"Mémoire sur l'expression analytique de élasticité et de la raideur des courbes à double courbure," J. Ecole poly., Cahier XVII, **10** (1815), 418-456.

Memoir on the analytical expression of the elasticity and stiffness of curves of double curvature

Read to the Institute on 22 August 1814

BY J. BINET

Translated by D. H. Delphenich

If an arbitrary cause determines a change of form in a material line with double curvature, whose length one imagines to be divided into infinitely-small pieces, then that change can be referred to three distinct types of variations for each of its points:

1. An extension or contraction of the element of the curve in the sense of its length.

2. An increase or decrease in the contingency angle that is formed by two consecutive infinitely-small elements, or a *flexure* of the curve.

3. An increase or decrease of the contingency angle that is defined by two consecutive osculating planes, when referred to that same point, and that can be called a *torsion.*

If the material is *elastic* (i.e., if it opposes the changes in form that the forces tend to impose upon it) then one can always consider that resistance at each point to be produced by three types of forces that oppose the three types of variations that we just spoke of. The force that is contrary to extension, or the longitudinal contraction of the elements of the curve, is called *tension*. The one that resists the opening up or reduction of the contingency angle is commonly called the *elasticity* of the angle of the curve, or more simply, the elasticity of the curve, because it is the only one besides tension that has been considered up to now. The third force tends to prevent the contingency angle between two consecutive osculating planes from changing. That new kind of elasticity is exerted by means of the torsion of the element of the curve. That type of force develops mainly in curves of double curvature, and up to the present geometers seem to have neglected its consideration entirely. While addressing the problem that we shall treat here, *Lagrange* arrived at some equations, which are undoubtedly exact, for curves that are endowed with only the first two types of elasticity, or for planar curves that are subjected to forces that are situated in their plane, but which are hardly convenient for the general problem of elastic curves of double curvature. Imagine, for example, a metallic filament that is bent into the form of a helix, like the springs that are called *coil springs.* If a force acts in such a manner that it brings the two extremities of that coil closer together or further apart then one will see well enough that the change in form that it experiences will take place everywhere at the expense of torsion in the metallic filament.

The three kinds of geometric elements that I just considered to be variable for an elastic curve will be constant for a stiff curve, and geometers know that it will result that the indefinite equations that these two problems provide must present themselves in absolutely the same form, or they must be capable of being converted into the same form, and they must differ only by their objective; i.e., by the things that they wish to determine. It is very singular that the author of *Mécanique analytique*, who insisted upon pointing that out on several occasions, neglected its application to the problem of elastic curves with double curvature. He was led to consider the type of torsion forces that present themselves naturally in that context, moreover. He had yet another advantage in his expression for the stiffness: He could avoid the forces that are produced by the indeterminates that *Lagrange* employed as multipliers for the variations of certain differential functions that must be constant along a rigid curve that is invariable in form. Upon examining what those forces are, one will see that two of them are infinite, one of first order and the other of second order. The reason for that extraordinary circumstance is found in the function that these forces are intended to fulfill. That is what one will see sufficiently well in the course of this paper. *Lagrange* did not look for the particular geometric significance of each of the three differential quantities that must be invariable, so he did not seem to recognize the inconvenience that I spoke of.

In order to invest a little clarity in a very delicate subject of analytical mechanics, I shall begin by treating the questions of equilibrium of several forces that act on polygons whose sides are straight and rigid rods. Here, I must predict that the type of coupling that I shall suppose for those rods hardly seems natural, and while reading that part of my work, one must not forget that it is quite preliminary and that it will be explained by what follows it. One must further recall that this paper is, above all, intended to complete and clarify some chapters in *Lagrange's* beautiful work, and that all of the details that one can might desire while reading it have been omitted, because they are presented in *Mécanique analytique.*

Having thus been led to consider some new elements in the curves and polygons that are constructed in an arbitrary manner in space, one will find some new expressions and some new formulas that relate to their geometry in my paper.

1. – Take several points in space that are denoted by the numbers 1, 2, 3, ..., and suppose that those points are joined by lines $\overline{12}$, $\overline{23}$, $\overline{34}$, ..., which form a continuous chain. Represent those successive lines by a, a', a'', ..., and the orthogonal coordinates of the first point by x, y, z ; those of the second one are x', y', z', those of third by x'', y'', z'', ... One knows that:

$$
\begin{aligned}
a^2 &= \Delta x^2 + \Delta y^2 + \Delta z^2, \\
a'^2 &= \Delta x'^2 + \Delta y'^2 + \Delta z'^2, \\
&\ldots,
\end{aligned}
$$

in which Δx represents the difference $x' - x$, and Δx^2 represents it square.

The coordinates of the points 2, 3, 4 relative to the first one, which is taken to be the origin, are:

$$x' - x,\ \ x'' - x,\ \ x''' - x, \ldots,$$
$$y' - y,\ \ y'' - y,\ \ y''' - y, \ldots,$$
$$z' - z,\ \ z'' - z,\ \ z''' - z, \ldots,$$

i.e.:

$$\Delta x, \quad \Delta x + \Delta x', \quad \Delta x + \Delta x' + \Delta x'', \ldots,$$
$$\Delta y, \quad \Delta y + \Delta y', \quad \Delta y + \Delta y' + \Delta y'', \ldots,$$
$$\Delta z, \quad \Delta z + \Delta z', \quad \Delta z + \Delta z' + \Delta z'', \ldots,$$

or rather:

$$\Delta x, \quad 2\Delta x + \Delta^2 x, \quad 3\Delta x + 3\Delta^2 x + \Delta^3 x, \ldots,$$
$$\Delta y, \quad 2\Delta y + \Delta^2 y, \quad 3\Delta y + 3\Delta^2 y + \Delta^3 y, \ldots,$$
$$\Delta z, \quad 2\Delta z + \Delta^2 z, \quad 3\Delta z + 3\Delta^2 z + \Delta^3 z, \ldots$$

The cosine of the angle aa' of the first two sides is:

$$\cos \widehat{aa'} = \frac{\Delta x\,\Delta x' + \Delta y\,\Delta y' + \Delta z\,\Delta z'}{a\,a'} .$$

The square of the sine will then be:

$$(f_1) \qquad \sin^2 \widehat{aa'} = \frac{(\Delta x^2 + \Delta y^2 + \Delta z^2)(\Delta x'^2 + \Delta y'^2 + \Delta z'^2) - (\Delta x\,\Delta x' + \Delta y\,\Delta y' + \Delta z\,\Delta z')^2}{a^2\,a'^2} .$$

Upon multiplying this by the product $a^2\,a'^2$ of the squares of the two sides of the angle, what will remain is the numerator, which is the square of the area of the parallelogram that is constructed on the first two sides $a\,a'$, and one can give that expression the form:

$$(\Delta y\,\Delta z' - \Delta z\,\Delta y')^2 + (\Delta z\,\Delta x' - \Delta x\,\Delta z')^2 + (\Delta x\,\Delta y' - \Delta y\,\Delta x')^2 .$$

The term $\Delta y\,\Delta z' - \Delta z\,\Delta y'$ is the projection of that rhombus onto the yz-plane. It is equal to $\Delta y\,\Delta^2 z - \Delta z\,\Delta^2 y$, so the expression for the area of that parallelogram is:

$$[(\Delta y\,\Delta^2 z - \Delta z\,\Delta^2 y)^2 + (\Delta z\,\Delta^2 x - \Delta x\,\Delta^2 z)^2 + (\Delta x\,\Delta^2 y - \Delta y\,\Delta^2 x)^2],$$

moreover.

2. – Consider three consecutive sides a, a', a'' and draw a plane through each of their extremities that passes through the other ones or is parallel to them. A parallelepiped will result from that construction, four edges of which will be equal and parallel to a, four others will be parallel to a', and the last four, to a''.

As one knows, the volume of that body will be expressed by:

$$\begin{aligned} &\Delta x\,(\Delta y + \Delta y')\,(\Delta z + \Delta z' + \Delta z'') - \Delta x\,(\Delta z + \Delta z')\,(\Delta y + \Delta y' + \Delta y'') \\ &+ \Delta y\,(\Delta z + \Delta z')\,(\Delta x + \Delta x' + \Delta x'') - \Delta y\,(\Delta x + \Delta x')\,(\Delta z + \Delta z' + \Delta z'') \\ &+ \Delta z\,(\Delta x + \Delta x')\,(\Delta y + \Delta y' + \Delta z'') - \Delta z\,(\Delta y + \Delta y')\,(\Delta x + \Delta x' + \Delta x''), \end{aligned}$$

and if one removes the terms that cancel each other, what will remain is:

$$\Delta x\,\Delta y'\,\Delta z'' + \Delta y\,\Delta z'\,\Delta x'' + \Delta z\,\Delta x'\,\Delta y''$$
$$- \Delta x\,\Delta z'\,\Delta y'' - \Delta y\,\Delta x'\,\Delta z'' - \Delta z\,\Delta y'\,\Delta x''.$$

Replace the values of Δx, $\Delta x'$, $\Delta x''$, Δy, $\Delta y'$, $\Delta y''$, ... with their values as functions of Δx, $\Delta^2 x$, $\Delta^3 x$, Δy, ..., and again remove the terms that cancel, and the latter expression for the volume will become:

$$\Delta x\,\Delta^2 y\,\Delta^3 z + \Delta y\,\Delta^2 z\,\Delta^3 x + \Delta z\,\Delta^2 x\,\Delta^3 y$$
$$- \Delta x\,\Delta^2 z\,\Delta^3 y - \Delta y\,\Delta^2 x\,\Delta^3 z - \Delta z\,\Delta^2 y\,\Delta^3 x,$$

about which, one can remark that, as in the expression for the area of the parallelogram that is constructed from a and a', the $\Delta x'$, $\Delta x''$, $\Delta y'$, $\Delta y''$, ..., are replaced with the $\Delta^2 x\,\Delta^3 x$, $\Delta^2 y\,\Delta^3 y$, ...; the reason for that will become obvious below in § **4**.

The square of that function can be put into the form:

$$(\Delta x^2 + \Delta y^2 + \Delta z^2) \times (\Delta^2 x^2 + \Delta^2 y^2 + \Delta^2 z^2) \times (\Delta^3 z^2 + \Delta^3 y^2 + \Delta^3 z^2)$$

$$+ \ 2\left\{\begin{array}{l} \quad (\Delta^2 x\,\Delta^3 x + \Delta^2 y\,\Delta^3 y + \Delta^2 z\,\Delta^3 z) \\ \times (\Delta^3 x\,\Delta x + \Delta^3 y\,\Delta y + \Delta^3 z\,\Delta z) \\ \times (\Delta x\,\Delta^2 x + \Delta y\,\Delta^2 y + \Delta z\,\Delta^2 z) \end{array}\right\}$$

$$- (\Delta^2 x\,\Delta^3 x + \Delta^2 y\,\Delta^3 y + \Delta^2 z\,\Delta^3 z)^2 \times (\Delta x^2 + \Delta y^2 + \Delta z^2)$$

$$- (\Delta^3 x\ \Delta x + \Delta^3 y\ \Delta y + \Delta^3 z\ \Delta z)^2 \times (\Delta^2 x^2 + \Delta^2 y^2 + \Delta^2 z^2)$$

$$- (\Delta x\ \Delta^2 x + \Delta y\ \Delta^2 y + \Delta z\ \Delta^2 z)^2 \times (\Delta^3 x^2 + \Delta^3 y^2 + \Delta^3 z^2).$$

(*See* the paper by *Lagrange* "Sur la Rotation des corps," (1773), Berlin, or the preceding volume of this journal.)

The three sides a, a', a'' are not in the same plane, and it will be useful for us to determine the angle that the plane of the first two forms with that of the second and third one. The sine of that angle is equal to the product of the volume of the parallelepiped by the length of its edge that is common to two planes, divided by the product of the areas of the two faces that are included in those planes. For the angle between the planes that cut along a', that sine will then be:

$$(f_2) \qquad \frac{\left\{\begin{array}{l} \quad \Delta x\,\Delta^2 y\,\Delta^3 z + \Delta y\,\Delta^2 z\,\Delta^3 x + \Delta z\,\Delta^2 x\,\Delta^3 y \\ - \Delta x\,\Delta^2 z\,\Delta^3 y - \Delta y\,\Delta^2 x\,\Delta^3 z - \Delta z\,\Delta^2 y\,\Delta^3 x \end{array}\right\}\sqrt{\Delta x'^2 + \Delta y'^2 + \Delta z'^2}}{\sqrt{\left\{\begin{array}{l} \quad [(\Delta y\,\Delta^2 z - \Delta z\,\Delta^2 y)^2 + (\Delta z\,\Delta^2 z - \Delta x\,\Delta^2 z)^2 + (\Delta x\,\Delta^2 y - \Delta y\,\Delta^2 x)^2] \\ \times [(\Delta y'\,\Delta^2 z' - \Delta z'\,\Delta^2 y')^2 + (\Delta z'\,\Delta^2 z' - \Delta x'\,\Delta^2 z')^2 + (\Delta x'\,\Delta^2 y' - \Delta y'\,\Delta^2 x')^2] \end{array}\right\}}}.$$

In order for the three sides a, a', a'' to be situated in the same plane, it is necessary that this quantity must be zero, or that:

$$\begin{aligned}0 = {} & \Delta x\,\Delta^2 y\,\Delta^3 z + \Delta y\,\Delta^2 z\,\Delta^3 x + \Delta z\,\Delta^2 x\,\Delta^3 y \\ & - \Delta x\,\Delta^2 z\,\Delta^3 y - \Delta y\,\Delta^2 x\,\Delta^3 z - \Delta z\,\Delta^2 y\,\Delta^3 x ,\end{aligned}$$

and that condition must be verified for each of the first $n - 3$ summits, where n is the total number of them if all of the polygon is in the same plane.

3. – Suppose that one has a curve of double curvature, and start from an arbitrary point. Take three consecutive infinitesimal elements, so everything that was just said about a polygon can apply to the curve, and the formulas will undergo only a change of the characteristic of the finite differences Δ into that d of infinitely-small differences. The area of the parallelogram that is constructed from the sides a, a' of the polygon will change into the area of one that is constructed from two consecutive infinitely-small sides, which will be infinitely-small of order three, since the two sides and their angle are infinitely small. Its expression is:

$$\sqrt{[(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2]}.$$

If one divides that area by the product of the lengths of the two consecutive elements or (what amounts to the same thing) by the square ds^2 of the element of arc length of the curve then one will get the sine of the contingency angle or the angle between two consecutive sides, which is what the radius of curvature depends upon ordinarily. That infinitely-small angle, which does not differ from its sine, will then be:

$$(f_3) \qquad \frac{\sqrt{[(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2]}}{ds^2}.$$

One will likewise get the angle between two consecutive osculating planes upon substituting differentials for differences in formula (f_2) and dropping the primes; the sine of that angle will then be:

$$(f_3') \qquad ds\,\frac{dx\,d^2y\,d^3z + dy\,d^2z\,d^3x + dz\,d^2x\,d^3y - dx\,d^2z\,d^3y - dy\,d^2x\,d^3z - dz\,d^2y\,d^3x}{(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2z)^2}.$$

It is precisely that angle that constitutes the *second curvature* of the curve. In order for that curve to be planar, it is necessary that the preceding function should be zero at all of its points, or that one should have:

$$0 = dx\,d^2y\,d^3z + dy\,d^2z\,d^3x + dz\,d^2x\,d^3y - dx\,d^2z\,d^3y - dy\,d^2x\,d^3z - dz\,d^2y\,d^3x .$$

(See *Calcul différentiel* by *Lacroix*, vol. I, pp. 631, new edition.)

4. – Go back to the polygon 1, 2, 3, ... that is defined by the sides a, a', a'', ... Extend the side a' towards the point 2, which will serve as origin, by a quantity that is equal to its length. One will then arrive at a point whose coordinates are $x' - \Delta x'$, $y' - \Delta y'$, $z' - \Delta z'$, or rather, $x - \Delta^2 x$, $y - \Delta^2 y$, $z - \Delta^2 z$. Call that point 3′. If we repeat the same construction with the summits 3, 4, ..., then we will have:

$$x' - \Delta^2 x',\ y' - \Delta^2 y',\ z' - \Delta^2 z',\ \ldots; \qquad x'' - \Delta^2 x'',\ y'' - \Delta^2 y'',\ z'' - \Delta^2 z'',\ \ldots$$

for the coordinates of the extremities of the extensions of the sides a'', a''', ... Call those points 3″, 3‴, 3⁗, ..., and join them to the points 1, 2, 3, 4, ..., by the straight lines b, b', b'', ... The lengths of those lines will be given by the equations:

$$\begin{aligned} b^2 &= \Delta^2 x^2 + \Delta^2 y^2 + \Delta^2 z^2, \\ b'^2 &= \Delta^2 x'^2 + \Delta^2 y'^2 + \Delta^2 z'^2, \\ b''^2 &= \Delta^2 x''^2 + \Delta^2 y''^2 + \Delta^2 z''^2, \\ &\ldots \end{aligned}$$

We draw a parallel through the point 3′ that we just constructed that is equal to b' and directed from 3′ in the same way that 3′ is directed towards 2. We will then arrive at a point 4′ whose coordinates will be $x - \Delta^3 x + \Delta^2 x'$, $y - \Delta^3 y + \Delta^2 y'$, $z - \Delta^3 z + \Delta^2 z'$, or $x + \Delta^3 x$, $y + \Delta^3 y$, $z + \Delta^3 z$; let c denote its distance to the point 1. Similarly, draw a parallel through the point 3″ that is equal to b'' and directed in the same sense. We will then arrive at a point 4″ whose coordinates will be $x' + \Delta^3 x'$, $y' + \Delta^3 y'$, $z' + \Delta^3 z'$, and its distance from the point 2 will be called c'. One will have:

$$\begin{aligned} c^2 &= \Delta^3 x^2 + \Delta^3 y^2 + \Delta^3 z^2, \\ c'^2 &= \Delta^3 x'^2 + \Delta^3 y'^2 + \Delta^3 z'^2, \\ &\ldots \end{aligned}$$

After having drawn a parallel through the point 4′ that is equal to c', a parallel through 4″ that is equal to c'', ..., and so on for the ultimate constructions that depend upon the number n of proposed points 1, 2, 3, etc, then we will have constructed $n - 2$ points that are denoted by 3′, 4′, 5′, ... and separated by the distances b', c', d', ..., $n - 3$ points that are denoted by 3″, 4″, 5″, ..., $n - 4$ points that are denoted by 3″, 4″, ..., and separated by distances b'', c'', d'', ..., b''', c''', d''', ... If we join the points 1, 2, to the $n - 2$ points 3′, 4′, 5′, ..., then we will have n points in all that are separated by distances a, a', b', c', ..., and form a polygon of $n - 1$ sides, like the original polygon. Upon commencing with point 2, the lines a', a'', b'', c'', ..., we will also define another polygon that has the same relationship to the polygon 2, 3, 4, ..., that the polygon 1, 2, 3′, 4′, 5′, ..., has to the proposed one. Things happen just the same for the point 3, and so on for the other ones.

The coordinates of the summits 1, 2, 3′, 4′, 5′, ..., of the new polygon that answers to the point 1 will then be:

$$x, \qquad x + \Delta x, \qquad x - \Delta^2 x, \qquad x + \Delta^3 x,\ \ldots,$$

$$
\begin{array}{llll}
y, & y + \Delta y, & y - \Delta^2 y, & y + \Delta^3 y, \ldots, \\
z, & z + \Delta z, & z - \Delta^2 z, & z + \Delta^3 z, \ldots
\end{array}
$$

Call those coordinates $x, x', x'', x''', \ldots, y, y', y'', y''', \ldots, z, z', z'', z''', \ldots$ Those of a polygon that has the same relationship with respect to the one that we address that the latter polygon has to the proposed polygon will also have the values:

$$
\begin{array}{llll}
x, & x + \Delta x, & x - \Delta^2 x, & x + \Delta^3 x, \ldots, \\
y, & y + \Delta y, & y - \Delta^2 y, & y + \Delta^3 y, \ldots, \\
z, & z + \Delta z, & z - \Delta^2 z, & z + \Delta^3 z, \ldots
\end{array}
$$

Now, upon letting i denote an arbitrary number:

$$\Delta^i x_{\prime} = x_{\prime}^{(i)} - i\, x_{\prime}^{(i-1)} + i\,\frac{i-1}{2}\, x_{\prime}^{(i-2)} - i\,\frac{i-1}{2}\cdot\frac{i-2}{3}\, x_{\prime}^{(i-3)} + \ldots$$

If one replaces the $x_{\prime}^{(i)}$, $x_{\prime}^{(i-1)}$, ... with their values in terms of x, Δx, $\Delta^2 x$, ... then one will have:

$$\Delta^i x_{\prime} = x\left[1 - i + i\,\frac{i-1}{2} - i\,\frac{i-1}{2}\cdot\frac{i-2}{3} + \text{etc.}\right] \mp \left[\Delta^i x + i\,\Delta^{(i-1)} x + i\,\frac{i-1}{2}\,\Delta^{(i-2)} x + \text{etc.} + i\,\Delta x\right].$$

The – sign will be used when the number i is even. The multiplier of x for that value will be $(1 - 1)^i = 0$, and the second part will be equal to $x^{(i)} - x$; one will then have $\Delta^i x_{\prime} = \mp (x^{(i)} - x)'$. It will then result that the coordinates $x_{\prime}$ above will come down to:

$$x_{\prime}, \qquad x_{\prime} + x' - x, \qquad x'_{\prime} + x'' - x, \ldots,$$

i.e., $x, x', x'', x''', \ldots$, and for y and z, one will also have $y, y', y'', \ldots, z, z', z'', \ldots$ One will then see that this new polygon is precisely the proposed polygon. That reciprocity is very singular. One can deduce that from a direct consideration of the construction by which one passes from one of those polygons to the other one. It will permit one to call the polygons that were derived in that manner *reciprocal* if one finds that their consideration can be useful. They have many other properties, and we shall perhaps develop the most remarkable of them, moreover.

The construction of the reciprocal polygon to the proposed polygon amounts to this: Prolong the last side along its length to the point that is denoted $n - 1$ and join the extremity of that prolongation to the point $n - 2$ by a line to which one will have to draw a parallel through the extremity of the prolongation of the penultimate side. Join that extremity to the point $n - 3$, as well as that of the parallel, and after having prolonged the antepenultimate side along its length, draw a parallel that is equal to the first of the preceding two lines that goes from the point $n - 3$, and draw a parallel through its extremity that is equal to the second one. After having once more joined the summits of that new polygon by lines to the point $n - 4$, we will continue the construction in the same manner up to the last side. One will have thus constructed the reciprocal polygons to the polygon that is composed of the last two sides of the proposed one, of its last three sides, of its last four sides, …, and finally, of all of its sides.

A polygon has two different reciprocal polygons that answer to each of its extremities and in the sense by which one follows its contour. The distance from the extremity of the last side of one of those reciprocal polygons to the origin of its first side is equal to the analogous line of the other one.

5. – Now take a system of three points 1, 2, 3, to which three forces P, P', P'' are applied, whose components we denote by X, Y, Z, X', Y', ..., in the sense of the three coordinate axes. We suppose that the rods a, a' that separate these points are capable of extension or contraction while remaining inflexible and straight, that the longitudinal forces on these rods will be A, A' when equilibrium has been established, and that no matter what the reaction of the third point to the first one is, we will suppose that it can be replaced by an internal force B that acts in the direction of a line b that joins the point 1 to the point 3′, which one finds by prolonging the inflexible side a' along its length. Upon supposing that this system is entirely free, the sum of the virtual moments of all those forces will be:

$$X\,\delta x + Y\,\delta y + Z\,\delta z + X'\,\delta x' + Y'\,\delta y' + Z'\,\delta z' + X''\,\delta x'' + Y''\,\delta y'' + Z''\,\delta z'' \\ + A\,\delta a + A'\,\delta a' + B\,\delta b,$$

and that sum will be zero for the equilibrium of those forces. However, one has:

$$\begin{aligned} a^2 &= \Delta x^2 + \Delta y^2 + \Delta z^2, \\ a'^2 &= \Delta x'^2 + \Delta y'^2 + \Delta z'^2, \\ b^2 &= \Delta^2 x^2 + \Delta^2 y^2 + \Delta^2 z^2, \end{aligned}$$

and consequently:

$$\delta a = \frac{\Delta x}{a}\,\delta\Delta x + \frac{\Delta y}{a}\,\delta\Delta y + \frac{\Delta z}{a}\,\delta\Delta z,$$

$$\delta a' = \frac{\Delta x'}{a'}\,\delta\Delta x' + \frac{\Delta y'}{a'}\,\delta\Delta y' + \frac{\Delta z'}{a'}\,\delta\Delta z',$$

$$\delta b = \frac{\Delta^2 x}{b}\,\delta\Delta^2 x + \frac{\Delta^2 y}{b}\,\delta\Delta^2 y + \frac{\Delta^2 z}{b}\,\delta\Delta^2 z\,.$$

Now:

$$\delta\Delta x = \delta x' - \delta x, \qquad \delta\Delta x' = \delta x'' - \delta x', \qquad \delta\Delta^2 x = \delta x'' - 2\delta x' + \delta x,$$

and the values of $\delta\Delta y$, $\delta\Delta y'$, $\delta\Delta^2 y$, $\delta\Delta z$, ... are formed in the same manner. Since the system is free, one must equate to zero the terms in the sum of the moments multiplied by each of the independent variations δx, $\delta x'$, ..., δy, ... One will then have the following nine equations for the equilibrium of those forces, which express the equilibrium of the three times three forces that act at each point:

$$0 = X - \frac{A}{a}\Delta x + \frac{B}{b}\Delta^2 x,$$

$$0 = Y - \frac{A}{a}\Delta y + \frac{B}{b}\Delta^2 y,$$

$$0 = Z - \frac{A}{a}\Delta z + \frac{B}{b}\Delta^2 z,$$

$$0 = X' + \frac{A}{a}\Delta x - \frac{A'}{a'}\Delta x' - 2\frac{B}{b}\Delta^2 x,$$

$$(f_5) \qquad 0 = Y' + \frac{A}{a}\Delta y - \frac{A'}{a'}\Delta y' - 2\frac{B}{b}\Delta^2 y,$$

$$0 = Z' + \frac{A}{a}\Delta z - \frac{A'}{a'}\Delta z' - 2\frac{B}{b}\Delta^2 z,$$

$$0 = X'' + \frac{A}{a}\Delta x' + \frac{B}{b}\Delta^2 x,$$

$$0 = Y'' + \frac{A}{a}\Delta y' + \frac{B}{b}\Delta^2 y,$$

$$0 = Z'' + \frac{A}{a}\Delta z' + \frac{B}{b}\Delta^2 z\,.$$

6. – Upon adding these equations three at a time – namely, the first with the fourth and the seventh, the second with the fifth and the eighth, the third with the sixth and the ninth – one will have:

$$X + X' + X'' = 0, \qquad Y + Y' + Y'' = 0, \qquad Z + Z' + Z'' = 0.$$

If one adds the fifth one, multiplied by Δz, to the sixth one, multiplied by $-\Delta y$, to the eighth one, multiplied by $\Delta z + \Delta z'$, and the ninth one, multiplied by $-(\Delta y + \Delta y')$ then one will find that the terms that depend upon internal forces A, A', B once more cancel, and that will give:

$$Y'\Delta z - Z'\Delta y + Y''(\Delta z + \Delta z') - Z''(\Delta y + \Delta y') = 0,$$

and similarly:

$$Z'\Delta x - X'\Delta z + Z''(\Delta x + \Delta x') - X''(\Delta z + \Delta z') = 0,$$

$$X'\Delta y - Y'\Delta x + X''(\Delta y + \Delta y') - Y''(\Delta x + \Delta x') = 0.$$

These six equations are the general relations for the external forces on the system. For a system of three points, one can replace the last three with some other ones that are simple to interpret. Multiply the first three equations by:

$$\Delta y\,\Delta^2 z - \Delta z\,\Delta^2 y, \qquad \Delta z\,\Delta^2 x - \Delta x\,\Delta^2 z, \qquad \Delta x\,\Delta^2 y - \Delta y\,\Delta^2 x,$$

respectively, and add them: The terms that are multiplied by A and B cancel, and one will have:

$$X\,(\Delta y\,\Delta^2 z - \Delta z\,\Delta^2 y) + Y\,(\Delta z\,\Delta^2 x - \Delta x\,\Delta^2 z) + Z\,(\Delta x\,\Delta^2 y - \Delta y\,\Delta^2 x),$$

which is an expression that express the idea that the force P, which is the resultant of X, Y, Z, is included in the plane of the three points 1, 2, 3. The other two expressions that one infers from the last six express the same condition in regard to the forces P', P''.

7. – If we eliminate B from the first three equations above (f_5) then we will arrive at the following three, two of which imply the third:

$$Y\,\Delta^2 z - Z\,\Delta^2 y = \frac{A}{a}(\Delta y\,\Delta^2 z - \Delta z\,\Delta^2 y),$$

$$Z\,\Delta^2 x - X\,\Delta^2 z = \frac{A}{a}(\Delta z\,\Delta^2 x - \Delta x\,\Delta^2 z),$$

$$X\,\Delta^2 y - Y\,\Delta^2 x = \frac{A}{a}(\Delta x\,\Delta^2 y - \Delta y\,\Delta^2 x),$$

and from this, upon taking the sum of the squares and extracting the root:

$$A = a\frac{\sqrt{[(Y\,\Delta^2 z - Z\,\Delta^2 y)^2 + (Z\,\Delta^2 x - X\,\Delta^2 z)^2 + (X\,\Delta^2 y - Y\,\Delta^2 x)^2]}}{\sqrt{[(\Delta y\,\Delta^2 z - \Delta z\,\Delta^2 y)^2 + (\Delta z\,\Delta^2 x - \Delta x\,\Delta^2 z)^2 + (\Delta x\,\Delta^2 y - \Delta y\,\Delta^2 x)^2]}},$$

and by the same process:

$$B = b\frac{\sqrt{[(Y\,\Delta z - Z\,\Delta y)^2 + (Z\,\Delta x - X\,\Delta z)^2 + (X\,\Delta y - Y\,\Delta x)^2]}}{\sqrt{[(\Delta y\,\Delta^2 z - \Delta z\,\Delta^2 y)^2 + (\Delta z\,\Delta^2 x - \Delta x\,\Delta^2 z)^2 + (\Delta x\,\Delta^2 y - \Delta y\,\Delta^2 x)^2]}}.$$

The numerator in the first expression can be put into the form:

$$\sqrt{[(X^2 + Y^2 + Z^2)\,(\Delta^2 x^2 + \Delta^2 y^2 + \Delta^2 z^2) - (X\,\Delta^2 x + Y\,\Delta^2 y + Z\,\Delta^2 z)^2]}\,.$$

However, X, Y, Z are the rectangular components of a force P; hence:

$$X^2 + Y^2 + Z^2 = P^2,$$

and one will have:

$$b^2 = \Delta^2 x^2 + \Delta^2 y^2 + \Delta^2 z^2 \quad \text{and} \qquad X\,\Delta^2 x + Y\,\Delta^2 y + Z\,\Delta^2 z = Pb \cos \widehat{Pb}\,,$$

moreover, in which one lets $\widehat{Pb}$ denote the angle that is formed by the direction of the force P with b, which is the direction of the tension B; the numerator above then amounts to $\sqrt{P^2 b^2 - P^2 b^2 \cos^2 \widehat{Pb}}$, or rather, to $Pb \sin \widehat{Pb}$.

As for the denominator, it is (cf., § **1**) the area of the parallelogram that is constructed on the lines a and a', and it is easy to see that it is equivalent to the area of the one that is constructed from the lines a, b or $ab \sin \widehat{ab}$. It will then follow that:

$$A = \frac{ab\,P \sin \widehat{Pb}}{ab \sin \widehat{ab}} = \frac{P \sin \widehat{Pb}}{\sin \widehat{ab}}\,,$$

and in the same way, one will also have:

$$B = P\,\frac{\sin \widehat{Pa}}{\sin \widehat{ab}}\,,$$

in such a way that the forces A and B will be the components of P in the sense of a and b, as they must be.

8. – Often, the action of the point 3 on the point 1 comes about by means of an elastic force on the angle 2 that one considers to be opposite to it, depending upon whether the angle 123 is open or closed. It seems to us that one can then regard that force as the tension in a filament that is constrained to preserve the form of an arc of a circle that is described with the point 2 that is the summit of the angle as its center, in such a manner that since the extremities of that filament are perpendicular to the two sides of the angle, that force of tension will also be perpendicular to the same sides at the points where the filament is attached to those rods a and a'; at least, that is how one commonly introduces the elasticity of an angle into the calculation. Upon denoting the tension in the filament, or the elasticity of the angle 2 by E, and letting e denote the supplement of the angle $\widehat{aa'}$ between the two rods, which is an angle that is measured by the length of the arc of the circle that is intercepted by its sides, and whose tension we have denoted by E. One will get $E\,\delta e$ for the virtual moment of the force E. Now, in order to determine the angle e, one will have:

$$\cos e = \frac{a^2 + a'^2 - b^2}{2aa'}$$

in the triangle $aa'b$, so:

$$-\sin e\cdot\delta e = \frac{\delta a}{2}\left[\frac{1}{a'}-\frac{a'}{a^2}+\frac{b^2}{a^2a'}\right]+\frac{\delta a'}{2}\left[\frac{1}{a}-\frac{a}{a'^2}+\frac{b^2}{a^2a'}\right]-\frac{b\,\delta b}{aa'}\,.$$

Upon dividing this by – sin e, the term that is multiplied by δa will become:

$$\frac{a^2+b^2-a'^2}{-2a\cdot aa'\sin e}\,.$$

However:

$$a\,a'\sin e = a\,a'\sin\widehat{a\,a'} = ab\sin\widehat{a\,b} \qquad \text{and} \qquad a^2+b^2-a'^2 = 2ab\cos\widehat{a\,b},$$

so that term will be equal to:

$$-\frac{\cos\widehat{ab}}{a\cdot\sin\widehat{ab}}\,.$$

The term that is multiplied by $\delta a'$ will also be:

$$-\frac{\cos\widehat{ab}}{a\cdot\sin\widehat{ab}}\,,$$

and the last one will become:

$$\frac{b\cdot\delta b}{aa'\sin\widehat{aa'}} = \frac{b\,\delta b}{b\cdot h} = \frac{\delta b}{h}\,,$$

in which h is the distance from the summit 2 to the side b. Hence:

$$\delta e = -\frac{\delta a}{a\tan\widehat{ab}}-\frac{\delta a'}{a'\tan\widehat{a'b}}+\frac{\delta b}{h}\,.$$

Let a and a' represent the tensions that exist in the rods a, a', resp. They will no longer be the same as the A, A' that we used above when we suppose that these two internal forces agree with the force B in order to produce equilibrium. The sum of the virtual moments of these new tensions and the elasticity E will be:

$$\mathrm{a}\;\delta a+\mathrm{a}'\;\delta a'+E\;\delta e,$$

or rather:

$$\left(\mathrm{a}-\frac{E}{a\tan\widehat{ab}}\right)\delta a+\left(\mathrm{a}'-\frac{E}{a'\tan\widehat{ab}}\right)\delta a'+\frac{E}{h}\delta b\,,$$

and that sum must replace the one $A\;\delta a+A'\;\delta a'+B\;\delta b$ of the internal forces in § **5**; one will then have:

$$A = \mathrm{a} - E : a \tan \widehat{ab}, \qquad A' = \mathrm{a}' - E : a' \tan \widehat{a'b}, \qquad B = E : h ;$$

one can also get this from decomposing the forces and comparing their moments.

9. – Once more, consider four points that are subject to external forces P, P', P'', P''' whose components are X, Y, Z ; X', Y', Z' ; X'', ... No matter what the reactions of those points on each other, one can always imagine that they are in equilibrium and replace their effect by that of:

1. Longitudinal tension in the sides a, a', a'' of the polygon that is formed from those four points, and we call those tensions A, A', A''.

2. The tensions in the two lines b, b' that join: first of all, the point 1 to the extremity $3'$ of a', prolonged along its length in the sense of $\overline{32}$, and secondly, b' to the extremity $3''$ of a'', prolonged in the sense of $\overline{43}$ along its entire length, and these new tensions will be called B, B', resp.

3. Finally, from the tension C in a line c that unites the point 1 to the well-defined point $4'$ while drawing a parallel that is equal to b' through the point $3'$.

One will see that the polygon that is formed from the points 1, 2, 3, 4 is then one that we called "the reciprocal polygon to the polygon 1234" above in § **4**. From what was shown there, we will have:

$$a^2 = \Delta x^2 + \Delta y^2 + \Delta z^2,$$

$$a'^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2,$$

$$a''^2 = \Delta x''^2 + \Delta y''^2 + \Delta z''^2,$$

$$b^2 = \Delta^2 x^2 + \Delta^2 y^2 + \Delta^2 z^2,$$

$$b'^2 = \Delta^2 x'^2 + \Delta^2 y'^2 + \Delta^2 z'^2,$$

$$c^2 = \Delta^3 x^2 + \Delta^3 y^2 + \Delta^3 z^2.$$

One might even presume that this hardly-natural constraint between the points 1, 2, 3, 4 is employed here only because it will serve to explain what follows, and that is also the only reason that makes us return to the question of the equilibrium of the forces that act on material polygons.

The sum of the virtual moments of all of those forces, whether external or internal, is:

$$X\,\delta x + Y\,\delta y + Z\,\delta z + X'\,\delta x' + Y'\,\delta y' + Z'\,\delta z' + \ldots +$$
$$+ A\,\delta a + A'\,\delta a' + A''\,\delta a'' + B\,\delta b + B'\,db' + C\,dc,$$

and if the system is entirely free, as we suppose, then that sum will be zero, no matter what the independent variations:

$$\delta x,\ \delta y,\ \delta z,\ \delta x',\ \delta y',\ \delta z',\ \ldots$$

are. Now:

$$\delta a = \frac{\Delta x}{a}\,\delta\Delta x + \frac{\Delta y}{a}\,\delta\Delta y + \frac{\Delta z}{a}\,\delta\Delta z,$$

$$\delta a' = \frac{\Delta x'}{a'}\,\delta\Delta x' + \frac{\Delta y'}{a'}\,\delta\Delta y' + \frac{\Delta z'}{a'}\,\delta\Delta z',$$

$$\delta a'' = \frac{\Delta x''}{a''}\,\delta\Delta x'' + \frac{\Delta y''}{a''}\,\delta\Delta y'' + \frac{\Delta z''}{a''}\,\delta\Delta z'',$$

$$\delta b = \frac{\Delta^2 x}{b}\,\delta\Delta^2 x + \frac{\Delta^2 y}{b}\,\delta\Delta^2 y + \frac{\Delta^2 z}{b}\,\delta\Delta^2 z,$$

$$\delta b' = \frac{\Delta^2 x'}{b'}\,\delta\Delta^2 x' + \frac{\Delta^2 y'}{b'}\,\delta\Delta^2 y' + \frac{\Delta^2 z'}{b'}\,\delta\Delta^2 z',$$

$$\delta c = \frac{\Delta^3 x}{c}\,\delta\Delta^3 x + \frac{\Delta^3 y}{c}\,\delta\Delta^3 y + \frac{\Delta^3 z}{c}\,\delta\Delta^3 z,$$

and

$$\delta\Delta x = \delta x' - \delta x, \qquad \delta\Delta y = \delta y' - \delta y, \qquad \delta\Delta z = \delta z' - \delta z,$$

$$\delta\Delta x' = \delta x'' - \delta x', \qquad \delta\Delta y' = \delta y'' - \delta y', \qquad \delta\Delta z' = \delta z'' - \delta z',$$

$$\delta\Delta x'' = \delta x''' - \delta x'', \qquad \delta\Delta y'' = \delta y''' - \delta y'', \qquad \ldots,$$

$$\delta\Delta^2 x = \delta x'' - 2\,\delta x' + \delta x, \qquad \delta\Delta^2 y'' = \delta y'' - 2\,\delta y' + \delta y, \qquad \ldots,$$

$$\delta\Delta^2 x' = \delta x''' - 2\,\delta x'' + \delta x', \qquad \delta\Delta^2 y' = \delta y''' - \ldots, \qquad \ldots,$$

$$\delta\Delta^3 x = \delta x''' - 3\delta x'' + 3\delta x' - \delta x, \qquad \delta\Delta^3 y = \delta y''' - \ldots, \qquad \ldots$$

Substitute these values for $\delta\Delta$ in those of δa, $\delta a'$, $\delta a''$, δb, …, and then substitute them in the sum of the moments. Upon equating the terms that are multiplied by the independent variations δx, δy, …, one will then have the equations:

$$0 = X - \frac{A}{a}\Delta x + \frac{B}{b}\Delta^2 x - \frac{C}{c}\Delta^3 x,$$

$$0 = Y - \frac{A}{a}\Delta y + \frac{B}{b}\Delta^2 y - \frac{C}{c}\Delta^3 y,$$

$$\begin{aligned}
0 &= Z - \frac{A}{a}\Delta z + \frac{B}{b}\Delta^2 z - \frac{C}{c}\Delta^3 z, \\
0 &= X' + \frac{A}{a}\Delta x - \frac{A'}{a'}\Delta x' - 2\frac{B}{b}\Delta^2 x + \frac{B'}{b'}\Delta^2 x' + 3\frac{C}{c}\Delta^3 x, \\
0 &= Y' + \frac{A}{a}\Delta y - \frac{A'}{a'}\Delta y' - 2\frac{B}{b}\Delta^2 y + \frac{B'}{b'}\Delta^2 y' + 3\frac{C}{c}\Delta^3 y, \\
0 &= Z' + \frac{A}{a}\Delta z - \frac{A'}{a'}\Delta z' - 2\frac{B}{b}\Delta^2 z + \frac{B'}{b'}\Delta^2 z' + 3\frac{C}{c}\Delta^3 z, \\
0 &= X'' + \frac{A'}{a'}\Delta x' - \frac{A''}{a''}\Delta x'' + \frac{B}{b}\Delta^2 x - 2\frac{B'}{b'}\Delta^2 x' - 3\frac{C}{c}\Delta^3 x, \\
0 &= Y'' + \frac{A'}{a'}\Delta y' - \frac{A''}{a''}\Delta y'' + \frac{B}{b}\Delta^2 y - 2\frac{B'}{b'}\Delta^2 y' - 3\frac{C}{c}\Delta^3 y, \\
0 &= Z'' + \frac{A'}{a'}\Delta z' - \frac{A''}{a''}\Delta z'' + \frac{B}{b}\Delta^2 z - 2\frac{B'}{b'}\Delta^2 z' - 3\frac{C}{c}\Delta^3 z, \\
0 &= X''' + \frac{A''}{a''}\Delta x'' + \frac{B'}{b'}\Delta^2 x' + 3\frac{C}{c}\Delta^3 x, \\
0 &= Y'' + \frac{A''}{a''}\Delta y'' + \frac{B'}{b'}\Delta^2 y' + 3\frac{C}{c}\Delta^3 y, \\
0 &= Z'' + \frac{A''}{a''}\Delta z'' + \frac{B'}{b'}\Delta^2 z' + 3\frac{C}{c}\Delta^3 z .
\end{aligned} \tag{f_9}$$

10. – Add the first one to the fourth one, the seventh one, and the tenth one. That will give:

$$0 = X + X' + X'' + X''',$$

and in the same manner:

$$0 = Y + Y' + Y'' + Y''',$$

$$0 = Z + Z' + Z'' + Z'''.$$

Multiply: the fifth by Δz and the sixth by $-\Delta y$,
the eighth by $\Delta z + \Delta z'$ and the ninth by $-(\Delta y + \Delta y')$

the eleventh by $\Delta z + \Delta z' + \Delta z''$ and the twelfth by $-(\Delta y + \Delta y' + \Delta y'')$

and add those products. All of the terms that depend upon internal forces *A*, *A*, … will cancel, and one will find that:

$$0 = Y'\Delta z + Y''\Delta (z + z') + Y'''\Delta (z + z' + z'') - Z'\Delta y - Z''\Delta (y + y') - Z'''\Delta(y + y' + y''),$$

and by an analogous process:

$$0 = Z'\Delta x + Z''\Delta (x + x') + Z'''\Delta (x + x' + x'') - X'\Delta z - X''\Delta (z + z') - X'''\Delta(x + x' + x''),$$

$$0 = X'\Delta y + X''\Delta (y + y') + X'''\Delta (y + y' + y'') - Y'\Delta x - Y''\Delta (x + x') - Y'''\Delta(x + x' + x''),$$

upon observing that:

$$\Delta (z + z') = z'' - z', \qquad \Delta (z + z' + z'') = z''' - z, \qquad \Delta (y + y') = y'' - y, \ldots$$

These equations amount to:

$$0 = Y'(z' - z) - Z'(y' - y) + Y''(z'' - z) - Z''(y'' - y) + Y'''(z''' - z) - Z'''(y''' - y),$$

$$0 = Z'(x' - x) - X'(z' - z) + Z''(x'' - x) - X''(z'' - z) + Z'''(x''' - x) - X'''(z''' - z),$$

$$0 = X'(y' - y) - Y'(x' - x) + X''(y'' - y) - Y''(x'' - x) + X'''(y''' - y) - Y'''(x''' - x) .$$

These are the known equations of the moments that must exist in the equilibrium of any free system.

To abbreviate, we shall let $(\Delta x, \Delta^2 y)$ denote the function $\Delta x\, \Delta^2 y - \Delta y\, \Delta^3 x$, let $(\Delta^3 x, \Delta y)$ denote the function $\Delta x\, \Delta^2 y - \Delta^3 y\, \Delta x$, and so on for all similar functions. The expression $(\Delta x, \Delta^2 y, \Delta^3 z)$ will represent:

$$\Delta x\, \Delta^2 y\, \Delta^3 z + \Delta y\, \Delta^2 z\, \Delta^3 x + \Delta z\, \Delta^2 x\, \Delta^3 y - \Delta x\, \Delta^2 z\, \Delta^3 y - \Delta y\, \Delta^2 x\, \Delta^3 z - \Delta z\, \Delta^2 y\, \Delta^3 x .$$

With that, one infers from the first three of equations (f_9) that:

$$A = \quad a\,[X\,(\Delta^2 y, \Delta^3 z) + Y\,(\Delta^2 z, \Delta^3 x) + Z\,(\Delta^2 x, \Delta^3 y)] : (\Delta x, \Delta^2 y, \Delta^3 z),$$

$$B = -\,b\,[X\,(\Delta^3 y, \Delta z) + Y\,(\Delta z, \Delta^2 x) + Z\,(\Delta^3 x, \Delta y)] : (\Delta x, \Delta^2 y, \Delta^3 z),$$

$$C = \quad c\,[X\,(\Delta y, \Delta^2 z) + Y\,(\Delta z, \Delta^2 x) + Z\,(\Delta x, \Delta^2 y)] : (\Delta x, \Delta^2 y, \Delta^3 z) .$$

In particular, consider the first of these quantities, namely, *A*. The coefficients of *X*, *Y*, *Z* in its numerator are (cf., § **1**) the projections onto the coordinate planes of a parallelogram that is constructed from the lines *b*, *c*. Let *M* be that parallelogram, and let *m* be the direction of a perpendicular to its plane. By the projection principle, one will have:

$$(\Delta^2 y, \Delta^3 z) = M \cos \widehat{mx}, \qquad (\Delta^2 z, \Delta^3 x) = M \cos \widehat{my}, \qquad (\Delta^2 x, \Delta^3 y) = M \cos \widehat{mz}.$$

The denominator $(\Delta x, \Delta^2 y, \Delta^3 z)$ is the volume of the parallelepiped that is constructed with that parallelogram as its base and the edge a. Hence:

$$(\Delta x, \Delta^2 y, \Delta^3 z) = Ma \sin \widehat{Ma},$$

in which $\sin \widehat{Ma}$ denotes the sine of the angle that the line a forms with the plane of M. Furthermore, $X = P \cos \widehat{xP}$, $Y = P \cos \widehat{yP}$, $Z = P \cos \widehat{zP}$. Making these substitutions in the value of A will give:

$$\begin{aligned} A &= P\,[\cos \widehat{xP} \cos \widehat{xm} + \cos \widehat{yP} \cos \widehat{ym} + \cos \widehat{zP} \cos \widehat{zm}] : \sin \widehat{Ma} \\ &= P \cos \widehat{Pm} : \sin \widehat{Ma} = P \sin \widehat{PM} : \sin \widehat{Ma}. \end{aligned}$$

The last expression is the component of the force P along the line a, while the other component of that same force will be in the plane M. One will obtain the same result for the forces B, C relative to the lines b, c in such a way that these three forces are the components of P along the directions a¸ b, c that concur at the point 1 where the force P is applied.

If one substitutes $\Delta x + \Delta^2 x$, $\Delta^2 x + \Delta^3 x$ for $\Delta x'$, $\Delta^2 x'$, resp., in the fourth of equations (f_9) then it will become:

$$0 = X' - \left(\frac{A'}{a'} - \frac{A}{a}\right)\Delta x + \left(\frac{B'}{b'} - 2\frac{B}{b} - \frac{A'}{a'}\right)\Delta^2 x + \left(\frac{B'}{b'} + 3\frac{C}{c}\right)\Delta^3 x,$$

and the following two will change into:

$$0 = Y' - \left(\frac{A'}{a'} - \frac{A}{a}\right)\Delta y + \left(\frac{B'}{b'} - 2\frac{B}{b} - \frac{A'}{a'}\right)\Delta^2 y + \left(\frac{B'}{b'} + 3\frac{C}{c}\right)\Delta^3 y,$$

$$0 = Z' - \left(\frac{A'}{a'} - \frac{A}{a}\right)\Delta z + \left(\frac{B'}{b'} - 2\frac{B}{b} - \frac{A'}{a'}\right)\Delta^2 z + \left(\frac{B'}{b'} + 3\frac{C}{c}\right)\Delta^3 z,$$

in the same manner. If one adds the first, second, and third of (f_9) to these equations, respectively, then they will become:

$$0 = X + X' - \frac{A'}{a'}\Delta x + \left(\frac{B'}{b'} - \frac{B}{b} - \frac{A'}{a'}\right)\Delta^2 x + \left(\frac{B'}{b'} + 2\frac{C}{c}\right)\Delta^3 x,$$

$$0 = Y + Y' - \frac{A'}{a'}\Delta y + \left(\frac{B'}{b'} - \frac{B}{b} - \frac{A'}{a'}\right)\Delta^2 y + \left(\frac{B'}{b'} + 2\frac{C}{c}\right)\Delta^3 y,$$

$$0 = Z + Z' - \frac{A'}{a'}\Delta z + \left(\frac{B'}{b'} - \frac{B}{b} - \frac{A'}{a'}\right)\Delta^2 z + \left(\frac{B'}{b'} + 2\frac{C}{c}\right)\Delta^3 z\,.$$

One will then infer that:

$$A' = a'\,\frac{(X + X')(\Delta^3 y, \Delta^3 z) + (Y + Y')(\Delta^3 y, \Delta^3 z) + (Z + Z')(\Delta^3 y, \Delta^3 z)}{(\Delta x, \Delta^2 y, \Delta^3 z)}\,.$$

One operates similarly on B' / b', when one already knows B / b, A' / a', and finally for A'' / a'', which one easily infers from a combination of the last equations in (f_9).

11. – If the polygon is invariable in form then the values of A, A', A'', B, B', C that are provided by the preceding formulas will be determined completely, and they will represent the efforts that are exerted in the sense of the lengths of the rods a, a', a'', and three others b¸ b', c that are coupled with them in a fixed manner, and which are capable of producing that invariability, along with them.

Upon supposing that the rods a, a', a'' are absolutely rigid, the angles that a' forms with a and a'' will be indeterminate, along with the inclination of the plane aa' with respect to that of $a'a''$. If, one gives the lengths of b and b' along with those of a, a', a'' then the angle between a and a' and the one between a' and a'' will be determinate, although the inclination of the planes of those angles will once more be arbitrary and capable of taking on all magnitudes. However, upon assigning the value of c, moreover, the form of the polygon will become entirely determinate, and the same thing will also be true for efforts of the forces that act at its angles in the sense of those lines, when one supposes (as we shall) that it is those links that allow those points to transmit the action that they receive from external forces. If the polygon that we consider is elastic (i.e., it is capable of changing in form under the action of external forces as a reaction) then we will have to consider that elasticity, so to be complete, we must operate at the expense of the six elements that we just indicated, namely, at the expense of the elongation of the three rods a, a', a'', and that elasticity will then take the name of *extensibility* or *contractibility*, at the expense of the variation of the angles between a and a' and between a' and a'', and finally, at the expense of the angle of inclination of the planes of these two angles, and the latter variation can itself take place only by virtue of a type of torsion of the side a', which is the edge of that inclination. The effects of the last three elements of elasticity of the polygon are the ones that mainly replace the efforts B, B', C of the rods b, b', c, which are supposed to be elastic in the sense of their length, whether extensible or contractible. As we have said (§ **8**), one measures the elasticity of an angle by the tension in a circular cord that have the summit of that angle for its center and a radius of a linear unit, and which is found between the sides of the angle and is attached to those same sides. In that same paragraph, we saw how one can replace the force of elasticity of the angle between the two sides with tensions. In order to replace the system of six forces of elasticity that we just enumerated with similar longitudinal tensions A, A', A''¸ B, B', C, we let i denote the inclination of the planes of aa' and $a'a''$, which cut along a', and let I denote the elastic force on that angle. We also let e, e' denote the angles $\widehat{aa'}$, $\widehat{a'a''}$, resp., and let E, E' denote the elastic forces on the those angles, and finally let a, a′, a″ denote the longitudinal tensions in the rods a, a', a'', resp., that are produced by these angular elastic forces. The sum of the virtual moments of those six forces will be:

$$\mathrm{a}\,\delta a + \mathrm{a}'\delta a' + \mathrm{a}''\,\delta a'' + E\,\delta e + E'\,\delta e' + I\,\delta i,$$

and that sum must be equal to:

$$A\,\delta a + A'\,\delta a' + A''\,\delta a'' + B\,\delta b + B'\,\delta b' + C\,\delta c\,.$$

Now, the angles e, e', i will be easy to determine as functions of a, a', a'', b, b', c by the usual methods of geometry. I shall not report on their values, due to the extreme complexity in that of i. Upon differentiating them using δ and substituting that in the first of those two sums, one will get an expression of the same form as the second one, and one can then compare the coefficients of the same variations δa, δb, δc, … That will provide the equations that are necessary for the determination of the tensions A, A', A'', B, B', C by means of the three forces of elasticity E, E', I, and the tensions a, a′, a″, or conversely. One sees that this transformation rests upon the theorem of analytical mechanics that when the sums of the virtual moments of the two systems of forces are equal, those two systems will be equivalent, and one of them can replace the other.

In these two systems of internal efforts, which are likewise appropriate to establish the equilibrium of the forces that act on the polygon, one will remark that the longitudinal tensions on the rods a, a', a'' are expressed by a, a′, a″, resp., when one considers the forces of elasticity E, E', I along with them and that the tensions in those lines will be A, A', A'' if one imagines that equilibrium is established with the aid of three other tensions B, B', C. Now, the A, A', A'' have expressions that are very different from a, a′, a″. However, in a polygon that is presently in equilibrium, which of the two systems of forces is the one that actually occurs? One can respond that it is neither of them, at least if the constitution of the polygon is materially just the one that was supposed in order to arrive at those formulas for equilibrium. However, both of them are suitable to establish equilibrium and to replace the one that provides the real action of contiguous molecules on each other in all cases.

If the polygon has four sides then one must consider three new internal forces that one denotes by A''', B'', C', while supposing that the new side a''' is coupled with the preceding ones in the first way – i.e., by means of longitudinal tensions in straight lines a''', b'', c' or by a‴, E'', I', if one employs two forces of elasticity along with the tension a‴ in the new side a''', one of which E'' relates to the angle $\widehat{a''a'''}$, and the other of which I' relates to the angle that is found between the plane of the sides a''', a'' and the plane of the sides a'', a'. One sees that each new side can introduce only three new forces for its coupling with the preceding ones, but one will also be led to three new equations for the determination of those forces.

12. – All that we just said about polygons with an arbitrary number of sides and the manner by which the action of the forces upon those polygons can be imagined to be transmitted or cancelled is obviously applicable to curves of double curvature, whether elastic or stiff. The elasticity of these curves can depend upon only the extensibility of their elements, the variation of the contingency angle between two consecutive elements at each point, and the variation of the contingency angle between two consecutive osculating planes at that point, which can be considered to intersect along the second of the three consecutive elements of the curve at that point, or rather, along the tangent that is its prolongation. The stiffness of the curve consists of the invariability of those three quantities for each point of the curve, so it must depend upon three forces that are capable of producing that invariability in each of those elements. We first suppose that the elasticity or stiffness is produced by forces that are analogous to the ones *A*, *B*, *C* that we employed in the equilibrium of polygons (§ **9**). They act along directions that form angles with the coordinate axes whose cosines are:

$$\frac{\Delta x}{\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}}, \qquad \frac{\Delta y}{\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}}, \qquad \frac{\Delta z}{\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}},$$

$$\frac{\Delta^2 x}{\sqrt{\Delta^2 x^2+\Delta^2 y^2+\Delta^2 z^2}}, \qquad \frac{\Delta^2 y}{\sqrt{\Delta^2 x^2+\Delta^2 y^2+\Delta^2 z^2}}, \qquad \frac{\Delta^2 z}{\sqrt{\Delta^2 x^2+\Delta^2 y^2+\Delta^2 z^2}},$$

$$\frac{\Delta^3 x}{\sqrt{\Delta^3 x^2+\Delta^3 y^2+\Delta^3 z^2}}, \qquad \frac{\Delta^3 y}{\sqrt{\Delta^3 x^2+\Delta^3 y^2+\Delta^3 z^2}}, \qquad \frac{\Delta^3 z}{\sqrt{\Delta^3 x^2+\Delta^3 y^2+\Delta^3 z^2}}.$$

For the arbitrary point of the curve whose coordinates are denoted by *x*¸ *y*, *z*, we then imagine three forces *A*, *B*, *C* that act along directions that form angles with *x*, *y*, *z* whose cosines are similarly:

$$\frac{dx}{\sqrt{dx^2+dy^2+dz^2}}, \qquad \frac{dy}{\sqrt{dx^2+dy^2+dz^2}}, \qquad \frac{dz}{\sqrt{dx^2+dy^2+dz^2}},$$

$$\frac{d^2 x}{\sqrt{d^2 x^2+d^2 y^2+d^2 z^2}}, \qquad \frac{d^2 y}{\sqrt{d^2 x^2+d^2 y^2+d^2 z^2}}, \qquad \frac{d^2 z}{\sqrt{d^2 x^2+d^2 y^2+d^2 z^2}},$$

$$\frac{d^3 x}{\sqrt{d^3 x^2+d^3 y^2+d^3 z^2}}, \qquad \frac{d^3 y}{\sqrt{d^3 x^2+d^3 y^2+d^3 z^2}}, \qquad \frac{d^3 z}{\sqrt{d^3 x^2+d^3 y^2+d^3 z^2}},$$

and the forces *A*, *B*, *C* that are placed with respect to the point *x*, *y*, *z* as we said in of § **9**.

Let *X*, *Y*, *Z* denote the intensity of the components of the force that acts upon the element *dm* of the curve that is situated at the point *x*, *y*, *z*, so *X dm*, *Y dm*, *Z dm* will be the absolute forces that are produced by its action upon that element. Upon observing that the internal forces *A*, *B*, *C* act in the directions of the three lines whose lengths are:

$$\sqrt{dx^2+dy^2+dz^2}\,, \qquad \sqrt{d^2x^2+d^2y^2+d^2z^2}\,, \qquad \sqrt{d^3x^2+d^3y^2+d^3z^2}\,,$$

the variations of those directions will be expressed by:

$$\frac{dx\,\delta dx+dy\,\delta dy+dz\,\delta dz}{\sqrt{dx^2+dy^2+dz^2}}, \quad \frac{d^2x\,\delta d^2x+d^2y\,\delta d^2y+d^2z\,\delta d^2z}{\sqrt{d^2x^2+d^2y^2+d^2z^2}}, \quad \frac{d^3x\,\delta d^3x+d^3y\,\delta d^3y+d^3z\,\delta d^3z}{\sqrt{d^3x^2+d^3y^2+d^3z^2}}.$$

If one abbreviates the quantities:

$$\frac{A}{\sqrt{dx^2+dy^2+dz^2}}, \qquad \frac{B}{\sqrt{dx^2+dy^2+dz^2}}, \qquad \frac{C}{\sqrt{dx^2+dy^2+dz^2}}$$

by a, b, c then one will have:

$$\begin{aligned} &\mathrm{S}\ a\,(dx\ \delta dx + dy\ \delta dy + dz\ \delta dz) \\ +&\mathrm{S}\ b\,(d^2x\ \delta d^2x + d^2y\ \delta d^2y + d^2z\ \delta d^2z) \\ +&\mathrm{S}\ c\,(d^3x\ \delta d^3x + d^3y\ \delta d^3y + d^3z\ \delta d^3z), \end{aligned}$$

for the sum of the virtual moments of those internal forces, in which the integration sign *S* must be extended over the entire curve. After integrating by parts and neglecting the terms that refer to the limits of the integral, that sum of the moments will be combined with that $\mathrm{S}\,(X\ \delta x + Y\ \delta y + Z\ \delta z)\ dm$ of the external force and reduce to:

$$\begin{aligned} &\mathrm{S}\ [X\,dm - d\,(a\,dx) + d^2\,(b\,d^2x) - d^3\,(c\,d^3x)]\ \delta x \\ +&\mathrm{S}\ [Y\,dm - d\,(a\,dy) + d^2\,(b\,d^2y) - d^3\,(c\,d^3y)]\ \delta y \\ +&\mathrm{S}\ [Z\,dm - d\,(a\,dz) + d^2\,(b\,d^2z) - d^3\,(c\,d^3z)]\ \delta z, \end{aligned}$$

and one knows that the terms that are multiplied by δx, δy, δz must be separately equal to zero, which will provide the three indefinite equations:

$$(f_{12}) \qquad \begin{aligned} 0 &= X\,dm - d\,(a\,dx) + d^2\,(b\,d^2x) - d^3\,(c\,d^3x), \\ 0 &= Y\,dm - d\,(a\,dy) + d^2\,(b\,d^2y) - d^3\,(c\,d^3y), \\ 0 &= Z\,dm - d\,(a\,dz) + d^2\,(b\,d^2z) - d^3\,(c\,d^3z)\,. \end{aligned}$$

13. – The object of these equations will be very different depending upon the problem that one is treating. It amounts to determining the figure that an elastic filament will take that initially has a known form and upon which the elastic elements will depend before the application of external forces, and some relations will exist between the internal forces *A*, *B*, *C*, and those elastic elements that one imagines can be deduced from what we said above (§ **11**) on the subject of elastic polygons; we shall return to that subject, moreover. However, in order to get an idea of the forces A, B, C, assume that the filament is perfectly stiff. Those forces will then be the resistances that are suitable to establish that stiffness in the filament when it cancels the external forces that act upon its particles. Since the figure of the filament is given by two equations in three coordinates, the equations above will be completely integrable, because the forces X, Y, Z can then depend upon only the variable that fixes the point of the curve to which they relate. One will then first infer these integrals:

$$0 = \int X\,dm - a\,dx + d\,(b\,d^2 x) - d^2\,(c\,d^3 x),$$

$$0 = \int Y\,dm - a\,dy + d\,(b\,d^2 y) - d^2\,(c\,d^3 y),$$

$$0 = \int Z\,dm - a\,dz + d\,(b\,d^2 z) - d^2\,(c\,d^3 z),$$

in which the $\int$ sign refers to an integration that is performed from one of the extremities of the curve up to the point at which one would like to calculate the efforts A, B, C, which are replaced in those equations by a, b, c. In addition, I suppose that the constants are attached to that $\int$ sign. If we eliminate a from those equations then we can form the following three, two of which will imply the third:

$$0 = dy \int Z\,dm + dy\,d\,(b\,d^2 z) - dy\,d^2\,(c\,d^3 z) - dz \int Y\,dm - dz\,d\,(b\,d^2 y) + dz\,d^2\,(c\,d^3 y),$$

$$0 = dz \int X\,dm + dz\,d\,(b\,d^2 x) - dz\,d^2\,(c\,d^3 x) - dx \int Z\,dm - dx\,d\,(b\,d^2 z) + dx\,d^2\,(c\,d^3 z),$$

$$0 = dx \int Y\,dm + dx\,d\,(b\,d^2 y) - dx\,d^2\,(c\,d^3 y) - dy \int X\,dm - dy\,d\,(b\,d^2 x) + dy\,d^2\,(c\,d^3 x).$$

One can further integrate them and arrive at the following equations, in which we have once more supposed that the constants are included in the new integration signs that were introduced:

$$\begin{aligned} 0 = y \int Z\,dm - z \int Y\,dm - \int (y\,Z - z\,Y)\,dm + b\,[dy\,d^2 z - dz\,d^2 y] \\ - c\,[(dy\,d^4 z - dz\,d^4 y) - (d^2 y\,d^3 z - d^2 z\,d^3 y)] - dc\,[dy\,d^3 z - dz\,d^3 y], \end{aligned}$$

$$\begin{aligned} 0 = z \int X\,dm - x \int Z\,dm - \int (z\,X - x\,Z)\,dm + b\,[dz\,d^2 x - dx\,d^2 z] \\ - c\,[(dz\,d^4 x - dx\,d^4 z) - (d^2 z\,d^3 x - d^2 x\,d^3 z)] - dc\,[dz\,d^3 x - dx\,d^3 z], \end{aligned}$$

$$\begin{aligned} 0 = x \int Y\,dm - y \int X\,dm - \int (x\,Y - y\,X)\,dm + b\,[dx\,d^2 y - dy\,d^2 x] \\ - c\,[(dx\,d^4 y - dy\,d^4 x) - (d^2 x\,d^3 y - d^2 y\,d^3 x)] - dc\,[dx\,d^3 y - dy\,d^3 x]. \end{aligned}$$

One can observe that upon multiplying these equations by dx, dy, dz, respectively, and adding the products b and dc, they will be collectively eliminated, and one will get the equation:

$$0 = (y\,dz - z\,dy)\int X\,dm + (z\,dx - x\,dz)\int Y\,dm + (x\,dy - y\,dx)\int Z\,dm$$

$$+ dx\int (y\,Z - z\,Y)\,dm + dy\int (z\,X - x\,Z)\,dm + dz\int (x\,Y - y\,X)\,dm$$

$$- c\,(dx\,d^2y\,d^3z + dy\,d^2z\,d^3x + dz\,d^2x\,d^3y - dx\,d^2z\,d^3y - dy\,d^2x\,d^3z - dz\,d^2y\,d^3x)$$

for the determination of c. If one adds these equations, after multiplying them by d^3x, d^3y, d^3z, respectively, then one will find that they have been eliminated, but b is preserved in them:

$$0 = (y\,d^3z - z\,d^3y)\int X\,dm + (z\,d^3x - x\,d^3z)\int Y\,dm + (x\,d^3y - y\,d^3x)\int Z\,dm$$

$$+ d^3x\int (yZ - zY)\,dm + d^3y\int (zX - xZ)\,dm + d^3z\int (xY - yX)\,dm$$

$$- b\,(dx\,d^2y\,d^3z + dy\,d^2z\,d^3x + dz\,d^2x\,d^3y - dx\,d^2z\,d^3y - dy\,d^2x\,d^3z - dz\,d^2y\,d^3x)$$

$$- c\,(dx\,d^3y\,d^4z + dy\,d^3z\,d^4x + dz\,d^3x\,d^4y - dx\,d^3z\,d^4y - dy\,d^3x\,d^4z - dz\,d^3y\,d^4x)\,.$$

b is eliminated by means of that equation, since c is eliminated already, but a enters into the first integrals with b and c, so it will be easy to obtain in its own right.

Before going any further, it might be good to point out that the preceding integrals of the three equations (f_{12}), which are of order three with respect to a, b, c, contain only the six arbitrary constants that are included in the integrals:

$$\int X\,dm, \quad \int Y\,dm, \quad \int Z\,dm, \quad \int (yZ - zY)\,dm, \quad \int (zX - xZ)\,dm, \quad \int (xY - yX)\,dm\,.$$

This is how a, b, c enter into these equations: They each contain only the differentials da, db, dc, d^2b, d^2c, d^3c, in such a way that they will have order three in c, order two in b, and first order in a. That consideration assures one that their complete integrals can contain only six arbitrary constants. If one imagines that one differentiates these equations three times in succession then nine equations will result that contain a, b, c, da, db, dc, d^2a, d^2b, d^2c, d^3a, d^3b, d^3c, d^4a, d^4b, d^4c, d^5a, d^5b, d^6c, and one principal variable. If one combines these nine equations with the original three then one can eliminate the eleven quantities a, b, da, db, d^2a, d^2b, d^3a, d^3b, d^4a, d^4b, d^5b, and what will result is a differential equation of order six in c. One knows that this completely-integrable equation will provide the complete integrals of the original three equations by eliminations, which will be integrals that contain only the six arbitrary constants of the equation that has sixth order in c.

14. – If one has *a*, *b*, *c* then the internal forces *A*, *B*, *C* that belong to each point will be given by the equations:

$$A = a\sqrt{dx^2 + dy^2 + dz^2}\,, \qquad B = b\sqrt{d^2x^2 + d^2y^2 + d^2z^2}\,, \quad C = c\sqrt{d^3x^2 + d^3y^2 + d^3z^2}\,.$$

Now, the quantity *c* that is found above has a first-order differential function for its numerator, and a sixth-order differential quantity for its denominator, in such a way that its value will be infinite of order five, and when one multiplies by $\sqrt{d^3x^2 + d^3y^2 + d^3z^2}$, it will give an expression for *C* that is infinite of order two. One sees in the same way that *b* has a sixth-order differential function for its denominator and another one of third order for its numerator. However, in order to get *B*, one must multiply that value by $\sqrt{d^2x^2 + d^2y^2 + d^2z^2}$, which is infinitely-small of order two. It will then be obvious that it will have an infinitely-small dimension that is lower in the numerator of *B* than in its denominator, and that it will be infinitely-large of order one. As for *A*, it will be a finite quantity. One will see the reason for that in the fact that there is a singularity in the magnitude of those forces when one compares with all that was said (§§ **5** and **9**) in relation to polygons. One will see in that way that the force *B* that opposes the opening-up of the angle between two infinitely-small contiguous sides will have its direction at a distance from the summit of that angle that is infinitely-small of order one, that the force *C* that mainly opposes the variation of the angle will be included in the planes of the first two elements of the curve, and the plane of the second and third ones, and that this force, I say, will be at a distance from the edge of the angle between those plane that is infinitely-small of order two. The magnitude of those forces must compensate for the way that their direction approaches the summits or edges of the angles of opening that oppose them.

15. – What we just said about the forces *A*, *B*, *C* that keep the curve in the form that one assumes to be preserved when it is considered to be stiff can be applied to the internal forces that exist in an elastic curve of double curvature when the curve when the curve has taken the form that it must have under the action of external forces that are in equilibrium by reacting with its elasticity. However, that quality of the curve cannot be expressed by means of the infinite forces that we just encountered. One can avoid the use of those infinite forces by introducing elements, whether elastic or stiff, that are more natural than the forces *A*, *B*, *C*. We will take them to be:

1. The tension that exists in the sense of the element of the curve.

2. The elasticity of the angle between two infinitely-small contiguous sides, when measured by the tension in a material line that joins the extremities of the two infinitely-small sides when each of them are prolonged to a unit length in the sense of the edges of the infinitely-small angle. The distance between those two points will be a measure of that angle.

3. The elasticity of the inclination between two consecutive osculating planes. The latter force is again measured by the tension in a line that joins the extremities of two lines that equal to one unit, situated in the two osculating planes, and perpendicular to their common intersection, or rather to the second of the three consecutive infinitely-small sides of the curve at the given point. Upon taking those two lines when starting from this second side in the curve in the sense for which the angle between them is infinitely-small, the distance between their extremities will be the measure of the infinitely-small angle between two consecutive osculating planes, and the tension in that line can be regarded as a force of torsion on the second side of the curve. Let i denote that angle, or rather, the infinitely-small line that measures it, and which we call the tension. From (f_3'), one will have:

$$i = ds\,\frac{dx\,d^2y\,d^3z + dy\,d^2z\,d^3x + dz\,d^2x\,d^3y - dx\,d^2z\,d^3y - dy\,d^2x\,d^3z - dz\,d^2y\,d^3x}{(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2}\,.$$

The square of the function that enters into the numerator of that value can be put into the form:

$$\begin{aligned}
&(dx^2 + dy^2 + dz^2)\,(d^2x^2 + d^2y^2 + d^2z^2)\,(d^3x^2 + d^3y^2 + d^3z^2)\\
&+ 2\,(d^2x\,dx^3 + d^2y\,dy^3 + d^2z\,dz^3)\,(d^3x\,dx + d^3y\,dy + d^3z\,dz)\,(dx\,d^2x + dy\,d^2y + dz\,d^2z)\\
&- (dx^2 + dy^2 + dz^2)\,(d^2x\,d^3x + d^2y\,d^3y + d^2z\,d^3z)^2\\
&- (d^2x^2 + d^2y^2 + d^2z^2)\,(d^3x\,dx + d^3y\,dy + d^3z\,dz)^2\\
&- (d^3x^2 + d^3y^2 + d^3z^2)\,(dx\,d^2x + dy\,d^2y + dz\,d^2z)^2\,.
\end{aligned}$$

However, if represents:

$$\begin{aligned}
&dx^2 + dy^2 + dz^2 &&\text{by } \alpha,\\
&d^2x^2 + d^2y^2 + d^2z^2 &&\text{by } \beta,\\
&d^3x^2 + d^3y^2 + d^3z^2 &&\text{by } \gamma,
\end{aligned}$$

as in *Mécanique analytique*, then one will have:

$$dx\,d\,x^2 + dy\,d^2y + dz\,d^2z = \frac{d\alpha}{2},$$

$$d^2x\,d\,x^2 + d^2y\,d^2y + d^2z\,d^2z = \frac{d\beta}{2},$$

$$d^3x\,dx + d^3y\,dy + d^3z\,dz \qquad = \frac{d^2\alpha}{2} - \beta.$$

The denominator:

$$(dy\,d^2z - dz\,d^2y)^2 + (dz\,d^2x - dx\,d^2z)^2 + (dx\,d^2y - dy\,d^2x)^2$$

amounts to:

$$(dx^2 + dy^2 + dz^2)\,(d^2x^2 + d^2y^2 + d^2z^2) - (dx\,d\,x^2 + dy\,d^2y + dz\,d^2z)^2.$$

One will then see that the differential function i depends upon only:

$$\alpha,\ \ \beta,\ \ \gamma, \qquad d\alpha, \qquad d\beta, \qquad d^2\alpha.$$

I shall dispense with writing it out, but it is obvious that the variation of that angle or δi will have the form:

$$o\,\delta\alpha + p\,\delta\beta + q\,\delta\gamma + o'\,\delta\,d\alpha + p'\,\delta\,d\beta + o''\,\delta\,d^2\alpha,$$

and that variation, when multiplied by ι, will be the virtual moment of that force ι, which will measure the torsion at the point considered on the curve.

From formula (f_3), the angle e between two consecutive elements of the curve is:

$$\frac{\sqrt{\left[(dx^2+dy^2+dz^2)(d^2x^2+d^2y^2+d^2z^2)-(dx\,d^2x+dy\,d^2y+dx\,d^2z)\right]}}{dx^2+dy^2+dz^2};$$

hence, that value will depend upon only α, β, $d\alpha$. Its variation will then have the form:

$$\delta e = m\,\delta\alpha + n\,\delta\beta + m'\,\delta\,d\alpha.$$

Call the elastic force of that angle e, which is measured in the way that we just spoke of; its moment will be $\varepsilon\,\delta e$. Finally, if σ is the force of tension in the element of the curve in the direction of ds then its moment will be $\sigma\,\delta ds$. Now, one has $\delta\,ds = \dfrac{\delta\alpha}{2\sqrt{\alpha}}$, so that moment will have the form $\sigma\,l\,\delta\alpha$. The sum of the moments of the forces at the point x, y, z will then be:

$$(X\,\delta x + Y\,\delta y + Z\,\delta z)\,dm + \sigma\,l\,\delta\alpha + \varepsilon\,(m\,\delta\alpha + n\,\delta\beta + m'\,\delta\,d\alpha)$$
$$+ l\,(o\,\delta\alpha + p\,\delta\beta + q\,\delta\gamma + o'\,\delta\,d\alpha + p'\,\delta\,d\beta + o''\,\delta\,d^2\alpha),$$

and upon supposing that the elastic or stiff filament is free along its length, but can have its extremities constrained, when the integral of that sum over the entire extent of the filament is added to the sum of the moments of the forces at the extremities, the result must be zero. Hence, upon abstracting from the latter forces, we propose only to find the indefinite equations:

$$\begin{aligned} 0 &= \mathrm{S}\,(X\,\delta x + Y\,\delta y + Z\,\delta z)\,dm \\ &+ \mathrm{S}\,[\sigma\, l + \varepsilon\, m + \iota\, o]\,\delta\alpha \\ &+ \mathrm{S}\,[\varepsilon\, n + \iota\, p]\,\delta\beta + \mathrm{S}\,\iota\, q\,\delta\gamma \\ &+ \mathrm{S}\,[\varepsilon\, m' + \iota\, o']\,d\,\delta\alpha + \mathrm{S}\,\iota\, p'\,d\,\delta\beta + \mathrm{S}\,\iota\, o''\,d^2\,\delta\alpha\,. \end{aligned}$$

The last three integrals can be partially performed and we can make the following replacements, in which we have again omitted the terms that relate to the limits, as in (§ **12**), namely:

$$\mathrm{S}\,[\varepsilon\, m' + \iota\, o']\,d\,\delta\alpha \qquad \text{with} \qquad -\mathrm{S}\,d\,[\varepsilon\, m' + \iota\, o']\,\delta\alpha,$$

$$\mathrm{S}\,\iota\, p'\,d\,\delta\beta \qquad \text{with} \qquad -\mathrm{S}\,d\,[\iota\, p']\,\delta\beta,$$

$$\mathrm{S}\,\iota\, o''\,d^2\,\delta\alpha \qquad \text{with} \qquad +\mathrm{S}\,d^2\,[\iota\, o'']\,\delta\alpha\,.$$

The integral above can then be replaced with:

$$\begin{aligned} &\mathrm{S}\,(X\,\delta x + Y\,\delta y + Z\,\delta z)\,dm \\ &+ \mathrm{S}\,\{\sigma\, l + \varepsilon\, m + \iota\, o - d\,[\varepsilon\, m' + \iota\, o'] + d^2\,[\iota\, o'']\}\,\delta\alpha \\ &+ \mathrm{S}\,\{\varepsilon\, n + \iota\, p - d\,[\iota\, p']\}\,\delta\beta \\ &+ \mathrm{S}\,\{\iota\, q\}\,\delta\gamma\,. \end{aligned}$$

However:

$$\delta\alpha = 2\,(dx\,d\,\delta x + dy\,d\,\delta y + dz\,d\,\delta z),$$

$$\delta\beta = 2\,(d^2x\,d^2\delta x + d^2y\,d^2\,\delta y + d^2z\,d^2\,\delta z),$$

$$\delta\gamma = 2\,(d^3x\,d^3\,\delta x + d^3y\,d^3\,\delta y + d^3z\,d^3\delta z)\,.$$

If one puts those values into the sum above then the result to which one will arrive will be identical in form to the one in (§ **12**), and one will deduce three indefinite equations that are entirely similar to the ones (f_{12}) in that article, but in which the quantities a, b, c are replaced with:

$$2\,\{\sigma\, l + \varepsilon\, m + \iota\, o - d\,[\varepsilon\, m' + \iota\, o'] + d^2\,[\iota\, o'']\},$$

$$2\,\{\varepsilon\, n + \iota\, p - d\,[\iota\, p']\},$$

$$2\,\{\iota\, q\},$$

respectively.

Once one has determined the values of a, b, c for a stiff rod with a given figure above (§ **13**), one can deduce the value of the form ι from the equation $c = 2\iota\, q$. On that subject, one can remark that q is a quantity that is infinite of order five, and we saw above that c has the same order, so ι will be a finite quantity, as we stated. If one knows ι then will get ε by means of the equation:

$$b = 2\,\{\,\varepsilon\, n + \iota\, p - d\,[\iota\, p]\},$$

in which b is infinite of order three, as well as n, $\iota\, p$, $d\,[\iota\, p']$; it will then follow that e is finite. The same thing will be true for σ, which is determined by means of a, and of ε and ι, which are presently known.

The value of the torsion ι is remarkable. One finds it by very simple substitutions:

$$\iota = \frac{y\,dz - z\,dy}{ds}\int X\,dm + \frac{z\,dx - x\,dz}{ds}\int Y\,dm + \frac{x\,dy - y\,dx}{ds}\int Z\,dm$$

$$+\ \frac{dx}{ds}\int (yZ - zY)\,dm + \frac{dy}{ds}\int (zX - xZ)\,dm + \frac{dz}{ds}\int (xY - yX)\,dm\,.$$

As is easy to see, that is the sum of the moments of all the external forces when one starts from one of the extremities of the rod and goes to the point x, y, z of the curve that relates to the tangent to the curve at that point. One will arrive at the general values of ε and σ only by tedious calculations whose results will seem quite complicated.

16. – Such is the manner of introducing the most complete elements of stiffness or elasticity into the indefinite equations of equilibrium of a stiff or elastic filament. These elements seem natural to us, and can, in addition, replace the ones that *Lagrange* introduced for stiffness to some advantage, using his beautiful method of indeterminates, which he appealed to as multipliers of the variations of certain functions α, β, γ whose special meaning he did not discuss. From our discussion, it is easy to see that the forces that result from those indeterminates will be nothing but the ones that we have denoted by A, B, C; i.e., two of them – viz., B, C – will have the convenience of being quantities that are infinite of various orders.

In all of this, I believe that one must not separate the problem of the equilibrium of a stiff rod from that of an elastic filament, because the indefinite equations have the same form: However, we have already said that they will differ essentially depending upon the use that one makes of those two questions. For the equilibrium of a stiff rod, one knows the form of the rod from two equations in three coordinates. If, for example, one supposes that it is entirely free then six arbitrary constants, and no more, will enter into those equations, from which, that curve can take all situation in space without changing form. Now, one sees from the discussion of that case in *Mécanique analytique*, page 163 (new edition) that one will be led to the following equations:

$$0 = \mathrm{S}\, X\,dm, \qquad 0 = \mathrm{S}\, Y\,dm, \qquad 0 = \mathrm{S}\, Z\,dm,$$

$$0 = \mathrm{S}\,(Yz - Zy)\,dm, \qquad 0 = \mathrm{S}\,(Zx - Xz)\,dm, \qquad 0 = \mathrm{S}\,(Xy - Yx)\,dm,$$

in which the sign S always represents an integral that is extended over the entire length of the filament. These definite integrations can be performed since the forces X, Y, Z will be functions that can depend upon only the position of the point on the curve at which they act, and due to the two given equations for the curve, that point will be determined by one of its coordinates. The six constants of the equations of the curve will enter into the results of those integrations, and they will be determined by the six equations that are formed from those integrals when they are equated to zero. Once those constants are known, the particular position of the curve will itself be determined, since nothing indeterminate will remain in its six equations. One substitutes the values of x, y, z, X, Y, Z as functions of the variable that one deems to be suitable for fixing the position of the point on the curve in the equations of § **13**. The integrations can pertain to only known functions of that variable, and one further considers them to have been performed. The elements a, b, c of the stiffness for each point of the curve (which replace *Lagrange's* indeterminates λ, μ, ν) will become known from those equations, and they will, in turn, provide the forces A, B, C that we have employed, or even better, the values of the forces σ, ε, ι that we have found to be convenient substitutes for them, in order to avoid the infinite magnitudes of the forces B, C. The evaluation of σ, ε, ι demands only some differentiations, so it will present no difficulty. The examinations of any other case will likewise leads to the entire determination of the constants that fix the rigid rod in its equilibrium.

However, when one addresses a filament that is endowed with the three types of elasticity that we represent by σ, ε, ι, or two of them or just one of them, those quantities will be given by the nature and constitution of the filament, and what one must know is the form that it must take under the action of the forces. For example, one might know that the filament has a certain form before being subjected to that action, and its constitution permits it to oppose the changes in that form by known forces for each of the three types of elasticity. One then knows the tension σ, the elasticity ε, and the torsion ι. Upon imagining that they are expressed by the same variable for each point of the curve, as well as the forces X, Y, Z that relate to that point, it will suffice to have those of these three indefinite equations upon which the form of the curve depends. However, an indeterminate will necessarily enter into those equations that itself depends upon the unknown length that the extensible filament must take under the action of forces, and it will only be when one knows the form that it takes on in its equilibrium that one can obtain that indeterminate. It must be eliminated beforehand, and that will reduce their number to two. The difficulty in their integration will be very great in any case. Since our goal is solely to point out the manner by which one can make the most complete elements of elasticity enter into consideration, the example that we just followed through seems sufficient to us, but each case to be treated can bring with it even more modifications of that path. As for the equations at the limits, we have constantly neglected to address them, because the method of variation and the applications that *Lagrange* developed in his *Mécanique* leave nothing in the dark in regard to that point.