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CHAPTER I

Curves of double curvature

Tangent and normal plane. – First curvature or flexure. – Osculating plane. – Principal trihedron. – Second curvature or torsion. – Frenet formulas. – Intrinsic equations. – Cylindrical helices. – Developable surfaces. – Channel surfaces. – Developable surfaces. – Rectifying developable and polar developable of a curve. – Edge of regression of the polar developable. – Evolvent and evolute. – Orthogonal trajectory of a simple infinitude of planes. – Combescure's transformation. – Curve of constant flexure and spherical ellipse. – Orthogonal trajectories of a simple infinitude of spheres. – Bertrand curve.

§ 1.

Tangent and normal plane.

In order to define a curve C analytically, refer to a system of orthogonal Cartesian axes Ox, Oy, Oz, and express the coordinates x, y¸ z of a moving point on the curve as functions of one parameter u:

$$x = x(u), \qquad y = y(u), \qquad z = z(u).$$

In regard to the functions $x(u)$, $y(u)$, $z(u)$, we shall say, once and for all, that they are assumed to be *finite and continuous, along with their first, second, and third derivatives, in the entire interval in which the independent variable u is defined, except perhaps for some special points.*

Any particular value u_1 that one attributes to the parameter u in the interval considered corresponds to a particular position M_1 of the point M (which is the *generator* of the curve), and when u varies continuously, the point M will move in space by a continuous law and thus describe a curve C. Suppose, as always, that the sense in which the generator point M moves when the parameter u increases is taken to be the *positive* sense of the curve C, and the opposite sense is the *negative* sense.

In most cases, one assumes that the parameter, or *auxiliary variable*, u is the arc length s of the curve C, as measured by starting from a fixed point (i.e., *origin*) on the curve. In any case, in order to define s as a function of u, one has the known formula:

$$\frac{ds}{du} = \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2},$$

in which one measures s in the direction of increasing u, so one must choose the positive sign for the radical.

Consider the tangent at a point M of the curve C, whose positive direction is assumed to agree with the positive sense of the curve. Then (as we shall constantly do in what follows), let α, β, γ denote the cosines of the positive direction of the tangent, so we shall have the formulas:

$$\alpha = \frac{dx/du}{\sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2+\left(\frac{dz}{du}\right)^2}}, \qquad \beta = \frac{dy/du}{\sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2+\left(\frac{dz}{du}\right)^2}},$$

$$\gamma = \frac{dz/du}{\sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2+\left(\frac{dz}{du}\right)^2}},$$

so:

$$\alpha = \frac{dx}{ds}, \qquad \beta = \frac{dy}{ds}, \qquad \gamma = \frac{dz}{ds}. \tag{1}$$

In these formulas (1), it is irrelevant whether one considers the right-hand sides to be differential quotients or derivatives that are taken with respect to the arc length.

The *normal plane* to the tangent at M is called the *normal plane to the curve*. It has the equation:

$$(X - x)\,\alpha + (Y - y)\,\beta + (Z - z)\,\gamma = 0,$$

in which X, Y, Z denote the current coordinates of the point.

§ 2.

First curvature or flexure.

We judge the more or less rapid deviation that the point that describes a curve experiences in the rectilinear direction to be the greater or lesser *flexure* of that curve. In order to make that concept more precise, and to render it susceptible to being measured, consider a point M on the curve and a neighboring point M_1. Divide the small angle $\Delta\varepsilon$ that is defined by the directions of the two tangents at M, M_1 by the arc length $MM_1 = \Delta s$. The quotient $\Delta\varepsilon / \Delta s$ will converge to a well-defined, finite limit when M_1 approaches M indefinitely (i.e., as Δs converges to zero), and we assume that it measures the *first curvature* or *flexure* of the curve at M. Denote that limit by $1 / \rho$, and its inverse ρ, which is interpreted as a length, will be called the *radius of first curvature.*

In order to prove the existence of that limit and, at the same time, find its expression, we make use of the following considerations:

Describe a sphere whose center is at the origin and whose radius is unity and intersect its surface with rays that are drawn parallel to the positive directions of the successive tangents to the curve. The locus of the extremes of those rays is called the *spherical indicatrix* of the tangents. Any position of the generator point $M \equiv (x, y, z)$ on the curve

C will correspond to a point $M' \equiv (x', y', z')$ on the spherical indicatrix C' of the tangents, and one will obviously have:

(2) $$x' = \alpha, \qquad y' = \beta, \qquad z' = \gamma.$$

Now, consider a point M_1 on the curve C that is close to M, so the angle $\Delta\varepsilon$ will be measured precisely by the arc length of the maximal circle on the representative sphere that unites the image points M', M_1'. In calculating the limit:

$$\frac{1}{\rho} = \lim_{\Delta s = 0} \left(\frac{\Delta\varepsilon}{\Delta s} \right),$$

one can replace $\Delta\varepsilon$ with the corresponding arc length of the indicatrix, which will be an infinitesimal whose ratio to $\Delta\varepsilon$ will tend to unity as $\Delta\varepsilon$ converges to zero. Let ds' denote the elementary arc length of the spherical indicatrix. One will certainly have:

$$\frac{1}{\rho} = \frac{ds'}{ds}$$

then, so, from (2):

(3) $$\frac{1}{\rho} = \sqrt{\left(\frac{d\alpha}{ds}\right)^2 + \left(\frac{d\beta}{ds}\right)^2 + \left(\frac{d\gamma}{ds}\right)^2}\,.$$

Now, if one assumes that the independent variable is s then, from (1), that formula can be written:

(3*) $$\frac{1}{\rho} = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}\,.$$

In that formula, *we agree to attribute only an absolute value to the first curvature*, so we always choose the positive value of the radical.

One observes immediately that a curve C cannot have zero flexure *along a segment* without being rectilinear along that segment. In fact, from the following equations:

$$\frac{d\alpha}{ds} = 0, \qquad \frac{d\beta}{ds} = 0, \qquad \frac{d\gamma}{ds} = 0,$$

one will get that α, β, γ are constant, and one will then have the formulas:

$$x = \alpha s + a, \quad y = \beta s + b, \quad z = \gamma s + c,$$

with a, b, c constant, which define a straight line.

§ 3.

Osculating plane.

Among all of the planes that pass through the point M of the curve C, there is only one of them that deviates the least from any other plane of the curve C in the vicinity of M, and it is called the *osculating* plane to the curve at M. Indeed, write the equation of any plane that is drawn through $M \equiv (x, y, z)$ in the form:

$$a\,(X - x) + b\,(Y - y) + c\,(Z - z) = 0, \tag{4}$$

in which a, b, c denote the cosines of the normal to the plane. Take the parameter u to be the arc length s of the curve and consider a point M' in the neighborhood of M that corresponds to the value $s + h$ of the arc length, in which h is regarded as a first-order infinitesimal. If Δx, Δy, Δz are the increments that correspond to x, y, z when s passes to $s + h$ then one will have:

$$\delta = a\,\Delta x + b\,\Delta y + c\,\Delta z$$

for the distance δ from the point M' to the plane (4).

One will have:

$$\left\{\begin{aligned} \Delta x &= \frac{dx}{ds}h + \frac{d^2x}{ds^2}\frac{h^2}{2} + \varepsilon_1, \\ \Delta y &= \frac{dy}{ds}h + \frac{d^2y}{ds^2}\frac{h^2}{2} + \varepsilon_2, \\ \Delta z &= \frac{dz}{ds}h + \frac{d^2z}{ds^2}\frac{h^2}{2} + \varepsilon_3, \end{aligned}\right. \tag{a}$$

in which ε_1, ε_2, ε_3 are *third*-order infinitesimals, and therefore:

$$\delta = \left(a\frac{dx}{ds} + b\frac{dy}{ds} + c\frac{dz}{ds}\right) h + \left(a\frac{d^2x}{ds^2} + b\frac{d^2y}{ds^2} + c\frac{d^2z}{ds^2}\right)\frac{h^2}{2} + \eta,$$

in which η is a third-order infinitesimal.

The plane is specified by the conditions:

$$a\frac{dx}{ds} + b\frac{dy}{ds} + c\frac{dz}{ds} = 0,$$
$$a\frac{d^2x}{ds^2} + b\frac{d^2y}{ds^2} + c\frac{d^2z}{ds^2} = 0,$$

the first of which expresses the idea that the given plane passes through the tangent, so it is the one that deviates less than the other planes of the curve in the neighborhood of M.

We have thus proved the existence of the osculating plane, whose equation, from the foregoing, can be written in the form of a determinant ([1]):

$$(5) \qquad \begin{vmatrix} X-x & Y-y & Z-z \\ \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \\ \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \end{vmatrix} = 0,$$

The osculating plane at a point M of the curve is generally unique and well-defined.

One will get exceptions to that for those points at which the three minors of the matrix:

$$\begin{Vmatrix} \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \\ \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \end{Vmatrix}$$

are simultaneously annulled.

However, since the formulas:

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1,$$

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0$$

imply that, from (3^*), the square of that matrix will equal the square of the flexure, one will see that the given singular points will be the ones at which the flexure is zero.

We also point out some other definitions of the osculating plane that always lead to equations (5), as one easily sees.

If one passes a plane through the tangent at M and a neighboring point M' on C then it will tend to the osculating plane as a limit as M' converges to M. Furthermore: If one passes a plane through M and two neighboring points M', M'' then it will tend to the osculating plane when M', M'' converge to M simultaneously (in such a way that the difference between the coordinates of M', M'' do not become higher-order infinitesimals than the corresponding differences with M). From that last property, one can also say,

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([1]) The osculating plane will obviously preserve the same form:

$$\begin{vmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0$$

for an arbitrary independent variable u, as well, in which the primes indicate differentiation with respect to u.

more briefly, that the osculating plane to a point is the plane through that point and two successive points on the curve.

§ 4.

Principal trihedron.

Among all of the normals to a curve through *M*, one calls the one that lies in the osculating plane – i.e., the line of intersection of the osculating plane with the normal plane – the *principal normal*. One then calls the normal to the osculating plane the *binormal* to the curve.

We have already fixed the positive direction of the tangent. We now agree to fix the positive direction of the principal normal and the binormal by a suitable convention. Those three positive directions specify a tri-rectangular trihedron that one calls the *principal trihedron* of the curve relative to the point *M* that one considers.

As a result, always let:

$$\xi, \quad \eta, \quad \zeta$$

denote the (positive) direction cosines of the principal normal, and let:

$$\lambda, \quad \mu, \quad \nu$$

denote those of the binormal. Equation (5) for the osculating plane immediately implies the proportions:

$$\lambda : \mu : \nu = \begin{vmatrix} \frac{dy}{ds} & \frac{dz}{ds} \\ \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \end{vmatrix} : \begin{vmatrix} \frac{dz}{ds} & \frac{dx}{ds} \\ \frac{d^2z}{ds^2} & \frac{d^2x}{ds^2} \end{vmatrix} : \begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} \\ \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} \end{vmatrix},$$

and consequently (since $\xi\alpha + \eta\beta + \zeta\gamma = 0$, $\xi\lambda + \eta\mu + \zeta\nu = 0$):

$$\xi : \eta : \zeta = \frac{d^2x}{ds^2} : \frac{d^2y}{ds^2} : \frac{d^2z}{ds^2},$$

so one will deduce from (3^*) that:

$$(6) \qquad \xi = \pm\rho\,\frac{d^2x}{ds^2}, \quad \eta = \pm\rho\,\frac{d^2y}{ds^2}, \quad \zeta = \pm\rho\,\frac{d^2z}{ds^2}.$$

Now, consider the plane of the tangent and the binormal, whose equation is:

$$(7) \qquad (X - x)\,\xi + (Y - y)\,\eta + (Z - z)\,\zeta = 0,$$

and calculate the distance *d* from that plane to a point M' on the curve that is close to *M*. With the notations of § 3, we will have:

$$\delta = x\,\Delta x + h\,\Delta y + z\,\Delta z,$$

and, as is well-known in analytic geometry, δ will be positive or negative according to whether M' is situated in the positive or negative side of the plane (7), resp.; i.e., in the region towards which the positive direction (ξ, η, ζ) of the normal to the plane points or the opposite one, resp. Now, if one applies formula (*a*), page 4, and observes (6) then one will get:

$$\delta = \pm\frac{h^2}{2\rho} + \eta, \tag{8}$$

in which η denotes a third-order infinitesimal. The sign of δ in this formula will result independently of the sign of h, so one sees that:

In the neighborhood of any point, the curve will lie completely in one part of the plane of the tangent and the binormal ([1]).

One assumes that the positive half of that plane is the one that the curve points to in the neighborhood of M, and consequently it will fix the positive direction of the principal normal. With that convention, the sign of δ in (8) must prove to be positive, and that is why the sign that is adopted in (6) is the upper one. One then has the definitive formulas:

$$\xi = \rho\,\frac{d^2x}{ds^2}, \qquad \eta = \rho\,\frac{d^2y}{ds^2}, \qquad \zeta = \rho\,\frac{d^2z}{ds^2}. \tag{9}$$

In conclusion, we agree to fix the positive direction (λ, μ, ν) of the binormal to be the one that lies with respect to the positive direction of the tangent and principal normal that was fixed already as the positive direction of Oz axis lies with respect to the axes Ox, Oy. The determinant:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \xi & \eta & \zeta \\ \lambda & \mu & \nu \end{vmatrix}$$

of the nine cosines of the three principal direction will prove to be equal to positive unity, and any of its elements will be equal to its algebraic complement. One will then have:

([1]) One must exclude the singular points at which the flexure is zero.

(10)
$$\begin{cases} \lambda = \beta\zeta - \gamma\eta = \rho\left(\dfrac{dy}{ds}\dfrac{d^2z}{ds^2} - \dfrac{dz}{ds}\dfrac{d^2y}{ds^2}\right), \\ \mu = \gamma\xi - \alpha\zeta = \rho\left(\dfrac{dz}{ds}\dfrac{d^2x}{ds^2} - \dfrac{dx}{ds}\dfrac{d^2z}{ds^2}\right), \\ \nu = \alpha\eta - \beta\xi = \rho\left(\dfrac{dx}{ds}\dfrac{d^2y}{ds^2} - \dfrac{dy}{ds}\dfrac{d^2x}{ds^2}\right). \end{cases}$$

§ 5.

Second curvature or torsion

The osculating plane of a curve C will generally vary as the point M of osculation varies, and the rapidity of its deviation – i.e., the wandering of the plane that the curve lies in – measures the *second curvature* or *torsion*.

In order to make that concept more precise, consider a point M on the curve and a neighboring point M_1. The two osculating planes at M, M_1 define a small angle $\Delta\sigma$ between them, and the quotient $\Delta\sigma / \Delta s$, in which Δs denotes the arc length MM_1, will converge to a well-defined, finite limit as Δs tends to zero, which one assume to measure the *torsion* of the curve, and (up to a convenient sign) which one denotes by $1 / T$. Its inverse T is called the *radius of second curvature*. In order to find the expression for $1 / T$, we begin with the observation that the angle Δs is measured by the angle between the two successive binormals at M, M_1 . One can then construct *the spherical indicatrix of the binormals* in a manner that is entirely analogous to § 3, in which the generator point has the coordinates:

$$x_1 = \lambda, \qquad x_2 = \mu, \qquad x_3 = \nu,$$

and one denotes its element of arc length by ds_1:

$$ds_1 = \sqrt{dx_1^2 + dy_1^2 + dz_1^2}\ ,$$

so one will obviously have:

(11)
$$\frac{1}{T} = \pm\frac{ds_1}{ds} = \pm\sqrt{\left(\frac{d\lambda}{ds}\right)^2 + \left(\frac{d\mu}{ds}\right)^2 + \left(\frac{d\nu}{ds}\right)^2}\ .$$

We shall fix the sign of the torsion in the most convenient way in the next §. In the meantime, observe that the only curves with zero torsion are plane curves. In fact, if $1 / T = 0$ then it will follow from (11) that λ, μ, ν are constants. If one then takes, for simplicity, the fixed direction of the binormal to be the z-axis then one will have:

$$\lambda = 0, \quad \mu = 0, \quad \nu = 1,$$

so $\gamma = 0$, and thus z = constant. Hence, the curve will be traced in a plane that is parallel to the xy-plane.

§ 6.

Frenet formulas

We now go on to establish the important formulas that express the derivatives (with respect to arc length) of the nine cosines of the three principal directions in terms of those cosines and the radii ρ, T of the first and second curvature. Three of them will result immediately from (9), § 4 (page 7), which give:

$$(a) \qquad \frac{d\alpha}{ds} = \frac{\xi}{\rho}, \qquad \frac{d\beta}{ds} = \frac{\eta}{\rho}, \qquad \frac{d\gamma}{ds} = \frac{\zeta}{\rho}.$$

Now, if one differentiates the two identities:

$$\begin{aligned} \alpha\lambda + \beta\mu + \gamma\nu &= 0, \\ \lambda^2 + \mu^2 + \nu^2 &= 0, \end{aligned}$$

with respect to s and observes the preceding equations (a) then one will deduce that:

$$\begin{cases} \alpha \dfrac{d\lambda}{ds} + \beta \dfrac{d\mu}{ds} + \gamma \dfrac{d\nu}{ds} = 0, \\ \lambda \dfrac{d\lambda}{ds} + \mu \dfrac{d\mu}{ds} + \nu \dfrac{d\nu}{ds} = 0, \end{cases}$$

and therefore the proportions:

$$\frac{d\lambda}{ds} : \frac{d\mu}{ds} : \frac{d\nu}{ds} = \begin{vmatrix} \beta & \gamma \\ \mu & \nu \end{vmatrix} : \begin{vmatrix} \gamma & \alpha \\ \nu & \lambda \end{vmatrix} : \begin{vmatrix} \alpha & \beta \\ \lambda & \mu \end{vmatrix},$$

i.e.:

$$\frac{d\lambda}{ds} : \frac{d\mu}{ds} : \frac{d\nu}{ds} = \xi : \eta : \zeta .$$

Hence, the three ratios:

$$\frac{1}{\xi}\frac{d\lambda}{ds}, \quad \frac{1}{\eta}\frac{d\mu}{ds}, \quad \frac{1}{\zeta}\frac{d\nu}{ds}$$

will be equal. Their absolute values are obviously equal to the value:

$$\sqrt{\left(\frac{d\lambda}{ds}\right)^2 + \left(\frac{d\mu}{ds}\right)^2 + \left(\frac{d\nu}{ds}\right)^2} = \pm \frac{1}{T}.$$

One fixes the sign of the torsion by assuming that:

$$\frac{1}{T} = \frac{1}{\xi}\frac{d\lambda}{ds} = \frac{1}{\eta}\frac{d\mu}{ds} = \frac{1}{\zeta}\frac{d\nu}{ds},$$

and one will have:

$$(a') \qquad \frac{d\lambda}{ds} = \frac{\xi}{T}, \qquad \frac{d\mu}{ds} = \frac{\eta}{T}, \qquad \frac{d\nu}{ds} = \frac{\zeta}{T}.$$

Hence, the torsion can be associated with not just an absolute value, but also an algebraic value, and it remains for us to examine briefly the geometric circumstances that correspond to the positive or negative sign of torsion.

Complete formulas (a), (a') with ones that relate to the derivatives:

$$\frac{d\xi}{ds}, \frac{d\eta}{ds}, \frac{d\zeta}{ds}.$$

If one observes that one has, e.g.:

$$\xi = \gamma\mu - \beta\nu$$

and differentiates then if one observes (a), (a'), one will get:

$$\frac{d\xi}{ds} = \frac{1}{\rho}(\xi\mu - \eta\nu) + \frac{1}{T}(\gamma\eta - \beta\zeta);$$

i.e.:

$$\frac{d\xi}{ds} = -\frac{\alpha}{\rho} - \frac{\lambda}{T},$$

and analogously for the other two derivatives.

Collecting all of the formulas that were obtained, one will have the following array:

$$(A) \qquad \begin{cases} \dfrac{d\alpha}{ds} = \dfrac{\xi}{\rho}, & \dfrac{d\beta}{ds} = \dfrac{\eta}{\rho}, & \dfrac{d\gamma}{ds} = \dfrac{\zeta}{\rho}, \\[2ex] \dfrac{d\xi}{ds} = -\dfrac{\alpha}{\rho} - \dfrac{\lambda}{T}, & \dfrac{d\eta}{ds} = -\dfrac{\beta}{\rho} - \dfrac{\mu}{T}, & \dfrac{d\zeta}{ds} = -\dfrac{\gamma}{\rho} - \dfrac{\nu}{T}, \\[2ex] \dfrac{d\lambda}{ds} = \dfrac{\xi}{T}, & \dfrac{d\mu}{ds} = \dfrac{\eta}{T}, & \dfrac{d\nu}{ds} = \dfrac{\zeta}{T}. \end{cases}$$

These are the FRENET formulas, which are more commonly known by the name of the SERRET formulas.

§ 7.

Sign of torsion

We now examine the geometric significance that the positive or negative sign of the torsion might have, and from the last line of formula (*A*), it already results independently of the positive sense of traversal along the curve *C*, since changing it to the opposite sense will not alter ξ, η, ζ, but λ, μ, ν will change in sign along with the positive sense of *s*.

Our goal is to calculate (up to third-order infinitesimals) the distance δ from a point M' on the curve close to *M* to the osculating plane at *M* with the equation:

$$(X - x)\,\lambda + (Y - y)\,\mu + (Z - z)\,\nu = 0.$$

With the usual notations of the preceding §§, we will have:

$$\delta = \lambda\,\Delta x + \mu\,\Delta y + \nu\,\Delta z.$$

Now, one has:

$$\begin{cases} \Delta x = \dfrac{dx}{ds}h + \dfrac{d^2x}{ds^2}\dfrac{h^2}{2} + \dfrac{d^3x}{ds^3}\dfrac{h^3}{6} + \varepsilon_1, \\[2ex] \Delta y = \dfrac{dy}{ds}h + \dfrac{d^2y}{ds^2}\dfrac{h^2}{2} + \dfrac{d^3y}{ds^3}\dfrac{h^3}{6} + \varepsilon_2, \\[2ex] \Delta z = \dfrac{dz}{ds}h + \dfrac{d^2z}{ds^2}\dfrac{h^2}{2} + \dfrac{d^3z}{ds^3}\dfrac{h^3}{6} + \varepsilon_3, \end{cases}$$

in which ε_1, ε_2, ε_3 are infinitesimals of order higher than three with respect to *h*. On the other hand, it will result from the Frenet formulas that:

$$\frac{dx}{ds} = \alpha, \qquad \frac{d^2x}{ds^2} = \frac{\xi}{\rho}, \qquad \frac{d^3x}{ds^3} = -\frac{1}{\rho}\left\{\frac{\alpha}{\rho} + \frac{\lambda}{T} + \frac{\xi}{\rho}\frac{d\rho}{ds}\right\},$$

and therefore:

$$\delta = -\frac{1}{6\rho T}h^3 + \eta,$$

in which η / h^3 is infinitesimal with *h*. If one supposes that the two curvatures $1/\rho$, $1/T$ are not zero at *M* then one will see that the sign of δ changes with the sign of *h*; i.e.: *The curve crosses the osculating plane at M*. More precisely, the preceding formula proves that if $1/T > 0$ then the generating point will move in the positive sense of the curve, passing from the positive face of the osculating plane to the negative one, and the opposite will happen when $1/T < 0$. It also results from this that the sign of the torsion is independent of the chosen positive sense along the curve, since reversing it will permute both of the two sides (viz., positive and negative) of the osculating plane.

In order to express our result more concisely, imagine an observer that is located at M on one or the other face of the osculating plane, and looks towards the positive direction of the principal normal. The curve passes from left to right or right to left at M when it *rises* with respect to the observer. In the former case, one says that the curve is *dextrous* at M, while it is *sinistrous* in the latter.

We will now fix, once and for all, that the positive direction of Oy lies to the left of Ox on the positive half of the xy-plane and immediately see that:

The torsion of a curve C, as calculated from the Frenet formulas, will be positive or negative according to whether the curve C is sinistrous or dextrous at the point considered, resp.

§ 8.

Intrinsic equations

We can immediately apply the Frenet formulas to the proof of an important theorem:

A skew curve C is determined completely by the form of the expressions for the two curvatures $1 / \rho$, $1 / T$ *as functions of the arc length.*

In other words, we say that if two curves C, C' of equal arc length have equal flexures, as well as torsions, then they can be superimposed. If we denote the quantities that relate to C' by primes then we will have, at the same time:

$$s' = s, \qquad \rho' = \rho, \qquad T' = T.$$

Now, move the curve C' in space in such a way that one of its points (i.e., the origin $s = 0$ of the arc length) overlaps with the corresponding point of C, and simultaneously its principal trihedron overlaps with that of C at the same point $s = 0$. One will then have:

$$\left.\begin{array}{lll} \alpha' = \alpha, & \beta' = \beta, & \gamma' = \gamma, \\ \xi' = \xi, & \eta' = \eta, & \zeta' = \zeta, \\ \lambda' = \lambda, & \mu' = \mu, & \nu' = \nu \end{array}\right\} \qquad \text{for } s = 0.$$

Write down the three Frenet formulas in the first column of the matrix (A) (page 10) for both the curve C and the curve C' :

$$\frac{d\alpha}{ds} = \frac{\xi}{\rho}, \qquad \frac{d\xi}{ds} = -\frac{\alpha}{\rho} - \frac{\lambda}{T}, \qquad \frac{d\lambda}{ds} = \frac{\xi}{T},$$
$$\frac{d\alpha'}{ds} = \frac{\xi'}{\rho}, \qquad \frac{d\xi'}{ds} = -\frac{\alpha'}{\rho} - \frac{\lambda'}{T}, \qquad \frac{d\lambda'}{ds} = \frac{\xi'}{T}.$$

Multiply the first three by α', ξ', λ', as usual, the second one by α, ξ, λ, and sum, and the right-hand side will be zero, so it will follow that:

$$\frac{d}{ds}(\alpha\alpha' + \xi\xi' + \lambda\lambda') = 0\;;$$

i.e.:

$$\alpha\alpha' + \xi\xi' + \lambda\lambda' = \text{constant}.$$

However, since the value of the left-hand side is unity initially (i.e., $s = 0$), one will have that for any value of s:

$$\alpha\alpha' + \xi\xi' + \lambda\lambda' = 1,$$

and due to the identities:

$$\alpha^2 + \xi^2 + \lambda^2 = 1,$$
$$\alpha'^2 + \xi'^2 + \lambda'^2 = 1,$$

that formula can be written:

$$(\alpha - \alpha')^2 + (\xi - \xi')^2 + (\lambda - \lambda')^2 = 0,$$

which implies that:

$$\alpha = \alpha', \qquad \xi = \xi', \qquad \lambda = \lambda'.$$

One similarly deduces that:

$$\beta = \beta', \qquad \eta = \eta', \qquad \mu = \mu'.$$
$$\gamma = \gamma', \qquad \zeta = \zeta', \qquad \nu = \nu',$$

and therefore:

$$\frac{d(x' - x)}{ds} = 0, \qquad \frac{d(y' - y)}{ds} = 0, \qquad \frac{d(z' - z)}{ds} = 0.$$

The differences $x' - x$, $y' - y$, $z' - z$ are then constants, and if they are initially zero then they will always be zero, which proves our theorem.

If the expressions $1 / \rho$, $1 / T$ are known as functions of s:

$$\frac{1}{\rho} = f(s), \qquad \frac{1}{T} = \varphi(s)$$

(in which, by the conventions that were made, the function $f(s)$ must always be positive) then the form of the curve will be specified without regard to its particular position in space. That is why the equations above are appropriately called the *intrinsic equations* of the curve.

§ 9.

Integration of the intrinsic equations

On the basis of the theorems that assure the existence of integrals of differential equations, one will easily see that:

If one is given the intrinsic equations of a curve:

$$\frac{1}{\rho} = f(s), \qquad \frac{1}{T} = \varphi(s)$$

arbitrarily then the corresponding curve will actually exist.

If one denotes three unknown functions of s by l, m, n then one can, in fact, write down the system of three homogeneous, linear equations:

$$\frac{dl}{ds} = m\,f(s), \qquad \frac{dm}{ds} = -\,l\,f(s) - n\,\varphi(s), \qquad \frac{dn}{ds} = m\,\varphi(s), \tag{12}$$

for which there will exist precisely three systems of integrals:

$$(\alpha, \xi, l), \qquad (\beta, \eta, \mu), \qquad (\gamma, \zeta, \nu)$$

from the Frenet formulas. One knows from the theory of differential equations that if one is given the initial values of the unknown functions l, m, n arbitrarily (e.g., for $s = 0$) then there will exist a system of integrals (l, m, n) of (12) that will reduce to the initially-assigned system (l_0, m_0, n_0). Moreover, if (12) are linear then the integrals l, m, n will exist and will be regular in all of the interval of s for which the assigned functions $f(s)$, $\varphi(s)$ are kept finite and continuous.

One further observes that if (l, m, n), (l', m', n') are two systems of integrals of (12), whether distinct or coincident, then it will result from the same differential equations that:

$$\frac{d}{ds}(ll' + mm' + nn') = 0,$$

and therefore:

$$ll' + mm' + nn' = \text{constant}.$$

Having assumed that, take nine constants:

$$\begin{matrix} l_0 & l'_0 & l''_0 \\ m_0 & m'_0 & m''_0 \\ n_0 & n'_0 & n''_0 \end{matrix}$$

which define the coefficients of an orthogonal substitution, and let:

$$(l, m, n), \qquad (l', m', n'), \qquad (l'', m'', n'')$$

denote three systems of integrals of (12), which will reduce to:

$$(l_0, m_0, n_0), \qquad (l'_0, m'_0, n'_0), \qquad (l''_0, m''_0, n''_0),$$

respectively, for $s = 0$.

It results from the observation above that *for all values of s,* one will have:

$$\begin{matrix} l & l' & l'' \\ m & m' & m'' \\ n & n' & n'' \end{matrix}$$

for the coefficients of an orthogonal substitution, and in particular, one will have:

$$l^2 + l'^2 + l''^2 = 1.$$

Then set:

$$x = \int l\, ds, \qquad y = \int l' ds, \qquad z = \int l''\, ds,$$

and interpret x, y, z as coordinates of a moving point M. The curve that is the locus of points M will obviously have s for its arc length and l, l', l'' for the direction cosines of the tangent. If, in addition, one takes the differential equations (12) into account, which are satisfied by (l, m, n), (l', m', n'), (l'', m'', n''), along with the Frenet formulas then one will see immediately that the two curvatures $1 / r$, $1 / T$ of the curve will have precisely the assigned values:

$$\frac{1}{\rho} = f(s), \qquad \frac{1}{T} = \varphi(s).$$

The last one shows, as Darboux did, that the integration of the system (12) will reduce to that of a differential equation of Ricatti type. For any system of integrals of (12), one will have:

$$l^2 + m^2 + n^2 = \text{constant},$$

and if one multiplies l, m, n by the same constant factor (which will produce a new system of integrals, due to homogeneity) then one can certainly suppose that:

$$l^2 + m^2 + n^2 = 1.$$

One then expresses l, m, n in terms of two angles θ, φ by the formula:

$$l = \sin\theta\cos\varphi, \qquad m = \sin\theta\sin\varphi, \qquad n = \cos\theta$$

and (12) will give the two simultaneous equation for θ, φ:

(13) $$\frac{d\theta}{ds}+\frac{\sin\varphi}{T} = 0, \qquad \frac{d\varphi}{ds}+\frac{\cot\theta\cos\varphi}{T}+\frac{1}{\rho} = 0.$$

Then introduce the complex function:

$$\sigma = \cot\frac{\theta}{2}\, e^{i\varphi}$$

as the single unknown [which we will recognize later on (Chap. III) to have the significance of the *complex variable* on the sphere]. The single equation for σ ([1]):

(14) $$\frac{d\sigma}{ds}+\frac{i\sigma^2}{2T}+\frac{i\sigma}{\rho}-\frac{i}{2T} = 0$$

will follow from (13), and conversely (13) will follow when one separates the real part from the imaginary:

The problem of determining a curve from its intrinsic equations reduces to the integration of equation (14), *which has Ricatti type.*

From known properties of equations of that type, knowing a particular solution is enough to give the general integral by quadrature.

As an example, consider the case of a *planar* curve, which is determined by its intrinsic equations:

$$\frac{1}{\rho} = f(s), \qquad \frac{1}{T} = 0.$$

(14) or (13) can then be integrated immediately. It is clear that θ will have a constant value, and with no loss of generality, one can take $\theta = \pi / 2$. One will then obtain φ from the quadrature:

(15) $$\varphi = -\int f(s)\, ds\,,$$

so one will finally get:

(15*) $$x = a + \int \cos\varphi\, ds, \qquad y = b + \int \sin\varphi\, ds, \qquad z = c$$

for the required curve, in which a, b, c are constants. If one then supposes that $f(s) = k / s$ (k constant) – i.e., one demands that the planar curve must have its radius of curvature proportional to the arc length – then it will follow immediately from (15*) that the required curve is a logarithmic spiral.

([1]) More simply, one calculates the logarithmic derivative of σ, while observing (13), and replaces sin φ, cos φ with their expressions as exponentials.

§ 10.

Cylindrical helices

We once more apply the Frenet formulas to the study of an important class of curves that are known by the name of *cylindrical helices*. One gives that name to the curves that are traced on an arbitrary cylindrical surface and cut the generators at a constant angle. If one rolls the cylindrical surface out onto a plane then a helix will turn into a straight line, and since the linear lengths will not be altered under that process, it will follow that another characteristic property of cylindrical helices consists of tracing out the shortest path between two points on a cylinder.

Locate the z-axis parallel to the generators of the cylinder, and one will get:

$$\gamma = \text{constant}$$

as a consequence.

It will then follow from the Frenet formulas:

$$\frac{d\gamma}{ds} = \frac{\zeta}{\rho}, \qquad \frac{d\zeta}{ds} = -\frac{\gamma}{\rho} - \frac{\nu}{T}, \qquad \frac{d\nu}{ds} = \frac{\zeta}{T}$$

that ([1]):

$$\zeta = 0, \quad \nu = \text{constant}, \quad \frac{\rho}{T} = -\frac{\gamma}{\nu} = \text{constant}.$$

One will then have:

1. *The principal normal at any point of a cylindrical helix will coincide with the normal to the cylinder.*

That property is obviously characteristic of the helix.

2. *The ratio of the two curvatures is constant for any cylindrical helix.*

By a theorem of Bertrand, the latter property is invertible:

Any curve that has a constant ratio of its two curvatures will be a cylindrical helix.

In order to prove that, suppose that one has:

$$\frac{\rho}{T} = k$$

for a curve C, in which k is constant. One deduces from the Frenet formulas that:

([1]) One intends that the case $1 / \rho = 0$, which pertains to just the straight line, should be excluded.

$$\frac{d\lambda}{ds} = \frac{\rho}{T}\frac{d\alpha}{ds}, \qquad \frac{d\mu}{ds} = \frac{\rho}{T}\frac{d\beta}{ds}, \qquad \frac{d\nu}{ds} = \frac{\rho}{T}\frac{d\gamma}{ds},$$

so:

$$\frac{d}{ds}(\lambda - k\alpha) = 0, \qquad \frac{d}{ds}(\mu - k\beta) = 0, \qquad \frac{d}{ds}(\nu - k\gamma) = 0,$$

which integrates to:

$$\lambda - k\alpha = A, \qquad \mu - k\beta = B, \qquad \nu - k\gamma = C,$$

in which A, B, C are three constants, for which, the sum of the squares will obviously be:

$$A^2 + B^2 + C^2 = 1 + k^2.$$

Then set:

$$a = \frac{A}{\sqrt{A^2 + B^2 + C^2}}, \quad b = \frac{B}{\sqrt{A^2 + B^2 + C^2}}, \quad c = \frac{C}{\sqrt{A^2 + B^2 + C^2}},$$

so a, b, c will be the cosines of a fixed direction in space, and from the foregoing, one will deduce that:

$$a\,\alpha + b\,\beta + c\,\gamma = -\frac{k}{\sqrt{1+k^2}}.$$

The tangents to the curve C then define a constant angle with that fixed direction, and therefore C is an ellipse on the cylinder that is defined by drawing parallels to that fixed direction through the points of C.

§ 11.

Special cylindrical helices

In order to establish the general formulas that relate to cylindrical helices, one takes the z-axis to be parallel to the generators of the cylinder, and supposes that the coordinates x, y of a point on the cross-section $z = 0$ of the cylinder are expressed as functions of the arc length u of that section by the formulas:

$$x = x(u), \qquad y = y(u).$$

One will see immediately that one will have ([1]):

$$x = x(u), \qquad y = y(u), \qquad z = u \cot \varepsilon,$$

([1]) For simplicity, one measures the arc length u from the point where the cross-section meets the helix.

in which ε denotes the constant angle of inclination (which one can assume to be acute) of the helix with respect to the generators of the cylinder.

If one applies the formulas of the preceding §§ then one will find that ([1]):

$$ds = \frac{du}{\sin\varepsilon}, \qquad s = \frac{u}{\sin\varepsilon},$$

$$\alpha = \sin\varepsilon\, x'(u), \qquad \beta = \sin\varepsilon\, y'(u), \qquad \gamma = \cos\varepsilon,$$

$$\frac{d\alpha}{ds} = \sin^2\varepsilon\, x''(u), \qquad \frac{d\beta}{ds} = \sin^2\varepsilon\, y''(u), \qquad \frac{d\gamma}{ds} = 0,$$

in which the primes indicate derivatives with respect to *u*. It will then follow that the flexure of the helix is:

$$\frac{1}{\rho} = \sin^2\varepsilon \sqrt{x''^2(u) + y''^2(u)},$$

or

(15)
$$\frac{1}{\rho} = \frac{\sin^2\varepsilon}{R},$$

in which $1 / R$ is the curvature of the cross-section. One finds the formula:

$$\frac{\rho}{T} = -\frac{\gamma}{\nu},$$

for the torsion $1 / T$, and if one observes that $\nu = \pi / 2 \pm \varepsilon$:

(16)
$$\frac{1}{T} = \pm\frac{\sin\varepsilon\cos\varepsilon}{R}.$$

The upper sign is true for left-wound helices, and the lower one for right-wound, which will result from the criteria that were established in § 4 or also from direct geometric considerations.

It follows from that formula that the radii of flexure and torsion are constants only for helices on right circular cylinders.

Such a helix is called *circular* ([2]), and its characteristic property (according to Puiseaux) of having two constant radii of curvature corresponds to the property that it has in common with lines and circles (which would follow from the fundamental theorem in § 8) that any of its subsets can be superimposed with itself.

([1]) One measures the arc length *s* by starting with the origin in the cross-section $z = 0$.

([2]) It can be considered to be generated by a point that slides in a uniform motion along a generator of the circular cylinder while it rotates in a uniform motion around the axis.

One sees from formulas (15), (16) that the problem of determining a helix from its intrinsic equations reduces to the analogous problem for the cross-section of the cylinder. Hence ([1]):

If one is given the intrinsic equations of a cylindrical helix then one can find it in finite terms by quadrature.

We apply those considerations to a second helix, which merits special mention, after the circular helix, and which is called the *cylindro-conical helix.*

One defines such a curve by the intrinsic equations:

$$\rho = as, \qquad T = bs,$$

with a, b constants. The cross-section of the cylinder on which that helix is described will have a radius of curvature that is proportional to the arc length, and from the observation at the end of § 9, it will then be a logarithmic spiral. It then follows that the equations of our helix can be put into the form:

$$x = A\, e^{kt} \cos t, \qquad y = A\, e^{kt} \sin t, \qquad z = B\, e^{kt},$$

in which t is the variable parameter that specifies the points on the curve, and A, B, h are constants. The helix is then traced on the surface:

$$x^2 + y^2 - \frac{A^2}{B^2} z^2 = 0,$$

which is a cone of revolution around the z-axis with its vertex at the origin. It cuts the generators of the cone at a constant angle; i.e., it is a *loxodrome* of the cone ([2]). Indeed, one finds that the direction cosines of the tangent to the helix are:

$$\alpha = \frac{A(h\cos t - \sin t)}{\sqrt{A^2 + h^2(A^2+B^2)}}, \qquad \beta = \frac{A(h\sin t + \cos t)}{\sqrt{A^2 + h^2(A^2+B^2)}}, \qquad \gamma = \frac{Bh}{\sqrt{A^2 + h^2(A^2+B^2)}},$$

and those of a generator of the cone are:

$$a = \frac{A\cos t}{\sqrt{A^2+B^2}}, \qquad b = \frac{A\sin t}{\sqrt{A^2+B^2}}, \qquad c = \frac{B}{\sqrt{A^2+B^2}},$$

([1]) The same thing will also follow immediately from the general observations of § 9, since one immediately knows *two* constant particular solutions of the fundamental Ricatti equation (14) in which T / ρ is constant (= k) that are the roots of the second-degree equation $\sigma^2 + 2k\,\sigma - 1 = 0$, namely:

$$\sigma_1 = -k + \sqrt{k^2+1}, \qquad \sigma_2 = -k - \sqrt{k^2+1}.$$

([2]) A line on an arbitrary surface of revolution that cuts the meridians at constant angles is called a *loxodrome*.

so

$$a\,\alpha + b\,\beta + c\,\gamma = \frac{h\sqrt{A^2+B^2}}{\sqrt{A^2+h^2(A^2+B^2)}} = \text{constant}.$$

One derives the name of *cylindro-conic helix* from that property ([1]).

§ 12.

Enveloping surface

It is useful to prefix the study of the ultimate properties of skew curves with some brief notions on *enveloping surfaces.*

Let:

(17) $$f(x, y, z, \alpha) = 0$$

be the equation of a surface that contains an arbitrary parameter α, which one assumes to vary continuously within an assigned interval. In addition, suppose that the function f of x, y, z, α is finite and continuous in the domain of variability that one assumes for those four variables, and assume that the partial derivatives:

$$\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z},\ \frac{\partial f}{\partial \alpha},\ \frac{\partial^2 f}{\partial \alpha^2}$$

are also finite and continuous. Any special value α_1 of α corresponds to a particular surface of the system ∞^1 (17). If α varies continuously then that surface will deform continuously in space.

Now, consider a special surface:

(18) $$f(x, y, z, \alpha_1) = 0$$

of the system and a neighborhood that corresponds to a variation h of the parameter:

$$f(x, y, z, \alpha_1 + h) = 0.$$

As h converges to zero, the intersection curve of those two surfaces (real or imaginary) will converge to a limiting position in the surface (18) that one calls (with Monge) the *characteristic* of the surface (18). In order to prove the existence of that limiting curve, one replaces the preceding two equations with the equivalent system:

$$f(x, y, z, \alpha_1) = 0, \qquad \frac{f(x,y,z,\alpha_1+h) - f(x,y,z,\alpha_1)}{h} = 0.$$

([1]) Observe that if one rolls the cone into a plane then the cylindro-conic helix will turn into a logarithmic spiral.

The second of these will converge to the limiting equation:

$$\left(\frac{\partial f}{\partial \alpha}\right)_{\alpha=\alpha_1} = 0$$

as h converges to zero, and one can prove rigorously that the curve is determined by the two simultaneous equations:

$$f(x, y, z, \alpha_1) = 0, \qquad \left(\frac{\partial f(x,y,z,\alpha)}{\partial \alpha}\right)_{\alpha=\alpha_1} = 0$$

and is precisely the limiting curve that is sought. The locus of all characteristics is a surface that takes the name of envelope, while any single surface of the system (17) is called an *envelopment*. The equation of the enveloping surface is obtained, from the foregoing, by eliminating α from the two equations:

$$f = 0, \quad \frac{\partial f}{\partial \alpha} = 0,$$

or, what amounts to the same thing, by deducing the value of α as a function of x, y, z from the second equation and substituting it in the first one.

Therefore, the equation:

$$f(x, y, z, \alpha) = 0,$$

which gives an envelopment for α constant, also represents the envelope when one replaces α with the function of x, y, z that one obtains by solving:

$$\frac{\partial f}{\partial \alpha} = 0$$

for α.

After that, one will see immediately that:

Any envelopment will touch the envelope along the entire characteristic.

In fact, the equation of the tangent plane to the envelope at a point (x, y, z) is:

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial x}\right)(X - x) + \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial y}\right)(Y - y) + \left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial z}\right)(Z - z) = 0.$$

However, since α makes $\frac{\partial f}{\partial \alpha} = 0$, what will remain is identical to the equation:

$$\frac{\partial f}{\partial x}(X-x)+\frac{\partial f}{\partial y}(Y-y)+\frac{\partial f}{\partial z}(Z-z)=0$$

of the tangent plane to the envelopment at (x, y, z). Q. E. D.

The characteristic of the surface (18) meets the neighboring surface:

$$f(x, y, z, \alpha_1 + h) = 0$$

at a certain discrete number of points, and when one varies h, they will move along the characteristic, and as h converges to zero, they will tend to certain limiting points that are determined in the following way:

The three simultaneous equations:

$$(a) \qquad f = 0, \quad \frac{\partial f}{\partial \alpha} = 0, \qquad f(x, y, z, \alpha + h) = 0$$

exist for each of the given points of intersection, the last of which can be replaced with:

$$f + h\frac{\partial f}{\partial \alpha} + \frac{h^2}{2}\frac{\partial^2 f}{\partial \alpha^2} + \eta = 0,$$

in which h is an infinitesimal of order higher than two with respect to h. The system (a) can be replaced with the equivalent one:

$$f = 0, \quad \frac{\partial f}{\partial \alpha} = 0, \quad \frac{\partial^2 f}{\partial \alpha^2} + \frac{2\eta}{h^2} = 0.$$

Now, the last of these will converge to the limiting equation:

$$\frac{\partial^2 f}{\partial \alpha^2} = 0$$

when h tends to zero.

The limiting points on the characteristic that we seek, which correspond to value α of the parameter, are then determined from three simultaneous equations:

$$f = 0, \quad \frac{\partial f}{\partial \alpha} = 0, \quad \frac{\partial^2 f}{\partial \alpha^2} = 0.$$

The locus of limit points of the various characteristics takes the name of the *edge of regression* on the enveloping surface. Its equations are obtained by eliminating α from the three above, or deducing α as a function of x, y, z from the third of them and substituting it in the first two. Since the first two represent the characteristic when α is constant, it will follow that any characteristic touches the edge of regression at the limit points ([1]). In summary, the edge of regression (when it exists) is the envelope of all characteristics on the enveloping surface.

§ 13.

Channel surface

As a first example, consider a sphere of constant radius a that moves in space and assumes a simple infinitude of positions and look for its enveloping surface. In order to define the system of ∞^1 spheres, it is enough to give the curve C that is the locus of centers, and we revert to the usual notations for it. The equation of the moving sphere will be:

$$(X-x)^2 + (Y-y)^2 + (Z-z)^2 = a^2, \tag{19}$$

and the variable parameter will presently be the arc length s of the curve C. In order to find the characteristic on the sphere (19), we agree to associate that equation with the one that is obtained by taking the first derivative with respect to s – i.e.:

$$(X-x)\,\alpha + (Y-y)\,\beta + (Z-z)\,\gamma = 0, \tag{20}$$

which is the equation of the normal plane to the curve C. Hence:

The characteristic of the moving sphere is its great circle that lies in plane that is normal to the curve C at the center.

The locus of that moving circle with constant radius a whose center describes the curve C and whose plane remains constantly normal to that curve is then the desired envelope. Such a surface is called a *channel surface.*

If we would now like to determine the enveloping channel surface to the edge of regression then we must differentiate (20) again and associate the new equation that is thus-obtained:

$$(X-x)\,\xi + (Y-y)\,\eta + (Z-z)\,\zeta = r \tag{21}$$

with (19), (20). Now, (21) is nothing but the equation of the normal plane to the principal normal that goes through that point of that normal that is at a distance of $M \equiv (x, y, z)$ and

---

([1]) If dx, dy, dz denote differentials that are taken along the edge of regression then one will have the equations:

$$\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}+\frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z}+\frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial z}\right) dz = 0,$$

$$\left(\frac{\partial^2 f}{\partial \alpha\partial x}+\frac{\partial^2 f}{\partial \alpha^2}\frac{\partial \alpha}{\partial x}\right) dx + \left(\frac{\partial^2 f}{\partial \alpha\partial y}+\frac{\partial^2 f}{\partial \alpha^2}\frac{\partial \alpha}{\partial y}\right) dy + \left(\frac{\partial^2 f}{\partial \alpha\partial z}+\frac{\partial^2 f}{\partial \alpha^2}\frac{\partial \alpha}{\partial z}\right) dz = 0,$$

which will reduce to the ones that determine the differentials that are taken along the characteristic when one sets $\frac{\partial f}{\partial \alpha} = 0$, $\frac{\partial^2 f}{\partial \alpha^2} = 0$.

whose positive part is precisely at a length that is equal to the radius ρ of first curvature (i.e., the center of curvature). The limiting points on the characteristic circle are then real and distinct when $\rho < a$, real and coincident when $\rho = a$, and imaginary when $\rho > a$.

The edge of regression on the channel surface will then be real and will consist of two distinct sheets for the region of the curve C for which $a > \rho$.

§ 14.

Developable surface

For the theory of skew curves, one can consider exclusively the case in which the simple infinitude of enveloping surfaces is composed of planes, in which case the enveloping surface will take on the name of *developable* for reasons that we shall now explain.

The characteristic of any plane of the system will obviously be a line, and all of the characteristic lines will be the tangents to the edge of regression. The developable is then the locus of the tangents to a curve that is the edge of regression. It is easy to see that the moving plane (viz., the envelopment) coincides with the osculating plane to the edge of regression. Indeed, if one retains the usual notations for that curve then the equation of the osculating plane will be:

$$(X - x)\,\lambda + (Y - y)\,\mu + (Z - z)\,\nu = 0,$$

and the characteristic on the osculating plane will be obtained by associating that equation with the one that one obtains by differentiating with respect to s; i.e.:

$$(X - x)\,\xi + (Y - y)\,\eta + (Z - z)\,\zeta = 0.$$

Now, the intersection of those two planes is precisely the tangent.

Moreover, observe that the edge of regression can reduce to a point, and then the developable will become a cone or a cylinder according to whether that point is at a finite distance or an infinite one.

Any enveloping plane touches the developable along the entire characteristic line (i.e., generator) and therefore the tangent planes to a developable constitute a simple infinitude, while for any other surface, the tangent planes will define a double infinitude ([1]).

The name of developable derives from the fact that if one assumes that the surface is flexible and inextensible then one can roll it out onto a plane without tearing or folding it. Conversely, any surface that is endowed with that property is necessarily a developable, as we will prove later.

One can consider three developables in relation to any skew curve that correspond to the three faces of the principal trihedron. The envelope of the osculating plane is nothing

([1]) According to the duality principle, any surface will correspond to a surface, except that a developable will correspond to a curve.

but the locus of the tangents to the given curve, as we saw above. The envelope of the normal planes bears the name of *polar developable* of the curve C, and the envelope of the normal planes to the principal normals bears that of *rectifying developable*.

§ 15.

Rectifying developable

We now treat those two developables, beginning with the second one.

The moving plane has the equation:

$$(X-x)\,\xi+(Y-y)\,\eta+(Z-z)\,\zeta=0, \tag{22}$$

and its characteristic is obtained by associating the equation that results from differentiating with respect to the parameter s:

$$(X-x)\left(\frac{\alpha}{\rho}+\frac{\lambda}{T}\right)+(Y-y)\left(\frac{\beta}{\rho}+\frac{\mu}{T}\right)+(Z-z)\left(\frac{\gamma}{\rho}+\frac{\nu}{T}\right)=0. \tag{23}$$

That second plane also passes through the point $M \equiv (x, y, z)$ of the curve C, and the characteristic is then a line that emanates from the point M of C and lies in the plane of the tangent and binormal; it is called the *rectifying* line. Our developable Σ will then pass through the curve C, and the principal normal to C will coincide with normal to the surface Σ.

Now, when a curve that is traced on a surface has a principal normal that coincides with the normal to the surface at any point, that curve will take the name of *geodetic line* on the surface. It indicates the shortest path (as we will prove much later) that one can follow on the surface from one of its points to another. The rectifying developable then contains the given curve C as a geodetic line. If one rolls out the developable onto a plane then the curve C will rectify in it, which is precisely why one says rectifying developable. It is clear that there is no other developable that contains the curve as a geodetic line.

The rectifying line that is the intersection of the two planes (22), (23) obviously has direction cosines that are proportional to the three binomials:

$$\frac{\alpha}{T}-\frac{\lambda}{\rho}, \qquad \frac{\beta}{T}-\frac{\mu}{\rho}, \qquad \frac{\gamma}{T}-\frac{\nu}{\rho},$$

and therefore, if one lets σ denote the angle of inclination of the rectifying line with respect to the tangent then one can set:

$$\cos\sigma = \frac{1/T}{\sqrt{\dfrac{1}{\rho^2}+\dfrac{1}{T^2}}}, \quad \sin\sigma = \frac{1/\rho}{\sqrt{\dfrac{1}{\rho^2}+\dfrac{1}{T^2}}},$$

i.e., one will have:

$$\tan\sigma = \frac{T}{\rho}. \tag{24}$$

As one sees, that angle σ is constant only for the cylindrical helix, in which case the rectifying developable is nothing but the cylinder on which the helix is described, and the rectifying lines will coincide with the generators.

If one differentiates (23) once more then one will obtain the elements that define the edge of regression of the rectifying developable. However, the inverse problem is more interesting:

Find all of the curves C_1 for which a given curve C is the edge of regression of the rectifying developable,

or, in other words:

Find all of the geodetic lines on a given developable surface.

That problem proves to be easy to solve by quadrature.

Indeed, if we let x_1, y_1, z_1 denote the coordinates of a point M_1 of the desired curve C_1 that is located in the tangent at $M \equiv (x, y, z)$ to the given curve C then we will have:

$$x_1 = x + t\,\alpha, \quad y_1 = y + t\,\beta, \quad z_1 = z + t\,\gamma, \tag{25}$$

in which t denotes the (algebraic) value of the segment MM_1. Our problem will then be that of determining t as a function of s in such a way that the curve C_1, which is described by the point (x_1, y_1, z_1) that is defined by (25) will have the direction (λ, μ, ν) of the binormal to the original curve for its principal normal. Now, if we differentiate (25) then we will get:

$$\frac{dx_1}{ds} = \left(1+\frac{dt}{ds}\right)\alpha + t\,\frac{\xi}{\rho}, \qquad \frac{dy_1}{ds} = \left(1+\frac{dt}{ds}\right)\beta + t\,\frac{\eta}{\rho},$$
$$\frac{dz_1}{ds} = \left(1+\frac{dt}{ds}\right)\gamma + t\,\frac{\zeta}{\rho},$$

and then, if we denote the elements that relate to the curve C_1 by the subscript 1 then:

$$\frac{ds_1}{ds} = \sqrt{\frac{t^2}{\rho^2}+\left(1+\frac{dt}{ds}\right)^2}$$

$$\alpha_1 = \alpha\cos\sigma + \xi\sin\sigma, \quad \beta_1 = \beta\cos\sigma + \eta\sin\sigma, \quad \gamma_1 = \gamma\cos\sigma + \zeta\sin\sigma, \tag{26}$$

in which, we have set:

$$\cos\sigma = \frac{1+\dfrac{dt}{ds}}{\sqrt{\dfrac{t^2}{\rho^2}+\left(1+\dfrac{dt}{ds}\right)^2}}, \qquad \sin\sigma = \frac{\dfrac{t}{\rho}}{\sqrt{\dfrac{t^2}{\rho^2}+\left(1+\dfrac{dt}{ds}\right)^2}};$$

i.e.:

(26*) $$\cot\sigma = \frac{\rho}{t}\left(1+\frac{dt}{ds}\right),$$

and σ then signifies the angle of inclination of C_1 with respect to the tangent to C.

Differentiate (26) once more and obtain a quantity in the left-hand side that is proportional to ξ_1, η_1, ζ_1; i.e., by hypothesis, to λ, μ, ν. One must then have:

(α) $$\frac{d}{ds}(\alpha\cos\sigma + \xi\sin\sigma) = K\lambda, \qquad \frac{d}{ds}(\beta\cos\sigma + \eta\sin\sigma) = K\mu,$$
$$\frac{d}{ds}(\gamma\cos\sigma + \zeta\sin\sigma) = K\nu,$$

in which K denotes a proportionality factor.

If one performs the differentiations then one will find that:

(β) $$\begin{cases} -\sin\sigma\left(\dfrac{d\sigma}{ds}+\dfrac{1}{\rho}\right)\alpha+\cos\sigma\left(\dfrac{d\sigma}{ds}+\dfrac{1}{\rho}\right)\xi-\dfrac{\sin\sigma}{T}\lambda = K\lambda, \\ -\sin\sigma\left(\dfrac{d\sigma}{ds}+\dfrac{1}{\rho}\right)\beta+\cos\sigma\left(\dfrac{d\sigma}{ds}+\dfrac{1}{\rho}\right)\eta-\dfrac{\sin\sigma}{T}\mu = K\mu, \\ -\sin\sigma\left(\dfrac{d\sigma}{ds}+\dfrac{1}{\rho}\right)\gamma+\cos\sigma\left(\dfrac{d\sigma}{ds}+\dfrac{1}{\rho}\right)\zeta-\dfrac{\sin\sigma}{T}\nu = K\nu, \end{cases}$$

and one must then have:

$$\frac{d\sigma}{ds} = -\frac{1}{\rho},$$

so one will obtain s by a quadrature:

(27) $$\sigma = c - \int\frac{ds}{\rho},$$

and, in turn, one will obtain t from (26*) by integrating a linear, first-order differential equation, which will give:

(28) $$t = \frac{1}{\sin\sigma}(c' - \int\sin\sigma\,ds).$$

Conversely, if one assumes that σ, t are given by those formulas then they will satisfy all of the required conditions.

Hence, one will solve the problem with two quadratures, which will introduce two arbitrary constants c, c'. One concludes that:

If one is given a developable surface then two quadratures (27), (28) *will suffice to find the double infinitude of its geodetic lines in finite terms.*

One should observe that (27), (28), which do not contain $1 / T$, depend upon only the flexure $1 / \rho$ of the edge of regression C of the developable. Hence, if one deforms the curve C anyway by twisting it without altering the flexure then the values (28) of t will always remain the same. Now, consider any two of those forms for C – say, C, C' – so one can roll the two corresponding developables Σ, Σ' onto each other in such a way that C, C' overlap at corresponding points, and the generators of Σ roll onto the corresponding ones of Σ' ([1]). With that, the observation above will take on the obvious significance that:

Any geodetic of Σ *will roll onto a geodetic of* Σ'.

That is a special case of a general theorem that we shall encounter in our studies later on.

In particular, if one takes C' to be the planar transform of C then the developable Σ' will become the plane of the curve, and its geodetics will be nothing but the lines in the plane. Furthermore, that will result analytically from (α), (β), page 28. In fact, (β) will give $K = 0$ for $1 / T = 0$, and therefore (α) will imply that:

$$\frac{d\alpha_1}{ds} = \frac{d\beta_1}{ds} = \frac{d\gamma_1}{ds} = 0.$$

Thus, any curve C_1 that has a fixed direction for its tangents is a straight line.

§ 16.

Polar developable

We now go on to treat the envelope of the normal planes to a curve C, and start with the equation of the normal plane:

(29) $$(X - x)\,\alpha + (Y - y)\,\beta + (Z - z)\,\gamma = 0,$$

which will give:

(30) $$(X - x)\,\xi + (Y - y)\,\eta + (Z - z)\,\zeta = \rho$$

([1]) Geometrically, it is obvious that one can consider any developable as being composed of successive strips that are included between two consecutive generators. The indicated deformation will rotate each strip around the generator that is common to the preceding one through a suitable angle; hence that will obviously not alter the flexure of the edge of regression.

when it is differentiated with respect to *s*.

That second plane is normal to the principal normal at the point M_1 that is situated in its positive direction at a distance of ρ from the point *M* on the curve. That point is called the *center of curvature*, while the circle that is described in the osculating plane with its center at M_1 and a radius of $MM_1 = \rho$ is called the *osculating circle* ([1]). The generator of the polar developable is then the normal to the osculating plane at the center of the osculating circle, or – as one says – *the axis of the osculating circle.*

In order to then determine the point M_0 where that axis touches the edge of regression of the polar developable, one must associate (29), (30) with the equation that results from (30) after another derivation, namely:

$$(X - x)\,\lambda + (Y - y)\,\mu + (Z - z)\,\nu = -T\,\frac{d\rho}{ds}.$$

When the coordinates x_0, y_0, z_0 of M_0 are substituted for *X*, *Y*, *Z* in (29), (30), (31), they must simultaneously satisfy them, which will give:

$$(32)\qquad \begin{cases} x_0 = x + \rho\xi - T\dfrac{d\rho}{ds}\lambda, \\ y_0 = y + \rho\eta - T\dfrac{d\rho}{ds}\mu, \\ z_0 = z + \rho\zeta - T\dfrac{d\rho}{ds}\nu \end{cases}$$

when one solves them.

The sphere that is described with its center at M_0 and a radius of M_0M is called the *osculating sphere* to the curve *C* at *M*, because one can prove (by a method that is entirely similar to the one that was used in § 3 to define the osculating plane) that among all of the spheres that go through *M*, it is the one that deviates the least from the curve in the vicinity of *M* ([2]). It is clear that the osculating circle is the intersection of the osculating sphere with the osculating plane.

Let *R* denote the radius of the osculating sphere, so one will have:

$$R^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2,$$

namely, from (32):

$$(33)\qquad R^2 = \rho^2 + T^2\left(\frac{d\rho}{ds}\right)^2.$$

([1]) In effect, among all of the circles through *M*, the osculating circle is the one that deviates the least from the curve in the neighborhood of *M*.

([2]) By an abuse of language, one can also define it to be the sphere that passes through *M* and three successive points of the curve.

§ 17.

Edge of regression of the polar developable

We now go on to study the curve C_0 that is the locus of points M_0 of the edge of regression of the polar developable of a given curve C, or the locus of centers of the osculating spheres.

In the first place, differentiate (32) with respect to s and get:

$$(32^*)\qquad \frac{dx_0}{ds} = -\left\{\frac{\rho}{T} + \frac{d}{ds}\left(T\frac{d\rho}{ds}\right)\right\}\lambda, \qquad \frac{dy_0}{ds} = -\left\{\frac{\rho}{T} + \frac{d}{ds}\left(T\frac{d\rho}{ds}\right)\right\}\mu,$$
$$\frac{dz_0}{ds} = -\left\{\frac{\rho}{T} + \frac{d}{ds}\left(T\frac{d\rho}{ds}\right)\right\}\nu.$$

First consider the special case in which the expression $\frac{\rho}{T} + \frac{d}{ds}\left(T\frac{d\rho}{ds}\right)$ is zero. From (32*), x_0, y_0, z_0 we then be constants; i.e., the center of the osculating sphere is immobile. Since differentiating (33) will give:

$$(33^*)\qquad R\frac{dR}{ds} = T\frac{d\rho}{ds}\left\{\frac{\rho}{T} + \frac{d}{ds}\left(T\frac{d\rho}{ds}\right)\right\},$$

in addition, one sees that $dR / ds = 0$ in this case; i.e., R is constant. The curve is then traced on a sphere of radius R. Conversely, if the curve is spherical then all of the normal planes to the curve will pass through the center of the sphere that is the osculator at all of its points. One then sees that:

The intrinsic equation:

$$\frac{\rho}{T} + \frac{d}{ds}\left(T\frac{d\rho}{ds}\right) = 0$$

is characteristic of the spherical curve ([1]).

If the curve C is not spherical then one will have an actual curve C_0 that is the locus of the centers of the osculating spheres, and if one denotes the elements of that curve by means of the subscript 0 then, from (32*), one will have:

([1]) One observes that if $\frac{\rho}{T} + \frac{d}{ds}\left(T\frac{d\rho}{ds}\right)$ is annulled at a point of an arbitrary curve C then the osculating sphere will be stationary there, and the curve C_0 will have a point of regression at the corresponding point.

(34) $$\frac{ds_0}{ds} = \varepsilon \left\{\frac{\rho}{T} + \frac{d}{ds}\left(T\frac{d\rho}{ds}\right)\right\},$$

in which ε denote positive or negative unity. If one agrees to count s_0 as something that increases with s then one must give ε the sign of $\frac{\rho}{T} + \frac{d}{ds}\left(T\frac{d\rho}{ds}\right)$.

After that, one will have:

$$\frac{ds_0}{\rho_0}\xi_0 = -\varepsilon\frac{ds}{T}\xi, \qquad \frac{ds_0}{\rho_0}\eta_0 = -\varepsilon\frac{ds}{T}\eta, \qquad \frac{ds_0}{\rho_0}\zeta_0 = -\varepsilon\frac{ds}{T}\zeta.$$

It will then follow that:

(36) $$\frac{ds_0}{\rho_0} = \varepsilon'\frac{ds}{T},$$

in which ε' denotes positive or negative unity according to whether $1 / T$ is positive or negative, resp.

We then have:

(37) $$\xi_0 = -\varepsilon\,\varepsilon'\xi, \quad \eta_0 = -\varepsilon\,\varepsilon'\eta, \quad \zeta_0 = -\varepsilon\,\varepsilon'\zeta,$$

and from this, along with (35), we deduce that:

(38) $$\lambda_0 = -\varepsilon'\alpha, \qquad \mu_0 = -\varepsilon'\beta, \qquad \nu_0 = -\varepsilon'\gamma,$$

so that, finally, upon differentiation, it will result from (37) that:

(39) $$\frac{ds_0}{T_0} = \varepsilon'\frac{ds}{\rho}.$$

The preceding formulas prove that for each of the two curves C, C_0, the tangent to one is parallel to the binormal of the other one, and their principal normals are parallel, which is a result that one can see quite easily *a priori* geometrically (if one ignores the precise determination of the sign).

Observe that the three formulas (34), (36), (39), viz.:

$$\frac{ds_0}{ds} = \varepsilon \left\{\frac{\rho}{T} + \frac{d}{ds}\left(T\frac{d\rho}{ds}\right)\right\}, \qquad \frac{ds_0}{\rho_0} = \varepsilon'\frac{ds}{T}, \qquad \frac{ds_0}{T_0} = \varepsilon'\frac{ds}{\rho},$$

already suffice to calculate (with one quadrature) the intrinsic equations of C_0 when one is given those of C. Hence, if C is a cylindrical helix then C_0 will also be a cylindrical helix. If C is a circular helix or a cylindro-conic helix, moreover, then the same will be true for C_0 .

The radius of the osculating sphere is constant in the case of a spherical curve. However, (33*) shows that there is a second case in which that radius is constant, namely, when $d\rho / ds = 0$, or when the curve has constant flexure. The third term in (32) then disappears, so the center of the osculating sphere will coincide with the center of the osculating circle, and for the curve C_0 that is the locus of those centers, for which $\varepsilon' = \varepsilon$, it will result from the preceding formulas that:

$$ds_0 = \varepsilon \frac{\rho}{T} ds,$$
$$\rho_0 = \rho, \qquad T_0 \, T = \rho^2,$$
$$\xi_0 = -\xi, \qquad \eta_0 = -\eta, \qquad \zeta_0 = -\zeta.$$

Hence:

The curve C_0 *that is the locus of centers of the osculating circles to a curve C with constant flexure* $1 / a$ *has the same flexure, and the product of the two torsions is equal to the square* $1 / a^2$ *of the common flexure. The relationship between C,* C_0 *is involutory, as well; i.e., C is the locus of centers of curvature of* C_0 .

§ 18.

Evolvents and evolutes

Consider the developable Σ of the tangents to a curve C and look for the curve C' that is traced on Σ that cuts all of the generators at a right angle; i.e., the tangents to C. Let $M \equiv (x, y, z)$ be an arbitrary point of C, and let $M' \equiv (x', y', z')$ be a point of one of those curves C' where C' intersects the tangent at M orthogonally. If one sets $\overline{MM'} = \tau$ then one will have:

$$x' = x + \tau\,\alpha, \quad y' = y + \tau\,\beta, \quad z' = z + \tau\,\gamma.$$

τ is a function of s that is determined in such a way that the tangent to C' is normal to the tangent to C. One now has:

$$\begin{cases} \dfrac{dx'}{ds} = \left(1 + \dfrac{d\tau}{ds}\right)\alpha + \tau\dfrac{\xi}{\rho}, \\ \dfrac{dy'}{ds} = \left(1 + \dfrac{d\tau}{ds}\right)\beta + \tau\dfrac{\eta}{\rho}, \\ \dfrac{dz'}{ds} = \left(1 + \dfrac{d\tau}{ds}\right)\gamma + \tau\dfrac{\zeta}{\rho}, \end{cases}$$

and since one must have:

$$\alpha\frac{dx'}{ds} + \beta\frac{dy'}{ds} + \gamma\frac{dz'}{ds} = 0,$$

it will then result that $d\tau / ds = -1$, i.e.:

$$\tau = c - s,$$

in which c indicates an (arbitrary) constant.

The formulas:

$$x' = x + (c - s)\,\alpha, \qquad y' = y + (c - s)\,\beta, \qquad z' = z + (c - s)\,\gamma \tag{40}$$

then define a simple infinitude of curves C' that are orthogonal to the tangents to C. Since $\overline{MM'} = c - s$ (or if one measures s from a suitable point on C, $MM' = - s$), one sees that if one winds a flexible and inextensible string on the curve C and unwinds it along that curve (starting from the origin of the arc length) in such a way that the string always remains tense then the rectilinear portion MM' of the unwound string will always stay tangent to the curve at M and equal in length to the arc length s. The free extremity M' of the string will then describe the curve C', which is called an *evolvent* (or developing) of C, for that reason, while C takes the name of *evolute* (or development) of the curve C'.

Therefore:

Any curve C has a simple infinitude of evolvents that are the orthogonal trajectories of the generators of the developable of the tangents to C.

One infers from differentiating (40) (and supposing that $c = 0$) that:

$$\frac{dx'}{ds} = -\frac{s}{\rho}\,\xi, \quad \frac{dy'}{ds} = -\frac{s}{\rho}\,\eta, \quad \frac{dz'}{ds} = -\frac{s}{\rho}\,\zeta,$$

and therefore, if s' is considered to increase with s then:

$$\frac{ds'}{ds} = \frac{s}{\rho},$$

and thus:

$$\alpha' = -\,\xi, \qquad \beta' = -\,\eta, \qquad \gamma' = -\,\zeta.$$

These formulas prove that: *The tangents to the evolvent are parallel and opposite in direction to the principal normal of the evolute.*

We now address the inverse problem: *Find all of the evolutes of a given curve C.*

Let C' denote any of the desired evolutes, so the edge of regression of a developable must define a simple infinitude of the normals to the curve C. If one refers the point M of the evolute C that lies in the normal plane at M to the evolvent C, the principal normal, and the binormal as moving auxiliary axes and denotes the relative coordinates of M by u, υ then one will have:

$$x' = x + u\,\xi + \upsilon\,\lambda, \qquad y' = y + u\,\eta + \upsilon\,\mu, \qquad z' = z + u\,\zeta + \upsilon\,\nu$$

for the usual coordinates x', y', z' of M'.

One now treats the determination of u, v as functions of s in such a way that the tangent to C'(which is locus of M') at M' is precisely the normal $M'M$ to C; i.e., we must express the idea that:

$$\frac{dx'}{ds}, \quad \frac{dy'}{ds}, \quad \frac{dz'}{ds}$$

are neatly proportional to:

$$u\,\xi + \upsilon\,\lambda, \qquad u\,\eta + \upsilon\,\mu, \qquad u\,\zeta + \upsilon\,\nu,$$

respectively.

If one performs the differentiations then one will immediately find the following conditions for u, υ:

$$u = \rho, \qquad \frac{1}{u}\left(\frac{du}{ds} + \frac{\upsilon}{T}\right) = \frac{1}{\upsilon}\left(\frac{d\upsilon}{ds} - \frac{u}{T}\right),$$

or

$$u = \rho, \qquad \frac{\rho\dfrac{d\upsilon}{ds} - \upsilon\dfrac{d\rho}{ds}}{\rho^2 + \upsilon^2} = \frac{1}{T},$$

the last of which integrates to:

$$\arctan\frac{\upsilon}{\rho} = \int_0^s \frac{ds}{T} + c,$$

i.e.:

$$\upsilon = \rho \tan(\tau + c),$$

in which c is an arbitrary constant, and one has set:

$$\tau = \int_0^s \frac{ds}{T}.$$

The problem that was posed is then solved by a quadrature by means of the formulas:

(41) $$x' = x + \rho\,\xi + \tan(\tau + c)\,\lambda, \quad y' = y + \rho\,\eta + (\tau + c)\mu, \quad z' = z + \rho\,\zeta + (\tau + c)\nu.$$

Observe the formulas:

(41*) $$\alpha' = \cos(\tau + c)\,\xi + \sin(\tau + c)\,\lambda, \qquad \beta' = \cos(\tau + c)\,\eta + \sin(\tau + c)\,\mu,$$
$$\gamma' = \cos(\tau + c)\,\zeta + \sin(\tau + c)\,\nu,$$

which follow immediately from the last ones. It then results that:

The angle that the tangent to the evolute forms with the principal normal is given by $\tau + c$.

Corresponding to the infinitude of values for the constant c in (41), one will have ∞^1 evolutes of the given curve C, which are traced on its polar developable. In addition, since the principal normal to any evolute is parallel to the tangent to the evolvent, and is therefore the normal to the polar developable, one sees that all of the evolutes are geodetics of the polar developable. If one rolls that developable onto a plane then the evolute will be changed into a system of lines, and as one can prove, more precisely, a sheaf of lines ([1]).

Observe that if the evolvent is a plane curve then it will have just one planar evolute, namely, the locus of its centers of curvature. The other evolutes are helices of the right cylinder (i.e., polar developable) that has the planar evolute for its base.

A theorem that is very important because of its applications results from the observation that was made above on the geometric significance of $\tau + c$ in (41), (41^*). If one considers two different evolutes that correspond to the values c_1, c_2 of the arbitrary constant c then the difference $(\tau + c_1) - (\tau + c_2) = c_1 - c_2$ will represent the angle subtended by the two tangents to the respective evolutes that emanate from the same point M of the evolvent, so one will have:

A) *The tangents to two different evolutes that emanate from the same point of the evolvent C define a constant angle between them along C.*

It is useful to state that theorem in the somewhat different form:

B) *If the generators of a developable surface rotate around respective points of intersection with one of their orthogonal trajectories in its normal plane with a constant angle then the locus of the new positions of the generators will be another developable surface.*

§ 19.

Orthogonal trajectories of a system of ∞^1 planes

The method that was used in the preceding § to find the evolute of a given curve can be applied to the solution of another problem: *Determine all curves C′ that are the loci of centers of the osculating spheres to an assigned curve C.* Since the osculating planes to C must be normal planes to C', the proposed search is obviously equivalent to the search

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([1]) Anticipating the notions of the next chapter II, observe that if one considers c to be variable in (41) and takes:

$$\frac{\rho}{\cos(\tau+c)} = R,$$

and constructs the square of the line element of the polar developable then one will find:

$$ds^2 = dx'^2 + dy'^2 + dz'^2 = dR^2 + R^2\, dc^2,$$

which is the line element of the plane in polar coordinates, which proves the property that was stated in the text.

for curves C' that cut an assigned series of ∞^1 planes orthogonally. If C' is such a curve, and M' is the point where it meets the osculating plane to C at M orthogonally then one will fix the position of M' in that plane by means of its orthogonal Cartesian coordinates u, v when referred to the tangent and principal normal as axes. One will then have:

$$(42) \qquad x' = x + u\,\alpha + v\,\xi, \qquad y' = y + u\,\beta + v\,\eta, \qquad z' = z + u\,\gamma + v\,\zeta$$

for the usual coordinates x', y', z' of M', and the condition that is imposed on C' that it must be the locus of M' will then imply that $\frac{dx'}{ds}, \frac{dy'}{ds}, \frac{dz'}{ds}$ must be proportional to λ, μ, ν. One will then find the equations:

$$\frac{dv}{ds} = -\frac{u}{\rho}, \qquad \frac{du}{ds} = \frac{v}{\rho} - 1$$

for the unknown functions u, v of s, from which, one will get the formulas:

$$(42^*) \qquad \frac{dx'}{ds} = -\frac{v}{T}\lambda, \quad \frac{dy'}{ds} = -\frac{v}{T}\mu, \quad \frac{dz'}{ds} = -\frac{v}{T}\nu .$$

If one changes the independent variable s by putting:

$$\sigma = -\int \frac{ds}{\rho}$$

then one will have:

$$u = -\frac{dv}{d\sigma}, \qquad \frac{du}{d\sigma} = v - \rho,$$

i.e.:

$$\frac{d^2v}{d\sigma^2} + v = \rho,$$

which will give:

$$v = c \cos \sigma + c' \sin \sigma - \cos \sigma \int \sin \sigma\, ds + \sin \sigma \int \cos \sigma\, ds,$$

in which c, c' are arbitrary constants. One will, in turn, have:

$$u = -\frac{dv}{d\sigma} = c \sin \sigma - c' \cos \sigma - \sin \sigma \int \sin \sigma\, ds - \cos \sigma \int \cos \sigma\, ds,$$

and if one substitutes those values for u, v in (42) then one will have determined the required curve C' by quadrature, and as is geometrically obvious, there will be a double infinitude of them.

Analogously to the problem in § 15 (cf., page 28), observe that the values of the unknowns u, v, and therefore those of $R = \sqrt{u^2 + v^2}$, which is the radius of the sphere, depend upon only the flexure $1 / \rho$. If one twists the curve C in any way that drags along its osculating planes then the locus of those points M' will always be a curve C' that is orthogonal to those planes. In particular, if one makes $1 / T = 0$ then (42*) will show that x', y', z' reduce to constants, i.e., the curve C' reduces to a point. With that, we can present the geometric solution to our problem (viz., Jamet's problem) in the following elegant form that is due to Cesàro ([1]):

Given a curve C, in order to construct all of a series of spheres that have their center on C and as a result osculate a curve C′, transform C while altering only the torsion into a plane curve Γ *and consider the spheres that have their centers on* Γ *and pass through a fixed point M′ of its plane. If the curve* Γ *again takes on the form C while dragging along the spheres rigidly then the new spheres will osculate a curve C′ that is the locus of the new positions of M′.*

§ 20.

Combescure transformation of curves

Given two curves C, C_1, if it is possible to establish a correspondence between the two curves that makes any point M of C correspond to a point M_1 of C_1 *in such a way that the tangents to two corresponding points always prove to be parallel* then we shall say that the two curves are obtained from each other by a *Combescure transformation* ([2]). In order for two curves C, C_1 to be Combescure transforms of each other, it is necessary and sufficient that they have a common spherical indicatrix of tangents, as would be the case for, e.g., two homothetic curves or two planar curves in parallel planes. Since it then results from the Frenet formulas that the tangent to the spherical indicatrix of a curve is parallel to the principal normal of the latter, one sees that:

If the curve C_1 *is obtained by a Combescure transformation of the curve C then the three principal direction at a point* M_1 *of* C_1 *will be parallel to the three corresponding principal directions at the corresponding point M of C, respectively.*

In particular, the osculating planes to C, C_1 at two corresponding points M, M_1 will be mutually parallel. Conversely, one sees immediately that if two curves correspond point-by-point in such way that the osculating planes to two corresponding points M, M_1 are parallel then the two curves will be transforms of each other under a Combescure transformation.

It results from those observations that the geometric construction by which one deduces any of the Combescure transforms of a given curve C is the following one:

([1]) *Lezioni di geometria intrinsica*, page. 146.

([2]) The Combescure transformation really applies to the triple systems of orthogonal surfaces that we shall study later on. When applied to curves, that will lead to precisely the geometric relations that are considered in the text, and it thus seems suitable to adopt that terminology.

Draw a plane that is parallel to any osculating plane to the curve C and at an arbitrary distance that varies continuously as the osculating plane to C varies. The enveloping developable of the new planes will have an edge of regression that is the Combescure transform C_1 of the given curve.

Keep the usual notations for the curve C and denote the corresponding elements of a Combescure transform C_1 by adding the index 1. The formulas that define C_1 will obviously be the following ones:

$$x_1 = \int \alpha f(s)\, ds, \qquad y_1 = \int \beta f(s)\, ds, \qquad z_1 = \int \gamma f(s)\, ds, \tag{43}$$

in which $f(s)$ denotes an *arbitrary* function of s ([1]). One will then have:

$$ds_1 = f(s)\, ds, \quad \frac{ds_1}{\rho_1} = \frac{ds}{\rho}, \qquad \frac{ds_1}{T_1} = \frac{ds}{T}$$

for the intrinsic equations of C_1; i.e.:

$$\frac{ds_1}{ds} = f(s), \qquad \rho_1 = \rho\, f(s), \qquad T_1 = T\, f(s). \tag{44}$$

In particular, one deduces that $\frac{\rho_1}{T_1} = \frac{\rho}{T}$, i.e., the property of Combescure transformations that they leave the ratio of the two curvatures unchanged.

§ 21.

Curves of constant flexure and spherical helices

One Combescure transform of any curve C will always be a curve of constant flexure with an assigned value for the first curvature; e.g., $\rho_1 = 1$. In order to obtain that particular transform, it is obviously enough to set $f(s) = 1 / \rho$. Hence, if one is given an arbitrary curve C then the quadratures:

$$x_1 = \int \frac{\alpha\, ds}{\rho}, \quad y_1 = \int \frac{\beta\, ds}{\rho}, \quad z_1 = \int \frac{\gamma\, ds}{\rho}$$

will give the Combescure transform C_1 that has constant flexure = 1. If one then observes that:

$$\frac{ds}{\rho} = \sqrt{d\alpha^2 + d\beta^2 + d\gamma^2} = d\sigma$$

([1]) If one takes a constant value for $f(s)$ then one will get a *homothetic* transformation of C.

then one will see that:

The formulas:

(45) $$x = \int \alpha\, d\sigma, \qquad y = \int \alpha\, d\sigma, \qquad z = \int \gamma\, d\sigma,$$

in which α, β, γ *are arbitrary functions of one parameter that are linked by the relations* $\alpha^2 + \beta^2 + \gamma^2 = 1$ *and:*

(45*) $$d\sigma = \sqrt{d\alpha^2 + d\beta^2 + d\gamma^2}\,,$$

define the most general curve of constant flexure $1 / \rho = 1$.

We further apply the Combescure transformation to the search for *spherical helices.*

In order to that, observe first of all that, in general, (44) will give the formulas:

$$\frac{d\rho_1}{ds_1} = \frac{1}{f}\,(\rho f)',$$

$$T_1\,\frac{d\rho_1}{ds_1} = T\,(\rho f)',$$

in which the primes refer to derivatives with respect to s. If one would then wish that the Combescure transform C_1 of C should be described on a sphere of radius = 1 then one would have to determine f from the equation [1]:

$$(\rho f)^2 + T^2\,(\rho f)'^2 = 1,$$

which integrates to:

$$\rho f = \sin(\tau + c), \qquad \tau = \int_0^s \frac{ds}{T}\,,$$

in which c is an arbitrary constant. The required spherical transform will then be defined by the quadratures:

(46) $$x_1 = \int \frac{\alpha \sin(\tau + c)\, ds}{\rho}, \quad y_1 = \int \frac{\beta \sin(\tau + c)\, ds}{\rho}, \quad z_1 = \int \frac{\gamma \sin(\tau + c)\, ds}{\rho}.$$

It is clear that the simple infinitude of spherical curves that are determined in that way will give the orthogonal trajectories of the system of great circles of the sphere that lie in the plane that goes through its center parallel to the normal plane of C. Hence, from the preceding formulas, the problem of finding the orthogonal trajectories to a system of ∞^1 great circles (i.e., geodetics) of the sphere is solved by quadrature.

[1] Strictly speaking, from the observations in § 17, the equations in the text are not characteristic of the spherical curves, but can also belong to a curve $\rho_1 = \rho f = 1$, which is excluded in the text by omitting the singular integral $\rho f = 1$.

In particular, apply the preceding result to the search for spherical helices – i.e., those curves that are traced on the sphere whose tangents are inclined at a constant angle above a fixed direction. They are the spherical transforms by a Combescure transformation of a circular helix or the orthogonal trajectories to a system of great circles on the tangent sphere to one of its minor circles. One starts with the equations of the circular helices:

$$x = \cos u, \qquad y = \sin u, \qquad z = u \cot \varepsilon$$

that are traced on a right cylinder of radius = 1, in which ε denotes the constant angle of inclination of the tangents from its generators. One will then have:

$$s = \frac{u}{\sin \varepsilon}, \qquad \frac{1}{\rho} = \sin^2 \varepsilon, \qquad \frac{1}{T} = \sin \varepsilon \cos \varepsilon,$$

so

$$\tau = \int \frac{ds}{T} = u \cos \varepsilon,$$

and if one applies (46), while setting $c = 0$ ([1]), and performs the quadrature then one will obtain the following formulas that define the spherical helices:

$$\begin{cases} x_1 = \dfrac{\sin^2 \varepsilon}{2} \left\{ \dfrac{\sin(u(1+\cos \varepsilon))}{1+\cos \varepsilon} - \dfrac{\sin(u(1-\cos \varepsilon))}{1-\cos \varepsilon} \right\}, \\ y_1 = \dfrac{\sin^2 \varepsilon}{2} \left\{ \dfrac{\cos(u(1+\cos \varepsilon))}{1+\cos \varepsilon} - \dfrac{\cos(u(1-\cos \varepsilon))}{1-\cos \varepsilon} \right\}, \\ z_1 = \sin \varepsilon \cos(u \cos \varepsilon). \end{cases}$$

Observe that if the angle ε corresponds to a rational value of $\cos \varepsilon$ then the spherical helix will be a rational algebraic curve.

§ 22.

Orthogonal trajectories of a system of ∞^1 spheres

We go on to treat a problem that was solved already in § 19 for a system of planes, but for a system of ∞^1 spheres, this time, namely, the problem of finding the ∞^2 curves that are their orthogonal trajectories. The results that are known are due to Darboux, who discussed them (in a different form) on page 35, *et seq.*, of his *Leçons sur les systèmes orthogonal et les coordinées curvilignes* (Volume I).

([1]) It is geometrically clear that if one gave c another value then one would obtain the same curve on the sphere, but displaced by a rotation around the polar axis (viz., the axis Oz).

In order to define the system of ∞^1 spheres ([1]), one gives the curve C that is the locus of centers of the spheres, for which one keeps the usual notations, and in addition, one gives the radius R of the spheres as a function of the arc length s of C. Any point $M' \equiv (x', y', z')$ of one of the spheres can be defined by the formulas:

$$(47) \qquad \begin{cases} x' = x + R\sin\theta\cos\varphi\,\alpha + R\sin\theta\sin\varphi\,\xi + R\cos\theta\,\lambda, \\ y' = y + R\sin\theta\cos\varphi\,\beta + R\sin\theta\sin\varphi\,\eta + R\cos\theta\,\mu, \\ z' = z + R\sin\theta\cos\varphi\,\gamma + R\sin\theta\sin\varphi\,\zeta + R\cos\theta\,\nu, \end{cases}$$

in which R, θ, φ are the polar coordinates of M, when referred to the principal trihedron to C at M. For a curve C' to be orthogonal to all of the spheres, one must have θ, φ as functions of s such that the three derivatives:

$$\frac{dx'}{ds}, \quad \frac{dy'}{ds}, \quad \frac{dz'}{ds}$$

prove to be proportional to the binomials:

$$x' - x, \qquad y' - y, \qquad z' - z.$$

A simple calculation will lead to the first-order differential equations for θ, φ:

$$(48) \qquad \begin{cases} \dfrac{d\theta}{ds} + \dfrac{\sin\varphi}{T} + \dfrac{\cos\theta\cos\varphi}{R} = 0, \\ \dfrac{d\varphi}{ds} + \dfrac{1}{\rho} + \dfrac{\cos\theta\cos\varphi}{T} - \dfrac{\sin\varphi}{\sin\theta}\dfrac{1}{R} = 0. \end{cases}$$

Just as we did above for (13), page 16, for a particular case in which $1 / R = 0$, we can replace them with a single equation of Ricatti type, and also introduce the complex variable on the spheres:

$$\sigma = R \cot\frac{\theta}{2}\, e^{i\varphi}.$$

We will then find the equation:

$$(48^*) \qquad \frac{d\sigma}{ds} = -\frac{i\,\sigma^2}{2T} + \frac{\sigma^2}{2R} + \left(\frac{1}{R}\frac{dR}{ds} - \frac{i}{\rho}\right)\sigma + \frac{i}{2T} - \frac{1}{2R},$$

from which, (48) will follow conversely when we separate the real and imaginary parts.

Hence:

([1]) We leave aside the case of a system of concentric spheres.

If one is given a system of ∞^1 *spheres then the determination of their orthogonal trajectories will depend upon the integration of a Ricatti equation. The problem will then be solved by quadrature as soon as one knows one of the orthogonal trajectories.*

One can easily confirm by direct calculation that if one of the orthogonal trajectories of the system of spheres is known then the Riccati equation will reduce to a linear one that can then be integrated by quadrature. Indeed, suppose that the given curve C is one of the orthogonal trajectories, and let $R = R(s)$ be the radius of the sphere. The coordinates x', y', z' of a point M' of the sphere can then be written in the form:

$$(49)\qquad \begin{cases} x' = x + R(1+\cos\theta)\,\alpha + R\sin\theta\cos\varphi\,\xi + R\sin\theta\sin\varphi\,\lambda, \\ y' = y + R(1+\cos\theta)\,\beta + R\sin\theta\cos\varphi\,\eta + R\sin\theta\sin\varphi\,\mu, \\ z' = z + R(1+\cos\theta)\,\gamma + R\sin\theta\cos\varphi\,\zeta + R\sin\theta\sin\varphi\,\nu. \end{cases}$$

If one desires that the point M' should describe an orthogonal trajectory of the spheres then one must take θ, φ to be functions of s such that:

$$\frac{dx'}{ds}, \quad \frac{dy'}{ds}, \quad \frac{dz'}{ds}$$

prove to be proportional to:

$$x' - x - R\alpha, \quad y' - y - R\beta, \quad z' - z - R\gamma,$$

respectively, which will then give two simultaneous equations for θ, φ:

$$(49^*)\qquad \begin{cases} \dfrac{d\theta}{ds} = \dfrac{R'+1}{R}\sin\theta - \dfrac{(1+\cos\theta)\cos\varphi}{\rho}, \\[2ex] \dfrac{d\varphi}{ds} = \dfrac{\cot\dfrac{\theta}{2}\sin\varphi}{\rho} + \dfrac{1}{T}. \end{cases}$$

If one again sets:

$$\sigma = R\cot\frac{\theta}{2}\,e^{i\varphi}$$

then one will find the equation for σ:

$$\frac{d\sigma}{ds} = \frac{\sigma^2}{R\,\rho} + \left(\frac{i}{T} - \frac{1}{R}\right)\sigma,$$

which is linear in $1 / \sigma$ and integrates to:

$$(50)\qquad \frac{1}{\sigma} = e^{\int\left(\frac{1}{R}-\frac{i}{T}\right)ds}\left\{C - e^{-\int\left(\frac{1}{R}-\frac{i}{T}\right)ds}\frac{ds}{R\,\rho}\right\},$$

in which C is an arbitrary (complex) constant. Therefore, the problem is solved precisely by quadrature, and if one varies the two arbitrary constants that are contained in C' then one will get the required double infinitude of orthogonal trajectories. In addition, one can deduce another elegant geometric property of the system of spheres from (50), which was observed by Darboux. If one applies that formula to two different spheres S, S_1 of the series and denotes the corresponding complex variables by s, s_1 then it will obviously follow from the elimination of C from σ, σ_1 that the two complex variables are linear functions of each other. The geometric interpretation of that fact (which will be shown in Chap. III, § 52) will give the theorem:

If one regards the points of two arbitrary spheres S, S_1 of the system as corresponding when they are points of intersection with the same orthogonal trajectory then the representation of one sphere on the other one will preserve angles and circles (viz., a Möbius circular affinity).

In conclusion, observe that if one would like to avoid the introduction of imaginaries in the calculations above then it would be enough to proceed in the following way: Assume that the unknown functions are:

$$\tan\frac{\theta}{2}\cos\varphi = L, \qquad \tan\frac{\theta}{2}\sin\varphi = M,$$

instead of θ, φ, so (49*) will assume the linear form:

$$\begin{cases} \dfrac{dL}{ds} - \dfrac{R'+1}{R}L + \dfrac{1}{T}M + \dfrac{1}{\rho} = 0, \\[2ex] \dfrac{dM}{ds} - \dfrac{R'+1}{R}M - \dfrac{1}{T}L = 0. \end{cases}$$

They reduce immediately by quadratures using the D'Alembert method of seeking a multiplier λ such that $L + \lambda M$ satisfy a linear equation. The equation for λ is:

$$\text{(50*)} \qquad \frac{d\lambda}{ds} = \frac{\lambda^2+1}{T},$$

which will give:

$$\lambda = \tan(\tau + c), \qquad \tau = \int_0^s \frac{ds}{T}$$

when it is integrated.

One can then take:

$$\lambda_1 = \tan\tau, \qquad \lambda_2 = -\cot\tau,$$

and the two unknowns:

$$L + \lambda_1 M = \frac{\tan\frac{\theta}{2}\cos(\tau-\varphi)}{\cos\tau},$$

$$L + \lambda_2 M = \frac{\tan\frac{\theta}{2}\sin(\tau - \varphi)}{\cos\tau}$$

will each be determined by a linear equation ([1]).

§ 23.

Darboux's method

For all of the analytical solutions to the problem that was treated in the preceding §, one can also know a second solution, which is more geometric and is precisely the one that was given by Darboux (*loc. cit.*), who had used it in the context of the Combescure transformation (§ 20).

For all of the ∞^1 spheres in the system that was given in the preceding § by assigning the curve C that is the locus of their centers and setting their radii $R = R(s)$, consider any other of them that one obtains by transforming the curve C that is the locus of the centers into another curve C_1 by means of a Combescure transformation whose formulas are (§ 20):

$$ds_1 = f(s)\,ds, \quad \rho_1 = \rho\,f(s), \qquad T_1 = T\,f(s),$$

and reduce the radii of the spheres by the same ratio by assuming that:

$$R_1 = R\,f(s).$$

One then sees immediately that equations (48) for θ, φ will remain the same, because when one integrates the problem of the orthogonal trajectories of the given system of spheres, one will simultaneously solve that same problem for all of the systems that are derived from it by a Combescure transformation. It is quite easy to give the geometric interpretation of that simple result if one observes that two points of the two corresponding spheres that are given by the same values of θ, φ will give two parallel radii when joined at their respective centers. Hence:

If one is given an orthogonal trajectory of the given system of spheres then one can deduce an orthogonal trajectory of the Combescure-transformed system by singling out the point M_1 of any sphere S_1 of the second system that corresponds to the point M where the original sphere S meets the orthogonal trajectory.

One will then see that if one Combescure transforms a system of ∞^1 spheres then their orthogonal trajectories will also be subjected to the Combescure transformation.

Now, the fundamental observation in Darboux's method is the following one:

([1]) If one takes λ_1, λ_2 to be the singular integrals $\lambda_1 = i$, $\lambda_2 = -i$ then one will return to the preceding formulas in imaginaries.

If one knows one of the orthogonal trajectories C of a system of spheres then one can, by quadrature, replace it with a Combescure-transformed system in which all of the spheres pass through a fixed point.

In order to prove this, take the usual notations for the curve C, so:

$$x_0 = x + R\,\alpha, \quad y_0 = y + R\,\beta, \quad z_0 = z + R\,\gamma$$

will be the coordinates of the center of the sphere. Draw a segment R_1 (which is determined as a function of s) that is parallel to the tangent to C at (x, y, z) through a fixed point – e.g., the origin O. One will have:

$$x_1 = R_1\,\alpha, \qquad x_2 = R_1\,\beta, \qquad x_3 = R_1\,\gamma$$

for the coordinates of its extreme, and in order for the spheres:

$$(X - x_1)^2 + (Y - y_1)^2 + (Z - z_1)^2 = R_1^2$$

to describe a Combescure-transformed system, one must determine R_1 as a function of s in such a way that one has:

$$\frac{dx_1}{ds} : \frac{dy_1}{ds} : \frac{dz_1}{ds} = \frac{dx_0}{ds} : \frac{dy_0}{ds} : \frac{dz_0}{ds}.$$

Those conditions immediately give the equation for R_1:

$$\frac{R_1'}{R_1} = \frac{1+R'}{R},$$

which integrates to:

$$R_1 = C\,R\,e^{\int \frac{ds}{R}} \qquad (C \text{ constant}).$$

However, conversely, the system that one obtains is effectively Combescure-transformed into the original one, since one can verify immediately that one has:

$$\frac{ds_1}{R_1} = \frac{ds_0}{R},$$

in which ds_0, ds_1 denote the elements of arc length for the curves that are loci of the respective centers.

One is then reduced to a system of spheres that pass through a fixed point, so an inversion by reciprocal radius vectors whose center in at that fixed point will change the spheres into a system of ∞^1 planes and the orthogonal trajectories of the spheres into orthogonal trajectories of planes, since, as is well-known, the given inversion will preserve angles. The general problem is then ultimately reduced to the special problem of § 19, and is then solved by quadrature.

§ 24.

Bertrand curves

We will conclude this first chapter by examining a problem that was posed and solved by Bertrand. We start with the result that was obtained at the end of § 17, according to which, the curves with constant flexure are presented as pairs that have their principal normals in common, and demand: *Determine all of the curves C that admit a second one C′ that has the same principal normals as C.* If we indicate the quantities that relate to C' by primes then we will have:

$$x' = x + k\,\xi, \quad y' = y + k\,\eta, \quad z' = z + k\,\zeta$$

for the coordinates x', y', z' of the point that corresponds to the point $M \equiv (x, y, z)$ of C, in which k denotes the length of the principal normal MM'. Now, if MM' is the normal to C' at M', by hypothesis, then we must have, in the first place:

$$\xi \frac{dx'}{ds} + \eta \frac{dy'}{ds} + \zeta \frac{dz'}{ds} = 0,$$

which will give:

$$\frac{dk}{ds} = 0, \qquad \text{or} \qquad k = \text{constant}.$$

In the second place, if one differentiates (51) and sets:

$$(52) \qquad \cos\sigma = \frac{1 - \dfrac{k}{\rho}}{\sqrt{\left(1 - \dfrac{k}{\rho}\right)^2 + \dfrac{k^2}{T^2}}}, \qquad \sin\sigma = \frac{-\dfrac{k}{\rho}}{\sqrt{\left(1 - \dfrac{k}{\rho}\right)^2 + \dfrac{k^2}{T^2}}}$$

then one will get:

$$(53) \qquad \alpha' = \alpha\cos\sigma + \lambda\sin\sigma, \qquad \beta' = \beta\cos\sigma + \mu\sin\sigma, \qquad \gamma' = \gamma\cos\sigma + \nu\sin\sigma,$$

in which σ denotes the angle between the two tangents at M, M'. If one differentiates (53) once more then from the Frenet formulas and the hypothesis that C' has the same principal normal as C, one will deduce that:

$$\sigma = \text{constant},$$

and therefore, from (52), one will have the theorem:

The curve C must have its two curvatures coupled by the linear relation:

(54) $$\frac{k\cos\sigma}{T} - \frac{k\sin\sigma}{\rho} + \sin\sigma = 0.$$

Conversely, suppose that a curve C has its curvatures coupled by the linear equation:

(54*) $$\frac{A}{T} + \frac{B}{\rho} + C = 0,$$

without both ρ, T being constants (viz., the case of a circular helix). If one identifies (54) with (54*) then one will find a second curve C with the same principal normals as C, which will be determined completely by the formulas:

(55) $$k = -\frac{B}{C}, \qquad \tan\sigma = -\frac{B}{A},$$

and it is clear that the two curvatures of C' will satisfy the same relation (54*). Moreover, in the case of circular helices, one can take k quite arbitrarily. Hence:

The orthogonal trajectories of the principal normals to a circular helix all have the principal normals in common.

The surface of the principal normals of a circular helix will be presented later on in this study as the *ruled helicoid of minimal area.*

§ 25.

Determination of all Bertrand curves

The curves that satisfy the intrinsic equation (54*) are called *Bertrand curves.* They are presented as pairs that have principal normals in common, and their tangents at two corresponding points make a constant angle σ.

It is already true for the curves of constant flexure in § 21 (and thus, for the general Bertrand curves), that one can give the explicit equations for those curves by quadrature upon applying a Combescure transformation. In fact, it is enough to observe that if one is given a curve C arbitrarily then it will admit a Bertrand curve C_1 for its Combescure transform whose curvatures satisfy the fixed linear relation:

$$\frac{A}{\rho_1} + \frac{B}{T_1} + C = 0,$$

and from (44), page 39, that will give:

$$f(s) = -\frac{1}{C}\left(\frac{A}{\rho} + \frac{B}{T}\right).$$

If one introduces that value for $f(s)$ in (43) then one will have precisely a Bertrand curve C_1 .

One can answer the same question in a second way: That is why we pose the problem:

Given a curve C, find a second one C′ that corresponds to C with equality of arc lengths and has its principal normal at any point parallel to the one at the corresponding point of C.

If we let σ denote the angle between the two corresponding tangents to C, C', resp., then we will have:

(56) $\alpha' = \alpha \cos \sigma + \lambda \sin \sigma, \qquad \beta' = \beta \cos \sigma + \mu \sin \sigma, \qquad \gamma' = \gamma \cos \sigma + \nu \sin \sigma,$

and by hypothesis:

(56*) $\xi' = \pm \xi, \qquad \eta' = \pm \eta, \qquad \zeta' = \pm \zeta.$

If one differentiates (56) then it will result immediately that:

$$\sigma = \text{constant},$$

and therefore the desired curve C' must be given by the formulas:

(57) $x' = \int (\alpha \cos \sigma + \lambda \sin \sigma)\, ds, \qquad y' = \int (\beta \cos \sigma + \mu \sin \sigma)\, ds,$

$$z' = \int (\gamma \cos \sigma + \nu \sin \sigma)\, ds .$$

However, conversely, when gives σ an arbitrary constant value, (57) will define (up to a translation) a curve C' that has the required relationship with C. One then deduces from (56), (56*) that:

(56**) $\lambda' = \pm \lambda \cos \sigma \mp \alpha \sin \sigma, \qquad \mu' = \pm \mu \cos \sigma \mp \beta \sin \sigma,$

$$\nu' = \pm \nu \cos \sigma \mp \gamma \sin \sigma,$$

and differentiating (56), (56*), (56**) will give the formulas:

$$\begin{cases} \dfrac{1}{\rho'} = \pm\left(\dfrac{\cos\sigma}{\rho} + \dfrac{\sin\sigma}{T}\right), \\[2ex] \dfrac{1}{T'} = \dfrac{\cos\sigma}{T} - \dfrac{\sin\sigma}{\rho}; \end{cases}$$

i.e., the two curvatures of C' are homogeneous, linear combinations of those of C. In addition, one observes that:

The sum of the squares of the two curvatures is an invariant for that transformation.

It is clear that the transform of any Bertrand curve is a Bertrand curve. In particular, any such curve has constant flexure. The most general Bertrand curve is then obtained by applying the present transformation to the curve of constant flexure that was determined already in § 21.

In conclusion, observe the particular case of out transformation in which one sets $\sigma = \pi / 2$.

The formulas:

$$x' = \int \lambda\, ds, \qquad y' = \int \mu\, ds, \qquad z' = \int \nu\, ds$$

define a curve C' that corresponds to C with equality of arc lengths. The two curvatures and the directions of the tangent and binormal are permuted by the transformation.