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Euclidian kinematics and non-Euclidian geometry. I, II.

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Contents

In the sequel, a systematic treatment of planar kinematics will be sketched out whose guiding principle is based in the ideas of E. Study in an Appendix to his *Geometrie der Dynamen*. For the case of the plane, one can clarify the essential group-theoretic viewpoint more easily than for space, and it is easier to organize the differential-geometric problems of the geometry of motion that are treated by the usual manner of representation within the scope of Study's kinematics. The present paper can thus also serve as an introduction to Study's kinematical methods.

The guiding thought of the investigation is the following one: *It will be shown that planar kinematics can be associated with the projective geometry of space. In this way, one comes to a projective metric that goes to the non-Euclidian, and indeed elliptic, metric under passing to the limit.* The geometry that is thus defined will be called *quasi-elliptic geometry*. Its absolute structure consists of a pair of conjugate-imaginary planes and a pair of conjugate-imaginary points on the line of intersection of these planes.

The passage from planar kinematics to this quasi-elliptic geometry is mediated by a map of ordered point-pairs in the plane to lines of space, a map that associated motions in the plane with points in space and transfers in the plane with planes in space.

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[*]) D.H.D.: Corrections to the text that were later published by Blaschke have also been incorporated.

I.

A map of the lines in space to ordered point-pairs in the plane. Quasi-elliptic geometry.

§ 1. Construction of the map.

We start with a (real) perpendicular axis cross and denote the inhomogeneous point coordinates that we would like to employ by x, y, and z. We call the plane $z = 0$ the *base plane*. We clarify that the *positive sense of rotation* has the sense of the shortest rotation that takes the semi-axis in the positive x direction to the positive y. Let the planes $z = -1$ and $z = +1$ be denoted by α_l and α_r, accordingly. By the symbols l and r, we intend this to means the words "left" and "right" that will be employed later on.

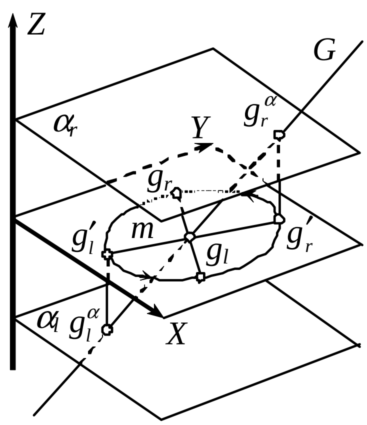

Figure 1.

Now, let G be a (real) line [1]), that does not intersect the ideal line C in the base plane. G is then a real line that does not run parallel to the base plane. We carry out the following construction (see Figure 1):

Extend the line G to its intersection with the planes α_l, α_r at the points g_l^α, g_r^α, and look for the normal piercing points (Normalriss) g_l', g_r' of these two intersection points on the base plane. One then rotates the ordered point-pair (g_l', g_r') around its midpoint m in the base plane through a positive right angle to (g_l, g_r), such that the point g_l' goes to the point g_l, and the point g_r' goes to g_r. [2])

In this way, one finds an ordered point pair (g_l, g_r) associated with every line C. On the other hand, if we choose an ordered pair of real, actual (not necessarily distinct) points (g_l, g_r) in the base plane then we can search for the line G that is associated with them in a unique way by inverting the given construction. One has thus exhibited an invertible relationship $G \leftrightarrow (g_l, g_r)$ between the lines of space that do not meet C and the ordered pairs of real, actual points in the base plane. What this association first attracts attention to is the following fact:

Theorem I. *Under our association, two lines G and G' that cut each other go to two ordered point-pairs (g_l, g_r) and (g_l^*, g_r^*) that relate to each other in such a way that the line segments $g_l g_l^*$ and $g_r g_r^*$ are equal to each other. Conversely, such point-pairs are associated with lines that lie in a plane.*

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[1]) We first restrict our reasoning to *real* elements throughout.

[2]) The map of lines G onto the ordered pair (g_l', g_r') finds an application in descriptive geometry, as E. Müller has discussed thoroughly in his lectures.

We likewise prove a more far-reaching theorem that subsumes the latter one:

Theorem IIa. *The* ∞^2 *lines* G *that go through a point* p *that does not line on* C *will be mapped to* ∞^2 *ordered pairs* (g_l, g_r) *under the association* $G \to (g_l, g_r)$ *whose starting points* g_l *are associated with the endpoints* g_r *under a motion* $g_l \to g_r$ *.*

We first assume that the point p is a finite point and has the coordinates x, y, z; its "base point" (Grundriss) p' then has the coordinates x, y, 0. We draw a line G through p (which does not meet C, as we have always assumed here) and apply our construction to it. (Cf., Fig. 2, in which the construction is presented at the base point and upper point.) The points p', g_l', and g_r' lie in a line, and one has (also up to sign):

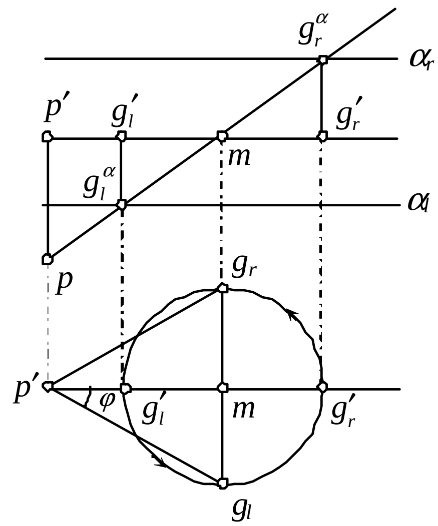

Figure 2.

$$p'g_l' : p'g_r' = (z+1) : (z-1)$$

and

$$p'm : mg_l' = z : 1.$$

If we now rotate the pair (g_l', g_r') around its midpoint m through a positive right angle to (g_l, g_r) then the segments $p'g_l$ and $p'g_r$ will be absolutely equal to each other, and for the angle 2φ (which is determined mod 2π, including the sign) between these two directed line segments, one finds:

(1) $$\cot \varphi = -z.$$

The transformation $g_l \to g_r$ is then a *rotation* around p' through the angle 2φ.

If p lies at infinity (but not on C) then a *translation* enters in place of the rotation, as one easily confirms.

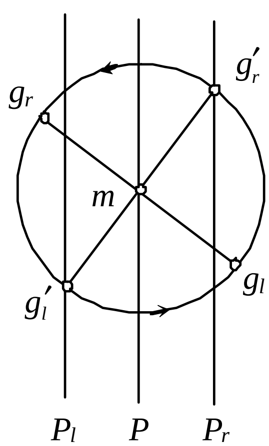

Figure 3.

By an analogous argument, we find:

Theorem IIb. *Under the association* $G \to (g_l, g_r)$*, the* ∞^2 *lines* G *that lie in a plane* π *and do not go through* C *are mapped to* ∞^2 *ordered point-pairs whose starting points* g_l *correspond to their end points* g_r *under a transfer.*

Thus, as usual, by the term *transfer*, we shall understand this to mean a point transformation in the base plane that can be generated by the composition of reflections through three lines of a plane.

One can see the validity of IIb in the following way: π cuts α_l and α_r at two parallel actual lines whose base points we would like to call P_l and P_r (Fig. 3). These two lines will be switched with each other under the reflection in the trace P of π on the base plane that is parallel to them. To an arbitrary line G, there now belongs a pair (g'_l, g'_r) whose starting point g'_l lies on P_l and whose end point g'_r lies on P_r . If one rotates this pair around its midpoint m from P to (g_l, g_r) through a positive right angle then the new pair is oriented such that one can take the starting point g_l to the end point g_r in such a way that one first reflects g_l through P and then translates it in the direction of P. The magnitude of the translation is equal to the distance from P_l to P_r, or $-2\cot\omega$, when we let ω denote the angle that the base plane subtends with π. If we then map all lines G in π onto the base plane then we obtain ∞^2 point-pairs (g_l, g_r) that determine a transfer $g_l \rightarrow g_r$ that is generated when one composes the reflection through the trace P of π with a translation through the line segment:

$$2\vartheta = -2\cot\omega \tag{2}$$

in the direction of P.

A *remark concerning the sign* is of importance here. If one orients the trace P – i.e., one singles out a sense of traversal on this line as the "positive" one – then a certain sign is established by this for the (determined mod π) angle ω, as well as for the magnitude of translation 2ϑ, when one agrees upon the following closely-related things: The quantity ϑ shall be counted as positive or negative according to whether the sense of the translation does not does not agree with the positive sense of traversal on P, respectively. In order to clarify the sign of ω, we further establish that the positive sense of rotation around P shall likewise imply the positive sense of traversal along that line, just as the positive sense of rotation in the base plane relates to the positive sense of traversal along the z-axis.

Formula (2) is correct with these assumptions, as one also might have chosen the positive sense on P. In fact, if one changes the orientation of P then ϑ and ω change their signs *simultaneously*.

We summarize the latter results neatly as follows:

Theorem IIIa. *The points p in space that do not lie on C are mapped to the motions $g_l \rightarrow g_r$ in the base plane in a one-to-one manner. Every actual point p corresponds to a rotation. The center of the rotation is the base point of p. The relation:*

$$\cot\varphi = -z \tag{1}$$ [1]

exists between the rotation angle 2φ and the distance z from the point p to the base plane.

Theorem IIIb. *The planes π in space that do not go through C are mapped to the transfers $g_l \rightarrow g_r$ in the base plane in a one-to-one manner. Any of these planes π correspond to a transfer whose centerline is the trace of π on the base plane. The relation:*

$$\cot\varphi = -\vartheta \tag{2}$$

exists between the translation magnitude 2ϑ and the angle the base plane makes with π.

[1]) R. Bricard employed (Nouvelles Annales de Mathématiques (4) 10, 1910) a similar arrangement between the points in space and the rotations of a plane.

Concerning the theorem on the right, we remark: Under *any* transfer in the plane, *a single* actual line will be transformed in such a way that its sense of traversal remains preserved; we call it the *centerline* of the transfer. One can compose the transfer from the reflection in its centerline and a translation that commutes with it through the segment 2ϑ in the direction of the centerline.

The analogy between the theorems on the left and right may be pushed somewhat further when one counts the translations along with the rotations and clarifies that the center of a translation is that ideal point to which the normals to the translation direction point.

The chosen association $G \to (g_l, g_r)$ is also useful in descriptive geometry. Furthermore, it represents a transference principle, in that it mediates the connection between theorems of spatial geometry and those of plane geometry. An example of this will be given later (§ 9). Here, we would next like to put our arrangement into formulas.

§ 2. Analytical representation of the map.

It is preferable here to introduce homogeneous coordinates. We set:

$$x = \frac{x_2}{x_1}, \qquad y = \frac{x_3}{x_1}, \qquad z = \frac{x_0}{x_1}. \tag{3}$$

In the base plane, we preserve the inhomogeneous coordinates, in order to give the formulas the most lucid form possible. Let the coordinates of g_l be x, y and let those of g_r be $\mathfrak{x}$, $\mathfrak{y}$.

For the homogeneous coordinate of the points g_l^α and g_r^α (cf., Fig. 1), one finds:

$$g_l^\alpha = \begin{cases} x_0 = -2, \\ x_1 = +2, \\ x_2 = +x + y + \mathfrak{x} - \mathfrak{y}, \\ x_3 = -x + y + \mathfrak{x} + \mathfrak{y}, \end{cases} \qquad g_r^\alpha = \begin{cases} x_0 = +2, \\ x_1 = +2, \\ x_2 = +x - y + \mathfrak{x} + \mathfrak{y}, \\ x_3 = -x + y - \mathfrak{x} + \mathfrak{y}. \end{cases} \tag{4}$$

The two-rowed determinants of the matrix:

$$\begin{Vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{Vmatrix}$$

are proportional to the Plücker line coordinates G_{ik} of G:

$$G_{ik} = \rho \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix}, \qquad \rho \neq 0.$$

One finds:

$$(5) \qquad \left.\begin{matrix} G_{01} & : G_{02} & : G_{03} \\ G_{23} & : G_{31} & : G_{12} \end{matrix}\right\} = \begin{cases} \quad 2 & : x + \mathfrak{x} \ : z + \mathfrak{z} \\ -\dfrac{(x^2 + y^2) - (\mathfrak{x}^2 + \mathfrak{y}^2)}{2} & : x - \mathfrak{x} \ : y - \mathfrak{y} \end{cases}$$

and from this, one has, conversely:

$$(6) \qquad \begin{gathered} x = \frac{G_{02} + G_{31}}{G_{01}}, \quad y = \frac{G_{03} + G_{12}}{G_{01}}; \\ \mathfrak{x} = \frac{G_{02} - G_{31}}{G_{01}}, \quad \mathfrak{y} = \frac{G_{03} - G_{12}}{G_{01}}. \end{gathered}$$

The assumption:

$$(7) \qquad G_{01} \neq 0$$

that is necessary in this states precisely that G shall not meet the ideal line C in the base plane. It has, in fact, the coordinates:

$$(8) \qquad \left.\begin{matrix} C_{01} & : C_{02} & : C_{03} \\ C_{23} & : C_{31} & : C_{12} \end{matrix}\right\} = \begin{cases} 0 : 0 : 0 \\ 1 : 0 : 0 \end{cases},$$

and one thus has:

$$G_{01}\, C_{23} + G_{02}\, C_{31} + G_{03}\, C_{12} + G_{23}\, C_{01} + G_{31}\, C_{02} + G_{12}\, C_{03} = \lambda\, G_{01}, \qquad \lambda \neq 0.$$

$G_{01} = 0$ is thus the necessary and sufficient condition for G and C to intersect. We take two lines G and G^* then we find under the assumption $G_{01} \neq 0$, $G^*_{01} \neq 0$:

$$(9) \qquad \left\{\begin{matrix} G_{01}G^*_{23} + G_{02}G^*_{31} + G_{03}G^*_{12} \\ + G_{23}G^*_{01} + G_{31}G^*_{02} + G_{12}G^*_{03} \end{matrix}\right\} = \chi \left\{\begin{matrix} +[(x - x^*)^2 + (y - y^*)^2 \\ -[(\mathfrak{x} - \mathfrak{x}^*)^2 + (\mathfrak{y} - \mathfrak{y}^*)^2 \end{matrix}\right\}.$$

This is the analytical formulation of our theorem I. [1])

Now, let the point p lie on the line G, which we think of as the connecting line two points x and y. The three-rowed determinants of the matrix:

$$\begin{Vmatrix} x_0 & x_1 & y_2 & y_3 \\ y_0 & y_1 & x_2 & x_3 \\ p_0 & p_1 & p_2 & p_3 \end{Vmatrix}$$

must all vanish. If we establish the assumption that $x_0\, y_1 - x_1\, y_0 \neq 0$ then it suffices to set two of these determinants equal to zero, and the other two must then also vanish. We have set the determinants equal to zero that arise from our matrix when one first deletes

[1]) Formula (5) and (9), as I have learned from personal communication, have also been found by Study. Cf., the last § of the next-appearing issue 1 of the lectures on selected topics in geometry (Leipzig, 1911).

the fourth column, and then deletes the third column. If one develops the last row then one finds:

$$p_0 G_{12} - p_1 G_{02} + p_2 G_{01} = 0, \quad p_0 G_{31} + p_1 G_{03} - p_3 G_{01} = 0, \tag{10}$$

or, when we invoke formulas (5):

$$p_0 (y - \mathfrak{y}) - p_1 (x + \mathfrak{x}) + 2 p_2 = 0, \quad p_0 (x - \mathfrak{x}) + p_1 (y - \mathfrak{y}) - 2 p_3 = 0. \tag{11}$$

From our assumption $G_{01} \neq 0$, p cannot lie on C, so p_0 and p_1 cannot be simultaneously zero, and $p_0^2 + p_1^2$ is therefore positive (thus, we always restrict ourselves to real elements here). One can therefore solve the formula (11) for $\mathfrak{x}$, $\mathfrak{y}$:

$$\begin{aligned} (p_0^2 + p_1^2)\mathfrak{x} &= (p_0^2 + p_1^2) x + 2 p_0 p_1 \cdot y + 2(p_1 p_2 - p_0 p_3), \\ (p_0^2 + p_1^2)\mathfrak{y} &= -2 p_0 p_1 \cdot x + (p_0^2 - p_1^2) y + 2(p_1 p_2 + p_0 p_3). \end{aligned} \tag{12}$$

In this, if one thinks of the parameter p_k as held fixed then these equations represent the motion $g_l(x, y) \to g_r(\mathfrak{x}, \mathfrak{y})$ that is associated with the motion (Theorem IIa). [1])

For $p_1 \neq 0$, we find the coordinates of the rotational center (cf., Theorem IIIa) from (11):

$$x = \mathfrak{x} = \frac{p_2}{p_1}, \quad y = \mathfrak{y} = \frac{p_0}{p_1}. \tag{13}$$

Our formula (1) for the rotational angle 2φ is also confirmed, since here we find:

$$\cot \varphi = -\frac{p_0}{p_1}. \tag{14}$$

If p goes to infinity ($p_0 \neq 0$, $p_1 = 0$) then the formulas (12) simplify to:

$$\mathfrak{x} = x - 2\frac{p_3}{p_0}, \quad \mathfrak{y} = y + 2\frac{p_2}{p_0}. \tag{15}$$

The magnitude of this displacement is:

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[1]) This parametric representation of the motions in the plane has already been found by E. Study, "Von den Bewegungen und Umlegungen," Math. Ann. **39** (1891), § 11, pp. 585, *et seq.*

(16) $$2\vartheta = \frac{2}{p_0}\sqrt{p_2^2 + p_3^2}\,.$$

We can present an analogous argument when we regard the line G as the line of intersection of two planes μ, ν. If p also goes through G then the three-rowed determinants of the following matrix must vanish:

$$\begin{Vmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 \\ \nu_0 & \nu_1 & \nu_2 & \nu_3 \\ \pi_0 & \pi_1 & \pi_2 & \pi_3 \end{Vmatrix}.$$

One finds, in particular:

(17) $$\begin{aligned} &\pi_0 G_{01} - \pi_2 G_{12} - \pi_3 G_{31} = 0, \\ &\pi_1 G_{01} + \pi_2 G_{02} + \pi_3 G_{01} = 0, \end{aligned}$$

and from this:

(18) $$\begin{aligned} &2\pi_0 - \pi_2(y - \mathfrak{y}) - \pi_3(x - \mathfrak{x}) = 0, \\ &2\pi_1 + \pi_2(x + \mathfrak{x}) + \pi_3(y + \mathfrak{y}) = 0, \end{aligned}$$

or:

(19) $$\begin{aligned} -(\pi_2^2 + \pi_3^2)\mathfrak{x} &= (\pi_2^2 - \pi_3^2)x + 2\pi_2\pi_3 \cdot y + 2(\pi_1\pi_2 - \pi_0\pi_3), \\ -(\pi_2^2 + \pi_3^2)\mathfrak{y} &= 2\pi_2\pi_3 \cdot x - (\pi_2^2 - \pi_3^2)y + 2(\pi_1\pi_3 - \pi_0\pi_2). \end{aligned}$$

For fixed values of the π_k , these equations represent the transfer $g_l(x, y) \to g_r(\mathfrak{x}, \mathfrak{y})$, which corresponds to the plane π. [1]) The second formula (18) yields the equation for the centerline:

(20) $$+ (\pi_1 + \pi_2\, x + \pi_3\, y) = - (\pi_1 + \pi_2\, \mathfrak{x} + \pi_3\, \mathfrak{y}) = 0.$$

For the magnitude of the displacement 2ϑ, one obtains:

(21) $$\vartheta = \frac{\pi_0}{\sqrt{\pi_2^2 + \pi_3^2}}\,,$$

when one sets:

(22) $$\cot w = -\frac{\pi_0}{\sqrt{\pi_2^2 + \pi_3^2}}\,.$$

§ 3. Composition of motions and transfers.

The formulas shall be repeated here in brief that Study found for the composition of motions and transfers in the plane. [2])

[1]) Formula (19) was also found by Study, *loc. cit.*
[2]) Math. Ann. 39 (1891), pp. 558-561. Cf., also the Monatschefte f. Math. und Phys. I (1900), pp. 352.

For a motion that takes the point (x, y) to (x', y'), we have found the parametric representation [Formula 12]:

$$(23)\qquad \begin{aligned}(p_0^2+p_1^2)x' &= (p_0^2+p_1^2)x + 2p_0p_1\cdot y + 2(p_1p_2 - p_0p_3),\\ (p_0^2+p_1^2)y' &= -2p_0p_1\cdot x + (p_0^2-p_1^2)y + 2(p_1p_2 + p_0p_3).\end{aligned}$$

If we follow this motion $(x, y) \to (x', y')$, which has the parameters p_i, with another motion $(x', y') \to (x'', y'')$ that has the parameters p_i', then we obtain a new motion under composition $(x, y) \to (x'', y'')$. For its parameters p_i'', one finds the expressions:

$$(24)\qquad \begin{cases}\rho p_0'' = p_0p_0' - p_1p_1',\\ \rho p_1'' = p_0p_1' - p_1p_0',\\ \rho p_2'' = p_0p_2' - p_1p_3' + p_2p_0' + p_3p_1',\\ \rho p_3'' = p_0p_3' + p_1p_2' - p_2p_1' + p_3p_0'.\end{cases}$$

For a transfer, $(x, y) \to (x', y')$, we have the formula (19):

$$(25)\qquad \begin{aligned}-(\pi_2^2+\pi_3^2)x' &= (\pi_2^2-\pi_3^2)x + 2\pi_2\pi_3\cdot y + 2(\pi_1\pi_2 - \pi_0\pi_3),\\ -(\pi_2^2+\pi_3^2)y' &= 2\pi_2\pi_3\cdot x - (\pi_2^2-\pi_3^2)y + 2(\pi_1\pi_3 + \pi_0\pi_2).\end{aligned}$$

If we compose two transfers $(x, y) \to (x', y')$, $(x', y') \to (x'', y'')$ with the parameters π_i and π_i', resp., then this produces a motion with the parameters:

$$(26)\qquad \begin{cases}\rho p_0'' = \qquad\qquad\qquad\quad -\pi_2\pi_2' - \pi_3\pi_3',\\ \rho p_1'' = \qquad\qquad\qquad\quad -\pi_2\pi_3' - \pi_3\pi_2',\\ \rho p_2'' = \pi_0\pi_2' - \pi_1\pi_3' + \pi_2\pi_0' + \pi_3\pi_1',\\ \rho p_3'' = \pi_0\pi_3' + \pi_1\pi_2' - \pi_2\pi_1' + \pi_3\pi_0'.\end{cases}$$

For the composition of a motion with a transfer, one finds, depending upon the order in which these transformations were performed, either:

$$(27)\qquad \begin{cases}\rho\pi_0'' = p_0\pi_0' - p_1\pi_1' - p_2\pi_2' - p_3\pi_3',\\ \rho\pi_1'' = p_0\pi_1' + p_1\pi_0' + p_2\pi_3' - p_3\pi_2',\\ \rho\pi_2'' = p_2\pi_2' - p_1\pi_3',\\ \rho\pi_3'' = p_0\pi_3' + p_1\pi_2',\end{cases}$$

or:

$$(28)\qquad \begin{cases} \rho\pi_0'' = \pi_0 p_0' - \pi_1 p_0' - \pi_2 p_2' - \pi_3 p_3', \\ \rho\pi_1'' = \pi_0 p_1' + \pi_1 p_0' + \pi_2 p_3' - \pi_3 p_2', \\ \rho\pi_2'' = \qquad\qquad\quad + \pi_2 p_0' + \pi_3 p_1', \\ \rho\pi_3'' = \qquad\qquad\quad - \pi_2 p_1' + \pi_3 p_0'. \end{cases}$$

From Study, one can summarize these formulas in an elegant way, when one appeals to higher complex numbers. We set:

$$(29)\qquad \begin{aligned} p &= p_0 e_0 + p_1 e_1 + p_2 \varepsilon e_2 + p_3 \varepsilon e_3, \\ \pi &= \pi_0 \varepsilon e_0 + \pi_1 \varepsilon e_1 + \pi_2 e_2 + \pi_3 e_3. \end{aligned}$$

In this, the e_k might mean the quaternion units, which, as is well-known, satisfy the multiplication rules:

$$(30)\qquad \begin{aligned} e_0 &= e_0 e_0 = - e_1 e_1 = - e_2 e_2 = - e_3 e_3, \\ e_1 &= e_0 e_1 = + e_1 e_0 = + e_2 e_3 = - e_3 e_2, \\ e_2 &= e_0 e_2 = - e_1 e_3 = + e_2 e_0 = + e_3 e_1, \\ e_3 &= e_0 e_3 = + e_1 e_2 = - e_2 e_1 = + e_3 e_0, \end{aligned}$$

and ε *means a unit that commutes with the quaternion units, and is subject to the rule of calculation:*

$$(31)\qquad \varepsilon^2 = 0.$$

By means of these complex relations, one write formulas (24), (26), (27), (28) for $\rho = 1$ in the form:

$$(24)^* \quad \dot{p}'' = \dot{p}\dot{p}', \qquad (26)^* \quad \dot{p} = \dot{\pi}\dot{\pi}',$$

$$(27)^* \quad \dot{\pi}'' = \dot{p}\,\dot{\pi}', \qquad (28)^* \quad \dot{\pi} = \dot{\pi}\,\dot{p}'.$$

We then introduce two notations. If:

$$\dot{p} = p_0\, e_0 + p_1\, e_1 + p_2\, \varepsilon\, e_2 + p_3\, \varepsilon\, e_3,$$

then we set:

$$(32)\qquad \bar{\dot{p}} = p_0\, e_0 - p_1\, e_1 - p_2\, \varepsilon\, e_2 - p_3\, \varepsilon\, e_3.$$

Likewise, from:

$$\dot{\pi} = \pi_0\, \varepsilon\, e_0 + \pi_1\, \varepsilon\, e_1 + \pi_2\, e_2 + \pi_3\, e_3,$$

one deduces the relation:

$$(33)\qquad \bar{\dot{\pi}} = \pi_0\, \varepsilon\, e_0 - \pi_1\, \varepsilon\, e_1 - \pi_2\, e_2 - \pi_3\, e_3.$$

We further define:

$$(34)\qquad \begin{aligned} N(\dot{p}) &= p_0^2 + p_1^2 = \dot{p}\,\bar{\dot{p}}, \\ N(\pi) &= \pi_2^2 + \pi_3^2 = \dot{\pi}\,\bar{\dot{\pi}}. \end{aligned}$$

The following *rules of calculation* are valid for the two symbols that were introduced, which one can confirm immediately by direct computation or by assuming the known formulas of the theory of quaternions. By small German symbols, we intend this to mean the formulas that follow by using complex quantities to replace the previous ones, so we now associate motions with $(\dot{p}, \dot{p}', \dot{p}'', \ldots)$ and transfers with $(\dot{\pi}, \dot{\pi}', \dot{\pi}'', \ldots)$.

From $\dot{\mathfrak{p}}'' = \dot{\mathfrak{p}}\dot{\mathfrak{p}}'$, *it follows that* $\overline{\dot{\mathfrak{p}}''} = \overline{\dot{\mathfrak{p}}}\,\overline{\dot{\mathfrak{p}}'}$. *Thus, from* $\dot{\mathfrak{p}}'' = \dot{\mathfrak{p}}\dot{\mathfrak{p}}'$, *it also follows that* $N(\dot{\mathfrak{p}}'') = N(\dot{\mathfrak{p}})N(\dot{\mathfrak{p}}')$.

The complex relations also enable one to combine formulas (23) and (25). One sets:

(35) $$\dot{g} = e_1 + x\,\varepsilon\,e_2 + y\,\varepsilon\,e_3 .$$

One then finds*:*

(23)* $$N(\dot{p}) \cdot \dot{g}' = \overline{\dot{p}}\,\dot{g}\,\dot{p}$$

for the *motions*, and:

(25)* $$N(\dot{\pi}) \cdot \dot{g}' = \overline{\dot{\pi}}\,\dot{g}\,\dot{\pi}$$

for the *transfers.*

§ 4. Left-parallel and right-parallel lines.

Let two lines G, G^* be given, that do not meet the ideal line C in the base plane. Let (g_l, g_r), (g_l^*, g_r^*) be their images. We pose the following *definition:*

Two lines G, G^* *are called* left-parallel (right-parallel, resp.) *when the associated points* g_l *and* g_l^* (g_r *and* g_r^*, resp.) *coincide.*

From the formulas (6), one deduces the following conditions for *left-parallelism:*

(36) $$G_{01} : G_{02} + G_{31} : G_{03} + G_{12} = G_{01}^* : G_{02}^* + G_{31}^* : G_{03}^* + G_{12}^*$$

and *right-parallelism:*

(37) $$G_{01} : G_{02} - G_{31} : G_{03} - G_{12} = G_{01}^* : G_{02}^* - G_{31}^* : G_{03}^* - G_{12}^* .$$

From the definition, one infers the following properties of this parallelism:

If one reflects a left (right, resp.)-parallel line in the base plane then this produces a right (left, resp.)-parallel line. In fact, the reflection corresponds to the exchange of the starting and ending points of the point-pair.

One and only one left (right, resp.)-parallel to any arbitrary chosen line G goes through each point that does not lie on C.

One and only one left (right, resp.)-parallel to any line G lies in any plane that does not go through C.

We now introduce *imaginary* elements, which have a distinguished relationship to our real figures. We then introduce two points c_l, c_r that belong to the absolute conic section (spherical circle) of the Euclidian metric, and lie in the base plane:

$$
(38) \qquad \begin{array}{ll} c_l & x_0 : x_1 : x_2 : x_3 = 0 : 0 : i : 1, \\ c_r & x_0 : x_1 : x_2 : x_3 = 0 : 0 : 1 : i, \end{array}
$$

and two planes:

$$
(39) \qquad \begin{array}{ll} \gamma_l & x_0 - i x_1 = 0, \\ \gamma_r & x_0 + i x_1 = 0, \end{array} \qquad (i = \sqrt{-1}).
$$

If we consider all lines G^* that run left-parallel to the fixed line G [equation (36)] then we see that they all meet two conjugate-imaginary lines M, $\bar{M}$ with the coordinates:

$$
(40) \qquad \begin{array}{l}
\left.\begin{array}{r} M_{01} : M_{02} : M_{03} : \\ M_{23} : M_{31} : M_{12} \end{array}\right\} = \left\{\begin{array}{ll} 0 & : -G_{01} : +iG_{01} : \\ (G_{02} + G_{31}) - i(G_{03} + G_{12}) & : -G_{01} : +iG_{01}, \end{array}\right. \\
\left.\begin{array}{r} \bar{M}_{01} : \bar{M}_{02} : \bar{M}_{03} : \\ \bar{M}_{23} : \bar{M}_{31} : \bar{M}_{12} \end{array}\right\} = \left\{\begin{array}{ll} 0 & : -G_{01} : -iG_{01} : \\ (G_{02} + G_{31}) + i(G_{03} + G_{12}) & : -G_{01} : -iG_{01}. \end{array}\right.
\end{array}
$$

We find the following coordinates for the intersection points g_l^γ, g_r^γ of G with γ_l, γ_r [cf., formulas (10)]:

$$
(41) \qquad \begin{array}{ll} g_l^\gamma & x_0 : x_1 : x_2 : x_3 = +i\,G_{01} : G_{01} : G_{02} - i\,G_{12} : G_{03} + i\,G_{31}, \\ g_r^\gamma & x_0 : x_1 : x_2 : x_3 = -i\,G_{01} : G_{01} : G_{02} + i\,G_{12} : G_{03} - i\,G_{31}. \end{array}
$$

One sees from this that M connects the points g_l^γ and c_l, while $\bar{M}$ connects the points γ_r^γ and c_r.

Theorem IV. *The ∞^2 lines that run left (right, resp.)-parallel to a fixed line G belong to an elliptic net – i.e., the totality of all lines that cut two conjugate-imaginary lines (viz., the guidelines of the net).*

The guidelines of a net of left-parallel lines are an imaginary line at γ_l through c_l and the conjugate-imaginary at γ_r through c_r.

The guidelines of a net of right-parallel lines are an imaginary line at γ_l through c_r and the conjugate-imaginary at γ_r through c_l.

We have proved the relation only for left-parallel lines, but it also follows for right-parallel lines when one applies a reflection in the base plane. One can, although we shall not do so here, also easily confirm the content of Theorem IV in a purely geometric way without the use of the formulas.

We call the ideal point the *normal* to the base plane p^0 (1 : 0 : 0 : 0) and formulate Theorem IV in a somewhat different way:

Theorem V. *Let* g_l^γ, g_r^γ *be the intersection points of a line G with* γ_l *and* γ_r. *The two conjugate-imaginary connecting lines* $[g_l^\gamma c_l p^0]$, $[g_r^\gamma c_r p^0]$ *cut the base plane at the real starting point* g_l *and the two conjugate-imaginary lines* $[g_l^\gamma c_r p^0]$, $[g_r^\gamma c_l p^0]$ *cut the base plane at the real endpoint* g_r *in the image* (g_l, g_r) *of G.*

In fact, from Theorem IV, e.g., the line of intersection of the former two planes is left-parallel to G, and since it is normal to the base plane, its base point must coincide with g_l on the base plane.

§ 5. The transformation groups of quasi-elliptic geometry.

The figure that consists of the points c_l, c_r, and the planes g_l, g_r shall be called the *quasi-absolute structure.*

Let there be given a real collineation or correlation of space that takes the quasi-absolute structure to itself. We would like to denote this transformation $G \to G^*$, since we would like to think of it as being applied to the lines in space. Under the association $G \to (g_l, g_r)$, the map $G \to G^*$ corresponds to an exchange of the ordered point-pairs (g_l, g_r) and (g_l^*, g_r^*) in the base plane. From the imaginary-geometric interpretation of "parallelism" that was given in the previous paragraphs, under the transformation $G \to G^*$, left (right, resp.)-parallel lines must again go to either left or right-parallel lines; i.e., under the corresponding map $(g_l, g_r) \to (g_l^*, g_r^*)$, point-pairs with a common starting (ending, resp.) point will again go to point-pairs that have a common starting or ending point.

The transformation $(g_l, g_r) \to (g_l^*, g_r^*)$ thus decomposes point-pairs into two point-transformations, which we will denote by either the symbols $g_l \to g_l^*$, $g_r \to g_r^*$ or $g_l \to g_r^*$, $g_r \to g_l^*$. One can say more once one knows how the points and planes of the quasi-absolute structure permute amongst themselves under the map $G \to G^*$.

About the point transformations $g_l \to g_l^*$, $g_r \to g_r^*$ (or $g_l \to g_r^*$, $g_r \to g_l^*$), we next remark that they exchange the actual points of the plane with each other in a non-singular, one-to-one, and continuous manner. Since the map $G \to G^*$ takes intersecting lines G and H again to intersecting lines G^*, H^*, from our Theorem I in § 1, the two point transformations in the base plane must have the following property: When two pairs of real, actual points g_l, h_l; g_r, h_r are given in such a way that the segments $g_l\, h_l$ andf $g_r\, h_r$ are absolutely equal to each other, the lines $g_l^* h_l^*$, $g_r^* h_r^*$ between the associated points always turn out to be equal to each other, as well.

However, from this, it follows that these point transformations must be *similarities with the same expansion ratio.*

Conversely, if we choose two real similarities $g_l \to g_l^*$, $g_r \to g_r^*$ (or $g_l \to g_r^*$, $g_r \to g_l^*$) with the same expansion ratio, but still completely arbitrary, then a transformation $G \to G^*$ of lines in space corresponds to them under our transformation $(g_l, g_r) \to G$ that is first explained only for the those lines that do not cut the connecting line C of c_l and c_r.

For these lines, the transformation is uniquely invertible and takes intersecting lines to intersecting lines. $G \to G^*$ is then a part of a collineation or a correlation of space. By means of our imaginary-geometric interpretation of the two types of parallelism, one further proves that the map $G \to G^*$ transforms the quasi-absolute structure into itself.

We have thus found:

Theorem VI. *Under the association $G \to (g_l, g_r)$, the collineations and correlations $G \to G^*$ of space that transform the quasi-absolute structure into itself will be mapped to pairs of real similarity transformations $g_l \to g_l^*$, $g_r \to g_r^*$ and $g_l \to g_r^*$, $g_r \to g_l^*$ with the same expansion ratio.*

The seven-parameter group of collineations and correlations of the quasi-absolute structure decomposes into eight continuous families.

In order to make a simple notation possible, we would like to, e.g., denote an actual (i.e., sense-preserving) similarity $g_l \to g_r^*$ by $\mathfrak{a}_{lr}^+$ and, e.g., an ideal (i.e., sense-reversing) similarity $g_l \to g_l^*$ by $\mathfrak{a}_{ll}^-$. The eight families $\mathfrak{G}_7$, $\mathfrak{H}_7^{\mathrm{I}}$, $\mathfrak{H}_7^{\mathrm{II}}, \mathfrak{H}_7^{\mathrm{III}}$, X_7, X_7^{I}, X_7^{II}, X_7^{III} of the automoprhic group of the quasi-absolute structure map to the base plane in the following way:

$$
\begin{aligned}
&\mathfrak{G}_7 \leftrightarrow \{\mathfrak{a}_{ll}^+, \mathfrak{a}_{rr}^+\}, \quad X_7 \leftrightarrow \{\mathfrak{a}_{ll}^+, \mathfrak{a}_{rr}^-\},\\
&\mathfrak{H}_7^{\mathrm{I}} \leftrightarrow \{\mathfrak{a}_{ll}^-, \mathfrak{a}_{rr}^-\}, \quad X_7^{\mathrm{I}} \leftrightarrow \{\mathfrak{a}_{ll}^-, \mathfrak{a}_{rr}^+\},\\
&\mathfrak{H}_7^{\mathrm{II}} \leftrightarrow \{\mathfrak{a}_{lr}^+, \mathfrak{a}_{rl}^+\}, \quad X_7^{\mathrm{II}} \leftrightarrow \{\mathfrak{a}_{lr}^+, \mathfrak{a}_{rl}^-\},\\
&\mathfrak{H}_7^{\mathrm{III}} \leftrightarrow \{\mathfrak{a}_{lr}^-, \mathfrak{a}_{rl}^-\}, \quad X_7^{\mathrm{III}} \leftrightarrow \{\mathfrak{a}_{lr}^-, \mathfrak{a}_{rl}^+\}.
\end{aligned}
\tag{42}
$$

Thus, $\mathfrak{G}_7$, $\mathfrak{H}_7^{\mathrm{I}}$, $\mathfrak{H}_7^{\mathrm{II}}, \mathfrak{H}_7^{\mathrm{III}}$ is the mixed group of collineations and X_7, X_7^{I}, X_7^{II}, X_7^{III} are the families of correlations of the quasi-absolute structures.

With the help of Theorem V (§ 4), one can decide how the points and planes of the quasi-absolute structure are permuted. The result will be elucidated by the following table:

$$
\begin{aligned}
&\mathfrak{G}_7 \begin{Bmatrix} c_l \to c_l, & c_r \to c_r, \\ \gamma_l \to \gamma_l, & \gamma_r \to \gamma_r \end{Bmatrix}, \qquad
&&X_7 \begin{Bmatrix} c_l \to \gamma_l, & c_r \to \gamma_r, \\ \gamma_l \to c_l, & \gamma_r \to c_r \end{Bmatrix},\\
&\mathfrak{H}_7^{\mathrm{I}} \begin{Bmatrix} c_l \to c_r, & c_r \to c_l, \\ \gamma_l \to \gamma_r, & \gamma_r \to \gamma_l \end{Bmatrix}, \qquad
&&X_7^{\mathrm{I}} \begin{Bmatrix} c_l \to \gamma_r, & c_r \to \gamma_l, \\ \gamma_l \to c_r, & \gamma_r \to c_l \end{Bmatrix},\\
&\mathfrak{H}_7^{\mathrm{II}} \begin{Bmatrix} c_l \to c_l, & c_r \to c_r, \\ \gamma_l \to \gamma_r, & \gamma_r \to \gamma_l \end{Bmatrix}, \qquad
&&X_7^{\mathrm{II}} \begin{Bmatrix} c_l \to \gamma_l, & c_r \to \gamma_r, \\ \gamma_l \to c_r, & \gamma_r \to c_l \end{Bmatrix},\\
&\mathfrak{H}_7^{\mathrm{III}} \begin{Bmatrix} c_l \to c_r, & c_r \to c_l, \\ \gamma_l \to \gamma_l, & \gamma_r \to \gamma_r \end{Bmatrix}, \qquad
&&X_7^{\mathrm{III}} \begin{Bmatrix} c_l \to \gamma_r, & c_r \to \gamma_l, \\ \gamma_l \to c_l, & \gamma_r \to c_r \end{Bmatrix}.
\end{aligned}
\tag{43}
$$

If one substitutes motions and transfers $\mathfrak{b}^+$, $\mathfrak{b}^-$ everywhere in the schema (42), in place of actual and ideal similarities $\mathfrak{a}^+$, $\mathfrak{a}^-$, then one obtains a six-parameter group, in place of the seven-parameter one, whose eight continuous families we denote by $\mathfrak{G}_6$, $\mathfrak{H}_6^{\mathrm{I}}$, $\mathfrak{H}_6^{\mathrm{II}}$, $\mathfrak{H}_6^{\mathrm{III}}$, X_6, X_6^{I}, X_6^{II}, X_6^{III}, corresponding to the eight families of the seven-parameter group. This six-parameter group is included in the seven-parameter one as an invariant subgroup.

According to Felix Klein, any group of geometric transformations belongs to a "geometry." The geometry that is associated with the continuous group $\mathfrak{G}_6$ shall be called *quasi-elliptic geometry*.

The group $\mathfrak{G}_6$ is composed of two commuting subgroups $\mathfrak{G}_3^l$, $\mathfrak{G}_3^r$ – the so-called *parameter groups* of the plane motions – each of which is holomorphically isomorphic to the group of motions in the plane. The groups $\mathfrak{G}_3^l$, $\mathfrak{G}_3^r$ map to the plane as:

$$\mathfrak{G}_3^l \to \{\mathfrak{b}_{ll}^+, \mathfrak{i}_{rr}\}, \qquad \mathfrak{G}_3^r \to \{\mathfrak{i}_{ll}, \mathfrak{b}_{rr}^+\}. \tag{44}$$

The symbol $\mathfrak{i}$ means the identity in this.

$\mathfrak{G}_6$ is called the group of *quasi-motions;* $\mathfrak{G}_3^l$, $\mathfrak{G}_3^r$ are the groups of *left-sided* and *right-sided quasi-translations*, resp.

Any transformation of $\mathfrak{G}_6$ is the product of two particular transformations of $\mathfrak{G}_3^l$ and $\mathfrak{G}_3^r$, which are immediately commutable. If we write only one index, instead of two of them l and r then the three transformations are the following ones:

$$\{\mathfrak{b}_l, \mathfrak{b}_r\}, \qquad \{\mathfrak{b}_l, \mathfrak{i}_r\}, \qquad \{\mathfrak{i}_l, \mathfrak{b}_r\}. \tag{45}$$

§ 6. Transformation formulas.

One can represent the transformations of the quasi-elliptic space in a simple way by means of the complex relationships that were introduced in § 3.

We think of a quasi-motion $p \to p^*$ as being applied to the points of space. Let $[\mathfrak{b}_l, \mathfrak{b}_r]$ be its image in the base plane; $\mathfrak{b}_l$ is a motion $g_l \to g_l^*$, and $\mathfrak{b}_r$ is a motion $g_r \to g_r^*$. The corresponding point p^* then belongs to the motion $\mathfrak{p}^* = g_l^* \to g_r^* = \mathfrak{b}_l^{-1}\ \mathfrak{p}\ \mathfrak{b}_r$.

We introduce the complex relationships (cf. § 3):

$$\begin{aligned}
\dot{p} &= p_0 e_0 + p_1 e_1 + p_2 \varepsilon e_2 + p_3 \varepsilon e_3, \\
\dot{p}^* &= p_0^* e_0 + p_1^* e_1 + p_2^* \varepsilon e_2 + p_3^* \varepsilon e_3, \\
\dot{g}_l &= e_1 + x \varepsilon e_2 + y \varepsilon e_3, \\
\dot{g}_l^* &= e_1 + x^* \varepsilon e_2 + y^* \varepsilon e_3, \\
\dot{g}_r &= e_1 + \mathfrak{x} \varepsilon e_2 + \mathfrak{y} \varepsilon e_3, \\
\dot{g}_r^* &= e_1 + \mathfrak{x}^* \varepsilon e_2 + \mathfrak{y}^* \varepsilon e_3, \\
\dot{b}_l &= b_0^{(l)} e_0 + b_1^{(l)} e_1 + b_2^{(l)} \varepsilon e_2 + b_3^{(l)} \varepsilon e_3, \\
\dot{b}_r &= b_0^{(r)} e_0 + b_1^{(r)} e_1 + b_2^{(r)} \varepsilon e_2 + b_3^{(r)} \varepsilon e_3,
\end{aligned} \tag{46}$$

then, from (23*), § 3:

(47) $$N(\dot p)\dot g_r = \dot p\,\dot g_l\,\dot p, \qquad N(\dot p^*)\dot g_r^* = \dot p^*\,\dot g_l^*\,\dot p^*,$$

(48) $$N(\dot p)\dot g_r = \dot p\,\dot g_l\,\dot p, \qquad N(\dot p^*)\dot g_r^* = \dot p^*\,\dot g_l^*\,\dot p^*,$$

and therefore (cf., the rule of calculation in § 3):

(49) $$\dot p^* = \dot b_l \dot p \dot b_r .$$

This is the transformation formula for the quasi-motion $p \to p^*$.

When one sets:

(50) $$\begin{aligned} \dot\pi &= \pi_0 \varepsilon e_0 + \pi_1 \varepsilon e_1 + \pi_2 e_2 + \pi_3 e_3, \\ \dot\pi^* &= \pi_0^* \varepsilon e_0 + \pi_1^* \varepsilon e_1 + \pi_2^* e_2 + \pi_3^* e_3, \end{aligned}$$

in plane coordinates π_k, one likewise finds the transformation formula:

(51) $$\dot\pi^* = \dot b_l \dot\pi \dot b_r .$$

In a similar way, one can also describe the remaining seven families of the six-parameter group $\mathfrak{G}_6$, $\mathfrak{H}_6^{\mathrm{I}}$, $\mathfrak{H}_6^{\mathrm{II}}$, $\mathfrak{H}_6^{\mathrm{III}}$, X_6, X_6^{I}, X_6^{II}, X_6^{III}. We would thus like to content ourselves with giving a discrete Abelian group of eight involutory transformations $\mathfrak{t}$, $\mathfrak{t}_1$, $\mathfrak{t}_2$, $\mathfrak{t}_3$, τ, τ_1, τ_2, τ_3 that correspondingly are contained in the mixed group, and through whose adjunction, the continuous group $\mathfrak{G}_6$ will be extended to the mixed group. We thus make the coordinates individually evident.

$\mathfrak{t}$ is the identity.

$$\mathfrak{t}_1 \begin{cases} x^* = x, & y^* = -y, \quad \mathfrak{x}^* = \mathfrak{x}, \quad \mathfrak{y}^* = -\mathfrak{y}, \\ p_0^* = -p_0, & p_1^* = p_1, \quad p_2^* = p_2, \quad p_3^* = -p_3, \\ \pi_0^* = -\pi_0, & \pi_1^* = \pi_1, \quad \pi_2^* = \pi_2, \quad \pi_3^* = -\pi_3, \end{cases}$$

$$\mathfrak{t}_2 \begin{cases} x^* = \mathfrak{x}, & y^* = \mathfrak{y}, \quad \mathfrak{x}^* = x, \quad \mathfrak{y}^* = y, \\ p_0^* = -p_0, & p_1^* = p_1, \quad p_2^* = p_2, \quad p_3^* = p_3, \\ \pi_0^* = -\pi_0, & \pi_1^* = \pi_1, \quad \pi_2^* = \pi_2, \quad \pi_3^* = \pi_3, \end{cases}$$

$$\mathfrak{t}_3 \begin{cases} x^* = \mathfrak{x}, & y^* = -\mathfrak{y}, \quad \mathfrak{x}^* = x, \quad \mathfrak{y}^* = -y, \\ p_0^* = p_0, & p_1^* = p_1, \quad p_2^* = p_2, \quad p_3^* = -p_3, \\ \pi_0^* = \pi_0, & \pi_1^* = \pi_1, \quad \pi_2^* = \pi_2, \quad \pi_3^* = -\pi_3, \end{cases}$$

(52)
$$\tau\begin{cases} x^* = \ \ x, \ \ y^* = \ \ y, \ \ \mathfrak{x}^* = \ \ \mathfrak{x}, \ \ \mathfrak{y}^* = -\mathfrak{y}, \\ p_0^* = -\pi_3, \ \ p_1^* = \ \ \pi_2, \ \ p_2^* = -\pi_1, \ \ p_3^* = \ \ \pi_0, \\ \pi_0^* = -p_3, \ \ \pi_1^* = \ \ p_2, \ \ \pi_2^* = -p_1, \ \ \pi_3^* = \ \ p_0, \end{cases}$$

$$\tau_1\begin{cases} x^* = \ \ x, \ \ y^* = -y, \ \ \mathfrak{x}^* = \ \ \mathfrak{x}, \ \ \mathfrak{y}^* = \ \ \mathfrak{y}, \\ p_0^* = \ \ \pi_3, \ \ p_1^* = \ \ \pi_2, \ \ p_2^* = -\pi_1, \ \ p_3^* = -\pi_0, \\ \pi_0^* = \ \ p_3, \ \ \pi_1^* = \ \ p_2, \ \ \pi_2^* = -p_1, \ \ \pi_3^* = -p_0, \end{cases}$$

$$\tau_2\begin{cases} x^* = \ \ \mathfrak{x}, \ \ y^* = \ \ \mathfrak{y}, \ \ \mathfrak{x}^* = \ \ \mathfrak{x}, \ \ \mathfrak{y}^* = -y, \\ p_0^* = -\pi_3, \ \ p_1^* = \ \ \pi_2, \ \ p_2^* = -\pi_1, \ \ p_3^* = -\pi_0, \\ \pi_0^* = -p_3, \ \ \pi_1^* = \ \ p_2, \ \ \pi_2^* = -p_1, \ \ \pi_3^* = -p_0, \end{cases}$$

$$\tau_3\begin{cases} x^* = \ \ \mathfrak{x}, \ \ y^* = -\mathfrak{y}, \ \ \mathfrak{x}^* = \ \ x, \ \ \mathfrak{y}^* = \ \ y, \\ p_0^* = \ \ \pi_3, \ \ p_1^* = \ \ \pi_2, \ \ p_2^* = -\pi_1, \ \ p_3^* = \ \ \pi_0, \\ \pi_0^* = \ \ p_3, \ \ \pi_1^* = \ \ p_2, \ \ \pi_2^* = -p_1, \ \ \pi_3^* = \ \ p_0. \end{cases}$$

One can verify the relations (42) and (43) that were cited in the previous paragraphs by means of these formulas.

§ 7. Passing to the limit from elliptic to quasi-elliptic geometry.

The formulas that were developed in the previous paragraphs make it possible for us to realize this passage to the limit in a simple way.

We start with the formulas (46) and no longer write the rule of calculation $\varepsilon^2 = 0$ for the imaginary unit ε that appears in it, but the new rule $\varepsilon^2 = \rho$ $\{\rho > 0\}$. From formula (49), the continuous group of automorphic collineations will then be represented by the regular surface of second order:

(53)
$$p_0^2 + p_1^2 + \rho p_2^2 + \rho p_3^2 = 0,$$

which is the same thing as the group of elliptic motions of non-Euclidian space that has this surface for the absolute structure.

If one now carries out the passage to the limit $\rho \to 0$ then the surface of second order ranges through a pencil and finally degenerates into the plane-pair γ_l , γ_r . The two families of generators of the surface become the four pencils of planes that lie in γ_l and γ_r and have their vertices at c_l and c_r . The parallelism that Clifford introduced, to which Study gave the name of *parataxy*, becomes the parallelism that was given in § 4 in the limit.

Study has given an association between the pairs of lines in elliptical space that correspond to each other under the absolute polarity, and the point-pairs whose points are divided between two planes with elliptical metrics. This association, which is uniquely invertible in the real domain, associates the motions of the elliptical space with pairs of motions in the two elliptical planes. [1]) One can now also carry out a passage to the limit, under which this association will go to our map $G \leftrightarrow (g_l, g_r)$.

In fact, if the G_{ik} mean Plücker coordinates in the elliptic space whose absolute surface is represented by equation (53) then we can express the Study association in formulas as follows: We set:

$$s_1 : s_2 : s_3 = G_{01} + \rho G_{23} : G_{02} + G_{31} : G_{03} + G_{12}, \tag{54}$$
$$\mathfrak{s}_1 : \mathfrak{s}_2 : \mathfrak{s}_3 = G_{01} - \rho G_{23} : G_{02} - G_{31} : G_{03} - G_{12}.$$

We intend the s_k , and likewise $\mathfrak{s}_k$, to mean homogeneous point coordinates of a plane. In the plane of the s_k, we must take the conic section:

$$s_1^2 + \rho s_2^2 + \rho s_3^2 = 0 \tag{55}$$

for the absolute structure, and in the plane of the $\mathfrak{s}_k$:

$$\mathfrak{s}_1^2 + \rho \mathfrak{s}_2^2 + \rho \mathfrak{s}_3^2 = 0. \tag{55}$$

If one now introduces inhomogeneous coordinates into the two planes:

$$\frac{s_2}{s_1} = x, \qquad \frac{s_3}{s_1} = y, \qquad \frac{\mathfrak{s}_2}{\mathfrak{s}_1} = \mathfrak{x}, \qquad \frac{\mathfrak{s}_3}{\mathfrak{s}_1} = \mathfrak{y}, \tag{56}$$

and passes to the limit $\rho \to 0$ then the elliptic space becomes quasi-elliptic, the two non-Euclidian planes become Euclidian, and the formulas (54) are converted into formulas (6), § 2. [2])

§ 8. Invariants of quasi-motions.

Two (real) points p, p' whose connecting line does not meet the line C of the quasi-absolute structure have an absolute invariant under $\mathfrak{G}$, namely, their double ratio with the conjugate imaginary planes γ_l, γ_r . We call the real quantity [*]):

$$\varphi = + \tfrac{i}{2} \ln Dv(p\, p' \gamma_l\, \gamma_r) \tag{57}$$

---

[1]) One might confer, say, E. Study: "Beiträge zur nichteuklischen Geometrie II," Am. Journ. of Math., Bd. XXIX (1906), page 116, *et seq.*

[2]) A similar passage to the limit is found in the dissertation of H. Beck, Bonn, 1905.

[*]) DHD: the notation "*Dv*" is from the German "Doppelverhältnis," which means "double ratio."

the *quasi-separation* between p and p'. φ can be expressed in terms of homogeneous coordinates p_k, p'_k of p, p' by way of:

(58) $$\tan \varphi = -\frac{p_0 p'_1 - p_1 p'_0}{p_0 p'_0 - p_1 p'_1}.$$

The invariant φ (as well as its sign) is defined up to multiples of π.

If the connecting line pp' cuts the line C [$p_0 p'_1 - p_1 p'_0 = 0$, $p_0 p'_0 + p_1 p'_1 \neq 0$] then a new invariant under $\mathfrak{G}_6$ appears in place of the vanishing quasi-separation that is defined only up to sign, namely:

(59) $$\chi = \frac{1}{p_0 p'_0 + p_1 p'_1}\sqrt{(p_0 p'_2 - p_2 p'_0)^2 + (p_0 p'_3 - p_3 p'_0)^2 + (p_3 p'_1 - p_1 p'_3)^2 + (p_1 p'_2 - p_2 p'_1)^2}.$$

We define the *quasi-angle* between two planes π, π' whose intersection does not cut C to be the quantity:

(57)* $$\psi = -\tfrac{i}{2}\ln Dv(\pi\,\pi' c_l\, c_r).$$

One has:

(58)* $$\tan \varphi = +\frac{\pi_2 \pi'_3 - \pi_3 \pi'_2}{\pi_2 \pi'_2 + \pi_3 \pi'_3};$$

ψ is invariant under $\mathfrak{G}_7$. In the excluded special case, we again find an invariant under $\mathfrak{G}_6$:

(59)* $$\omega = \frac{1}{\pi_2 \pi'_2 + \pi_3 \pi'_3}\sqrt{(\pi_0 \pi'_2 - \pi_2 \pi'_0)^2 + (\pi_0 \pi'_3 - \pi_3 \pi'_0)^2 + (\pi_3 \pi'_1 - \pi_1 \pi'_3)^2 + (\pi_1 \pi'_2 - \pi_2 \pi'_1)^2}.$$

A point p and a plane π determine the *quasi-distance* (p does not line on C and π does not go through C), which is invariant under $\mathfrak{G}_6$:

(60) $$\vartheta = \frac{p_0\pi_0 + p_1\pi_1 + p_2\pi_2 + p_3\pi_3}{\sqrt{p_0^2 + p_1^2}\sqrt{\pi_2^2 + \pi_3^2}};$$

ϑ is only defined up to sign.

We take a point p that does not lie on C and a line G that does not meet C. Both determine an invariant ρ under G_6 that we would likewise like to call the *quasi-distance*. It agrees with the (previously-defined) quasi-distance from the point p to that plane through G that is perpendicular to the connecting line of G with p $\left(\varphi = \frac{\pi}{2}\right)$. One finds:

(61) $$\rho = \frac{\sqrt{\{p_0 G_{12} - p_1 G_{02} + p_2 G_{01}\}^2 + \{p_0 G_{31} + p_1 G_{03} - p_3 G_{01}\}^2}}{G_{01}\sqrt{p_0^2 + p_1^2}}.$$

(For this, one confers the formulas (10) in § 2). If one introduces the coordinates x, y and $\mathfrak{x}$, $\mathfrak{y}$ of the image points g_l, g_r of G then one gets:

(62) $$\rho = \frac{1}{2}\frac{\sqrt{\{p_0(y-\mathfrak{y}) - p_1(x+\mathfrak{x}) + 2p_2\}^2 + \{p_0(x-\mathfrak{x}) + p_1(y-\mathfrak{y}) - 2p_3\}^2}}{\sqrt{p_0^2 + p_1^2}}.$$

Dual to this, we define the *quasi-angle* between a line G with the plane π:

(61)* $$\tau = \frac{\sqrt{\{\pi_0 G_{01} - \pi_2 G_{12} + \pi_2 G_{31}\}^2 + \{\pi_1 G_{01} + \pi_1 G_{02} - \pi_3 G_{03}\}^2}}{G_{01}\sqrt{\pi_2^2 + \pi_3^2}}$$

or:

(62)* $$\tau = \frac{1}{2}\frac{\sqrt{\{2\pi_0 - \pi_0(y-\mathfrak{y}) + \pi_1(x-\mathfrak{x})\}^2 + \{2\pi_3 - \pi_2(x+\mathfrak{x}) + \pi_3(y+\mathfrak{y})\}^2}}{\sqrt{\pi_2^2 + \pi_3^2}}.$$

The line that meets C $\{G_{01} = 0\}$ also determines an invariant under $\mathfrak{G}_6$ with a point, and likewise with a plane, although we shall not go deeper into these quantities since we will make no further use of them.

Under the transformations of the family X_6^{I} (cf., § 5), the invariants φ, χ, ψ, ω, ϑ, ρ, τ will be permuted into the invariants ψ, ω, φ, χ, ϑ, τ, ρ, resp. *Our metric is dual to itself*.

§ 9. The ruled surface of second order.

Here, we consider only real surfaces of second order with real rectilinear generators. We next give a classification of the surfaces that are invariant under the groups $\mathfrak{G}_7$ $\mathfrak{H}_7^{\mathrm{I}}$ $\mathfrak{H}_7^{\mathrm{II}}$ $\mathfrak{H}_7^{\mathrm{III}}$ (cf., § 5).

Regular surfaces:

1. The surface contacts either C or γ_l, γ_r. It does not go through c_l, c_r.
2. The surface contacts C, but not γ_l, γ_r. It thus does not go through c_l, c_r, either.
3. The surface contacts γ_l, γ_r, and does not go through c_l, c_r.

3*. The surface goes through c_l, c_r and does not contact γ_l, γ_r.

4. The surface contacts γ_l, γ_r, and goes through c_l, c_r, but not through C.
5. The surface goes through C and has tangential planes at c_l, c_r that are different from γ_l, γ_r.
6. The surface goes through C and contacts c_l, c_r, in the planes γ_l, γ_r.

Of the singular surfaces, we consider only:

7. Irreducible conics, which have their double points on C and contact γ_l, γ_r along generators.

In the cases 1 to 4, the reciprocal polars to C relative to the surface will be called its *quasi-axes.*

All points of one of the surfaces of the 4th family have equal quasi-distance from their quasi-axis, and all tangential planes to the surface subtend equal quasi-angles with the quasi-axis. The converse is also true, which emerges from our formulas (61) and (61)*: All points that have an equal quasi-distance $\rho \neq 0$ from a line G does not meet C fill out a surface of the fourth family and all planes that subtend the quasi-angle $\tau = \rho$ with this line contact *this very* surface.

A surface of the 7th family consists of all points that have the same quasi-distance $\vartheta \neq 0$ from a plane: the "center plane" of the surface. We call each such surface a *quasi-sphere.* The center plane is associated with the line C relative to the polarity defined by the quasi-sphere.

Although we shall make no use of the fact later on, we shall nonetheless briefly discuss what the two ruled families of a surface of second order are associated with in the base plane under the transformation $G \to (g_l, g_r)$.

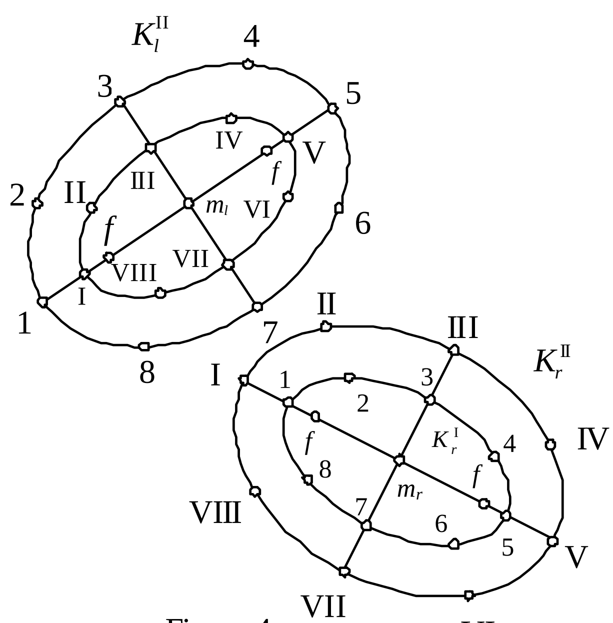

Figure 4.

We begin with a *surface of the first family*. A family of generators G^{I} is mapped onto the plane as follows: g_l^{I} and g_r^{I} both range across the conic sections K_l^{I}, K_r^{I} when G^{I} ranges through the family. K_l^{I}, K_r^{I} are either both ellipses (as in our Figure 4) or both hyperbolas with equal eccentricities; they are thus ellipses when the surface meets the line C in imaginary points and hyperbolas when the intersection points are real and coincident. The relation $g_l^{\mathrm{I}} \to g_r^{\mathrm{I}}$ between K_l^{I}, K_r^{I} is affine (in the same or opposite sense), and indeed in such a way that the endpoints of the principal axes, and also the endpoints of the secondary axis, correspond to each other. In the figure, associated points are characterized by equal Arabic symbols.

The second family of generators G^{II} is mapped to the same surface in an analogous way. g_l^{II} ranges over the conic section K_l^{II}, which is confocal to K_l^{I} and congruent to K_r^{I}, and likewise g_r^{II} ranges over the conic section K_r^{II}, which is confocal to K_r^{I} and congruent to K_l^{I}. The relationship $g_l^{\mathrm{II}} \to g_r^{\mathrm{II}}$ between the two curves $K_l^{\mathrm{II}}, K_l^{\mathrm{II}}$ is again affine and of the same type as the one between $K_l^{\mathrm{I}}, K_l^{\mathrm{I}}$ (cf., the figure).

Let the common midpoint of K_l^{I}, K_l^{II} be m_l and let that of K_r^{I}, K_r^{II} be m_r. (m_l, m_r) is the image of the quasi-axis of our surface.

Since each generator of the first family cuts a generator of the second one, there always exists the relationship:

$$g_l^{\mathrm{I}} g_l^{\mathrm{II}} = g_r^{\mathrm{I}} g_r^{\mathrm{II}},$$

between corresponding points g_l^{I}, g_r^{I}, g_l^{II}, g_r^{II}, which one confirms by means of a well-known property of confocal conics. In the figure, e.g., the two segments that are denoted by 2 III have equal length.

We now make a few assumptions about the *surfaces of the remaining families:* In the case 2, the conic sections K_l^{I}, K_r^{I}, K_l^{II}, K_r^{II} are parabolas. In the case 3, K_l^{I}, K_r^{I}, and likewise also K_l^{II}, K_r^{II}, are circles that are similar in the same sense, and in case 3^*, they are circles that relate to each other in a similar, but opposite sense.

A surface of the 4th family includes a family of left-parallel generators G^{I}. g_l^{I} thus remains fixed, while g_r^{I} rages over a circle K_r^{I}. The generators G^{II} of the other family are right-parallel. The associated point g_r^{II} remains fixed at the center of K_r^{I}. g_l^{II} runs over the circle K_l^{II} with the center g_l^{I} that has the same radius as K_r^{I}.

In the cases 5 and 6, the one family of generators cuts the line C, so it is passed over by our map $G \to (g_l, g_r)$. In case 5, the other families map to two lines that relate to each other as similar (but not congruent) to each other under the association $g_l \to g_r$. Finally, in the 6th case, the representable generators are either left-parallel or right-parallel, so either g_l remains fixed, while g_r ranges over a line, or conversely, g_r remains fixed, while g_l ranges over a line.

The surfaces of the families 3 and 3^* admit one-parameter groups in $\mathfrak{G}_6$, under which the quasi-axes of the surface remain at rest. For the surfaces 3, the quasi-axis then remains point-wise at rest, while the groups of automorphic quasi-motions of the plane through the quasi-axis that belong to case 3^* remains individually at rest.

Any surface of the 4th family can be regarded as a limiting case of the surfaces in 3, as well as the surfaces in 3^*. It admits a two-dimensional group of quasi-motions, namely, the group of all transformations in $\mathfrak{G}_6$ that take the quasi-axis of the surface to itself. The real points of the surface will be permuted with each other transitively by the transformations of the group. Such a surface behaves similarly to a cylinder of rotation in Euclidian geometry.

A quasi-sphere admits the three-parameter group of all quasi-motions that take the center plane of the sphere to itself.

II.

Group-theoretic foundations of kinematics in the Euclidian plane.

§ 10. Positive and negative somas.

The association that was treated in § 1 and § 2 of points and planes of space, on the one hand, and motions and transfers in the plane, on the other, shall be given a somewhat different interpretation here.

We have assumed that there is a right-angled axis cross in the base plane for the purpose of enumerating the inhomogeneous coordinates $x = p_2 / p_1$, $y = p_3 / p_1$. We call this axis cross σ_0 . If we perform a motion in the base plane then we obtain a new axis cross σ that is oriented with the same sense as σ_0 . Conversely, the motion [1]) that takes σ_0 to σ is determined uniquely by the given of the axis cross σ, which consists of an – in a certain sense – *ordered* pair of mutually perpendicular *oriented* lines.

An axis cross in the base plane that is oriented with the same sense as σ_0 shall be called a *positive soma*, as long as it can be regarded as the representative of a rigid planar field.

We remark that a certain motion belongs to any positive soma σ, namely, the one that takes the *ur-soma* σ_0 to σ. Previously (§ 1), we associated motions in the base plane with points of space, and we can, moreover, associate the positive somas with points of space in a uniquely invertible way. Only the points of the line C of the quasi-absolute structure are associated with no somas, for the moment.

The ur-soma σ_0 corresponds to the ("ideal" in the Euclidian sense) point p^0 with the coordinates:

$$p_0 : p_1 : p_2 : p_3 = 1 : 0 : 0 : 0.$$

If we choose the axis cross $\mathfrak{s}$ to be oriented in the same sense as σ_0, so it therefore comes from σ_0 under a transfer, then we analogously find a one-to-one relationship between these oppositely-oriented axis crosses, which we would like to call *negative somas*, and the planes of space that do not go through C.

Let two somas of the same type be given, so either both of them are positive σ, σ' or both of them are negative $\mathfrak{s}$, $\mathfrak{s}'$. The half-angle of the rotation that brings the first soma over the second one shall be called the *angle between the somas*, here. One finds (cf., the formulas of § 3 and the formulas (58), (58)* in § 8):

The angle between two positive (negative, resp.) somas σ, σ' ($\mathfrak{s}$, $\mathfrak{s}'$, resp.) *is equal to the quasi-separation (quasi-angle, resp.) between the associated points* p, p' *(planes* π, π'*, resp.).*

If a translation figures instead of a rotation then we set the rotation angle equal to zero. The half-translation magnitude is determined by the formula (59) or (59)*.

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[1]) By a "motion," here, we always mean a single point-transformation, and not perhaps a continuous family of such transformations.

Let a positive soma σ and a negative one $\mathfrak{s}$ be given. The half-translation magnitude of the transfer (cf., the remark on Theorem IIIb in § 1) that brings σ to $\mathfrak{s}$ shall be called the *distance between the somas*. One further finds [cf., (60), § 8]:

The distance from a positive soma σ to a negative one $\mathfrak{s}$ is equal to the quasi-distance ϑ from the associated point p to the associated plane π.

When $\vartheta = 0$, one finds that p and π are *united*, and we would also like to say of σ and $\mathfrak{s}$, which are permuted with each other under reflection in a line, that they, too, are united.

§ 11. Elementary equivalence concepts in kinematics. [1])

Let there be given an assemblage of positive and negative somas $\sigma_1, \sigma_2, \sigma_3, \ldots; \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \ldots$ The corresponding figure in space consists of points and planes $p_1, p_2, p_3, \ldots; \pi_1, \pi_2, \pi_3, \ldots$ The totality can also be included in a continuous manifold. It is close at hand that one can introduce a notion of equivalence, under which one allows two different types of transformations that yield new assemblages when they are applied to our assemblage to be regarded as equivalent to the original one. The two types of transformations are the following ones:

I. One subjects the assemblage $\sigma_1, \sigma_2, \sigma_3, \ldots; \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \ldots$ to one and the same motion.

II. One chooses an arbitrary positive soma σ_0^*. The motion (transfer, resp.) that takes the ur-soma σ_0 to σ_k ($\mathfrak{s}_k$, resp.) takes σ_0^* to a soma σ_k^* ($\mathfrak{s}_k^*$, resp.). We replace the assemblage $\sigma_1, \sigma_2, \sigma_3, \ldots; \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \ldots$ with the new one $\sigma_1^*, \sigma_2^*, \sigma_3^*, \ldots; \mathfrak{s}_1^*, \mathfrak{s}_2^*, \mathfrak{s}_3^*, \ldots$

The corresponding transformations of space are:

I. The figure $p_1, p_2, p_3 \ldots; \pi_1, \pi_2, \pi_3, \ldots$ is subjected to a *right-sided* translation, and:

II. It is subjected to a *left-sided translation* (cf., the end of § 5).

The two groups of translations generate the group $\mathfrak{G}_6$ of quasi-elliptic motions. We thus find the result:

Theorem VII. *The natural equivalence in planar kinematics corresponds in space to the equivalence under the group $\mathfrak{G}_6$ of quasi-motions.*

We are now also close to carrying over the geometry of other groups that include $\mathfrak{G}_6$ to kinematics.

One obtains a notion of "natural" equivalence that is not very different from the above when one allows the following transformations:

[1]) On this subject, cf., Study, *Geometrie der Dynamen*, pp. 589, *et seq.*

I. One subjects the present assemblage of somas to an actual or ideal (i.e., the same or opposite sense) *similarity transformation.*

II. One takes an arbitrary positive or negative soma σ_0^* or $\mathfrak{s}_0^*$ and constructs the new somas σ_k^*, $\mathfrak{s}_k^*$ with its help in a manner that is analogous to before. The associated group of transformations in space is the mixed group ($\mathfrak{G}_7$, $\mathfrak{H}_7^{\mathrm{I}}$, X_7, X_7^{I}).

In order to also clarify the kinematic meaning of the remaining four families of groups $\mathfrak{G}_7$, $\mathfrak{H}_7^{\mathrm{I}}$, $\mathfrak{H}_7^{\mathrm{II}}$, $\mathfrak{H}_7^{\mathrm{III}}$, X_7, X_7^{I}, X_7^{II}, X_7^{III}, one must introduce a transformation of somas that one refers to as the "inversion process."

Let σ ($\mathfrak{s}$, resp.) be a positive (negative, resp.) soma that arises from the ur-soma σ_0 by a certain motion (transfer, resp.). Under the inverse transformation, σ_0 goes to a soma σ^* ($\mathfrak{s}^*$, resp.). One calls the exchange of somas that replaces every soma σ ($\mathfrak{s}$, resp.) with the soma σ^* ($\mathfrak{s}^*$, resp.) that is constructed in this way the *inversion process*. The corresponding transformation of space is, as emerges from Theorem III, § 1, the reflection in the base plane.

The group ($\mathfrak{G}_7$, $\mathfrak{H}_7^{\mathrm{I}}$, X_7, X_7^{I}) will be extended to the group ($\mathfrak{G}_7$, $\mathfrak{H}_7^{\mathrm{I}}$, $\mathfrak{H}_7^{\mathrm{II}}$, $\mathfrak{H}_7^{\mathrm{III}}$, X_7, X_7^{I}, X_7^{II}, X_7^{III}) by adjunction of this transformation to the family $\mathfrak{H}_7^{\mathrm{II}}$.

§ 12. The projective geometry of somas.

In order to make the notion of equivalence in the projective geometry of space useful for planar kinematics, we must first dispense with the assumption that points and planes appear as positive and negative somas under the map. When one introduces *ideal somas,* one can *formally* arrive at a continuum of ∞^1 positive somas and a second continuum of ∞^1 negative ideal somas. One maps this in a one-to-one way onto the points and planes of space that are united with the line C of the absolute structure.

We have chosen an axis cross [1]) to be the geometric representative of an actual soma. One can also introduce geometric representatives for the ideal somas, such as the points of the doubly-covered ideal lines. One then has the notion (that was introduced at the end of § 10) of united objects, but extended to ideal somas, as well, and indeed, in such a way that two somas of unequal type that are united in space always correspond to united points and planes. [2])

It is now possible (at least, theoretically) to interpret all of projective geometry in space in our base plane. At least, we consider the somas to be structures that are associated with the *basic structures* of projective geometry, the sequence of points, the pencil of planes, the planar point field, and the bundle of planes, and we classify these structures under $\mathfrak{G}_6$.

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[1]) One can just as well introduce, e.g., an oriented line element for this purpose.

[2]) One can also make the association $G \leftrightarrow (g_l, g_r)$ that was treated in § 1.2 non-singularly one-to-one by the introduction of a continuum of ideal point-pairs. Following through on this line of thought is, however, quite laborious.

Thus, we next consider what the line is associated with when one regards it as a locus of points (planes, resp.) Under the group of quasi-motions, there are three classes of lines (when one restricts oneself to the real domain, as we always do):

1. Lines, G that do not cut C,
2. Lines that cut C, and
3. The line C itself.

Correspondingly, in the base plane we obtain:

1. The "line M_1" of positive (negative, resp.) somas, which contains no actual soma.

All of the somas of M_1 arise from an arbitrary one of them when one subjects it to all rotations around the point g_r that correspond to the line G under the map $G \to (g_l, g_r)$.

2. The line M_1 of positive (negative, resp.) somas that does contain an ideal soma.

All actual somas of this M_1 arise from an arbitrary one of them when one subjects it to all translations in a certain direction.

3. The line M_1 of positive (negative, resp.) ideal somas.

Under the group of quasi-motions, there are only two different classes of points (planes, resp.), namely, the ones that are not united with C and the ones that are united with C.

Accordingly, one must distinguish:

1) The "plane M_2" of positive (negative, resp.) somas with a single ideal soma. This consists of all somas that are united with the actual negative (positive, resp.) soma that correspond to the plane (point, resp.) of space that appears as the carrier of the plane point-field (pencil of planes, resp.).

One thus obtains all actual somas of such a planar M_2 when one reflects a fixed actual soma through all actual lines in the base plane.

2) The plane M_2 with ∞^1 ideal somas.

All actual somas of this M_2 emerge from an arbitrary one of them when one subjects it to all translations.

The projective geometry of somas shall not be constructed any further here, since this undertaking has already been commenced elsewhere [1]), although we shall make just one closely-related remark on the geometry in a plane M_2 of the first class. We take, e.g., such a plane M_2 of positive somas σ that correspond to the points p of a plane π. On the other hand, one can map the somas σ of M_2 to the axis S of the reflections, through which

[1]) De Saussure, Exposé resumé de la geometrie des feuillets. Genf 1910 and Bricard, Nouv. ann. (4) X, 1910; cf., on this, also the critique of Study, Jahresbericht der d. math. Ver., XIX, 1910, pp. 205-263.

they will be taken to the fixed soma $\mathfrak{s}$ that corresponds to the plane π. One can assign the ideal soma of M_2 with an ideal line in the base plane. However, in this way the field of lines S in the base plane is further related to the field of points p, *and this relation is projective.* Under this correlation, the absolute points of the base plane correspond to the line of intersection of π with the planes γ_l, γ_r of the quasi-absolute structure. From this, one further deduces that the three-parameter group of transformations that $\mathfrak{G}_6$ provokes among the somas σ of M_2 is mapped to the group of motions of the line S.

§ 13. The inversion geometry of somas.

In the previous paragraphs, we considered the collineation group of space. Now, we would like to add another group of point-transformations of space that likewise includes $\mathfrak{G}_6$, and which one can all the *inversion group of quasi-elliptic space*, since it defines an analogy to inversion geometry (or the sphere geometry of Möbius) in Euclidian space.

In § 9, it was explained what is meant by the term "quasi-sphere." We would now like to present all point-transformations that generally associate quasi-spheres with other quasi-spheres. It happens that these maps are, as a rule, (2, 2)-valued, such that one can, however, make then (1, 1)-valued transformations by means of a certain *orientation process.*

One must doubly cover the totality of points in projective space with the manifold of "oriented" points, and indeed in such a way that (in the real domain) only the line C is a branching manifold. Analytically, it happens that one picks one of the two values of the square root:

$$p_4 = \sqrt{p_0^2 + p_1^2}\,,$$

and takes the ratios $p_0 : p_1 : p_2 : p_3 : p_4$ as coordinates of one of the two oriented points that cover the point $p_0 : p_1 : p_2 : p_3$. The five homogeneous coordinates of the oriented points thus satisfy a quadratic condition:

$$p_0^2 + p_1^2 - p_4^2 = 0 \tag{63}$$

whose coefficient matrix has rank 3.

Dual to this, one introduces the *oriented planes*, whose five coordinates π_k have the relation:

$$\pi_0^2 + \pi_1^2 - \pi_4^2 = 0. \tag{63*}$$

One can now also clarify the *quasi-distance* (cf. (60) in § 8) from an oriented point to a plane, including its sign, by the formula:

$$\vartheta = \frac{p_0\pi_0 + p_1\pi_1 + p_2\pi_2 + p_3\pi_3}{p_4\pi_4}\,. \tag{64}$$

The totality of all oriented points that have equal quasi-distances ($\vartheta < 0$ or $\vartheta > 0$) from a fixed plane that does not go through C shall be called an *oriented quasi-sphere.*

The desired group of inversions consists of single-valued transformations of the oriented points that permute the oriented quasi-spheres with each other. In order to put this group into simple formulas, one introduces more suitable coordinates t that are expressed in terms of the p by:

$$p_4 + p_1 = t_1^2, \quad p_4 - p_1 = t_2^2, \quad p_0 = t_1\, t_2, \quad p_2 = t_{11}, \quad p_3 = t_{22}\,. \tag{65}$$

If one regards the t as being homogeneous in the sense that one considers the two systems of values t_1, t_2, t_{11} t_{22}, and $\rho\, t_1$, $\rho\, t_2$, $\rho^2\, t_{11}$, $\rho^2\, t_{22}$ for $\rho \neq 0$ as equivalent then any system of values of the t, with the single exception of the one that contains only zeroes, corresponds to one and only one oriented point, and also conversely, any oriented point corresponds to one, and essentially only one, system of t.

The group of inversions is expressed in terms of these nine coordinates thusly:

$$\begin{aligned} t_1^* &= a_{11}t_1 + a_{12}t_2, \qquad d = a_{11}a_{22} - a_{12}a_{21} \neq 0, \\ t_2^* &= a_{21}t_1 + a_{22}t_2, \qquad \delta = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \neq 0, \\ t_{11}^* &= d\{a_{11}t_1 + a_{12}t_2\} + b_{11}t_1^2 + 2b_{12}t_1t_2 + b_{22}t_2^2, \\ t_{22}^* &= d\{a_{21}t_1 + a_{22}t_2\} + c_{11}t_1^2 + 2c_{12}t_1t_2 + c_{22}t_2^2. \end{aligned} \tag{66}$$

With no restriction on generality, one can set $d = \pm 1$. The group includes 13 essential parameters and decomposes in the real domain into four separate continuous systems that are characterized by the various combinations of the conditions $d = \pm 1$, $\delta < 0$, $\delta > 0$.

How can one now carry over all of this to kinematics? First, one must define the *oriented soma.* In the first place, we choose a *positive* actual soma σ. From formula (58) in § 8, one has for the angle φ_0 (§ 10) between the ur-soma σ_0 and σ:

$$\tan \varphi_0 = -\frac{p_1}{p_0}.$$

From this, it follows, after an arbitrary decision concerning the sign:

$$\sin \varphi_0 = -\frac{p_1}{p_4}, \qquad \cos \varphi_0 = +\frac{p_0}{p_4}, \qquad p_4 = \sqrt{p_0^2 + p_1^2}\,.$$

If we thus give the angle φ, as well as its sign, up to a multiple of 2π then p_4 is determined uniquely from these formulas, and our soma σ is then oriented. One can then orient an actual positive soma σ when one gives the angle $2\varphi_0$ of rotation that takes σ_0 to σ up to a multiple of 4π (and not merely up to a multiple of 2π).

In the second place, we take a *negative* actual soma $\mathfrak{s}$. One can orient it when one gives the half-displacement magnitude (cf., (64)):

$$\vartheta_0 = \frac{\pi_0}{\pi_4}, \qquad \pi_4 = \sqrt{\pi_0^2 + \pi_1^2}$$

of the transfer that takes σ_0 to $\mathfrak{s}$, including its sign. The centerline of this transfer is also oriented in a definite way by this. For the angle ψ_0 with the x_2 / x_1 – axis, which is defined mod 2π, one has, in fact, the formula:

$$\sin \psi_0 = -\frac{\pi_2}{\pi_4}, \qquad \cos \psi_0 = +\frac{\pi_3}{\pi_4}.$$

One can, moreover, orient the centerline of both equal and opposite transfers that exchange two actual oriented somas σ and $\mathfrak{s}$ with each other when one establishes that they should subtend the angle $\varphi_0 + \psi_0$ with the x_2 / x_1 – axis (defined mod 2π). The distance between two such somas (§ 11) is defined by the orientation of the centerline, along with its sign. An *oriented soma sphere* contains all oriented, positive somas that have the same distance $\vartheta < 0$ or $\vartheta > 0$ from an oriented, negative, actual soma.

These preparations suffice to make it possible to execute the transfer of spatial inversion geometry to kinematics.

On this, we make the following remark: If one restricts his considerations to the oriented somas σ of an oriented quasi-sphere exclusively then one can map this in a single-valued way to the oriented centerline of the transfers that make the oriented "middle soma" $\mathfrak{s}$ of the sphere cover the associated soma σ. The oriented somas σ of the sphere will be permuted with each other under the 13-parameter, and likewise, also the 12-parameter, group (66) in a seven-parameter manner. The group of these permutations is mapped to a group of transformations of the oriented lines and *this is identical with the so-called extended Laguerre group in the plane.* [1])

Here, I would like to content myself with these few facts about the inversion geometry of somas, although it amounts to no hardship to present this little theory in a similar way to the geometry of oriented lines in the plane.

It is trivial that this inversion geometry is opposed to another dual one, in which the oriented plane appears as the spatial element and the oriented negative somas appear in the kinematics of the base plane.

The application of our map $G \leftrightarrow (g_l , g_r)$ to the questions of differential geometry shall be treated in a continuation of this treatise. As an example of the results that one comes to in this way, we give the following one:

One maps the tangents G to a curved line M_1 in quasi-elliptic space that does not lie in a plane through C then g_l and g_r range along two mutually isometric curves, which take the form of the "pole path" and "pole curve" of the "motion" that is associated with M_1. – The "motion" that the joint of a linked quadrilateral corresponds to in space is a

[1]) On this, one can confer the author's treatise: "Zur Geometrie der Speere in der Euclidischen Ebene," Monatshefte f. Math., XXI, 1910.

winding curve of fourth order: viz., the intersection of two surfaces of second order from the family 4 (cf., § 9).

Perhaps the following result should be mentioned concerning the treatment of the two-dimensional families of somas M_2 : Those M_2 for which a curve that is rigidly coupled to a soma (i.e., axis cross) that is moving in M_2 always contacts a fixed curve in the plane corresponds to a family of surfaces in space that, under the passage to the limit of § 7, emerge from the family of surfaces with zero curvature measure in elliptic space that L. Bianchi has examined. – The general M_2 have a remarkable relationship to the transformations of oriented lines in the plane.

The behavior of kinematics in the sphere is somewhat simpler than kinematics in the plane, and one can map to the geometry of elliptic space by employing a process that C. Stephanos gave. From that point onward, one proceeds by means of a transference principle of Study to kinematics in Euclidian space.

Graz, Easter 1911.

Corrections to the treatise "Euclidian kinematics and non-Euclidian geometry I. II." [1])

BY WILHELM BLASCHKE in Greifswald

One might excuse the large number of mistakes and printing errors that are included in my paper by the fact that only one proofreader was available to me.

I have since then noticed the following errors:

Page		in place of	one must set:
63.	Line 9 from bottom	$(1 + z) : (1 - z)$	$(z + 1) : (z - 1)$
69.	Formulas (21), (22)	q	π
70.	Formula (28)*	$\dot{\pi}$	$\dot{\pi}''$
71.	Line 13 from top	$\bar{\hat{p}}$	$\bar{\dot{p}}$
73.	Line 9 from top	γ_l^{γ}	g_l^{γ}
78.	Line 15, 16, where I am speaking of Study's investigations, one must strike the words "in connection with the papers of an Italian geometer." The dissertation of G. Fubini (Il parallelismo di Clifford negli spazii ellitici, Pisa 1900), in which this situation is also treated appeared simultaneously with Study's examination (Über nichteuklidische und Liniengeometrie, Greifswald 1900).		

Page		in place of	one must set:
82.	Line 4 from bottom	in the latter case	in the sixth case
87.	Line 5 "	collineation	correlation
87.	Line 4 "	α_l , α_r	γ_l , γ_r

Finally, let it be mentioned that simultaneous to my paper, which I submitted to this journal for editing on Easter of 1911, a comprehensive treatise on the same situation appeared that had the recently-deceased J. Grünwald for its author. (Ein Abbildingsprinzip, welches die ebene Geometrie und Kinematik mit der räumlichen Geometrie verknüpft. Presented to the Vienna Academy at the session on 4 May 1911).

Greifswald, in November 1911.

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[1]) pp. 61-91 of v. 60 (1911) of this Journal.