ x_n = 0,\ p_i = 0,\ p_{ik} = 0,$$

one will deduce that:

$$p^0_{nn-1,ijk,\ldots} = 0.$$

It will then result that when that holomorphic integral is developed in powers of x_n, it will contain only terms of degree three. *There is then an infinitude of integrals that contain the singular multiplicity* M^2_{n-1}.

The arbitrary functions $\varphi_3, \varphi_4, \ldots$ correspond precisely to the arbitrary functions that enter into the definition of the singular multiplicities $M^3_{n-1}, M^4_{n-1}, \ldots$

CONCLUSIONS

If one is given the second-order partial differential equation:

$$f(x_1, \ldots, x_n, z, p_1, \ldots, p_n, p_{ik}) = 0$$

then an integral of that equation is generally defined by a multiplicity M_{n-1}^1; there will be an exception when the multiplicity is singular; i.e., if the equations:

$$\frac{\partial z}{\partial x_i} = p_i + p_n \frac{\partial x_n}{\partial x_i}, \qquad \frac{\partial p_\rho}{\partial x_i} = p_{\rho i} + p_{\rho n} \frac{\partial x_n}{\partial x_i} \qquad \begin{pmatrix} i = 1, 2, \ldots, n-1 \\ \rho = 1, 2, \ldots, n \end{pmatrix},$$

$$\sum_{\rho=1}^{n-1}\sum_{i=1}^{n-1} \frac{\partial f}{\partial p_i} \frac{\partial x_n}{\partial x_i} \frac{\partial x_n}{\partial x_\rho} - \sum_{\rho=1}^{n-1} \frac{\partial f}{\partial p_i} \frac{\partial x_n}{\partial x_\rho} + \frac{\partial f}{\partial p_{nn}} = 0,$$

$$\sum_{\rho=1}^{n-1}\sum_{i=1}^{n-1} \frac{\partial f}{\partial p_i}\left(\frac{\partial p_{\rho n}}{\partial x_i} - \frac{\partial x_n}{\partial x_i}\frac{\partial p_{nn}}{\partial x_\rho}\right) + \sum_{\rho=1}^{n-1} \frac{\partial f}{\partial p_{\rho n}} \frac{\partial p_{nn}}{\partial x_\rho} + \sum_{\rho=1}^{n} \frac{\partial f}{\partial p_\rho} p_{\rho n} + \frac{\partial f}{\partial z} p_n + \frac{\partial f}{\partial x_n} = 0,$$

$$f = 0$$

are verified.

One then obtains a singular multiplicity M_{n-1}^2 that is contained in an infinitude of integrals.

Any order p will correspond to singular multiplicities M_{n-1}^p that enjoy the same property.

Any singular multiplicity M_{n-1}^p corresponds to an infinitude of singular multiplicities M_{n-1}^{p+1} that depend upon a first-order partial differential equation. As a result, if two singular multiplicities M_{n-1}^p correspond to the same singular multiplicity M_{n-1}^{p-1}, and if they have an element of order p in common at a point then the same thing will be true all along a curve that passes through that point.

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