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Exterior differential calculus for spinor forms and its application to the general pure gravitational radiation field

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Summary. – An exterior differential calculus for spinor forms that is quite analogous to the well-known tensorial Cartan calculus is developed that renders possible the direct spinorial calculation of the Weyl and Ricci tensors. As an application, the metric and curvature spinors of the most general space-time admitting a normal, sheer-free, null congruence are derived. The rather modest amount of calculation illustrates the advantages of the new method as compared to the techniques used thus far (*).

I. Spinor algebra

Let T be the metric tensor algebra over the vector space of complex pairs that is endowed with the symplectic metric:

$$\xi^A = \varepsilon^{AC}\,\xi_C\,, \quad \xi_A = \xi^C\,\varepsilon_{CA}\,, \quad \varepsilon_{AB} = -\,\varepsilon_{BA} = \varepsilon^{AB}, \qquad \varepsilon_{01} = 1.$$

Its elements are component matrices $T^{AB\dots}{}_{CD\dots}$ Let T' be a second exemplar of that algebra whose elements will be written with primes in order to distinguish them. The spinor algebra S is the tensorial product of the two algebras. S is, in a natural way, also a metric space whose elements are component matrices $T^{A\dots P'\dots}_{B\dots Q'\dots}$. We would like to list some structural properties of S that we will need in what follows (**)([1],[2]).

(*) **Remark by the editor:** R. DEBEVER, M. CAHEN, and L. DEFRISE have, according to an oral commutation from one of them, developed a calculus for the calculation of curvature spinors that likewise works with complex forms. A paper on that by those authors shall appear in J. Math. Phys.

(**) Which means the part that is skew in the index *pairs*; correspondingly, one will find 1. and 3. in, e.g., ([1, 2]). One will get 2. when one goes to the duals for S of formula (1.6) in ([1]).

([1]) R. PENROSE, Ann. Phys. **10** (1960), 171.
([2]) P. JORDAN, J. EHLERS, and R. SACHS, Wiss. Mainz, Abhandl. Math.-naturw. Kl. (1961), no. 1.

1. $T_{[\underline{AP'BQ'}]} = \frac{1}{2}[T_{C(P'}{}^{C}{}_{Q')}\,\varepsilon_{AB} + T_{(AR'B)}{}^{R'}\,\varepsilon_{P'Q'}]$. (I.1)

2. The following are equivalent:

$$S_{[\underline{AP'BQ'CR'}]} = 0 \qquad \text{and} \qquad S^{CR'}{}_{[\underline{CP'AR'}]} = 0. \tag{I.2}$$

3. One associates every spinor with its complex conjugate by means of (e.g.) $S^{AP'}{}_{B} \to \bar{S}^{AP'}{}_{B} \equiv \overline{S^{AP'}{}_{B}}$.

A spinor is called *real* when it is equal to its complex conjugate. (It will then have the same number of primed and unprimed indices in the same position.) The real spinors $S^{AP'}$ define a real metric vector space that is isometric to Minkowski space with the signature – 2.

According to PENROSE ([1]), one appeals to that isometry in general relativity in the following way: At any point x of the normal-hyperbolic V^4 with signature – 2 that one considers there, one chooses an isometry σ_x of the tangent space T_x to that space. It can be described in a coordinate system x^a by a numerical matrix $\sigma_a{}^{AP'}(x)$. With its help, one associates every tensor field with a real spinor field $k^a \to k^{AP'} = k^a\,\sigma_a{}^{AP'}$. That prescription is not unique; nonetheless, the following connection exists: If one chooses another isometry $\bar{\sigma}$ then there will be a unimodular complex 2×2 matrix $A^A{}_B(x)$ with an inverse $A_A{}^B(x)$ such that (*):

$$\tilde{\sigma}_a{}^{AP'} = A^A{}_B \bar{A}^{P'}{}_{Q'}\,\sigma_a{}^{BQ'}. \tag{I.3}$$

The spinor field that corresponds to the tensor field (e.g.) $T^a{}_b$ is connected with it by:

$$\tilde{T}^{AP'}{}_{BQ'} = A^A{}_C \bar{A}^{P'}{}_{R'}\,T^{CR'}{}_{DS'}\,A_B{}^D \bar{A}_{Q'}{}^{S'}. \tag{I.4}$$

Now, one generally defines a spinor field to be a function from V^4 to S that transforms analogously to (I.4) under an isometry (I.3).

We would like to extend that concept somewhat to a *spinorial form.* We understand that to mean a differential form on V^4 with values in S that transforms according to (I.3), (I.4). It will be a "spin whose components are differential forms." Some examples are spinorial 0-forms, which are spinor fields, and the so-called structure form $\Theta^{AP'} = dx^a\,\sigma_a{}^{AP'}$. With its help, the metric will take on the form (**):

$$Q = g_{ab}\,dx^a\,dx^b = \Theta^{AP'}\,\Theta^{BQ'}\,\varepsilon_{AB}\,\varepsilon_{P'Q'} = 2\,\Theta^{00'}\,\Theta^{11'} - \Theta^{01'}\,\Theta^{10'}. \tag{I.5}$$

(*) Strictly speaking, the statement of existence is correct only when the Lorentz transformation $\tilde{\sigma}^{-1}\sigma$ is actually orthochronous. In the event that V^4 is given an orientation and a time-orientation, one can refine one's choice of the σ's in such a way that this is the case. In what follows, we will then assume that V^4 has that additional structure.

(**) $\Theta^{AP'}$ is a real form: $\Theta^{00'}$, $\Theta^{11'}$ real, $\Theta^{01'} = \overline{\Theta^{10'}}$.

One gets another important example of a spinorial form from the structure form: From (I.1), the spinorial form $\Theta^{AP'} \wedge \Theta^{BQ'}$ can be decomposed into:

$$\Theta^{AP'} \wedge \Theta^{BQ'} = \Theta^{AB}\,\varepsilon^{P'Q'} + \bar{\Theta}^{P'Q'}\,\varepsilon^{AB} \qquad \text{with} \qquad \Theta^{AB} = \tfrac{1}{2}\,\Theta^{AP'} \wedge \Theta^{BQ'}\,\varepsilon_{P'Q'}\,. \tag{I.6}$$

Θ^{AB} is a spinorial 2-form that is symmetric in AB. One easily calculates from it (*) that with $\vartheta = \Theta^{00'} \wedge \Theta^{01'} \wedge \Theta^{10'} \wedge \Theta^{11'}$, one will have:

$$-\Theta^{AB} \wedge \Theta^{CD} = \varepsilon^{A(C}\,\varepsilon^{D)B}\,\vartheta, \qquad \Theta^{AB} \wedge \bar{\Theta}^{P'Q'} = 0. \tag{I.7}$$

The Θ^{AB}, together with their conjugates $\Theta^{P'Q'}$, define a basis for the space of 2-forms. Another example of a spinorial form will follow in the next section.

II. Spinor analysis

Since spinors should just as often serve as abbreviated descriptions of tensors, it is clear (at least, for real spinor fields) how one must define their covariant derivatives, namely, by translating the tensorial covariant derivative:

$$T^{AB'\cdots} \to T^{a\cdots} \to T^{a\cdots}{}_{;\,b} \to T^{AP'\cdots}{}_{;\,BQ'}\,. \tag{II.1}$$

The goal of this section is to generalize that definition to arbitrary spinor fields and spinorial forms and to derive a handy expression for it that will spare one the obviously-tedious translation in (II.1) Hence, we next define the *covariant differential* of a real spinor field with the help of (II.1) by:

$$D\,T^{AP'\cdots} = T^{AP'\cdots}{}_{;\,BQ'}\,\Theta^{BQ'},$$

and for real spinorial forms (e.g.) $\Phi^{AP'} = F^{AP'}{}_{BQ'}\,\Theta^{BQ'}$ by $D\Phi^{AP'} = DF^{AP'}{}_{BQ'} \wedge \Theta^{BQ'}$. One easily translates the following properties of the covariant differential:

1. One has the sum and Leibniz product rule (**):

$$D\,(\Phi^{S} \wedge \mathrm{X}^{\mathrm{T}}) = D\,\Phi^{S} \wedge \mathrm{X}^{\mathrm{T}} + (-1)^{\deg\Phi}\,\Phi^{S} \wedge D\,\mathrm{X}^{\mathrm{T}}\,.$$

2. The covariant differential commutes with contraction.

3. For a scalar function f, one has $D\,f = df$.

4. Taking the covariant differential commutes with going to complex-conjugate spinors (*).

(*) One represents $\Theta^{AP'} \wedge \Theta^{BQ'} \wedge \Theta^{CR'} \wedge \Theta^{DS'} = i\,\vartheta\,\eta^{AP'PQ'CR'DS'}$ and applies it to formula (I.6) as in ([1]).

(**) S, T are multi-indices. For 0-forms (i.e., spinor fields), one understands "$\wedge$" to mean ordinary multiplication.

5. D is torsion-free; i.e., one has $D\,\Theta^{AP'} = 0$ and $D\Phi = d\Phi$ for a scalar form Φ.

6. D commutes with a shift of indices.

Now, one has the theorem (**) that the covariant differential of a real spinorial form is defined uniquely by those six properties, and that there is precisely one continuation to all spinorial forms under which all of them also remain valid: One next translates, as usual (***)(3), the fact that the properties 1 to 4 are equivalent to the representation (iv):

$$\left.\begin{aligned} D\Phi^{A\cdots P'\cdots}_{B\cdots\cdots\cdots} &= d\Phi^{A\cdots P'\cdots}_{B\cdots\cdots\cdots} + \omega^A_{\ C}\wedge\Phi^{A\cdots P'\cdots}_{B\cdots\cdots\cdots}+\cdots+\bar{\omega}^{P'}_{\ Q'}\wedge\Phi^{A\cdots Q'\cdots}_{B\cdots\cdots\cdots}+\cdots-\omega^D_{\ B}\wedge\Phi^{A\cdots P'\cdots}_{D\cdots\cdots\cdots}-\cdots \\ &\text{and}\quad D\Phi = d\Phi \quad \text{for a scalar form } \Phi, \end{aligned}\right\} \tag{II.2}$$

in which the $\omega^A{}_B$ are certain Pfaffian (i.e., 1-) forms. Those forms will be defined by properties 5 and 6:

Theorem:

Each of the two systems of equations:

$$\omega^A{}_A = 0, \qquad d\Theta^{AP'} + \omega^A{}_B\wedge\Theta^{BP'} + \bar{\omega}^{P'}{}_{Q'}\wedge\Theta^{AQ'} = 0, \tag{II.3}$$

$$\omega^A{}_A = 0, \qquad d\Theta^{AB} + \omega^A{}_C\wedge\Theta^{CB} + \omega^B{}_C\wedge\Theta^{AC} = 0 \tag{II.4}$$

possesses precisely one solution of 1-forms $\omega^A{}_B$. Those two solutions will coincide. The differential D that is defined by (II.2) *using ω has the properties* 1 *to* 6 *and is therefore* ***the*** *covariant differential.*

Remark: Due to (II.2), one can subsequently rewrite (II.3) [(II.4), resp.] as $D\varepsilon_{AB} = D\Theta^{AP'} = 0$ [$D\varepsilon_{AB} = D\Theta^{AB} = 0$, resp.].

Proof. If ω is a solution of (II.3) then D will obviously have properties 5 and 6, as well. One reads off from (I.6) that (II.4) will also be true then. Now, let ω be a solution of (II.4). With that, the D that is defined by (II.2) will also commute with shift of indices then, and due to (I.6), one will also have $D\,(\Theta^{AP'}\wedge\Theta^{BQ'}) = 0$. By an application of the product rule 1, one will then see that $D\Theta^{AP'}\wedge\Theta^{AP'} = 0$, and therefore (v) $D\Theta^{AP'} = \Phi\wedge\Theta^{AP'}$ for some Φ. If one substitutes that in $D\,(\Theta^{AP'}\wedge\Theta^{BQ'}) = 0$ then it will follow that $\Phi = 0$. Any solution of (II.4) will be also a solution of (II.3) then. In order to show that (II.4) [and therefore (II.3)] has precisely one solution, we set:

(*) That statement is vacuous for real forms; cf., however, below.

(**) Cf. (2), Appendix, where the theorem for the derivative is formulated. I would like to thank Dr. J. EHLERS for the idea that the covariant differential can be constructed from those six properties.

(***) Cf., e.g., (3), Chap. I, § 9.

(3) P. JORDAN, *Schwerkraft und Weltall*, Braunschweig, 1955.

(iv) See footnote (**), pp. 3.

(v) On the basis of covariance, Φ does not depend upon AP'.

$$\omega^A{}_B = \Gamma^A{}_{BCP'}\,\Theta^{CP'}, \tag{II.5}$$

$$d\Theta^{AB} = T^{AB}{}_{CP'DQ'ER'}\,\Theta^{CP'} \wedge \Theta^{DQ'} \wedge \Theta^{ER'}, \tag{II.6}$$

in which T is completely skew in the lower index pairs. The second equation in (II.4) then assumes the form:

$$(T^{AB}{}_{CP'DQ'ER'} + \tfrac{1}{2}\delta^B_E\,\Gamma^A{}_{DCP'}\,\varepsilon_{Q'R'} + \tfrac{1}{2}\delta^A_E\,\Gamma^B{}_{DCP'}\,\varepsilon_{Q'R'})\,\Theta^{CP'} \wedge \Theta^{DQ'} \wedge \Theta^{ER'} = 0.$$

One can now apply this to the spinor in brackets in (I.2) and get:

$$2T^{ABDR'}{}_{DQ'ER'} = \tfrac{1}{2}\delta^{(B}_E\,\Gamma^{A)\,D}{}_{D\;Q'} + \Gamma^{(A\,D)}{}_{E\;Q'}.$$

If one now uses the first equation $\Gamma^A{}_{ACP'} = 0$ then upon contracting this equation by δ^E_B one will get:

$$\Gamma^A{}_{[BC]Q'} = \tfrac{1}{2}T^{AEDR'}{}_{DQ'ER'}\,\varepsilon_{BC},$$

and then, since $\Gamma^{(A\;B)}{}_{C\;Q'} = \Gamma_C{}^{(A\,B)}{}_{Q'}$:

$$\Gamma^A{}_{(BC)\,Q'} = \tfrac{1}{2}T_{BC}{}^{DR'}{}_{DQ'}{}^A{}_{R'} + \delta^A_{(C}\,T_{B)}{}^{EDR'}{}_{DQ'ER'}.$$

The theorem is then proved with that.

The proof likewise yields the solution of (II.3) and (II.4). In special cases in which the metric, and therefore Θ^{AB}, has a simple form, one can get the solution more simply in the following way: Exterior multiply the three complex 3-form equations (II.4) by the coordinate differentials dx^a and get the twelve complex equations $D\Theta^{AB} \wedge dx^a = 0$. It is often simpler to solve them than to calculate from the representation (II.6).

The form $\omega^A{}_B$ is called the *connection form* (*)([4]). It is hardly a spinorial form, since it transforms under an isometry (I.3) analogously to the Ricci symbols:

$$\tilde{\omega}^A{}_B = A^A{}_C\,\omega^C{}_D\,A_B{}^D + A^C{}_B\,dA_C{}^A, \tag{II.7}$$

as one can calculate from (II.2), (II.3), or (II.4).

Remark. At this point, we would like to draw a first comparison to the tensorial Cartan formalism. In the latter, in order to determine the connection $\omega^a{}_b$, one must solve the equations $\omega_{(ab)} = 0 = D\Theta^a$, which are analogous to (II.3) (*). They are 24 equations for the 24 real components of $\omega^a{}_b$, while we are dealing with 12 equations for 12 complex quantities in (II.3) or (II.4). In both cases, one has the freedom to calculate the components of ω in terms of the Θ or the coordinate differentials, according to the situation. In any event, only the forms ω will be important for all of what follows. There exists no difference in

(*) For the relationship between the formalism that is presented here and the theory of linear connections – in particular, the tensorial Cartan formalism – cf., ([4]).

([4]) K. BICHTELER, "Cartanformalismus für Spinoren," Preprint, Hamburg 1962.

effort between the two formalisms then, as long as one writes out the complex functions that appear in their real and imaginary components. However, that is unnecessary in many cases – e.g., in the case of the general pure radiation field that is treated below; one can get along with one name for a complex quantity. In our case, one can arrive at a further reduction in the amount of writing that one must do by a factor of 2/3 by the introduction of complex differentiation with respect to complex coordinate functions.

It is also untrue that one can employ the Cartan formalism only for the calculation of the Riemann tensor (curvature spinors, resp.), as one reads here and there, and must calculate the Γ-symbols for other examinations of a field – say, its completeness and Killing vectors – and will then do better to calculate the Riemann tensor with their help. The fact that one can also solve such problems (*)(5) with the tensorial or spinorial Cartan formalism is clear on the basis of the introduction and characterization of the ω. The actual advantage of the spinorial formalism seems to me to lie in the fact that one can calculate the reduced curvature spinors by multiplication along with no development, as will be shown in the next section. That will lighten the investigation of the radiation properties of a field considerably.

III. The curvature spinors

We would now like to apply D twice in succession to a spinorial form; we will get (e.g.) (**):

$$DD\ \Phi^{AP'}_{\dots\dots} = \Omega^A{}_B \wedge \Phi^{BP'}_{\dots\dots} + \bar{\Omega}^{P'}{}_{Q'} \wedge \Phi^{AQ'}_{\dots\dots} - \dots, \tag{III.1}$$

with

$$\Omega^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B\,. \tag{III.2}$$

$\Omega^A{}_B$ is a spinorial 2-form; one calls it the *curvature form.* We develop it into:

$$\left.\begin{aligned} \Omega^A{}_B &= -\mathrm{X}^A{}_{BCD}\Theta^{CD} - \tfrac{1}{2}\Sigma^A{}_{BP'Q'}\bar{\Theta}^{P'Q'} \\ &\text{with} \\ \mathrm{X}^A{}_{B[CD]} &= \Sigma^A{}_{B[P'Q']} = 0, \end{aligned}\right\} \tag{III.3}$$

and assert:

Theorem:

$\mathrm{X}^A{}_{BCD}$ is the dual-symmetric part of the Riemann tensor, $\Sigma^A{}_{BP'Q'}$ is the dual-skew-symmetric part – i.e., the reduced Ricci spinor.

Proof:

From the definition of a covariant differential, one has:

$$DD\ k^{AP'} = D\,(k^{AP'}{}_{;\,BQ'}\ \Theta^{BQ'}) = k^{AP'}{}_{;[\underline{BQ'CR'}]}\,\Theta^{CR'} \wedge \Theta^{BQ'}.$$

If one compares this with the result (III.1) for the 0-form (**), while considering (I.6) and (III.3), then one will find that:

(*) Cf., (5) for the treatment of holonomy groups by that calculus.
(5) W. BEIGLBÖCK, "Holonomiegruppen," Preprint Hamburg, 1962 (appearing in this Zeitschrift).
(**) Cf., the footnote to (II.2).

$$k^{AP'}{}_{;[\underline{BQ'CR'}]} = \tfrac{1}{4}(\delta^{P'}_{S'}\,\mathrm{X}^{A}{}_{EBC}\,\varepsilon_{Q'R'} + \delta^{A}_{E}\,\mathrm{X}^{P'}{}_{S'Q'R'}\,\varepsilon_{BC} + \delta^{P'}_{S'}\,\Sigma^{A}{}_{EQ'R'}\,\varepsilon_{BC} + \delta^{A}_{E}\,\Sigma^{P'}{}_{S'BC}\,\varepsilon_{Q'R'})\,k^{ES'} .$$

Obviously, it is just the Riemann spinor (*) that appears in the brackets on the right-hand side, and indeed with just the decomposition by which the spinors that were mentioned in the theorem were defined (**).

One reads off from (I.7) and (III.3) that one obtains the curvature spinors from $\Omega^A{}_B$ by exterior multiplication:

$$-\,2\,\Omega^A{}_B \wedge \Theta_{CD} = \mathrm{X}^A{}_{BCD}\;\vartheta, \qquad 2\,\Omega^A{}_B \wedge \bar{\Theta}_{P'Q'} = \Sigma^A{}_{BP'Q'}\;\vartheta. \tag{III.4}$$

Remark. There is only a very weak tensorial analogue of the very simple calculation prescription (III.4) for the curvature spinor. Upon exterior-multiplying the tensorial curvature form $\Omega^A{}_B$ by $\Theta_c \wedge \Theta_d$, one will in fact get the dual of the Riemann tensor, but one will then have much to calculate again in order to calculate the Riemann tensor itself from that dual, and thus, the irreducible components. Furthermore, one will see the radiation properties of a field more easily in terms of the Weyl spinor than in terms of the Weyl tensor.

IV. The properties of the curvature spinors

We would now like to derive briefly the known properties of the curvature spinors with calculus that we have developed up to now.

a) *Ricci identity.* – With (I.6), it follows, as in the proof above, that:

$$DD\,\xi^A = \xi^A{}_{;\,CP'DQ'}\,\Theta^{DQ'} \wedge \Theta^{CP'} = \xi^A{}_{;\,(C}{}^{R'}{}_{D)\,R'}\,\Theta^{CD} - \xi^A{}_{;\,E\,(P'}{}^{E}{}_{Q')}\,\bar{\Theta}^{P'Q'} ,$$

$$DD\,\xi^A = \Omega^A{}_B\,\xi^B = -\tfrac{1}{2}\mathrm{X}^A{}_{BCD}\,\xi^B\,\Theta^{CD'} - \tfrac{1}{2}\Sigma^A{}_{BP'Q'}\,\xi^B\,\bar{\Theta}^{P'Q'} ,$$

and from a comparison of coefficients:

$$\xi^A{}_{;\,(C\,R'D)}{}^{R'} = \tfrac{1}{2}\mathrm{X}^A{}_{BCD}\,\xi^B, \qquad \xi^A{}_{;\,E\,(P'}{}^{E}{}_{Q')} = \tfrac{1}{2}\Sigma^A{}_{BP'Q'}\,\xi^B.$$

b) *Symmetry properties of* X *and* Σ. – If one applies D to $D\,\Theta^{AP'} = 0$ then it will follow with (III.1) that:

$$\Omega^A{}_B \wedge \Theta^{BP'} + \bar{\Omega}^{P'}{}_{Q'} \wedge \Theta^{AQ'} = 0,$$

$$(\delta^{P'}_{S'}\,\mathrm{X}^{A}{}_{BCD}\,\varepsilon_{Q'R'} + \delta^{P'}_{S'}\,\Sigma^{A}{}_{BQ'R'}\,\varepsilon_{CD} + \delta^{A}_{B}\,\bar{\mathrm{X}}^{P'}{}_{S'Q'R'}\,\varepsilon_{CD} + \delta^{A}_{B}\,\bar{\Sigma}^{P'}{}_{S'CD}\,\varepsilon_{Q'R'})\;\Theta^{BS'} \wedge \Theta^{CQ'} \wedge \Theta^{DR'} = 0.$$

We apply (I.2) to the spinor in brackets and get:

(*) The factors are chosen in order to arrive at some link to the conventions of ([2, 3]).
(**) Cf., ([1]), § 2.

$$\mathrm{X}_{AB}{}^{C}{}_{D}\,\varepsilon_{Q'P'} + \varepsilon_{AD}\bar{\mathrm{X}}_{P'S'}{}^{R'}{}_{Q'} - \Sigma_{ADP'Q'} + \bar{\Sigma}_{P'Q'AD} = 0.$$

A consideration of the symmetric (antisymmetric, resp.) part of the equation will teach us that:

$$\Sigma_{ADP'Q'} = \bar{\Sigma}_{P'Q'AD},$$
$$\Lambda = \mathrm{X}_{AB}{}^{AB} = \bar{\mathrm{X}}_{P'Q'}{}^{P'Q'} = \bar{\Lambda}.$$

Σ is then real and irreducible, while X permits the reduction:

$$\mathrm{X}_{ABCD} = \Gamma_{ABCD} - \tfrac{1}{3}\Lambda\,\varepsilon_{A(C}\,\varepsilon_{D)B}\,.$$

Γ is the completely-symmetric Weyl spinor, while Λ is one-fourth of the curvature scalar *R*. From (I.7), one has:

$$\Lambda\,\vartheta = -\,2\Omega_{AB}\wedge\Theta^{AB}.$$

c) *Bianchi identity.* – We apply the operator D to the Ricci identity $DD\,\xi^{A} = \Omega^{A}{}_{B}\,\xi^{B}$ and get, on the one hand:

$$DDD\,\xi^{A} = \Omega^{A}{}_{B}\wedge D\xi^{B} + D\Omega^{A}{}_{B}\,\xi^{B},$$

from the product rule, and on the other:

$$DDD\,\xi^{A} = \Omega^{A}{}_{B}\wedge D\xi^{B},$$

from (III.1). It then follows that $D\Omega^{A}{}_{B} = 0$, and with (III.3), (I.6):

$$(\mathrm{X}^{A}{}_{BCD;ER'}\,\varepsilon_{P'Q'} + \Sigma^{A}{}_{BP'Q';ER'}\,\varepsilon_{CD})\,\Theta^{ER'}\wedge\Theta^{CP'}\wedge\Theta^{DQ'} = 0.$$

Now, (I.2) yields the Bianchi identity directly:

$$\mathrm{X}^{A}{}_{BCD;}{}^{D}{}_{P'} = \Sigma^{A}{}_{BP'R';C}{}^{R'}.$$

d) *Conformal invariance of the Weyl spinor.* – We perform a conformal transformation $Q \to e^{2U}\,Q \Leftrightarrow \Theta^{AB} \to e^{2U}\,\Theta^{AB}$ and denote the new connection by $\hat{\omega}^{A}{}_{B} = \omega^{A}{}_{B} + \pi^{A}{}_{B}$. From the theorem in the second section, it is determined by $\hat{\omega}^{A}{}_{B} = 0$ and $\hat{D}(e^{2U}\,\Theta^{AB}) = 0$. One will get the equation for $\pi^{A}{}_{B}$ from $D\,\Theta^{AB} = 0$:

$$2\,dU\wedge\Theta^{AB} + \pi^{A}{}_{C}\wedge\Theta^{CB} + \pi^{B}{}_{C}\wedge\Theta^{AC} = 0.$$

The proof of that theorem immediately implies that:

$$\pi^A{}_B = \tfrac{1}{4}(\delta^A_B\, dU + \delta^A_B\, U_{;BQ'}\,\Theta^{DQ'} + 3U_{;}{}^A{}_{Q'}\varepsilon_{DB}\,\Theta^{DQ'}) .$$

From (III.2), the new curvature form is:

$$\hat{\Omega}^A{}_B = \Omega^A{}_B + D\,\pi^A{}_B + \pi^A{}_C \wedge \pi^C{}_B .$$

One sees that any coefficient in the development of $D\,\pi^A{}_B$ in terms of Θ^{CD} will contain ε's that will consequently vanish in the part that is completely-symmetric in *ABCD*. A brief calculation will show that the same thing is true for $\pi^A{}_C \wedge \pi^C{}_B$.

V. The optical scalar. Transformation law

The calculus that we have presented up to now allows us to calculate the irreducible components of the curvature spinors for a given metric Q in a purely spinorial way (and, we believe, with relatively minor effort): (I.5) supplies one with suitable a $\Theta^{AP'}$, then (II.3) or (II.4) then gives the $\omega^A{}_B$, and from that, (III.2) and (III.4) will yield the curvature spinors. However, the problem is frequently a different one: The g_{ab} are not given, but certain physical or geometric properties of a field, and one must derive all other properties, and above all, those of the curvature spinors. The first step is therefore always to put the basic metric form of the field into a simple form. In this and the next section, it will be shown how the calculus can also be systematically employed to do that. We will assume that the field contains a preferred null congruence and show how one can systematically derive suitable coordinates from information about special properties of the congruence.

For a given congruence of null lines, we choose a spinor $\kappa^A = \underset{o}{\kappa}{}^A$ such that the null vector k^a that belongs to by means of $\sigma_{AP'}{}^a$ is tangential to it, and another spinor $\mu^A = \underset{o}{\mu}{}^A$ with $\kappa_A\,\mu^A = 1$. A null tetrad is defined by that in a known way (*). With an isometry (I.3) $\leftrightarrow$ (I.4), we now arrive at the fact that the two spinors have the components $\underset{A}{\kappa}{}^B = \delta^B_A$. With that choice, one finds from (II.5) and (II.2) that:

$$\underset{A}{\kappa}{}_{D;EQ'}\,\underset{B}{\kappa}{}^D\,\underset{C}{\kappa}{}^E\,\underset{P'}{\bar{\kappa}}{}^{Q'} = \Gamma_{ABCP'} . \tag{V.1}$$

The functions (V.1) are called *spin coefficients* (**)([6]) (also: *optical scalars*, cf. *infra*). We would like to designate them individually according to the following schema [cf., (II.5)]:

(*) Cf., e.g., ([2]), A.2.3.

(**) The terminology goes back to ([6]). The formalism that is developed there is very similar to the one here. It is also shown there how one derives spinors while bypassing (II.1) and calculates the curvature spinors from the connection that one gets. The skew-symmetrization of the covariant derivative enters in place of taking the exterior differential.

([6]) E. NEWMAN and R. PENROSE, J. Math. Phys. **3** (1962), 3.

$$\left.\begin{aligned} a)\quad & \omega^0{}_0 = \mu\,\Theta^{00'} + \alpha\,\Theta^{01'} + \tilde{\alpha}\,\Theta^{10'} + \tilde{\mu}\,\Theta^{11'}, \\ b)\quad & \omega^0{}_1 = -\tilde{\Omega}\,\Theta^{00'} + \tilde{z}\,\Theta^{01'} + \tilde{\sigma}\,\Theta^{10'} + \tilde{k}\,\Theta^{11'}, \\ c)\quad & \omega^1{}_0 = -k\,\Theta^{00'} - \sigma\,\Theta^{01'} - z\,\Theta^{10'} + \Omega\,\Theta^{11'}. \end{aligned}\right\} \tag{V.2}$$

One then has, e.g., $\tilde{\Omega} = -\Gamma^0{}_{100'} = -\mu_{B;\,CP'}\,\mu^B\,\kappa^C\,\bar{\kappa}^{P'}$, etc. Many of the quantities (V.1) describe properties of the ray congruence in the sense of geometric optics. We would like to discuss the meanings of some of them without going into a derivation of them, which can be found in e.g. ([2, 6]).

$k = 0$ means that the congruence is geodetic, while $k = \mathrm{lm}\,(z) = 0$ means that it is also hypersurface-orthogonal. Re $(z) \equiv \Theta$ is the relative increase in area in the shadows that the congruence makes, σ is its strain (i.e., shear), and Ω is the angular velocity by which the ray observer seems to rotate in the km-plane. $\tilde{\Omega} = m = 0$ means that the tetrad is parallel-displaced along the congruence. The quantities with a tilde each have the same meaning for the congruence that is generated by $m^a \leftrightarrow \mu^A\,\bar{\mu}^{P'}$.

So far, the prescription at the beginning of the section is not unique. Along with κ^A and μ^A, two primed spinors κ'^A and μ'^A that arise from κ and μ by a transformation:

$$[\kappa', \mu'] = [\kappa, \mu]\begin{bmatrix} \lambda & \Lambda \\ 0 & \lambda^{-1} \end{bmatrix} \qquad (\lambda, \Lambda \text{ complex}) \tag{V.3}$$

will also have the required properties. The transformation law for the connection form is (II.7). When one develops it in terms of $\Theta^{AP'}$, with the notations (V.2) and under the special transformation (V.3), it will read as follows [for Λ_k, Λ_l, cf., (VI.1), *et seq.*]:

$$\left.\begin{aligned} &\begin{pmatrix} \mu' & \alpha' & -\tilde{\Omega}' & \tilde{z}' \\ \tilde{\alpha}' & \tilde{\mu}' & \tilde{\sigma}' & \tilde{k}' \\ -k' & -\sigma' & -\mu' & -\alpha' \\ -z' & \Omega' & -\tilde{\alpha}' & -\tilde{\mu}' \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -\lambda\Lambda & 0 \\ \lambda^{-1}\Lambda & \lambda^{-2} & -\Lambda^2 & -\lambda^{-1}\Lambda \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda\Lambda & 1 \end{pmatrix} \begin{pmatrix} \mu & 0 & -\tilde{\Omega} & \tilde{z} \\ \tilde{\alpha} & \lambda^{-2} & \tilde{\sigma} & \tilde{k} \\ -k & -\sigma & -\mu & -\alpha \\ -z & \Omega & -\tilde{\alpha} & -\tilde{\mu} \end{pmatrix} \begin{pmatrix} \lambda\bar{\lambda} & \lambda\bar{\Lambda} & \lambda\Lambda & \Lambda\bar{\Lambda} \\ 0 & \lambda\bar{\lambda}^{-1} & 0 & \bar{\lambda}^{-1}\Lambda \\ 0 & 0 & \lambda^{-1}\bar{\lambda} & \lambda^{-1}\bar{\Lambda} \\ 0 & 0 & 0 & \lambda^{-1}\bar{\lambda}^{-1} \end{pmatrix} \\ &+ \begin{pmatrix} 1 & 0 & -\lambda\Lambda & 0 \\ \lambda^{-1}\Lambda & \lambda^{-2} & -\Lambda^2 & -\lambda^{-1}\Lambda \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda\Lambda & 1 \end{pmatrix} \begin{pmatrix} \lambda_k & \lambda_t & \Lambda_k & \Lambda_t \\ \lambda_{\bar{t}} & \lambda_m & \Lambda_{\bar{t}} & \Lambda_m \\ 0 & 0 & -\lambda^{-2}\lambda_k & -\lambda^{-2}\lambda_t \\ 0 & 0 & -\lambda^{-2}\lambda_{\bar{t}} & -\lambda^{-2}\lambda_m \end{pmatrix} \begin{pmatrix} \bar{\lambda} & \bar{\Lambda} & 0 & 0 \\ 0 & \bar{\lambda}^{-1} & 0 & 0 \\ 0 & 0 & \bar{\lambda} & \bar{\Lambda} \\ 0 & 0 & 0 & \bar{\lambda}^{-1} \end{pmatrix}. \end{aligned}\right\} \tag{V.5}$$

One has:

$$\begin{pmatrix} \Theta^{k'} & \Theta^{t'} \\ \Theta^{\bar{t}'} & \Theta^{m'} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & -\Lambda \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \Theta^k & \Theta^t \\ \Theta^{\bar{t}} & \Theta^m \end{pmatrix} \begin{pmatrix} \bar{\lambda}^{-1} & 0 \\ -\bar{\Lambda} & \bar{\lambda} \end{pmatrix}. \tag{V.5}$$

VI. Explicit formulas. Coordinates that are adapted to the properties of a null congruence

We first denote the index-pair AP' according to the prescription:

$$\begin{array}{cccc} 00' & 01' & 10' & 11' \\ \downarrow & \downarrow & \downarrow & \downarrow \\ k & t & \bar{t} & m \end{array} \tag{VI.1}$$

with lowercase Latin symbols k, t, …, such that, e.g., $\Theta^{01'} = \Theta^t$ and $df = f_{,AP'}\,\Theta^{AP'} = f_k\,\Theta^k + f_t\,\Theta^t + \ldots$ With the notations (V.2), (II.3) will then read as follows:

$$\left.\begin{array}{llll}
a) & d\Theta^k & +(\alpha+\bar{\tilde{\alpha}}+\bar{\tilde{\Omega}})\,\Theta^t\wedge\Theta^k & +(\bar{\alpha}+\tilde{\alpha}+\tilde{\Omega})\,\Theta^{\bar{t}}\wedge\Theta^k \\
 & & +(\tilde{\mu}+\bar{\tilde{\mu}})\,\Theta^m\wedge\Theta^k & +(\tilde{z}-\bar{\tilde{z}})\,\Theta^{\bar{t}}\wedge\Theta^k \\
 & & +\bar{\tilde{k}}\,\Theta^m\wedge\Theta^t & +\tilde{k}\,\Theta^{\bar{t}}\wedge\Theta^k \quad = 0, \\
 & & & \\
b) & d\Theta^t & +(\mu-\bar{\mu}+\tilde{z})\,\Theta^k\wedge\Theta^t & +(\bar{\tilde{\mu}}+\tilde{\mu}+z)\,\Theta^t\wedge\Theta^m \\
 & & +(\bar{\Omega}+\tilde{\Omega})\,\Theta^t\wedge\Theta^k & +(\bar{\alpha}-\tilde{\alpha})\,\Theta^t\wedge\Theta^{\bar{t}} \\
 & & +\bar{\sigma}\,\Theta^k\wedge\Theta^{\bar{t}} & +\tilde{\sigma}\,\Theta^{\bar{t}}\wedge\Theta^m \quad = 0, \\
 & & & \\
c) & d\Theta^m & +(\alpha+\bar{\tilde{\alpha}}+\Omega)\,\Theta^m\wedge\Theta^t & +(\bar{\alpha}+\tilde{\alpha}+\bar{\Omega})\,\Theta^m\wedge\Theta^{\bar{t}} \\
 & & (\mu+\bar{\mu})\,\Theta^m\wedge\Theta^k & +(z-\bar{z})\,\Theta^t\wedge\Theta^{\bar{t}} \\
 & & +k\,\Theta^t\wedge\Theta^k & +\bar{k}\,\Theta^{\bar{t}}\wedge\Theta^k \quad = 0.
\end{array}\right\} \tag{VI.2}$$

For the evaluation of (VI.2), one appeals to the well-known theorem of FROBENIUS ([7,8]), which says that for r forms $\Theta^1, \ldots, \Theta^r$, if $d\Theta^s \wedge \Theta^1 \wedge\ldots\wedge \Theta^r = 0$ then there will be precisely r functions $x^1, \ldots, x^r$, such that those forms depend upon the differentials dx^k linearly ($\Theta^s = a^s{}_k\,dx^k$; $s, k = 1, \ldots, r$). The more that the combinations of spin coefficients that appear in (VI.2) vanish, or the more that the spin dyads can be annulled by a suitable choice (V.3), (V.4), the simpler that the basic metric form can be made according to (I.5).

As an *example*, we would like to consider a gravitational field that admits a normal and shear-free null congruence. From $k = \mathrm{Im}(z) = 0$ and (VI.2c), it will then follow that $d\Theta^m \wedge \Theta^m = 0$, and with FROBENIUS's theorem $\Theta^m = F\,du$. A real stretching (V.3) with $\Lambda = 0$ and $\lambda = F^{1/2}$ makes $\Theta^m = du$. It follows from $\sigma = 0$ and (VI.2b) that $d\Theta^t \wedge \Theta^m \wedge \Theta^t = 0$, and $\Theta^t = p\,e^{2ir}\,(dz + B\,du)$, in which p and r are real and B and z are complex. From (V.5), the rotation (V.3), with $\Lambda = 0$ and $\lambda = e^{ir}$, will imply that $\Theta^t = p\,(dz + B\,du)$.

([7]) A. LICHNEROWICZ, *Théorie globale des Connexions et des groupes d'Holonomie*, Paris, Dunod, 1955.

([8]) G. DE RHAM, *Variétés differentiables*, Paris, Hermann, 1955.

Conversely, one also sees from (VI.2b) that one can arrive at this form for Θ^t only for $\sigma = 0$. Let s, along with u, $x = \mathrm{Re}\,(z)$, and $y = -\,i\,\mathrm{Im}\,(z)$ be four coordinates. That result then says that the congruence will map two wave surfaces $u = c$, $s = c'$, and $u = s$, $s = c''$ to each other conformally by throwing shadows (*durch Schattenwurf*) precisely when $\sigma = 0$ (c, c', c'' are constants). A real null rotation (V.3) with $\lambda = 1$ and Λ = real will then imply that $\tilde{z} - \ddot{\tilde{z}} = 0$, and (VI.2a) will yield $\Theta^k = \overline{e}^{\,C}\,ds + H\,du$. One will further arrive at $d\Theta^k \wedge \Theta^k = 0$, $\Theta^k = dr + H\,du$ by way of the null rotation $\Lambda = p^{-1}\int e^C C_{,\bar{z}}\,ds$ (*). With consideration given to (I.5), we summarize [cf., ([9], [10])]:

Theorem:

The metric form of a field that admits a normal and shear-free null congruence can be put into the form:

$$Q = -\,2p^2\,[dz + B\,du]^2 + 2\,dr\,du + 2\,H\,du^2.$$

VII. The curvature spinors of the general pure radiation field

A pure gravitational radiation field will be defined [above all, on the basis of the analogy with the electromagnetic case, cf., e.g., ([9])] by the existence of a normal, shear-free, null ray congruence, along with the vacuum equations. In the last section, we put the structure form of such a field into the form:

$$\Theta^m = du, \qquad \Theta^t = p\,(dz + B\,du), \qquad \Theta^t = dz + H\,du.$$

Here, we would like to give the connection form and curvature spinors in these coordinates, since those formulas will be necessary for the investigation of such fields; e.g., the determination of special solutions or the propagation properties of the curvature spinors (**).

With (II.4), one gets the following expressions for the connection form:

$$\omega^0{}_0 = G\,du + \tfrac{1}{2}(P_z - \tfrac{1}{2}p^2\overline{B}_r)\,dz - \tfrac{1}{2}(P_{\bar{z}} + \tfrac{1}{2}p^2 B_r)\,d\bar{z}\,,$$

$$\omega^0{}_1 = E\,du + F\,dz - p\,B_{\bar{z}}\,d\bar{z} - \tfrac{1}{2}p\,B_r\,dr,$$

$$\omega^1{}_0 = \tfrac{1}{2}p^{-1}\overline{b}_r\,du + p_r\,d\bar{z}\,,$$

(*) The complex derivative is defined by $dF = F_s\,ds + F_z\,dz + F_{\bar{z}}\,d\bar{z} + F_u\,du$. It behaves like a real one.

([9]) P. JORDAN, *Problems of Gravitation*, Air Force report 1961.
W. KUNDT, Zeit. Phys. **163** (1962).

([10]) I. ROBINSON and A. TRAUTMAN, "Some Spherical Grav. Waves in Gen. Rel.," Proc. Roy. Soc. (London) **265** (1962).

(**) Cf., ([5]).

in which:

$$P = \ln p, \qquad p = p^2 B, \qquad F = p\,\Phi, \qquad \Phi = P_u - H\,P_r - p^{-2}\,\mathrm{Re}\,(b_z),$$

$$G = \tfrac{1}{2}(H + p^{-2}\,\mathrm{Im}\,(b_z) - \tfrac{1}{2}\,p^2 (B\overline{B})_r\,),$$

$$E = p^{-1} H_{\bar z} - p(\overline{B}\,B_{\bar z} + \tfrac{1}{2} H\,B_r) + BF.$$

With (III.2), (I.6), and (III.4), one gets the following components for the curvature spinors:

$$\Gamma_{0000} = 0,$$

$$\Gamma_{0001} = p_r\,\overline{B}_r + \tfrac{1}{2}\,p\,B_{rr} - p^{-1}\,P_{rz}\,,$$

$$\Gamma_{0011} = \tfrac{2}{3}[\,p^2 B_r B_r + p^{-2}(\,p^2 \overline{B}_r)_{\bar z} + 2\,\mathrm{Re}(B\,P_{rz}) - p^{-2} P_{z\bar z} - \tfrac{1}{2}\,p^{-2}(\,p^2 B_r)_z - \tfrac{1}{2} H_{rr} + (H\,P_r)_r - P_{ru}\,],$$

$$\begin{aligned}\Gamma_{0111} = {} & p^{-1}(P_{\bar z}\overline{B} + \tfrac{1}{2}\,\overline{B}_{\bar z} - P_u + H\,P_r)_{\bar z} + \tfrac{3}{2}\,p\,\overline{B}_r B_{\bar z} \\ & + \tfrac{1}{2}\,p^{-1}[2B\,P_{z\bar z} + \overline{B}(\,p^2 B_r)_{\bar z} + B(\,p^2 B_r)_z - (\,p^2 B_r)_u - p^{-2}(\,p^2 B_{\bar z})_z + H(\,p^2 B_r)_r\,] - (\,p^{-1} H_{\bar z})_r\,,\end{aligned}$$

$$\begin{aligned}\Gamma_{1111} = {} & 2\,p^{-2}[B(\,p^2 B_{\bar z})_z + \overline{B}(\,p^2 B_{\bar z})_{\bar z}\,] - 4P_u B_{\bar z} - 2B_{u\bar z} + 2B_{\bar z}(2H\,P_r + 2\overline{B}_{\bar z} - H_r) \\ & + 2(H\,B_r - p^{-2} H_{\bar z})_{\bar z}\,,\end{aligned}$$

$$\Lambda = 2\,p^{-3}\,\mathrm{Re}\,[2p^3\,B\,P_{rz} + (p\,b_z)_r] + 2\,p^{-2} P_{z\bar z} + H_{rr} - 4\,p^{-3/2}[\,p^{3/2}(P_u - H\,P_r)]_r - \tfrac{1}{2}\,p^2 B_r \overline{B}_r\,,$$

$$\Sigma_{000'0'} = 2\,p^{-1}\,p_{rr}\,,$$

$$\Sigma_{001'1'} = 2P_r \overline{B}_z + \overline{B}_{rz} + \tfrac{1}{2}\,p^2 \overline{B}_r^2\,,$$

$$\Sigma_{010'0'} = p^{-1} P_{r\bar z} + 2\,p_r B_r + \tfrac{1}{2}\,pB_{rr}\,,$$

$$\begin{aligned}\Sigma_{011'1'} = {} & p^{-1}(P_u + \tfrac{1}{2}\,\overline{B}_{\bar z} - \tfrac{1}{2}\,B_z - \overline{B}\,P_{\bar z} - BP_z - H_r)_z + \tfrac{1}{2}\,p(B\overline{B}_z - B_u)_r + \tfrac{1}{2}\,p(\overline{B\,B_{r\bar z}} + H\overline{B}_{rr}) \\ & + p^{-1}(2\overline{B}_z P_{\bar z} - HP_{rz}) + p\,\overline{B}_r(2\overline{B}\,P_{\bar z} + 2H\,P_r + 2B\,P_z + \tfrac{1}{2}\,\overline{B}_{\bar z} + B_z - 2P_u)\,,\end{aligned}$$

$$\Sigma_{111'1'} = 2[\overline{B}_z B_{\bar z} + p^{-1}(F_u - H\,F_r) + \Phi\,H_r\,] - 2\,\mathrm{Re}(2B\,p^{-1} F_z - B_r H_z + \Phi B_z) - 2\,p^{-2} H_{z\bar z}\,.$$

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