"Die Existenzbedingungen eines von den ersten und zweiten Differentialquotienten der Coordinaten abhängigen kinetischen Potentials," J. reine angew. Math. 121 (1900), 124-140.

The existence conditions for a kinetic potential that depends upon the first and second differential quotients of the coordinates. (1)

(By Herrn **Karl Boehm** in Heidelberg)

Translated by D. H. Delphenich

Once the theorem on the existence conditions for a kinetic potential that **Helmholtz** posed (2) was first proved in the treatise by **Königsberger** on the principles of mechanics (3), **A. Mayer** showed in the Berichten der Kgl. Sächsischen Gesellschaft der Wissenschaften (4) that the proof of the aforementioned theorem could be carried out in an especially-simple way with the help of a principle from the calculus of variations, and at the same time, expressed the suspicion that the method that he applied might also make it possible to answer the question that **Königsberger** asked regarding the existence conditions for a kinetic potential that depends upon the first ν derivatives of the coordinates.

In fact, such a generalization will create no appreciable difficulties. Everything comes down to converting the problem into that of treating a system of linear partial differential equations. The way by which one can achieve that shall next be shown in the present article for case of $\nu = 2$. To that end, before we take up the actual problem, we must derive two auxiliary formulas.

Let $p_1, p_2, \ldots, p_n$ be any functions of t, and let V be a function of t ; $p_1, p_2, \ldots, p_n$; $p_1', p_2', \ldots, p_n'$; $p_1^{(\nu)}, p_2^{(\nu)}, \ldots, p_n^{(\nu)}$.

If one lets δV denote the variation that V experiences when one assigns the arbitrary variations δp to the variables p then one will always have:

$$\delta\left(\frac{d^\rho V}{dt^\rho}\right) \equiv \frac{d^\rho}{dt^\rho}(\delta V) ,$$

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(1) The present work was produced in the Summer of 1897 and submitted to the editors of this journal. In the meantime, **Arthur Hirsch** had solved the problem that was treated here in a more general way in volume 50 of Mathematischen Annalen.

(2) "Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung," J. für Math. **100** (1886), pp. 166. *Wissenschaftliche Abhandlungen*, Bd. III, Leipzig 1895, pp. 236.

(3) Mathematical and natural-scientific communications to the Sitzungsberichten of Königl. Preuss. Akademie der Wissenschaften zu Berlin. Report from 30 July 1896, pp. 387.

(4) Berichte der mathematisch-physikalische Klasse. Session on 7 December 1896.

$$\sum_{\lambda=1}^{n}\left\{\frac{\partial\left(\dfrac{d^{\rho}V}{dt^{\rho}}\right)}{\partial p_{\lambda}}\delta p_{\lambda}+\frac{\partial\left(\dfrac{d^{\rho}V}{dt^{\rho}}\right)}{\partial p'_{\lambda}}\frac{d}{dt}(\delta p_{\lambda})+\cdots+\frac{\partial\left(\dfrac{d^{\rho}V}{dt^{\rho}}\right)}{\partial p_{\lambda}^{(\nu+\rho)}}\frac{d^{(\nu+\rho)}}{dt^{(\nu+\rho)}}(\delta p_{\lambda})\right\}$$

$$\equiv \frac{d^{\rho}}{dt^{\rho}}\sum_{\lambda=1}^{n}\left\{\frac{\partial V}{\partial p_{\lambda}}\,\delta p_{\lambda}+\cdots+\frac{\partial V}{\partial p_{\lambda}^{(\sigma)}}\,\delta p_{\lambda}^{(\sigma)}+\cdots+\frac{\partial V}{\partial p_{\lambda}^{(\nu)}}\,\delta p_{\lambda}^{(\nu)}\right\}$$

$$\equiv \frac{d^{\rho-1}}{dt^{\rho-1}}\sum_{\lambda=1}^{n}\left\{\frac{d}{dt}\left(\frac{\partial V}{\partial p_{\lambda}}\right)\delta p_{\lambda}+\cdots+\left[\frac{\partial V}{\partial p_{\lambda}^{(\sigma-1)}}+\frac{d}{dt}\left(\frac{\partial V}{\partial p_{\lambda}^{(\nu)}}\right)\right]\frac{d^{\sigma}}{dt^{\sigma}}\delta p_{\lambda}+\cdots+\frac{\partial V}{\partial p_{\lambda}^{(\nu)}}\,\frac{d^{(\nu+1)}}{dt^{(\nu+1)}}(\delta p_{\lambda})\right\}$$

$$\equiv \frac{d^{\rho-2}}{dt^{\rho-2}}\sum_{\lambda=1}^{n}\left\{\frac{d^{2}}{dt^{2}}\left(\frac{\partial V}{\partial p_{\lambda}}\right)\delta p_{\lambda}+\cdots+\left[\frac{\partial V}{\partial p_{\lambda}^{(\sigma-2)}}+2\frac{d}{dt}\left(\frac{\partial V}{\partial p_{\lambda}^{(\sigma-1)}}\right)+\frac{d^{2}}{dt^{2}}\left(\frac{\partial V}{\partial p_{\lambda}^{(\sigma)}}\right)\right]\frac{d^{\sigma}}{dt^{\sigma}}\delta p_{\lambda}+\cdots+\frac{\partial V}{\partial p_{\lambda}^{(\nu)}}\,\frac{d^{(\nu+2)}}{dt^{(\nu+2)}}(\delta p_{\lambda})\right\}.$$

One now easily foresees what the coefficient of $\dfrac{d^{\sigma}}{dt^{\sigma}}(\delta p_{\lambda})$ will be after performing all ρ differentiations with respect to *t*, namely:

For $\sigma\leq\rho$:

$$\binom{\rho}{\sigma}\frac{d^{\rho-\sigma}}{dt^{\rho-\sigma}}\left(\frac{\partial V}{\partial p_{\lambda}}\right)+\binom{\rho}{\sigma-1}\frac{d^{\rho-\sigma+1}}{dt^{\rho-\sigma+1}}\left(\frac{\partial V}{\partial p'_{\lambda}}\right)+\cdots+\binom{\rho}{1}\frac{d^{\rho-1}}{dt^{\rho-1}}\left(\frac{\partial V}{\partial p_{\lambda}^{(\sigma-1)}}\right)+\frac{d^{\rho}}{dt^{\rho}}\left(\frac{\partial V}{\partial p_{\lambda}^{(\sigma)}}\right),$$

and for $\sigma\geq\rho$:

$$\frac{\partial V}{\partial p_{\lambda}^{(\sigma-\rho)}}+\binom{\rho}{\rho-1}\frac{d}{dt}\left(\frac{\partial V}{\partial p_{\lambda}^{(\sigma-\rho+1)}}\right)+\binom{\rho}{1}\frac{d^{\rho-1}}{dt^{\rho-1}}\left(\frac{\partial V}{\partial p_{\lambda}^{(\sigma-1)}}\right)+\cdots+\frac{d^{\rho}}{dt^{\rho}}\left(\frac{\partial V}{\partial p_{\lambda}^{(\sigma)}}\right).$$

When we set one of those expressions equal to the coefficient of $\dfrac{d^{\sigma}}{dt^{\sigma}}(\delta p_{\lambda})$ on the left-hand side of our first formula, we will get the two relations:

For $\sigma\leq\rho$:

(A) $$\frac{\partial}{\partial p_{\lambda}^{(\sigma)}}\left(\frac{d^{\sigma}V}{dt^{\sigma}}\right)\equiv\binom{\rho}{\sigma}\frac{d^{\rho-\sigma}}{dt^{\rho-\sigma}}\left(\frac{\partial V}{\partial p_{\lambda}}\right)+\binom{\rho}{\sigma-1}\frac{d^{\rho-\sigma+1}}{dt^{\rho-\sigma+1}}\left(\frac{\partial V}{\partial p'_{\lambda}}\right)+\cdots+\frac{d^{\rho}}{dt^{\rho}}\left(\frac{\partial V}{\partial p_{\lambda}^{(\sigma)}}\right),$$

and for $\sigma\geq\rho$:

(B) $$\frac{\partial}{\partial p_{\lambda}^{(\sigma)}}\left(\frac{d^{\sigma}V}{dt^{\sigma}}\right)\equiv\frac{\partial V}{\partial p_{\lambda}^{(\sigma-\rho)}}+\binom{\rho}{\sigma-1}\frac{d}{dt}\left(\frac{\partial V}{\partial p_{\lambda}^{(\sigma-\rho+1)}}\right)+\cdots+\frac{d^{\rho}}{dt^{\rho}}\left(\frac{\partial V}{\partial p_{\lambda}^{(\sigma)}}\right).$$

They define the generalization of formulas (A), (B), (C) of the aforementioned treatise of **Mayer**. If one defines equation (A) for a function R $(t, p_1, p_2, \ldots, p_n)$, which therefore does not depend upon the derivatives of p, then one will get the formula that **Königsberger** started with in his work on the principles of mechanics:

$$\frac{\partial}{\partial p_\lambda^{(\sigma)}}(R^{(\rho)}) \equiv \binom{\rho}{\sigma} \frac{d^{\rho-\sigma}}{dt^{\rho-\sigma}}\left(\frac{\partial R}{\partial p_\lambda}\right) \equiv \frac{\rho(\rho-1)\cdots(\rho-\sigma+1)}{1\cdot 2\cdots\sigma} \frac{\partial(R^{(\rho-\sigma)})}{\partial p_\lambda}.$$

In the present article, formulas (A) and (B) will be the basis for all calculations, such that it seems superfluous to me to point that out in the specific places. The investigation into which we shall now enter is concerned with the following question:

Let $p_1, p_2, \ldots, p_n$ be functions of the independent variable t. We seek to exhibit the conditions that the n functions $P_1, P_2, \ldots, P_n$ of:

$$t, \quad p_\iota, \quad \frac{dp_\iota}{dt} = p'_\iota, \quad \frac{d^2 p_\iota}{dt^2} = p_\iota^{(2)}, \quad \frac{d^3 p_\iota}{dt^3} = p_\iota^{(3)}, \quad \frac{d^4 p_\iota}{dt^4} = p_\iota^{(4)} \qquad (\iota = 1, 2, \ldots, n)$$

must satisfy in order for them to be expressible in terms of a single function H of t ; $p_1, p_2, \ldots, p_n$; $p'_1, p'_2, \ldots, p'_n$; $p_1^{(2)}, p_2^{(2)}, \ldots, p_n^{(2)}$ in the following way:

$$(1) \qquad P_\iota = -\left\{\frac{\partial H}{\partial p_\iota} - \frac{d}{dt}\left(\frac{\partial H}{\partial p'_\iota}\right) + \frac{d^2}{dt^2}\left(\frac{\partial H}{\partial p_\iota^{(2)}}\right)\right\} \qquad (\iota = 1, 2, \ldots, n).$$

If one sets:

$$(2) \qquad \frac{\partial H}{\partial p_\iota^{(2)}} = \varphi_\iota ,$$

$$(3) \qquad \frac{\partial H}{\partial p'_\iota} - \frac{d\varphi_\iota}{dt} = \psi_\iota ,$$

by which (1) will go to:

$$(4) \qquad -\frac{\partial H}{\partial p_\iota} + \frac{d\psi_\iota}{dt} = P_\iota ,$$

then one must examine the conditions under which $2n + 1$ functions φ_ι, ψ_ι, H can be determined from equations (2), (3), (4).

It next follows from (4) that since H is independent of $p_\kappa^{(3)}$ and $p_\kappa^{(4)}$, and ψ_ι is independent of $p_\kappa^{(4)}$, we will have:

$$(5)\qquad \begin{cases} a) & \dfrac{\partial P_\iota}{\partial p_\kappa^{(4)}} = \dfrac{\partial \psi_\iota}{\partial p_\kappa^{(3)}}, \\[2ex] b) & \dfrac{\partial P_\iota}{\partial p_\kappa^{(3)}} = \dfrac{\partial \psi_\iota}{\partial p_\kappa^{(2)}} + \dfrac{d}{dt}\left(\dfrac{\partial \psi_\iota}{\partial p_\kappa^{(3)}}\right). \end{cases}$$

We then move on to exhibit the integrability conditions for (2), (3), (4). In order for a function H of the p'_ι and $p_\iota^{(2)}$ to exist that satisfies equations (2) and (3), it is necessary and sufficient for us to determine the ψ_ι as functions of p'_ι, $p_\iota^{(2)}$, $p_\iota^{(3)}$ and the φ_ι as functions of the p'_ι and $p_\iota^{(2)}$ in such a way that the equations:

$$(6)\qquad \begin{cases} a) & \dfrac{\partial \psi_\iota}{\partial p_\kappa^{(3)}} = \dfrac{\partial \psi_\kappa}{\partial p_\iota^{(3)}}, \\[2ex] b) & \dfrac{\partial \psi_\iota}{\partial p_\kappa^{(2)}} + \dfrac{\partial \psi_\kappa}{\partial p_\iota^{(2)}} = \dfrac{d}{dt}\left(\dfrac{\partial \psi_\iota}{\partial p_\kappa^{(3)}} + \dfrac{\partial \psi_\kappa}{\partial p_\iota^{(3)}}\right), \\[2ex] c) & \dfrac{\partial \psi_\iota}{\partial p'_\kappa} - \dfrac{\partial \psi_\kappa}{\partial p'_\iota} = -\dfrac{\partial}{\partial p'_\kappa}\left(\dfrac{d\varphi_\iota}{dt}\right) + \dfrac{\partial}{\partial p'_\kappa}\left(\dfrac{d\varphi_\iota}{dt}\right), \end{cases}$$

and

$$(7)\qquad \begin{cases} a) & \dfrac{\partial \varphi_\iota}{\partial p_\kappa^{(2)}} = -\dfrac{\partial \psi_\kappa}{\partial p_\iota^{(3)}}, \\[2ex] b) & \dfrac{\partial \varphi_\iota}{\partial p'_\kappa} - \dfrac{\partial \varphi_\kappa}{\partial p'_\iota} = -\dfrac{1}{2}\left(\dfrac{\partial \psi_\iota}{\partial p_\kappa^{(2)}} - \dfrac{\partial \psi_\kappa}{\partial p_\iota^{(2)}}\right) \end{cases}$$

are fulfilled. They can also be derived from the $\frac{1}{2}n(n-1)+n^2+\frac{1}{2}n(n-1)$ integrability conditions:

$$\frac{\partial\left(\dfrac{\partial H}{\partial p_\iota^{(2)}}\right)}{\partial p_\kappa^{(2)}} = \frac{\partial\left(\dfrac{\partial H}{\partial p_\kappa^{(2)}}\right)}{\partial p_\iota^{(2)}}, \qquad \frac{\partial\left(\dfrac{\partial H}{\partial p_\kappa^{(2)}}\right)}{\partial p'_\iota} = \frac{\partial\left(\dfrac{\partial H}{\partial p_\iota^{(2)}}\right)}{\partial p'_\kappa}, \qquad \frac{\partial\left(\dfrac{\partial H}{\partial p'_\iota}\right)}{\partial p'_\kappa} = \frac{\partial\left(\dfrac{\partial H}{\partial p'_\kappa}\right)}{\partial p'_\iota},$$

with the help of formulas (A) and (B), which correspond to equations (7), (8), (9), (10), (11) in the treatise of **A. Mayer**. In addition, H shall now satisfy equations (4) as a function of the p_ι. One must then append the $n^2+n^2+\frac{1}{2}n(n-1)$ conditions:

$$\frac{\partial\left(\dfrac{\partial H}{\partial p_\iota}\right)}{\partial p_\kappa^{(2)}} = \frac{\partial\left(\dfrac{\partial H}{\partial p_\kappa^{(2)}}\right)}{\partial p_\iota}, \qquad \frac{\partial\left(\dfrac{\partial H}{\partial p_\iota}\right)}{\partial p'_\kappa} = \frac{\partial\left(\dfrac{\partial H}{\partial p'_\kappa}\right)}{\partial p_\iota}, \qquad \frac{\partial\left(\dfrac{\partial H}{\partial p_\iota}\right)}{\partial p_\kappa} = \frac{\partial\left(\dfrac{\partial H}{\partial p_\kappa}\right)}{\partial p_\iota},$$

which can be put into the following form:

$$(8)\qquad \begin{cases} a) & \dfrac{\partial}{\partial p_\kappa^{(3)}}\left(\dfrac{d\psi_\iota}{dt} - P_\iota\right) = \dfrac{\partial \varphi_\kappa}{\partial p_\iota}, \\ b) & \dfrac{\partial}{\partial p_\kappa'}\left(\dfrac{d\psi_\iota}{dt} - P_\iota\right) = \dfrac{\partial}{\partial p_\iota}\left(\dfrac{d\varphi_\kappa}{dt} + \psi_\kappa\right), \\ c) & \dfrac{\partial}{\partial p_\kappa}\left(\dfrac{d\psi_\iota}{dt} - P_\iota\right) = \dfrac{\partial}{\partial p_\iota}\left(\dfrac{d\psi_\kappa}{dt} - P_\kappa\right). \end{cases}$$

Our problem is then reduced to that of giving conditions that the P_ι must be subject to in order for equations (5), (6), (7), and (8) to be satisfied by $2n$ functions φ_ι and ψ_ι . That is because if that were the case then there would, in any event, exist a function H that possesses the required properties, and indeed it can be determined by mere quadratures from equations (2), (3), and (4) when the φ_ι and ψ_ι are found.

In order to solve the reduced problem, we first convert equations (6) and (8) with the use of (5) and (7). From (6.a), (6.b), (6.c), one gets the relations:

$$(9)\qquad \frac{\partial P_\iota}{\partial p_\kappa^{(4)}} = \frac{\partial P_\kappa}{\partial p_\iota^{(4)}},$$

$$(10)\qquad \frac{\partial P_\iota}{\partial p_\kappa^{(3)}} + \frac{\partial P_\kappa}{\partial p_\iota^{(3)}} = 2\frac{d}{dt}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(4)}} + \frac{\partial P_\kappa}{\partial p_\iota^{(4)}}\right),$$

$$(11)\qquad \frac{\partial \psi_\iota}{\partial p_\kappa'} - \frac{\partial \psi_\kappa}{\partial p_\iota'} + \frac{\partial \varphi_\iota}{\partial p_\kappa} - \frac{\partial \varphi_\kappa}{\partial p_\iota} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right),$$

from which, one can conversely derive equations (6) with the help of (5) and (7). If one then further takes the sum and difference of equation (8.a) or:

$$\frac{\partial P_\iota}{\partial p_\kappa^{(2)}} = -\frac{\partial \varphi_\iota}{\partial p_\kappa} - \frac{\partial \psi_\iota}{\partial p_\kappa'} + \frac{d}{dt}\left(\frac{\partial \psi_\iota}{\partial p_\kappa^{(2)}}\right)$$

and the ones that emerge when one switches ι and κ in them then that will give:

$$(12)\qquad \frac{\partial P_\iota}{\partial p_\kappa^{(2)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(2)}} = \frac{3}{2}\frac{d}{dt}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)$$

and

$$(13)\qquad \frac{\partial P_\iota}{\partial p_\kappa^{(2)}} + \frac{\partial P_\kappa}{\partial p_\iota^{(2)}} = -\frac{\partial \varphi_\iota}{\partial p_\kappa} - \frac{\partial \varphi_\kappa}{\partial p_\iota} + \frac{\partial \psi_\iota}{\partial p_\kappa'} - \frac{\partial \psi_\kappa}{\partial p_\iota'} + 2\frac{d^2}{dt^2}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(4)}}\right),$$

and in conjunction with (5), (7), (9), (10), (11), they are equivalent to equation (8.a). When the same process is applied to (8.b) or:

$$\frac{\partial P_\iota}{\partial p'_\kappa} = \frac{\partial \psi_\iota}{\partial p_\kappa} - \frac{\partial \psi_\kappa}{\partial p_\iota} + \frac{d}{dt}\left(\frac{\partial \psi_\iota}{\partial p'_\kappa}\right) - \frac{d}{dt}\left(\frac{\partial \varphi_\kappa}{\partial p_\iota}\right),$$

that will show us that the latter equation can be replaced with:

$$\text{(14)} \qquad \frac{\partial P_\iota}{\partial p'_\kappa} + \frac{\partial P_\kappa}{\partial p'_\iota} = \frac{d}{dt}\left(\frac{\partial P_\iota}{\partial p^{(2)}_\kappa} + \frac{\partial P_\kappa}{\partial p^{(2)}_\iota}\right) - 2\frac{d^3}{dt^3}\left(\frac{\partial P_\iota}{\partial p^{(4)}_\kappa}\right)$$

and

$$\text{(15)} \qquad \frac{\partial P_\iota}{\partial p'_\kappa} - \frac{\partial P_\kappa}{\partial p'_\iota} = 2\left(\frac{\partial \psi_\iota}{\partial p_\kappa} - \frac{\partial \psi_\kappa}{\partial p_\iota}\right) + \frac{1}{2}\frac{d^2}{dt^2}\left(\frac{\partial P_\iota}{\partial p^{(3)}_\kappa} - \frac{\partial P_\kappa}{\partial p^{(3)}_\iota}\right).$$

Finally, in combination with the foregoing, equation (8.c) is equivalent to:

$$\text{(16)} \qquad 2\left(\frac{\partial P_\iota}{\partial p_\kappa} - \frac{\partial P_\kappa}{\partial p_\iota}\right) = \frac{d}{dt}\left(\frac{\partial P_\iota}{\partial p'_\kappa} - \frac{\partial P_\kappa}{\partial p'_\iota}\right) - \frac{1}{2}\frac{d^3}{dt^3}\left(\frac{\partial P_\iota}{\partial p^{(3)}_\kappa} - \frac{\partial P_\kappa}{\partial p^{(3)}_\iota}\right),$$

such that we will arrive at the following result:

In order for a function H to exist that fulfills the requirements (1), the existence of the identities (9), (10), (12), (14), (16) is necessary. Thus, when the $2n$ functions φ_ι and ψ_ι can be determined from equations (5), (7), (11), (13), and (15), the given condition will also be sufficient. The fact that this is the case will be shown in what follows.

The equations:

$$\text{(11)} \qquad \frac{\partial \varphi_\iota}{\partial p_\kappa} - \frac{\partial \varphi_\kappa}{\partial p_\iota} + \frac{\partial \psi_\iota}{\partial p'_\kappa} - \frac{\partial \psi_\kappa}{\partial p'_\iota} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial P_\iota}{\partial p^{(3)}_\kappa} - \frac{\partial P_\kappa}{\partial p^{(3)}_\iota}\right)$$

and

$$\text{(13)} \qquad -\frac{\partial \varphi_\iota}{\partial p_\kappa} - \frac{\partial \varphi_\kappa}{\partial p_\iota} + \frac{\partial \psi_\iota}{\partial p'_\kappa} + \frac{\partial \psi_\kappa}{\partial p'_\iota} = \frac{\partial P_\iota}{\partial p^{(2)}_\kappa} + \frac{\partial P_\kappa}{\partial p^{(2)}_\iota} - 2\frac{d^2}{dt^2}\left(\frac{\partial P_\iota}{\partial p^{(4)}_\kappa}\right)$$

can be replaced with their sum and difference, so by the equations:

$$\text{(17)} \qquad \frac{\partial \varphi_\iota}{\partial p_\kappa} - \frac{\partial \psi_\kappa}{\partial p'_\iota} = -\frac{1}{2}\left(\frac{\partial P_\iota}{\partial p^{(2)}_\kappa} + \frac{\partial P_\kappa}{\partial p^{(2)}_\iota}\right) + \frac{1}{4}\frac{d}{dt}\left(\frac{\partial P_\iota}{\partial p^{(3)}_\kappa} - \frac{\partial P_\kappa}{\partial p^{(3)}_\iota}\right) + \frac{d^2}{dt^2}\left(\frac{\partial P_\iota}{\partial p^{(4)}_\kappa}\right)$$

and

$$\text{(18)} \qquad \frac{\partial \varphi_\kappa}{\partial p_\iota} - \frac{\partial \psi_\iota}{\partial p'_\kappa} = -\frac{1}{2}\left(\frac{\partial P_\iota}{\partial p^{(2)}_\kappa} + \frac{\partial P_\kappa}{\partial p^{(2)}_\iota}\right) + \frac{1}{4}\frac{d}{dt}\left(\frac{\partial P_\kappa}{\partial p^{(3)}_\iota} - \frac{\partial P_\iota}{\partial p^{(3)}_\kappa}\right) + \frac{d^2}{dt^2}\left(\frac{\partial P_\iota}{\partial p^{(4)}_\kappa}\right),$$

which must each be true for $\iota = 1, 2, \ldots, n$; $\kappa = 1, 2, \ldots, n$. Now since, with consideration given to the identities (9), (18) will emerge from (17) by switching ι and κ, the $\frac{1}{2}\,n\,(n-1)$ equations (18) will be identical to the $\frac{1}{2}\,n\,(n-1)$ equations (17), so they can be dropped.

The $2n$ functions φ_ι and ψ_ι must then be integrals of the following system of simultaneous first-order partial differential equations:

$$\left\{\begin{array}{ll}
(7.a) & \dfrac{\partial\varphi_\iota}{\partial p_\kappa^{(2)}} = -\dfrac{\partial P_\kappa}{\partial p_\iota^{(4)}}, \\[2ex]
(7.b) & \dfrac{\partial\varphi_\iota}{\partial p'_\kappa} - \dfrac{\partial\varphi_\kappa}{\partial p'_\iota} = -\dfrac{1}{2}\left(\dfrac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \dfrac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right), \\[2ex]
(5.a) & \dfrac{\partial\psi_\iota}{\partial p_\kappa^{(3)}} = \dfrac{\partial P_\kappa}{\partial p_\iota^{(4)}}, \\[2ex]
(5.b) & \dfrac{\partial\psi_\iota}{\partial p_\kappa^{(2)}} = \dfrac{\partial P_\kappa}{\partial p_\iota^{(3)}} - \dfrac{d}{dt}\left(\dfrac{\partial P_\iota}{\partial p_\kappa^{(4)}}\right), \\[2ex]
(17) & \dfrac{\partial\varphi_\iota}{\partial p_\kappa} - \dfrac{\partial\psi_\kappa}{\partial p'_\iota} = -\dfrac{1}{2}\left(\dfrac{\partial P_\iota}{\partial p_\kappa^{(2)}} + \dfrac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right) + \dfrac{1}{2}\left(\dfrac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \dfrac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right) + \dfrac{d^2}{dt^2}\left(\dfrac{\partial P_\iota}{\partial p_\kappa^{(4)}}\right), \\[2ex]
(15) & \dfrac{\partial\psi_\iota}{\partial p_\kappa} - \dfrac{\partial\psi_\kappa}{\partial p_\iota} = \dfrac{1}{2}\left(\dfrac{\partial P_\iota}{\partial p'_\kappa} - \dfrac{\partial P_\kappa}{\partial p'_\iota}\right) - \dfrac{1}{4}\dfrac{d^2}{dt^2}\left(\dfrac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \dfrac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right).
\end{array}\right.$$

In order to show that this system is integrable, we will derive a series of further relations from the identities:

$$\left\{\begin{array}{ll}
(9) & \dfrac{\partial P_\iota}{\partial p_\kappa^{(4)}} \equiv \dfrac{\partial P_\kappa}{\partial p_\iota^{(4)}}, \\[2ex]
(10) & \dfrac{\partial P_\iota}{\partial p_\kappa^{(3)}} + \dfrac{\partial P_\kappa}{\partial p_\iota^{(3)}} \equiv 4\,\dfrac{d}{dt}\left(\dfrac{\partial P_\iota}{\partial p_\kappa^{(4)}}\right), \\[2ex]
(12) & \dfrac{\partial P_\iota}{\partial p_\kappa^{(2)}} - \dfrac{\partial P_\kappa}{\partial p_\iota^{(2)}} \equiv \dfrac{3}{2}\dfrac{d}{dt}\left(\dfrac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \dfrac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right), \\[2ex]
(14) & \dfrac{\partial P_\iota}{\partial p'_\kappa} + \dfrac{\partial P_\kappa}{\partial p'_\iota} \equiv \dfrac{d}{dt}\left(\dfrac{\partial P_\iota}{\partial p_\kappa^{(2)}} + \dfrac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right) - 2\,\dfrac{d^3}{dt^3}\left(\dfrac{\partial P_\iota}{\partial p_\kappa^{(4)}}\right), \\[2ex]
(16) & \dfrac{\partial P_\iota}{\partial p_\kappa} - \dfrac{\partial P_\kappa}{\partial p_\iota} \equiv \dfrac{1}{2}\dfrac{d}{dt}\left(\dfrac{\partial P_\iota}{\partial p'_\kappa} - \dfrac{\partial P_\kappa}{\partial p'_\iota}\right) - \dfrac{1}{4}\dfrac{d^3}{dt^3}\left(\dfrac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \dfrac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right),
\end{array}\right.$$

which are now regarded as having been fulfilled.

Since the fifth differential quotients of the p do not figure, it will follow from (10) that:

(19) $$\frac{\partial^2 P_\iota}{\partial p_\kappa^{(4)}\,\partial p_\lambda^{(4)}} \equiv 0\,.$$

Likewise, when we differentiate (14) and (16) with respect to $p_\lambda^{(6)}$, we will get:

(20) $$\frac{\partial^2 P_\iota}{\partial p_\kappa^{(4)}\,\partial p_\lambda^{(3)}} \equiv 0\,,$$

(21) $$\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right) \equiv 0\,,$$

and furthermore, when we differentiate (14) with respect to $p_\lambda^{(5)}$, (12) with respect to $p_\lambda^{(4)}$, and (10) with respect to $p_\lambda^{(3)}$:

(22) $$\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}} + \frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right) - 2\,\frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)}\,\partial p_\kappa^{(4)}} \equiv 0\,,$$

(23) $$\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right) \equiv 0\,,$$

(24) $$\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} + \frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right) \equiv 4\,\frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)}\,\partial p_\kappa^{(4)}}\,,$$

and in conjunction with (21) and (24):

(24.a) $$\frac{\partial^2 P_\iota}{\partial p_\lambda^{(3)}\,\partial p_\kappa^{(3)}} - 2\,\frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)}\,\partial p_\kappa^{(4)}} \equiv 0\,.$$

Four further identities will be obtained by differentiating equations (16), (14), (12), (10) with respect to $p_\lambda^{(5)}$, $p_\lambda^{(4)}$, $p_\lambda^{(3)}$, $p_\lambda^{(2)}$, resp., with the use of the formulas that were derived before:

(25) $$\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa} - \frac{\partial P_\kappa}{\partial p'_\iota}\right) - \frac{1}{2}\,\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right) \equiv 0\,,$$

from which it will immediately follow from (9) that:

(25.a) $$\frac{\partial}{\partial p_\iota^{(2)}}\left(\frac{\partial P_\kappa}{\partial p_\lambda^{(3)}} - \frac{\partial P_\lambda}{\partial p_\kappa^{(3)}}\right) + \frac{\partial}{\partial p_\kappa^{(2)}}\left(\frac{\partial P_\lambda}{\partial p_\iota^{(3)}} - \frac{\partial P_\iota}{\partial p_\lambda^{(3)}}\right) + \frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right) \equiv 0\,,$$

$$
(26)\qquad \left\{ \begin{aligned} &\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\lambda}+\frac{\partial P_\kappa}{\partial p'_\iota}\right)-\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}}+\frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right)-\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}}+\frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right)\right] \\ &\qquad\qquad +2\frac{\partial^2 P_\iota}{\partial p'_\lambda\,\partial p_\lambda^{(4)}}+6\frac{d}{dt}\left[\frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)}\,\partial p_\kappa^{(4)}}\right]\equiv 0, \end{aligned}\right.
$$

or from (22):

$$
(26.a)\qquad \frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\lambda}+\frac{\partial P_\kappa}{\partial p'_\iota}\right)-\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}}+\frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right)+2\frac{\partial^2 P_\iota}{\partial p'_\lambda\,\partial p_\lambda^{(4)}}+4\frac{d}{dt}\left[\frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)}\,\partial p_\kappa^{(4)}}\right]\equiv 0\,,
$$

$$
(27)\qquad \frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right)\equiv\frac{3}{2}\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\equiv 3\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right),
$$

$$
(28)\qquad \frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}+\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\equiv 4\frac{\partial^2 P_\iota}{\partial p'_\lambda\,\partial p_\lambda^{(4)}}+4\frac{d}{dt}\left[\frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)}\,\partial p_\kappa^{(4)}}\right].
$$

When we subtract from equation (25) the ones that emerge from it by switching λ and κ, and in so doing consider the relations (9) and (27), we will get:

$$
(29)\qquad \frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)}\,\partial p_\kappa^{(3)}}-\frac{\partial^2 P_\iota}{\partial p_\kappa^{(2)}\,\partial p_\lambda^{(3)}}-\frac{\partial}{\partial p_\iota^{(4)}}\left(\frac{\partial P_\kappa}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\kappa}\right)\equiv 0\,.
$$

The following four formulas (30), (31), (32), and (33), which are obtained from the identities (16), (14), (12), (10) when they are differentiated with respect to $p_\lambda^{(4)}$, $p_\lambda^{(3)}$, $p_\lambda^{(2)}$, p'_λ, resp., are not as simple as the foregoing ones.

$$
(30)\qquad \left\{ \begin{aligned} &2\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\kappa}-\frac{\partial P_\kappa}{\partial p_\iota}\right)-\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)-\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)\right] \\ &\qquad\qquad +\frac{1}{2}\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)+\frac{3}{2}\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]\equiv 0, \end{aligned}\right.
$$

or with the use of (27):

$$
(30.a)\qquad 2\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\kappa}-\frac{\partial P_\kappa}{\partial p_\iota}\right)-\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)+\frac{1}{2}\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)+2\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)\right]\equiv 0\,,
$$

(31)
$$\left\{\begin{aligned} &\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p_\kappa}+\frac{\partial P_\kappa}{\partial p_\iota}\right)-\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}}+\frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right)-\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}+\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]\\ &\qquad +2\frac{\partial^2 P_\iota}{\partial p_\lambda\,\partial p_\kappa^{(4)}}+6\frac{d}{dt}\left[\frac{\partial^2 P_\iota}{\partial p'_\lambda\,\partial p_\kappa^{(4)}}\right]+6\frac{d^2}{dt^2}\left[\frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)}\,\partial p_\kappa^{(4)}}\right]\equiv 0, \end{aligned}\right.$$

or with the use of (26.a):

(31.a)
$$\left\{\begin{aligned} &\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}+\frac{\partial P_\kappa}{\partial p'_\iota}\right)-\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}}+\frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right)+2\frac{\partial^2 P_\iota}{\partial p_\lambda\,\partial p_\kappa^{(4)}}\\ &-\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}+\frac{\partial P_\kappa}{\partial p'_\iota}\right)-4\frac{\partial^2 P_\iota}{\partial p'_\lambda\,\partial p_\kappa^{(4)}}\right]+2\frac{d^2}{dt^2}\left[\frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)}\,\partial p_\kappa^{(4)}}\right]\equiv 0, \end{aligned}\right.$$

(32)
$$\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right)-\frac{3}{2}\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)-\frac{3}{2}\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]\equiv 0\,,$$

(33)
$$\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}+\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)-4\frac{\partial^2 P_\iota}{\partial p_\lambda\,\partial p_\kappa^{(4)}}-4\frac{d}{dt}\left[\frac{\partial^2 P_\iota}{\partial p'_\lambda\,\partial p_\kappa^{(4)}}\right]\equiv 0\,,$$

or

(33.a)
$$\left\{\begin{aligned} &2\frac{\partial}{\partial p_\kappa^{(3)}}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)-\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)+\frac{\partial}{\partial p'_\iota}\left(\frac{\partial P_\lambda}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\lambda^{(3)}}\right)\\ &\qquad -4\frac{\partial}{\partial p_\kappa^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\lambda}-\frac{\partial P_\lambda}{\partial p_\iota}\right)-4\frac{d}{dt}\left[\frac{\partial}{\partial p_\kappa^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)\right]\equiv 0. \end{aligned}\right.$$

It follows from (32) with the use of (25.a) that:

(34)
$$\frac{\partial}{\partial p'_\iota}\left(\frac{\partial P_\kappa}{\partial p_\lambda^{(3)}}-\frac{\partial P_\lambda}{\partial p_\kappa^{(3)}}\right)+\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\lambda}{\partial p_\iota^{(3)}}-\frac{\partial P_\iota}{\partial p_\lambda^{(3)}}\right)+\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\kappa^{(3)}}\right)\equiv 0\,,$$

such that (33.a) can be brought into a somewhat simpler form:

(33.b)
$$2\frac{\partial}{\partial p_\kappa^{(3)}}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)+\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\lambda}{\partial p_\iota^{(3)}}-\frac{\partial P_\iota}{\partial p_\lambda^{(3)}}\right)-4\frac{\partial}{\partial p_\kappa^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\lambda}-\frac{\partial P_\lambda}{\partial p_\iota}\right)-4\frac{d}{dt}\left[\frac{\partial}{\partial p_\kappa^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)\right]\equiv 0\,.$$

Although the identities that were just derived will not suffice for our purposes, their number will be increased when their necessity is exhibited in the course of the following investigation.

We shall next address equations (7.a), and with the help of (19) and (20), we will see that their right-hand sides are free of the third and fourth derivatives of *p*, such that from (23), we will further have:

$$\frac{\partial}{\partial p_\iota^{(4)}}\left(\frac{\partial P_\kappa}{\partial p_\lambda^{(2)}}-\frac{\partial P_\lambda}{\partial p_\kappa^{(2)}}\right)\equiv 0\ ,$$

i.e., that the *n* equations that the functions φ_ι must satisfy are compatible with each other. With the help of mere quadratures, that will then give φ_ι as the sum of a well-defined function of all *p*, as well as their first and second differential quotients and an arbitrary function of the *p* and their first differential quotients, which we would like to denote in the following way:

$$\varphi_\iota = \chi_\iota(p, p', p^{(2)})+\omega_\iota(p, p')\ .$$

The fact that χ_ι, as well as ω_ι, cannot depend explicitly upon the independent variable *t* (according to whether that is or is not true of the P_ι, resp.) is of no interest to us here, and for that reason it shall not be expressed in the notation that is employed either.

In order to determine the arbitrary functions φ_ι, we substitute the expression for φ_ι that was just found in equations (7.b):

(7.b)
$$\frac{\partial\omega_\iota}{\partial p'_\kappa}-\frac{\partial\omega_\kappa}{\partial p'_\iota}=-\left(\frac{\partial\chi_\iota}{\partial p'_\kappa}-\frac{\partial\chi_\kappa}{\partial p'_\iota}\right)-\frac{1}{2}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\equiv\Omega_{\iota\kappa}\ ,$$

$$\frac{\partial\Omega_{\iota\kappa}}{\partial p_\lambda^{(4)}}\equiv-\frac{1}{2}\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\equiv 0 \qquad \text{[from (20)]},$$

$$\frac{\partial\Omega_{\iota\kappa}}{\partial p_\lambda^{(3)}}\equiv-\frac{1}{2}\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\equiv 0 \qquad \text{[from (21)]},$$

$$\begin{aligned}\frac{\partial\Omega_{\iota\kappa}}{\partial p_\lambda^{(2)}}&\equiv-\frac{\partial^2\varphi_\iota}{\partial p'_\kappa\,\partial p_\lambda^{(2)}}+\frac{\partial^2\varphi_\kappa}{\partial p'_\iota\,\partial p_\lambda^{(2)}}-\frac{1}{2}\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\\&\equiv\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)-\frac{1}{2}\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\equiv 0 \text{[from (25)]},\end{aligned}$$

$$\frac{\partial\Omega_{\iota\kappa}}{\partial p'_\lambda}+\frac{\partial\Omega_{\kappa\lambda}}{\partial p'_\iota}+\frac{\partial\Omega_{\lambda\iota}}{\partial p'_\kappa}\equiv 0 \qquad \text{[from (34)]}.$$

The integrability conditions for equations (7.b) are then fulfilled, and as a result the functions φ_ι can be determined up to arbitrary functions of the *p*.

The integrability of equations (5.a), (5.b), (17), and (15) shall now be examined similarly.

Since the right-hand sides of equations (5.a) are free of the third and fourth differential quotients of the p, from (19) and (20), we will get the n functions ψ_ι in the form (upon introducing a notation that is similar to the one that was just used for φ_ι):

$$\psi_\iota = \mu_\iota(p, p', p^{(2)}, p^{(3)}) + \nu_\iota(p, p', p^{(2)}),$$

in which the ν_ι are arbitrary functions. We substitute the expression thus-found in (5.b) and get:

$$\frac{\partial \nu_\iota}{\partial p_\kappa^{(2)}} = -\frac{\partial \mu_\iota}{\partial p_\kappa^{(2)}} + \frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{d}{dt}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(4)}}\right) \equiv X_{\iota\kappa},$$

$$\frac{\partial X_{\iota\kappa}}{\partial p_\lambda^{(4)}} \equiv 0 \qquad \text{[from (19) and (20)]},$$

$$\frac{\partial X_{\iota\kappa}}{\partial p_\lambda^{(4)}} \equiv -\frac{\partial^2 \psi_\iota}{\partial p_\kappa^{(2)}\,\partial p_\lambda^{(3)}} + \frac{\partial^2 P_\iota}{\partial p_\kappa^{(3)}\,\partial p_\lambda^{(3)}} - \frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)}\,\partial p_\kappa^{(4)}} \equiv 0 \quad \text{[from (19) and (20)]}$$

Furthermore, we have:

$$\frac{\partial X_{\iota\kappa}}{\partial p_\lambda^{(2)}} - \frac{\partial X_{\iota\lambda}}{\partial p_\kappa^{(2)}} \equiv \frac{\partial^2 P_\iota}{\partial p_\kappa^{(3)}\,\partial p_\lambda^{(2)}} - \frac{\partial^2 P_\iota}{\partial p_\kappa^{(2)}\,\partial p_\lambda^{(3)}} - \frac{\partial}{\partial p_\iota^{(4)}}\left(\frac{\partial P_\kappa}{\partial p_\lambda'} - \frac{\partial P_\lambda}{\partial p_\kappa'}\right) \equiv 0 \qquad \text{[from (23) and (29)]}.$$

Those equations are also integrable, and we then get:

$$\nu_\iota = \rho_\iota(p, p', p^{(2)}) + \sigma_\iota(p, p'),$$
$$\psi_\iota = \mu_\iota(p, p', p^{(2)}, p^{(3)}) + \rho_\iota(p, p', p^{(2)}) + \sigma_\iota(p, p')$$

by simple quadratures, in which the σ_ι are arbitrary functions, and equations (17) and (15) will serve to determine them.

From (17), one sets:

$$\frac{\partial \sigma_\kappa}{\partial p'} = -\frac{\partial(\mu_\kappa + \rho_\kappa)}{\partial p_\iota'} + \frac{\partial \varphi_\iota}{\partial p_\kappa} + \frac{1}{2}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}} + \frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right) - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right) - \frac{d^2}{dt^2}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(4)}}\right) \equiv \Phi_{\iota\kappa}.$$

Based upon the identity (22), that will give:

$$\frac{\partial \Phi_{\iota\kappa}}{\partial p_\lambda^{(4)}} \equiv \frac{1}{2}\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}} + \frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right) - \frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)}\,\partial p_\kappa^{(4)}} \equiv 0.$$

Likewise, as a result of (25) and (26.a):

$$\frac{\partial \Phi_{\iota\kappa}}{\partial p_\lambda^{(3)}} \equiv \frac{\partial^2 P_\lambda}{\partial p'_\iota \, \partial p_\kappa^{(4)}} + \frac{1}{2} \frac{\partial}{\partial p_\lambda^{(3)}} \left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}} + \frac{\partial P_\kappa}{\partial p_\iota^{(2)}} \right) - \frac{1}{4} \frac{\partial}{\partial p_\lambda^{(2)}} \left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}} \right) - \frac{\partial^2 P_\iota}{\partial p'_\lambda \, \partial p_\kappa^{(4)}} - 2 \frac{d}{dt} \left[\frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)} \, \partial p_\kappa^{(4)}} \right] \equiv 0 \, .$$

Ultimately, one finds that:

$$\frac{\partial \Phi_{\iota\kappa}}{\partial p_\lambda^{(2)}} \equiv - \frac{\partial}{\partial p'_\iota} \left[\frac{\partial P_\kappa}{\partial p_\lambda^{(3)}} - \frac{d}{dt} \left(\frac{\partial P_\kappa}{\partial p_\lambda^{(4)}} \right) \right] - \frac{\partial^2 P_\lambda}{\partial p_\kappa \, \partial p_\iota^{(4)}} + \frac{1}{2} \frac{\partial}{\partial p_\lambda^{(2)}} \left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}} + \frac{\partial P_\kappa}{\partial p_\iota^{(2)}} \right) - \frac{1}{4} \frac{\partial}{\partial p'_\lambda} \left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}} \right)$$

$$- \frac{1}{4} \frac{d}{dt} \left[\frac{\partial}{\partial p_\lambda^{(2)}} \left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}} \right) \right] - \frac{\partial^2 P_\iota}{\partial p_\lambda \, \partial p_\kappa^{(4)}} - 2 \frac{d}{dt} \left[\frac{\partial}{\partial p'_\lambda} \left(\frac{\partial P_\iota}{\partial p_\kappa^{(4)}} \right) \right] - \frac{d^2}{dt^2} \left[\frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)} \, \partial p_\kappa^{(4)}} \right] ,$$

or with the use of (25) and (10):

$$\frac{\partial \Phi_{\iota\kappa}}{\partial p_\lambda^{(2)}} \equiv - \frac{\partial^2 P_\kappa}{\partial p'_\iota \, \partial p_\lambda^{(3)}} + \frac{1}{4} \frac{\partial}{\partial p'_\iota} \left(\frac{\partial P_\lambda}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\lambda^{(3)}} \right) - \frac{1}{4} \frac{\partial}{\partial p'_\lambda} \left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}} \right) - \frac{\partial^2 P_\lambda}{\partial p_\kappa \, \partial p_\iota^{(3)}} - \frac{d}{dt} \left[\frac{\partial^2 P_\lambda}{\partial p'_\kappa \, \partial p_\iota^{(4)}} \right]$$

$$+ \frac{1}{2} \frac{\partial}{\partial p_\lambda^{(2)}} \left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}} + \frac{\partial P_\kappa}{\partial p_\iota^{(2)}} \right) + \frac{1}{2} \frac{d}{dt} \left[\frac{\partial}{\partial p_\lambda^{(4)}} \left(\frac{\partial P_\iota}{\partial p'_\kappa} + \frac{\partial P_\kappa}{\partial p'_\iota} \right) \right] - \frac{\partial^2 P_\iota}{\partial p_\lambda \, \partial p_\kappa^{(4)}} - 2 \frac{d}{dt} \left[\frac{\partial}{\partial p'_\lambda} \left(\frac{\partial P_\iota}{\partial p_\kappa^{(4)}} \right) \right]$$

$$- \frac{d^2}{dt^2} \left[\frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)} \, \partial p_\kappa^{(4)}} \right] ,$$

and with consideration given to (34) and (10):

$$\frac{\partial \Phi_{\iota\kappa}}{\partial p_\lambda^{(2)}} \equiv - \frac{1}{2} \frac{\partial}{\partial p_\lambda^{(3)}} \left(\frac{\partial P_\iota}{\partial p'_\kappa} + \frac{\partial P_\kappa}{\partial p'_\iota} \right) + \frac{1}{2} \frac{\partial}{\partial p_\lambda^{(2)}} \left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}} + \frac{\partial P_\kappa}{\partial p_\iota^{(2)}} \right) + \frac{1}{2} \frac{d}{dt} \left[\frac{\partial}{\partial p_\lambda^{(4)}} \left(\frac{\partial P_\iota}{\partial p'_\kappa} + \frac{\partial P_\kappa}{\partial p'_\iota} \right) \right] - \frac{\partial^2 P_\iota}{\partial p_\lambda \, \partial p_\kappa^{(4)}}$$

$$- 2 \frac{d}{dt} \left[\frac{\partial^2 P_\iota}{\partial p'_\lambda \, \partial p_\kappa^{(4)}} \right] - \frac{d^2}{dt^2} \left[\frac{\partial^2 P_\iota}{\partial p_\lambda^{(2)} \, \partial p_\kappa^{(4)}} \right] .$$

One can see immediately that this expression vanishes when one subtracts the total differential of equation (26.a) with respect to t from (31).

In order to further show that the n differential equations whose common solution shall be the function σ_κ are mutually compatible, we form the expression:

$$\frac{\partial\Phi_{\iota\kappa}}{\partial p'_\lambda}-\frac{\partial\Phi_{\lambda\kappa}}{\partial p'_\iota}\equiv\frac{\partial}{\partial p_\kappa}\left(\frac{\partial\varphi_\iota}{\partial p'_\lambda}-\frac{\partial\varphi_\lambda}{\partial p'_\iota}\right)+\frac{1}{2}\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p^{(2)}_\kappa}+\frac{\partial P_\kappa}{\partial p^{(2)}_\iota}\right)-\frac{1}{2}\frac{\partial}{\partial p'_\iota}\left(\frac{\partial P_\lambda}{\partial p^{(2)}_\kappa}+\frac{\partial P_\kappa}{\partial p^{(2)}_\lambda}\right)$$

$$-\frac{1}{4}\frac{\partial}{\partial p'_\lambda}\left(\frac{d}{dt}\left[\frac{\partial P_\iota}{\partial p^{(3)}_\kappa}-\frac{\partial P_\kappa}{\partial p^{(3)}_\iota}\right]\right)+\frac{1}{4}\frac{\partial}{\partial p'_\iota}\left(\frac{d}{dt}\left[\frac{\partial P_\lambda}{\partial p^{(3)}_\kappa}-\frac{\partial P_\kappa}{\partial p^{(3)}_\lambda}\right]\right)$$

$$-2\frac{d}{dt}\left[\frac{\partial}{\partial p^{(4)}_\kappa}\left(\frac{\partial P_\iota}{\partial p_\lambda}-\frac{\partial P_\lambda}{\partial p_\iota}\right)\right]-\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p^{(4)}_\kappa}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)\right],$$

or with the use of (7.b) and (12):

$$\equiv-\frac{1}{2}\frac{\partial}{\partial p_\kappa}\left(\frac{\partial P_\iota}{\partial p^{(3)}_\lambda}-\frac{\partial P_\lambda}{\partial p^{(3)}_\iota}\right)+\frac{1}{3}\frac{\partial}{\partial p^{(2)}_\kappa}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)+\frac{2}{3}\left(\frac{\partial^2 P_\kappa}{\partial p'_\lambda\,\partial p^{(2)}_\iota}-\frac{\partial^2 P_\kappa}{\partial p'_\iota\,\partial p^{(2)}_\lambda}\right)$$

$$-2\frac{d}{dt}\left[\frac{\partial}{\partial p^{(4)}_\kappa}\left(\frac{\partial P_\iota}{\partial p_\lambda}-\frac{\partial P_\lambda}{\partial p_\iota}\right)\right]-\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p^{(4)}_\kappa}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)\right].$$

If one differentiates (14) with respect to $p^{(2)}_\lambda$ and subtracts the identity that is obtained from the one that emerges from it by switching ι and λ and considers (23) then that will give:

$$(35)\qquad\left\{\begin{aligned}&\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p^{(2)}_\lambda}-\frac{\partial P_\lambda}{\partial p^{(2)}_\iota}\right)-\frac{\partial}{\partial p^{(2)}_\kappa}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)-2\left(\frac{\partial^2 P_\kappa}{\partial p'_\lambda\,\partial p^{(2)}_\iota}-\frac{\partial^2 P_\kappa}{\partial p'_\iota\,\partial p^{(2)}_\kappa}\right)\\&\qquad-\frac{d}{dt}\left[\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p^{(3)}_\lambda}-\frac{\partial P_\lambda}{\partial p^{(3)}_\iota}\right)\right]+6\frac{d}{dt}\left[\frac{\partial}{\partial p^{(4)}_\kappa}\left(\frac{\partial P_\iota}{\partial p_\lambda}-\frac{\partial P_\lambda}{\partial p_\iota}\right)\right]\\&\qquad+6\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p^{(4)}_\kappa}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)\right]\equiv 0,\end{aligned}\right.$$

or in conjunction with (32) and (25):

$$(36)\qquad\left\{\begin{aligned}&\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p^{(2)}_\lambda}-\frac{\partial P_\lambda}{\partial p^{(2)}_\iota}\right)-\frac{\partial}{\partial p^{(2)}_\kappa}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)-2\left(\frac{\partial^2 P_\kappa}{\partial p'_\lambda\,\partial p^{(2)}_\iota}-\frac{\partial^2 P_\kappa}{\partial p'_\iota\,\partial p^{(2)}_\kappa}\right)\\&\qquad-\frac{3}{2}\frac{d}{dt}\left[\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p^{(3)}_\lambda}-\frac{\partial P_\lambda}{\partial p^{(3)}_\iota}\right)\right]+6\frac{d}{dt}\left[\frac{\partial}{\partial p^{(4)}_\kappa}\left(\frac{\partial P_\iota}{\partial p_\lambda}-\frac{\partial P_\lambda}{\partial p_\iota}\right)\right]\\&\qquad+3\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p^{(4)}_\kappa}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)\right]\equiv 0.\end{aligned}\right.$$

If one subtracts that equation from the following one, which is derived from (12):

$$(37) \qquad \frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p_\lambda^{(2)}} - \frac{\partial P_\lambda}{\partial p_\iota^{(2)}}\right) - \frac{3}{2}\frac{\partial}{\partial p_\kappa}\left(\frac{\partial P_\iota}{\partial p_\lambda^{(3)}} - \frac{\partial P_\lambda}{\partial p_\iota^{(3)}}\right) - \frac{3}{2}\frac{d}{dt}\left[\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p_\lambda^{(3)}} - \frac{\partial P_\lambda}{\partial p_\iota^{(3)}}\right)\right] \equiv 0$$

then one will find that the left-hand side of the relations that arises is equal to three times the expression for $\frac{\partial \Phi_{\iota\kappa}}{\partial p'_\lambda} - \frac{\partial \Phi_{\lambda\kappa}}{\partial p'_\iota}$ that was found above and that the integrability conditions for equations (17) :

$$\frac{\partial \Phi_{\iota\kappa}}{\partial p'_\lambda} - \frac{\partial \Phi_{\lambda\kappa}}{\partial p'_\iota} \equiv 0$$

will also be fulfilled then. We then obtain the n functions σ_ι in the form:

$$\sigma_\iota = \tau_\iota(p, p') + \zeta_i(p) \ ,$$

in which the ζ_ι are arbitrary functions of p. In regard to that, it should be remarked that the τ_ι also depend upon the arbitrary functions of the p that enter into the $\Phi_{\iota\kappa}$ when we substitute the expressions for φ_ι that are determined from equations (7.a) and (7.b), as we had to do above. Those functions will also remain undetermined in what follows. By contrast, we will try to arrange the ζ_ι in such a way that the expressions that are obtained for the ψ_ι :

$$\psi_\iota = \mu_\iota(p, p', p^{(2)}, p^{(3)}) + \rho_\iota(p, p', p^{(2)}) + \tau_\iota(p, p') + \zeta_i(p)$$

will satisfy equations (15), i.e., that:

$$\frac{\partial \zeta_\iota}{\partial p_\kappa} - \frac{\partial \zeta_\kappa}{\partial p_\iota} = -\left\{\frac{\partial(\mu_\iota + \rho_\iota + \tau_\iota)}{\partial p_\kappa} - \frac{\partial(\mu_\kappa + \rho_\kappa + \tau_\kappa)}{\partial p_\iota}\right\} + \frac{1}{2}\left(\frac{\partial P_\iota}{\partial p_\kappa} - \frac{\partial P_\kappa}{\partial p_\iota}\right) - \frac{1}{4}\frac{d^2}{dt^2}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right) \equiv \Theta_{\iota\kappa} \, ,$$

$$\frac{\partial \Theta_{\iota\kappa}}{\partial p_\lambda^{(4)}} \equiv \frac{1}{2}\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa} - \frac{\partial P_\kappa}{\partial p'_\iota}\right) - \frac{1}{4}\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right) \equiv 0 \quad \text{[from (25)]},$$

$$\frac{\partial \Theta_{\iota\kappa}}{\partial p_\lambda^{(3)}} \equiv -\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial \psi_\iota}{\partial p_\kappa} - \frac{\partial \psi_\kappa}{\partial p_\iota}\right) + \frac{1}{2}\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa} - \frac{\partial P_\kappa}{\partial p'_\iota}\right) - \frac{1}{4}\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right) - \frac{1}{2}\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]$$

$$\equiv -\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\kappa} - \frac{\partial P_\kappa}{\partial p_\iota}\right) + \frac{1}{2}\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa} - \frac{\partial P_\kappa}{\partial p'_\iota}\right) - \frac{1}{4}\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}} - \frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right) - \frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa} - \frac{\partial P_\kappa}{\partial p'_\iota}\right)\right]$$

[from (5.a) and (25)],

$$\frac{\partial\Theta_{\iota\kappa}}{\partial p_\lambda^{(3)}} \equiv 0 \qquad \text{[from (30.a)]}$$

$$\frac{\partial\Theta_{\iota\kappa}}{\partial p_\lambda^{(2)}} \equiv -\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial\psi_\iota}{\partial p_\kappa}-\frac{\partial\psi_\kappa}{\partial p_\iota}\right)+\frac{1}{2}\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)-\frac{1}{4}\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)-\frac{1}{2}\frac{d}{dt}\left[\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]$$
$$-\frac{1}{4}\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]$$

$$\equiv -\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p_\kappa}-\frac{\partial P_\kappa}{\partial p_\iota}\right)+\frac{1}{2}\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)-\frac{1}{4}\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)$$
$$+\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\kappa}-\frac{\partial P_\kappa}{\partial p_\iota}\right)\right]-\frac{1}{2}\frac{d}{dt}\left[\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]-\frac{1}{4}\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right].$$

If we introduce the value for the sum of the first three terms that we get by differentiating equation (16) with respect to $p_\lambda^{(3)}$ then we will get, with the use of (25):

$$\frac{\partial\Theta_{\iota\kappa}}{\partial p_\lambda^{(2)}} \equiv \frac{d}{dt}\left[-\frac{1}{2}\frac{\partial}{\partial p_\lambda^{(3)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)+\frac{1}{4}\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)+\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\kappa}-\frac{\partial P_\kappa}{\partial p_\iota}\right)\right]+\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)\right].$$

Thus:

$$\frac{\partial\Theta_{\iota\kappa}}{\partial p_\lambda^{(2)}} \equiv 0 \qquad \text{[from (33.b)]}$$

$$\frac{\partial\Theta_{\iota\kappa}}{\partial p'_\lambda} \equiv -\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial\psi_\iota}{\partial p_\kappa}-\frac{\partial\psi_\kappa}{\partial p_\iota}\right)+\frac{1}{2}\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)-\frac{1}{2}\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]-\frac{1}{4}\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]$$

$$\equiv \frac{\partial}{\partial p_\iota}\left\{\frac{\partial\varphi_\lambda}{\partial p_\kappa}+\frac{1}{2}\left(\frac{\partial P_\lambda}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\lambda}\right)-\frac{1}{6}\left(\frac{\partial P_\lambda}{\partial p_\kappa^{(2)}}-\frac{\partial P_\kappa}{\partial p_\lambda^{(2)}}\right)-\frac{d^2}{dt^2}\left[\frac{\partial P_\lambda}{\partial p_\kappa^{(4)}}\right]\right\}$$
$$-\frac{\partial}{\partial p_\kappa}\left\{\frac{\partial\varphi_\lambda}{\partial p_\iota}+\frac{1}{2}\left(\frac{\partial P_\lambda}{\partial p'_\iota}-\frac{\partial P_\iota}{\partial p'_\lambda}\right)-\frac{1}{6}\left(\frac{\partial P_\lambda}{\partial p_\iota^{(2)}}-\frac{\partial P_\iota}{\partial p_\lambda^{(2)}}\right)-\frac{d^2}{dt^2}\left[\frac{\partial P_\lambda}{\partial p_\iota^{(4)}}\right]\right\}$$
$$+\frac{1}{2}\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)-\frac{1}{2}\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]-\frac{1}{4}\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right],$$

$$\equiv \frac{1}{3}\left(\frac{\partial^2 P_\lambda}{\partial p_\iota\,\partial p_\kappa^{(2)}}-\frac{\partial^2 P_\lambda}{\partial p_\kappa\,\partial p_\iota^{(2)}}\right)+\frac{2}{3}\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\kappa}{\partial p_\iota}-\frac{\partial P_\iota}{\partial p_\kappa}\right)+\frac{1}{2}\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)$$
$$-\frac{1}{2}\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]-\frac{1}{4}\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]+\frac{d^3}{dt^3}\left[\frac{\partial}{\partial p_\lambda^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\kappa}-\frac{\partial P_\kappa}{\partial p_\iota}\right)\right].$$

In order to prove that $\partial\Theta_{\iota\kappa}/\partial p'_\lambda$ vanishes because of the identities (9) to (16), we differentiate (14) with respect to p'_λ, switch ι and λ in that, and subtract the equation that results from the first one:

$$(38)\quad \left\{\begin{aligned} &\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)-\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right)+\frac{\partial}{\partial p_\iota}\left(\frac{\partial P_\lambda}{\partial p_\kappa^{(2)}}-\frac{\partial P_\kappa}{\partial p_\lambda^{(2)}}\right)\\ &\qquad-\frac{d}{dt}\left[\frac{\partial}{\partial p_\kappa^{(2)}}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)\right]-\frac{d}{dt}\left[\frac{\partial^2 P_\kappa}{\partial p'_\lambda\,\partial p_\iota^{(2)}}-\frac{\partial^2 P_\kappa}{\partial p'_\iota\,\partial p_\lambda^{(2)}}\right]\\ &\qquad+6\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p_\kappa^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\lambda}-\frac{\partial P_\lambda}{\partial p_\iota}\right)\right]+2\frac{d^3}{dt^3}\left[\frac{\partial}{\partial p_\kappa^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)\right]\equiv 0\,. \end{aligned}\right.$$

Now, it will easily follow from (37), with the help of (34), that:

$$(37.a)\quad \left\{\begin{aligned} \frac{\partial^2 P_\kappa}{\partial p'_\lambda\,\partial p_\iota^{(2)}}-\frac{\partial^2 P_\kappa}{\partial p'_\iota\,\partial p_\lambda^{(2)}}\equiv\ &-\frac{3}{2}\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)+\frac{3}{2}\frac{\partial}{\partial p_\iota}\left(\frac{\partial P_\lambda}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\lambda^{(3)}}\right)\\ &+\frac{\partial}{\partial p_\kappa^{(2)}}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)-\frac{3}{2}\frac{d}{dt}\left[\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p_\lambda^{(3)}}-\frac{\partial P_\lambda}{\partial p_\iota^{(3)}}\right)\right], \end{aligned}\right.$$

so, from (12):

$$(37.b)\quad \left\{\begin{aligned} \frac{d}{dt}\left[\frac{\partial^2 P_\kappa}{\partial p'_\lambda\,\partial p_\iota^{(2)}}-\frac{\partial^2 P_\kappa}{\partial p'_\iota\,\partial p_\lambda^{(2)}}\right]\equiv\ &-\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(2)}}\right)+\frac{\partial}{\partial p_\iota}\left(\frac{\partial P_\lambda}{\partial p_\kappa^{(2)}}-\frac{\partial P_\kappa}{\partial p_\lambda^{(2)}}\right)\\ &+\frac{d}{dt}\left[\frac{\partial}{\partial p_\kappa^{(2)}}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)\right]-\frac{3}{2}\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p_\lambda^{(3)}}-\frac{\partial P_\lambda}{\partial p_\iota^{(3)}}\right)\right], \end{aligned}\right.$$

such that one can give equation (38) the following form:

$$
(38.a)\quad \left\{
\begin{aligned}
&2\left(\frac{\partial^2 P_\kappa}{\partial p_\iota\,\partial p_\lambda^{(2)}}-\frac{\partial^2 P_\kappa}{\partial p_\lambda\,\partial p_\iota^{(2)}}\right)+\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)-2\,\frac{d}{dt}\left[\frac{\partial}{\partial p_\kappa^{(2)}}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)\right]\\
&\qquad +6\,\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p_\kappa^{(4)}}\left(\frac{\partial P_\iota}{\partial p_\lambda}-\frac{\partial P_\lambda}{\partial p_\iota}\right)\right]+\frac{3}{2}\,\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p_\lambda^{(3)}}-\frac{\partial P_\lambda}{\partial p_\iota^{(3)}}\right)\right]\\
&\qquad\qquad +2\,\frac{d^3}{dt^3}\left[\frac{\partial}{\partial p_\kappa^{(4)}}\left(\frac{\partial P_\iota}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\iota}\right)\right]\equiv 0.
\end{aligned}\right.
$$

If one switches κ and λ in that identity, multiplies it by 1/2, and subtracts it from the equation below that is obtained by differentiating (16) with respect to $p_\lambda^{(2)}$:

$$
(39)\quad \left\{
\begin{aligned}
&2\,\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p_\kappa}-\frac{\partial P_\kappa}{\partial p_\iota}\right)-\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)-\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda^{(2)}}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)\right]\\
&\qquad +\frac{3}{2}\,\frac{d}{dt}\left[\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]+\frac{3}{2}\,\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p'_\kappa}\left(\frac{\partial P_\iota}{\partial p_\lambda^{(3)}}-\frac{\partial P_\lambda}{\partial p_\iota^{(3)}}\right)\right]\\
&\qquad\qquad +\frac{1}{2}\,\frac{d^3}{dt^3}\left[\frac{\partial}{\partial p_\lambda^{(\lambda)}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\lambda}{\partial p_\iota^{(3)}}\right)\right]\equiv 0
\end{aligned}\right.
$$

then one will get a formula whose right-hand side is zero, while the left-hand side agrees with three times the expression that was found for $\partial\Theta_{\iota\kappa}/\partial p'_\lambda$. Therefore:

$$
\frac{\partial\Theta_{\iota\kappa}}{\partial p'_\lambda}\equiv 0\,,
$$

and in order to complete our investigation, we still have only to show that the integrability conditions for equations (15), i.e., the identities:

$$
\frac{\partial\Theta_{\iota\kappa}}{\partial p_\lambda}+\frac{\partial\Theta_{\kappa\lambda}}{\partial p_\iota}+\frac{\partial\Theta_{\lambda\iota}}{\partial p_\kappa}\equiv 0
$$

are fulfilled.

To that end, we differentiate equation (16) with respect to p'_λ:

$$
(40)\quad \left\{
\begin{aligned}
&2\,\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa}-\frac{\partial P_\kappa}{\partial p_\iota}\right)-\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)-\frac{d}{dt}\left[\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)\right]\\
&\qquad +\frac{3}{2}\,\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]+\frac{1}{2}\,\frac{d^3}{dt^3}\left[\frac{\partial}{\partial p'_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]\equiv 0\,,
\end{aligned}\right.
$$

and add that equation to the two that arise from it by cyclic permutation of ι, κ, λ. The third term in the sum drops out automatically, the last drops out because of (34), and what will remain is:

$$-3\frac{\partial}{\partial p_\iota}\left(\frac{\partial P_\kappa}{\partial p'_\lambda}-\frac{\partial P_\lambda}{\partial p'_\kappa}\right)-3\frac{\partial}{\partial p_\kappa}\left(\frac{\partial P_\lambda}{\partial p'_\iota}-\frac{\partial P_\iota}{\partial p'_\lambda}\right)-3\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p'_\kappa}-\frac{\partial P_\kappa}{\partial p'_\iota}\right)$$
$$+\frac{3}{2}\frac{d^2}{dt^2}\left[\frac{\partial}{\partial p_\iota}\left(\frac{\partial P_\kappa}{\partial p_\lambda^{(3)}}-\frac{\partial P_\lambda}{\partial p_\kappa^{(3)}}\right)+\frac{\partial}{\partial p_\kappa}\left(\frac{\partial P_\lambda}{\partial p_\iota^{(3)}}-\frac{\partial P_\iota}{\partial p_\lambda^{(3)}}\right)+\frac{\partial}{\partial p_\lambda}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(3)}}-\frac{\partial P_\kappa}{\partial p_\iota^{(3)}}\right)\right]\equiv 0\,.$$

However, as we easily see, that is precisely the identity that we seek.

With that, we have show that among the assumptions that were made about the P_ι, there will always exist n functions ψ_ι of the $p,\ p',\ p^{(2)},\ p^{(3)}$ and n functions φ_ι of the $p,\ p',\ p^{(2)}$ that satisfy the partial differential equations (7.a), (7.b), (5.a), (5.b), (17), and (15), as well as include the independent variable t when it occurs explicitly in the P_ι.

The desired function H will then be determined by mere quadratures from equations (2), (3), and (4), whose integrability conditions (6), (7), and (8) for the φ_ι and ψ_ι are fulfilled.

In order for a kinetic potential H in the first and second differential quotients of the coordinates p that is defined by equations (1) *to exist, it is necessary and sufficient that the* P_ι *should be functions of* $p,\ p',\ p^{(2)},\ p^{(3)},\ p^{(4)}$ *that satisfy the identities* (9), (10), (12), (14), *and* (16).

In conclusion, we might give the form of the necessary and sufficient conditions for the existence of the most-general kinetic potential.

The function H shall then depend upon the first n differential quotients of the coordinates and be defined by the equations:

$$-\left\{\frac{\partial H}{\partial p_\iota}-\frac{d}{dt}\left(\frac{\partial H}{\partial p'_\iota}\right)+\cdots+(-1)^\nu\frac{d^\nu}{dt^\nu}\left(\frac{\partial H}{\partial p_\iota^{(\nu)}}\right)\right\}=P_\iota\,.$$

The P_ι must then be functions of the $p_\iota,\ p'_\iota,\ \ldots,\ p_\iota^{(2\nu)}$, and satisfy the $(2\nu+1)$ equations:

$$\frac{\partial P_\iota}{\partial p_\kappa^{(\tau)}}-\binom{\tau+1}{1}\frac{d}{dt}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(\tau+1)}}\right)+\binom{\tau+2}{2}\frac{d^2}{dt^2}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(\tau+2)}}\right)-\cdots+(-1)^{2\nu-\tau}\binom{2\nu}{2\nu-\tau}\frac{d^{2\nu-\tau}}{dt^{2\nu-\tau}}\left(\frac{\partial P_\iota}{\partial p_\kappa^{(2\nu)}}\right)$$
$$=(-1)^\tau\frac{\partial P_\kappa}{\partial p_\iota^{(\tau)}}\qquad(\tau=0,1,2,\ldots 2\nu-1,2\nu)$$

identically. When those relations are defined for $\nu=2$, they will coincide with formulas (9), (10), (12), (14), (16) of the foregoing article completely.
