"Interpretation de l'équation de Dirac…" Séminaire L. de Broglie. Théories physiques **25** (1955-1956), rep. no. 8, 1-10.

Interpretation of the Dirac equation as a linear approximation to the equation of a wave that propagates in a chaotically-agitated vorticial fluid of Dirac ether type

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Translated by D. H. Delphenich

PART ONE

1. Introduction. – A causal interpretation of the mechanics of the electron is proposed that is based upon the hypothesis that the electron is a particular object that follows a continuous and well-defined trajectory $\xi(t)$ that is accompanied by a real physical field $\Psi(\mathbf{x}, t)$ ([1]).

In order to obtain the results of the usual interpretation, it will suffice to suppose that:

1) Ψ satisfies one of the usual linear wave equations.
2) The corpuscle follows one of the streamlines that are associated with that wave equation.
3) An ensemble of such corpuscular objects is necessarily distributed with the density $P = \text{const.} \mid \Psi \mid^2$.

In a previous paper, two of us ([2]) showed that if one adopts the hydrodynamical representation of the wave equation, and one considers the corpuscular object to be a singularity in that field then it will suffice to assume that the quantum fluid is endowed with chaotic agitation in order to prove the hypothesis 3).

Here, we propose to develop that model and to show that the wave equations themselves are a consequence of the preceding hypothesis, provided that one assumes a supplementary hypothesis that relates to the structure and the nature of the energy that propagates in the quantum fluid.

Start with the idea that the vacuum is comparable to a continuous, relativistic fluid that is endowed with a perpetual chaotic agitation.

By "chaotic agitation," we mean that the density and the current of the fluid fluctuate in a very complex fashion in space and time around a rest state.

More precisely, if one considers a relative maximum of the density then one will find another one at a distance that we shall assume to be small compared to 10^{-13} cm, in such a way that the density will be constant, in the mean, over a spatial slab of that dimension, while that mean value itself will remain constant in time. Furthermore, we assume that

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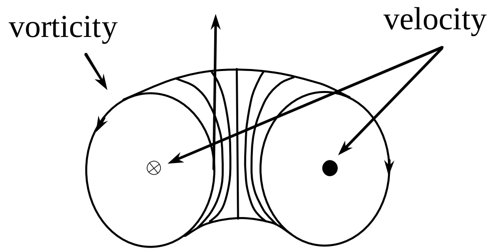
([1]) It amounts to the real physical wave *without* singularities that de BROGLIE introduced.
([2]) D. BOHM and J. P. VIGIER, Phys. Rev. **96** (1954), 208.

the geometric sum of the velocity vectors at the points of such a spatial slab is zero and that the orientation of the velocity vector at each point fills up the light cone in the course of time.

Having said that, we assume, in addition, that those motions are generally vorticial, and that there exist vorticial structures of very small dimensions inside of that fluid.

In the classical theory of vortices, one knows that such structures effectively exist.

The entire world knows about the classical vortex rings. One can likewise cite the following example:



Consider a torus whose meridian sections are vortex lines. The quantity of fluid that is enclosed by that surface will be conserved in the course of time by virtue of Helmholtz's theorem. The fluid turns around the axis of the torus, and its vortex moment is directed along that axis and will be conserved in the course of motion.

Such structures can be stable only if they are small compared to the dimensions of the fluctuations, because it is necessary that the latter carry them along without breaking them.

We assume that there exist a very large number of such vortices on each of the slabs that were previously envisioned.

By reason of the chaotic agitation of the vacuum, one can then suppose that their distribution is uniform in the mean in space and time, that the geometric sum of the moments of the vortices that are distributed on a spatial slab of 10^{-13} cm is zero, and that the orientation of each vortex varies chaotically in the course of time.

If one so desires, one can roughly compare those vortices to colloidal micelles that are endowed with kinetic moments, uniformly distributed in a liquid, and carried along by molecular agitation. If one supposes that each spatial element, no matter how small it might be (down to 10^{-13} cm), contains a large number of small vortices then the fluid will appear to be endowed with an internal angular momentum density.

2. Relativistic theory of bodies in rotation, described in terms of spinors. – We commence by describing the behavior of one of those vortices by supposing that it is stable enough that one can consider it to be a rotating rigid body.

In classical mechanics, the Lagrangian of a rotating body (in the absence of a field) is written:

$$L = \tfrac{1}{2} I^{ij} \omega_i \omega_j ,$$

in which:

I^{ij} is the inertia tensor,

ω_i is the angular velocity vector, which is dual to a second-rank tensor.

In relativity, we introduce an antisymmetric tensor $\omega_{\alpha\beta}$ that represents the spatial angular velocity of a rigid body and its acceleration (temporal angular velocity).

The invariance of the Lagrangian then imposes a fourth-rank inertia tensor $I^{\alpha\beta\gamma\delta}$ that one can suppose, with no loss of generality, to be symmetric in the pairs ($\alpha\beta$) and ($\gamma\delta$) and antisymmetric separately in α, β and γ, δ.

The Lagrangian that we will adopt will then be ([3]):

$$\boxed{L = \tfrac{1}{4} I^{[\alpha\beta][\gamma\delta]} \omega_{[\alpha\beta]} \omega_{[\gamma\delta]}.}$$

Then set:

$$K^{[\alpha\beta]} = \tfrac{1}{2} I^{[\alpha\beta][\gamma\delta]} \omega_{[\gamma\delta]}.$$

$K^{\alpha\beta}$ is the kinetic moment tensor of the rigid body, and we shall, in turn, use the Lagrangian in the form:

$$\boxed{L = \tfrac{1}{4} K^{[\alpha\beta]} \omega_{[\alpha\beta]}.}$$

We suppose, moreover, that the tensor $K^{[\alpha\beta]}$ is calculated with respect to the center-of-gravity of the system; i.e., at the point that is the center-of-mass in the Lorentz reference system in which the spatial components of the *total* impulse are annulled. One then knows (a theorem of Möller, Annales de la I.H.P., 1949) that the temporal components of $K^{\alpha\beta}$ are annulled in that system.

$K^{\alpha\beta}$ will then be the proper kinetic moment of the system in the Möller sense, and it is a special case of the Weyssenhoff tensor $S^{\alpha\beta}$.

We shall not use the Euler angles to express the angular velocity $\omega_{\alpha\beta}$. Indeed, we shall ultimately have to describe a continuous field of rotating bodies. Now, the Euler angles cannot vary continually during a continuous variation of the state of rotation, which is of little importance for an isolated body, and will not prevent one from writing the classical theory of gyroscope in terms of the Euler angles. By contrast, those angles cannot truly describe a field of rotating bodies. In order to do that, we shall take the Cayley-Klein parameters, which do not present that inconvenience, or – what amounts to the same thing – the spinors ([4]).

In the Appendix, one will find a theory of relativistic Cayley-Klein parameters that we have included in order to permit us to write the calculations that follow.

Calculate the expression for $\omega_{\alpha\beta}$ in terms of spinors:

From the expression:

([3]) One should note that, for example:

$A^{[\alpha\beta]}$: antisymmetric tensor

$B^{(\alpha\beta)}$: symmetric tensor

(Translator's note: I have changed the original notation to the present one for ease of typesetting.)

([4]) BOHM, TIOMNO, SCHILLER, Nuovo Cim. (Suppl.) **1** (1955).
H. WEYL, *Theory of Groups and Quantum Mechanics*, 1928, pp. 180.
SMIRNOV, *Course in Higher Mathematics*, v. III, 1. (Russian).
MURNAGHAN, *The Theory of Group Representations*, pp. 318.

$$l = 1 + \frac{1}{2}\,\delta\boldsymbol{\gamma}\cdot\boldsymbol{\alpha} - \frac{i}{2}\,\delta\boldsymbol{\beta}\cdot\boldsymbol{\sigma}$$

for the infinitesimal Lorentz transformation, one see that a variation of a spinor can be written:

$$\delta\Psi = (\frac{1}{2}\,\delta\boldsymbol{\gamma}\cdot\boldsymbol{\alpha} - \frac{i}{2}\,\delta\boldsymbol{\beta}\cdot\boldsymbol{\sigma})\,\Psi,$$

so:

$$\delta\Psi^* = \Psi^*\,(\frac{1}{2}\,\delta\boldsymbol{\gamma}\cdot\boldsymbol{\alpha} - \frac{i}{2}\,\delta\boldsymbol{\beta}\cdot\boldsymbol{\sigma}).$$

Now write the combinations:

$$i\,(\delta\Psi^*\,\beta\,\sigma_j\,\Psi - \Psi\,\beta\,\sigma_j\,\delta\Psi)$$

and

$$(\delta\Psi^*\,\beta\,\alpha_j\,\Psi - \Psi\,\beta\,\alpha_j\,\delta\Psi).$$

One sees that:

$$\beta\,\sigma_j = i\,\alpha_k\,\alpha_l\,\alpha_4 = \Gamma_{kl} \qquad (\text{with } \delta_{jkl} = +1),$$
$$\beta\,\alpha_j = \alpha_4\,\alpha_j = i\,\Gamma_{j4}\,,$$

in which Γ_{kl} and Γ_{j4} define the second-rank antisymmetric tensor that is associated with the spinor.

If we divide the two expressions by the proper-time interval and denote the derivative $\delta/\delta\tau$ by a dot then we can set:

(1) $$\boxed{\omega_{[\alpha\beta]} = i(\dot{\Psi}^*\,\Gamma_{[\alpha\beta]}\Psi - \Psi^*\,\Gamma_{[\alpha\beta]}\dot{\Psi}).}$$

The explicit calculation of that is not difficult, and gives the expressions:

(1′) $$\omega_{ij} = -\dot{\theta}_k\,\Omega_1 - \dot{\gamma}_k\,\Omega_2 \qquad (\delta_{ijk} = +1),$$
$$\omega_{4j} = \dot{\theta}_j\,\Omega_2 - \dot{\gamma}_j\,\Omega_1\,,$$

in which $\dot{\theta}$ is the angular velocity, and:

$$\dot{\gamma}_j = \frac{d}{d\gamma}\left(\arctan\frac{v_j}{c}\right) = \frac{\dot{v}_j/c}{1 - v_j^2/c^2}\,.$$

$$\Omega_1 = \cos A \qquad \text{and} \qquad \Omega_2 = \sin A$$

are the invariant and the pseudo-invariant that are coupled with the spinor. If one supposes that $A = \pi$ then one will have:

$$\boxed{\omega_{ij} = \dot{\theta}_k} \qquad (\delta_{ijk} = 1),$$

$$\boxed{\omega_{4j} = \dot{\gamma}_j.}$$

In the non-relativistic approximation, ω_{4j} must then be negligible compared to ω_j .

The spatial components of $K^{\alpha\beta}$ tend to the classical kinetic moment, and our Lagrangian will tend to the classical Lagrangian of the rotating rigid body. If $A \neq \pi$ then we can write the Lagrangian in terms of spinors:

$$\text{(2)} \qquad \boxed{L = \tfrac{1}{2} K^{\alpha\beta}\omega_{\alpha\beta} = \frac{i}{2}[\dot{\Psi}^* K^{\alpha\beta}\Gamma_{\alpha\beta}\Psi - \Psi^* K^{\alpha\beta}\Gamma_{\alpha\beta}\dot{\Psi}].}$$

Now calculate the quantities that are canonically conjugate to the components Ψ_n of the spinor, namely:

$$\Pi_n^* = \frac{\partial L}{\partial \dot{\Psi}_n},$$

$$\Pi_n = \frac{\partial L}{\partial \dot{\Psi}_n}.$$

We set:

$$\Pi^* = (\Pi_1^*, \Pi_2^*, \Pi_3^*, \Pi_4^*) = -\frac{i}{2}\Psi^* K^{\alpha\beta}\Gamma_{\alpha\beta},$$

$$\Pi = \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{pmatrix} = \frac{i}{2} K^{\alpha\beta}\Gamma_{\alpha\beta}\Psi.$$

One will then have:

$$\Pi^*\Psi = \Pi_1^*\Psi_1 + \Pi_2^*\Psi_2 + \Pi_3^*\Psi_3 + \Pi_4^*\Psi_4 = -\frac{i}{2}\Psi^* K^{\alpha\beta}\Gamma_{\alpha\beta}\Psi,$$
$$= -\frac{i}{2} K^{\alpha\beta}\Psi^*\Gamma_{\alpha\beta}\Psi = -\frac{i}{2} K^{\alpha\beta} m_{\alpha\beta},$$

$$\Psi^*\Pi = \Psi_1^*\Pi_1 + \Psi_2^*\Pi_2 + \Psi_3^*\Pi_3 + \Psi_4^*\Pi_4 = \frac{i}{2}\Psi^* K^{\alpha\beta}\Gamma_{\alpha\beta}\Psi,$$
$$= \frac{i}{2} K^{\alpha\beta}\Psi^*\Gamma_{\alpha\beta}\Psi = \frac{i}{2} K^{\alpha\beta} m_{\alpha\beta}.$$

Now, $K^{\alpha\beta}$ and $m_{\alpha\beta}$ are tensors, so $\Pi^*\Psi$ and $\Psi^*\pi$ are invariants, and one will see that Π^* and Π are *spinors* that transform like $\Psi^*\beta$ and $\beta\Psi$, respectively.

The Lagrangian can then be written:

(4) $$\boxed{L = \dot{\Psi}^*\Pi + \Pi^*\dot{\Psi}.}$$

From the mode of transformation of Π^* and Π, one will immediately deduce the quantities:

(5) $$M_{ij} = \Psi^*\alpha_i\,\alpha_j\,\Pi - \Pi^*\alpha_i\,\alpha_j\,\Psi,$$

(6) $$M_{k4} = \Psi^*\alpha_k\,\Pi + \Pi^*\alpha_k\,\Psi,$$

which are the components of a second-rank tensor $M_{\alpha\beta}$.

Upon specifying Π and Π^*, one will get:

$$\begin{aligned} M_{ij} &= \frac{i}{2}\Psi^* K^{\alpha\beta}(\alpha_i\,\alpha_j\,\Gamma_{\alpha\beta} + \Gamma_{\alpha\beta}\,\alpha_i\,\alpha_j)\,\Psi \\ &= i\,\Psi^* \underset{l>k}{K^{kl}}\,(\alpha_i\,\alpha_j\cdot i\,\alpha_k\,\alpha_l\,\alpha_4 + \alpha_k\,\alpha_l\,\alpha_4\,\alpha_i\,\alpha_j)\,\Psi \\ &\quad + i\,\Psi^* K^{k4}\,(\alpha_i\,\alpha_j\cdot i\,\alpha_k\,\alpha_4 + i\,\alpha_k\,\alpha_4\,\alpha_i\,\alpha_j)\,\Psi, \end{aligned}$$

and the relations between the α give:

$$\begin{aligned} \tfrac{1}{2}M_{ij} &= \frac{i}{2}\Psi^* K^{\alpha\beta}(\alpha_k\,\Gamma_{\alpha\beta} - \Gamma_{\alpha\beta}\,\alpha_k) \\ &= i\,\Psi^* \underset{l>k}{K^{kl}}\,(\alpha_k\cdot i\,\alpha_i\,\alpha_j\,\alpha_4 - i\alpha_i\,\alpha_j\,\alpha_4\,\alpha_k)\,\Psi \\ &\quad + i\,\Psi^* K^{l4}\,(\alpha_k\cdot i\,\alpha_l\,\alpha_4 - i\,\alpha_l\,\alpha_4\,\alpha_k)\,\Psi, \end{aligned}$$

which implies, after a simple calculation:

$$\tfrac{1}{2}M_{k4} = \Omega_1\,K_{k4} - \Omega_2\,K_{ij} \qquad (\delta_{ijk} = +\,1).$$

One can then write, in tensorial form, that:

(7) $$\boxed{\tfrac{1}{2}M_{\alpha\beta} = \Omega_1 K_{\alpha\beta} - \tfrac{1}{2}\Omega_2\,\delta_{\alpha\beta\gamma\delta}K^{\gamma\delta}.}$$

Consider the system that is linked with the body that is defined by $\theta_i = \gamma_i = 0$ (i = 1, 2, 3). In that system, one will have:

$$\Psi = \begin{pmatrix} 0 \\ \cos\dfrac{A}{2} \\ 0 \\ -i\sin\dfrac{A}{2} \end{pmatrix}, \qquad \Psi^* = \left(0,\ \cos\frac{A}{2},\ 0,\ i\,\sin\frac{A}{2}\right),$$

$$K^{14} = K^{24} = K^{34} = 0 \qquad \text{(from the starting hypothesis)},$$

$$K^{23} = T_1 , \qquad K^{31} = T_2 , \qquad K^{12} = T_3 ,$$

will be invariants that are the projections of the proper kinetic moment into the principal axes of the body.

If one takes the relations (5) and (6) into account, along with $\Omega_1 = \cos A$ and $\Omega_2 = \sin A$ then the relation (7) can be written in the proper system as:

(7, cont.)
$$\begin{cases} M_{23} = \ i(\Pi_1^* - \Pi_1)\cos\dfrac{A}{2} + (\Pi_3^* + \Pi_3)\sin\dfrac{A}{2} = 2T_1\cos A, \\ M_{31} = -(\Pi_1^* + \Pi_1)\cos\dfrac{A}{2} + i(\Pi_3^* - \Pi_3)\sin\dfrac{A}{2} = 2T_2\cos A, \\ M_{12} = -i(\Pi_2^* - \Pi_2)\cos\dfrac{A}{2} - (\Pi_4^* + \Pi_4)\sin\dfrac{A}{2} = 2T_3\cos A, \\ M_{14} = \ i(\Pi_3^* - \Pi_3)\cos\dfrac{A}{2} - i(\Pi_1^* - \Pi_1)\sin\dfrac{A}{2} = -2T_1\sin A, \\ M_{24} = \ i(\Pi_3^* + \Pi_3)\cos\dfrac{A}{2} - i(\Pi_1^* - \Pi_1)\sin\dfrac{A}{2} = -2T_2\sin A, \\ M_{34} = -i(\Pi_4^* - \Pi_4)\cos\dfrac{A}{2} + i(\Pi_2^* + \Pi_2)\sin\dfrac{A}{2} = -2T_3\sin A. \end{cases}$$

Those expressions are not covariant. In order to obtain a covariant form, we express the left-hand sides with the aid of the invariants that are linked to the spinors Π and Ψ.

We have four of them:

$$\Pi^*\Psi, \qquad \Pi^*\gamma_5\Psi, \qquad \Pi^*\tilde{\Psi}, \qquad \Pi^*\gamma_5\tilde{\Psi},$$

in which one has:

$$\gamma_5 = i\,\alpha_1\,\alpha_2\,\alpha_3$$

and

$$\tilde{\Psi} = i\,\alpha_2\,\alpha_4 \begin{pmatrix} \tilde{\Psi}_1^* \\ \tilde{\Psi}_2^* \\ \tilde{\Psi}_3^* \\ \tilde{\Psi}_4^* \end{pmatrix} \qquad \text{(= “charge conjugate”).}$$

One easily finds six invariant expressions that are identical to (7, cont.) in the proper system.

(8)
$$\begin{cases} \Pi^*\tilde{\Psi} + \tilde{\Psi}^*\Pi = 2T_1\sin A, \\ i(\tilde{\Psi}^*\gamma_5\Pi - \Pi^*\gamma_5\tilde{\Psi}) = 2T_1\cos A, \\ i(\Pi^*\tilde{\Psi} - \tilde{\Psi}^*\Pi) = 2T_3\sin A, \\ \Pi^*\gamma_5\tilde{\Psi} + \tilde{\Psi}^*\gamma_5\,\Pi = 2T_2\cos A, \\ \Pi^*\gamma_5\Psi + \Psi^*\Pi_5\,\pi = 2T_3\sin A, \\ i(\Psi^*\Pi - \Pi^*\Psi) = 2T_3\cos A. \end{cases}$$

One has the two relations:

(9) $$\begin{cases} \Pi^*\Psi + \Psi^*\Pi = 0, \\ \Pi^*\gamma_5\Psi - \Psi^*\gamma_5\Pi = 0, \end{cases}$$

moreover, which couple the Cayley-Klein parameters with each other.

3. The vorticial model of the ether. – We must now remind ourselves that we do not have to describe a rigid body, but a vortex; i.e., a rotating fluid mass.

In that case, once it attains its equilibrium state, there will be no spin precession around the rotational axis.

We then suppose that in the proper system the kinetic moment is aligned along one of the principal axes of the body, which is written:

$$T_1 = T_2 = 0,$$

and one can set $T_3 = k$.

Calculation then shows immediately that one must have:

$$\Pi^* = ik\,\Psi^+ = K\,\Psi^*\,\beta,$$
$$\Pi = -\,ik\,\beta\Psi.$$

The tensor $M_{\alpha\beta}$ that we have introduced can then be written:

$$M_{\alpha\beta} = 2\,\Psi^*\,\Gamma_{\alpha\beta}\,\Psi = 2\,m_{\alpha\beta}\,,$$

in which $m_{\alpha\beta}$ is the second-rank tensor that is linked with the spinor.

The relation (7) is then written:

(7_1) $$m_{\alpha\beta} = \Omega_1\,K_{\alpha\beta} - \tfrac{1}{2}\Omega_2\;\delta_{\alpha\beta\gamma\delta}\,K^{\gamma\delta},$$

but one of the Pauli-Kofinck identities is written:

(7_2) $$m_{\alpha\beta} = \tfrac{1}{2}\Omega_1\delta_{\alpha\beta\gamma\delta}\,(u^\gamma s^\delta - u^\delta s^\gamma) - \Omega_2\,(u_\alpha\,s_\beta - u_\beta\,s_\alpha),$$

in which one has (*because* Ψ *is normalized*):

$$u_\mu = \Psi^*\alpha_\mu\,\Psi, \qquad s_\mu = \Psi^*\sigma_\mu\,\Psi,$$

and

$$u_\mu\,u^\mu = 1, \qquad s_\mu\,s^\mu = -\,1.$$

The right-hand sides of (7_1) and (7_2) must be equal, no matter what the angle A is.

Now $K^{\alpha\beta}$, u^γ, and s^δ do not depend upon A. One will then have:

(10) $$\boxed{K_{\alpha\beta} = k\,\delta_{\alpha\beta\gamma\delta}\,u^\gamma s^\delta = k\,\overline{u_\alpha\,s_\beta - u_\beta\,s_\alpha}.}$$

That result is known already in Dirac's theory ([5]).

If one takes the expressions for Π and Π^* into account then the Lagrangian can be written:

(11) $$\boxed{L = ik\,(\Psi^+\dot{\Psi} - \dot{\Psi}^+\Psi).}$$

However, one also has that $L = \frac{1}{2}K^{\alpha\beta}\,\omega_{\alpha\beta}$, which can also be written:

$$L = k\ \omega^0_{12}\,.$$

($\omega^0_{\alpha\beta}$ denotes the values of $\omega_{\alpha\beta}$ in the proper system.)

The relation (1′) will then give:

$$\omega^0_{12} = -\ \Omega_1\,\dot{\theta}^0_3 - \Omega_2\,\dot{\gamma}^0_3\,.$$

If we suppose that *the energy of translation of the body is negligible compared to the energy of rotation* then the Lagrangian can be written (upon setting $\chi = k\ \theta^0_3$):

(12) $$\boxed{L = -\chi\,\Omega_1.}$$

Upon adding (11) and (12), the Lagrangian will take the form:

(13) $$\boxed{L = i\frac{k}{2}(\Psi^+\dot{\Psi} - \dot{\Psi}^+\Psi) - \chi\,\Omega_1.}$$

One will recognize the great formal analogy with the Dirac Lagrangian.

([5]) Cf., HALBWACHS, LOCHAK, VIGIER, C. R. Acad. Sc. **241** (1955), pp. 276.

"Interpretation de l'équation de Dirac…" Séminaire L. de Broglie. Théories physiques **25** (1955-1956), App. to rep. no. 8, 12-18.

Appendix

Expressions for the relativistic Cayley-Klein parameters as functions of the Euler angles

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1. Rotation matrices in space-time. – Let the following matrices describe the rotations in the coordinate planes in space-time (†):

A	rotation through an angle	α	in the	tOz plane
B	" " "	β	"	tOy "
C	" " "	γ	"	tOx "
D	" " "	ψ	"	xOy "
E	" " "	χ	"	xOz "
F	" " "	θ	"	yOz "
G	" " "	φ	"	xOy "

$$A = \begin{pmatrix} \operatorname{ch}\alpha & 0 & 0 & -\operatorname{sh}\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\operatorname{sh}\alpha & 0 & 0 & \operatorname{ch}\alpha \end{pmatrix}, \quad B = \begin{pmatrix} \operatorname{ch}\beta & 0 & -\operatorname{sh}\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\operatorname{sh}\alpha & 0 & \operatorname{ch}\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} \operatorname{ch}\gamma & -\operatorname{sh}\gamma & 0 & 0 \\ -\operatorname{sh}\gamma & \operatorname{ch}\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & \sin\psi & 0 \\ 0 & -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\chi & 0 & -\sin\chi \\ 0 & 0 & 1 & 0 \\ 0 & \sin\chi & 0 & \cos\chi \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The general rotation about a time axis is written $\Omega_L = ABC$; it is a Lorentz transformation without rotation of the spatial axes.

A rotation about a spatial axis can be written in two ways:

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(†) Translator's note: This list of matrices is clearly redundant, since the Lorentz group is six-dimensional. In particular, the matrices D and G describe the same basic rotation. However, this Appendix also had an unedited character to it, so one must occasionally correct things that do not seem right.

a) $\Omega_R = DEF$: successive rotations around spatial axes
b) $\Omega_E = DFG$: rotations through the Euler angles.

The general rotation can then be written:

either: $\Omega = \Omega_L \cdot \Omega_R = ABCDEF$
or: $\Omega = \Omega_L \cdot \Omega_E = ABCDFG$

2. Expressions for the rotations as functions of the Cayley-Klein parameters. – We briefly summarize the theory of spinors:

Consider two two-dimensional planes A_2 and $\hat{A}_2$ in a complex, four-dimensional affine space A_4 that have *just one point in common.*

Let the affine frames have two basis vectors in A_2 and two of them in $\hat{A}_2$, and consider the unimodular transformations that are defined by:

$$e_{\lambda'} = \alpha_{\lambda'}^{\lambda}\, e_{\lambda}, \quad \lambda = 1, 2,$$
$$e_{\hat{\mu}} = \alpha_{\hat{\mu}'}^{\hat{\mu}}\, e_{\hat{\mu}}, \quad \hat{\mu} = 1, 2,$$

with

$$\text{Det} \left\| \alpha_{\lambda'}^{\lambda} \right\| = \text{Det} \left\| \alpha_{\hat{\mu}'}^{\hat{\mu}} \right\| = 1.$$

One introduces the second-rank spin tensor $c^{\lambda\hat{\mu}}$, such that:

$$c^{\lambda\mu} = c^{\hat{\lambda}\hat{\mu}} = 0, \qquad c^{\lambda\hat{\mu}} = c.$$

That definition is invariant under the transformation law:

$$c^{\lambda'\hat{\mu}'} = \alpha_{\lambda}^{\lambda'}\, \alpha_{\hat{\mu}}^{\hat{\mu}'}\, c^{\lambda\hat{\mu}}.$$

One then sets:

$$c^{\lambda\hat{\mu}} = \begin{pmatrix} 0 & 0 & c^{1\hat{1}} & c^{1\hat{2}} \\ 0 & 0 & c^{2\hat{1}} & c^{2\hat{2}} \\ c^{\hat{1}1} & c^{\hat{1}2} & 0 & 0 \\ c^{\hat{2}1} & c^{\hat{2}2} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x^3 - x^0 & x^1 - ix^2 \\ 0 & 0 & x^1 + ix^2 & -x^3 - x^0 \\ x^3 - x^0 & x^1 + ix^2 & 0 & 0 \\ x^1 - ix^2 & -x^3 - x^0 & 0 & 0 \end{pmatrix},$$

in which x^0, x^1, x^2, x^3 are the contravariant components of a space-time vector. Since the transformations are unimodular, one verifies that:

$$c^{1\hat{1}} c^{2\hat{2}} - c^{1\hat{2}} c^{2\hat{1}} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

is invariant, which shows the correspondence between the spinorial transformations and the rotation group.

Moreover, one has:

$$c^{\lambda\hat{\mu}} = (\overline{c^{\hat{\mu}\lambda}}),$$

which implies that:

$$\alpha_{\hat{\lambda}}^{\hat{\lambda}'} = \overline{\alpha_{\hat{\lambda}}^{\hat{\lambda}'}}.$$

The spinorial transformations of the contravariant components of Ψ are then written:

$$\Psi^{1'} = k\,\Psi^1 + l\,\Psi^2, \qquad \Psi^{\hat{1}} = \bar{k}\,\Psi^{\hat{1}} + \bar{l}\,\Psi^{\hat{2}},$$
$$\Psi^{2'} = m\,\Psi^1 + n\,\Psi^2, \qquad \Psi^{2'} = \bar{m}\,\Psi^{\hat{1}} + \bar{n}\,\Psi^{\hat{2}},$$

and those of the covariant components are written:

$$\Psi_1,\ \Psi_2,\ ,\ \Psi_{\hat{1}},\ \Psi_{\hat{2}} = \Psi^2, -\Psi^1,\ \Psi^{\hat{2}}, -\Psi^{\hat{1}},$$

and one will always have the condition:

$$\begin{vmatrix} k & l \\ m & n \end{vmatrix} = 1,$$

in which k, l, m, n are the *Cayley-Klein parameters*.

The transformation $c^{\lambda'\tilde{\mu}'} = \alpha_{\lambda}^{\lambda'}\,\alpha_{\tilde{\mu}}^{\tilde{\mu}'}\,c^{\lambda\tilde{\mu}}$ is then written:

$$\begin{pmatrix} x^{3'} - x^{0'} & x^{1'} - ix^{2'} \\ x^{1'} + ix^{2'} & -x^{3'} - x^{0'} \end{pmatrix} = \begin{pmatrix} k & m \\ l & n \end{pmatrix} \begin{pmatrix} x^3 - x^0 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 - x^0 \end{pmatrix} \begin{pmatrix} \bar{k} & \bar{m} \\ \bar{l} & \bar{n} \end{pmatrix},$$

from which one infers that:

$$2x^{0'} = x^0(k\bar{k} + l\bar{l} + m\bar{m} + n\bar{n}) + x^1(-\bar{k}l - k\bar{l} - m\bar{n} - \bar{m}n) + ix^2(k\bar{l} - \bar{k}l + m\bar{n} - \bar{m}n) + x^3(-k\bar{k} + l\bar{l} - m\bar{m} + n\bar{n}),$$

$$2x^{1'} = x^0(-k\bar{m} - \bar{k}m - \mathrm{l}\bar{\mathrm{n}} - \bar{l}n) + x^1(k\bar{n} + \bar{k}n + l\bar{m} + \bar{l}m) + ix^2(\bar{k}n - k\bar{n} + l\bar{m} - \bar{l}m) + x^3(k\bar{m} + k\bar{m} - l\bar{n} - \bar{l}n),$$

$$2x^{2'} = x^0(k\bar{m} - \bar{k}m + l\bar{n} - \bar{l}n) + x^1(\bar{k}n - k\bar{n} + \bar{l}m - \bar{l}m) + ix^2(k\bar{n} + \bar{k}n - l\bar{m} - \bar{l}m) + x^3(\bar{k}m - k\bar{m} + l\bar{n} - \bar{l}n),$$

$$2x^{3'} = x^0(-k\bar{k} - l\bar{l} + m\bar{m} + n\bar{n}) + x^1(\bar{k}l + k\bar{l} - m\bar{n} - \bar{m}n) + ix^2(k\bar{l} - \bar{k}l + m\bar{n} - \bar{m}n) + x^3(k\bar{k} - l\bar{l} - m\bar{m} + n\bar{n}).$$

3. Calculation of the Cayley-Klein parameters as functions of the angles. – We shall now calculate the parameters that correspond to the rotation through an angle θ in xOt. One solves the system of 17 equations:

$$(0): \quad kn - lm = 1,$$

$$\begin{cases} (1) & \bar{k}l + k\bar{l} + \bar{m}n + m\bar{n} = 0, \\ (2) & \bar{k}l - k\bar{l} + \bar{m}n - m\bar{n} = 0, \\ (3) & -k\bar{k} - m\bar{m} + l\bar{l} + n\bar{n} = 0, \\ (4) & k\bar{m} + \bar{k}m + l\bar{n} + \bar{l}n = 0, \\ (5) & l\bar{m} - \bar{l}m + \bar{k}n - k\bar{n} = 0, \\ (6) & k\bar{m} + \bar{k}m - l\bar{n} - \bar{l}n = 0, \\ (7) & k\bar{m} - \bar{k}m + l\bar{n} - \bar{l}n = 0, \\ (8) & -l\bar{m} + \bar{l}m - \bar{k}n + k\bar{n} = 0, \end{cases} \qquad \begin{cases} (9) & k\bar{k} - m\bar{m} + l\bar{l} - n\bar{n} = 0, \\ (10) & \bar{k}l + k\bar{l} - \bar{m}n - m\bar{n} = 0, \\ (11) & k\bar{k} + l\bar{l} + m\bar{m} + n\bar{n} = 2, \\ (12) & l\bar{m} + m\bar{l} + \bar{k}n + k\bar{n} = 2, \\ (13) & k\bar{n} + \bar{k}n - l\bar{m} - \bar{l}m = 2\cos\theta, \\ (14) & \bar{k}m - k\bar{m} + l\bar{n} - \bar{l}n = 2i\sin\theta, \\ (15) & \bar{k}l - k\bar{l} + m\bar{n} - \bar{m}n = 2i\sin\theta, \\ (16) & \bar{k}k - l\bar{l} - m\bar{m} + n\bar{n} = 2\cos\theta. \end{cases}$$

One will then have:

$$\begin{cases} (11)+(16): & (a) \quad k\bar{k} + n\bar{n} = 2\cos^2\dfrac{\theta}{2}, \\ (11)-(16): & (a') \quad l\bar{l} + m\bar{m} = 2\sin^2\dfrac{\theta}{2}, \end{cases} \qquad \begin{cases} (3)+(9): & (b) \quad l\bar{l} - m\bar{m} = 0, \\ (3)-(9): & (b') \quad -k\bar{k} + n\bar{n} = 0, \end{cases}$$

so

$$\text{I} \quad \begin{cases} k\bar{k} = n\bar{n} = \cos^2\dfrac{\theta}{2}, \\ l\bar{l} = n\bar{n} = \sin^2\dfrac{\theta}{2}. \end{cases}$$

Moreover:

$$\begin{cases} (12)+(13): & (c) \quad k\bar{n} + \bar{k}n = 2\cos^2\dfrac{\theta}{2}, \\ (12)-(13): & (c') \quad l\bar{m} + \bar{l}m = 2\sin^2\dfrac{\theta}{2}, \end{cases} \qquad \begin{cases} (5)+(8): & (d) \quad \bar{k}n - k\bar{n} = 0, \\ (5)-(8): & (d') \quad l\bar{m} - \bar{l}m = 0, \end{cases}$$

so

$$\text{II} \quad \begin{cases} k\bar{n} = \bar{k}n = \cos^2\dfrac{\theta}{2}, \\ l\bar{m} = \bar{l}m = \sin^2\dfrac{\theta}{2}. \end{cases}$$

Upon comparing I and II, one will get:

$$\boxed{k = n \quad \text{and} \quad l = m.}$$

The relation (0) will then become:

$$(0'): \qquad k^2 - l^2 = 1.$$

On the other hand:

$$(1) + (2): \quad \bar{k}l + \bar{m}n = 0 \quad \text{and} \quad (2) + (10): \quad \bar{k}l - \bar{m}n = 0,$$

so

$$(\alpha) \quad \bar{m}n + m\bar{n} = 0 \quad \text{and} \quad (\beta) \quad \bar{k}l + k\bar{l} = 0.$$

Multiply (β) by k : $k\bar{k}l + k^2\bar{l} = 0$. From (0′), let:

$$l\cos^2\frac{\theta}{2} + \bar{l}\,(1 + l^2) = 0, \qquad \text{so} \qquad l\cos^2\frac{\theta}{2} + \bar{l} + l\sin^2\frac{\theta}{2} = 0,$$

so:

$$l + \bar{l} = 0,$$

hence:

$$\boxed{l = m = \pm i\sin\frac{\theta}{2},}$$

which gives:

$$\boxed{k = \pm\cos\frac{\theta}{2}.}$$

Now, if $\theta = 0$ then one will recover the identity matrix, so:

$$\boxed{k = n = \cos\frac{\theta}{2}.}$$

In order to get the sign of l, one substitutes $l = m = \pm\, i \sin\, \theta / 2$ into (14), for example.

After simplifying, that will become: $2i\,(\pm)\sin\theta = 2i\sin\theta$, so:

$$\boxed{l = m = i\sin\frac{\theta}{2}.}$$

Some analogous calculations will lead to expressions for the other rotation matrices. One will find the seven matrices:

$$a = \begin{pmatrix} \mathrm{ch}\frac{\alpha}{2} & \mathrm{sh}\frac{\alpha}{2} \\ \mathrm{sh}\frac{\alpha}{2} & \mathrm{ch}\frac{\alpha}{2} \end{pmatrix} = \exp\left(\sigma_1'\frac{\alpha}{2}\right), \qquad b = \begin{pmatrix} \mathrm{ch}\frac{\beta}{2} & -i\,\mathrm{sh}\frac{\beta}{2} \\ i\,\mathrm{sh}\frac{\beta}{2} & \mathrm{ch}\frac{\beta}{2} \end{pmatrix} = \exp\left(\sigma_2'\frac{\beta}{2}\right),$$

$$c = \begin{pmatrix} e^{\gamma/2} & 0 \\ 0 & e^{-\gamma/2} \end{pmatrix} = \exp\left(\sigma_3'\frac{\gamma}{2}\right), \qquad d = \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} = \exp\left(i\sigma_3'\frac{\psi}{2}\right),$$

$$e = \begin{pmatrix} \cos\frac{\chi}{2} & \sin\frac{\chi}{2} \\ -\sin\frac{\chi}{2} & \cos\frac{\chi}{2} \end{pmatrix} = \exp\left(\sigma_1' \frac{\chi}{2}\right), \qquad f = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \exp\left(i\sigma_1' \frac{\theta}{2}\right),$$

$$g = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} = \exp\left(i\sigma_3' \frac{\varphi}{2}\right),$$

in which σ_1', σ_2', σ_3' are the Pauli matrices.

4. The Dirac equation. – One introduces the contravariant spin tensor:

$$D^{\lambda\hat{\mu}} = \begin{pmatrix} 0 & 0 & \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^0} & \frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2} \\ 0 & 0 & \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^0} \\ \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^0} & \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} & 0 & 0 \\ \frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^0} & 0 & 0 \end{pmatrix}.$$

Note that the invariant of this spinor is the d'Alembertian.

Now consider the spinor $\tilde{\Psi}^{\lambda} = D^{\lambda\hat{\mu}}\Psi_{\hat{\mu}}$. Upon imposing the demand that $\tilde{\Psi}^{\lambda} = \frac{mc}{\hbar}\Psi^{\lambda}$, one will get:

$$\begin{cases} \tilde{\Psi}^1 = \frac{\partial\Psi_{\hat{1}}}{\partial x^3} + \frac{\partial\Psi_{\hat{1}}}{\partial x^0} + \frac{\partial\Psi_{\hat{2}}}{\partial x^1} - i\frac{\partial\Psi_{\hat{2}}}{\partial x^2} = -\frac{mc}{\hbar}\Psi_2, \\ \tilde{\Psi}^2 = \frac{\partial\Psi_{\hat{1}}}{\partial x^1} + i\frac{\partial\Psi_{\hat{1}}}{\partial x^0} - \frac{\partial\Psi_{\hat{2}}}{\partial x^3} + \frac{\partial\Psi_{\hat{2}}}{\partial x^0} = \frac{mc}{\hbar}\Psi_1, \\ \tilde{\Psi}^2 = \frac{\partial\Psi_1}{\partial x^3} + \frac{\partial\Psi_1}{\partial x^0} + \frac{\partial\Psi_2}{\partial x^1} + i\frac{\partial\Psi_2}{\partial x^2} = -\frac{mc}{\hbar}\Psi_{\hat{2}}, \\ \tilde{\Psi}^4 = \frac{\partial\Psi_1}{\partial x^1} - i\frac{\partial\Psi_1}{\partial x^2} - \frac{\partial\Psi_2}{\partial x^3} + \frac{\partial\Psi_2}{\partial x^0} = -\frac{mc}{\hbar}\Psi_{\hat{1}}. \end{cases}$$

Introduce the spinor that is defined by:

$$\varphi_1 = \tfrac{1}{2}(i\Psi_{\hat{2}} - \Psi_1),$$
$$\varphi_2 = -\tfrac{1}{2}(\Psi_2 + i\Psi_{\hat{1}}),$$
$$\varphi_3 = \tfrac{1}{2}(\Psi_1 + i\Psi_{\hat{2}}),$$

$$\varphi_4 = -\tfrac{1}{2}(\Psi_2 - i\Psi_{\hat{1}}).$$

The equations can then be written:

$$\begin{cases}
\dfrac{1}{c}\dfrac{\partial\varphi_1}{\partial t} = \left(\dfrac{\partial}{\partial x^1} + i\dfrac{\partial}{\partial x^2}\right)\varphi_4 + \dfrac{\partial\varphi_3}{\partial x^3} - i\dfrac{mc}{\hbar}\varphi_1, \\
\dfrac{1}{c}\dfrac{\partial\varphi_2}{\partial t} = \left(\dfrac{\partial}{\partial x^1} - i\dfrac{\partial}{\partial x^2}\right)\varphi_4 - \dfrac{\partial\varphi_4}{\partial x^3} - i\dfrac{mc}{\hbar}\varphi_2, \\
\dfrac{1}{c}\dfrac{\partial\varphi_3}{\partial t} = \left(\dfrac{\partial}{\partial x^1} + i\dfrac{\partial}{\partial x^2}\right)\varphi_2 + \dfrac{\partial\varphi_1}{\partial x^3} + i\dfrac{mc}{\hbar}\varphi_3, \\
\dfrac{1}{c}\dfrac{\partial\varphi_4}{\partial t} = \left(\dfrac{\partial}{\partial x^1} - i\dfrac{\partial}{\partial x^2}\right)\varphi_1 - \dfrac{\partial\varphi_2}{\partial x^3} + i\dfrac{mc}{\hbar}\varphi_4.
\end{cases}$$

5. Calculating the linear transformation of a Dirac spinor as a function of the Cayley-Klein parameters. – One has:

$$\begin{cases}
\Psi_1 = \varphi_3 - \varphi_1, \\
\Psi_2 = \varphi_4 - \varphi_2, \\
\Psi_{\hat{1}} = i(\varphi_4 + \varphi_2), \\
\Psi_{\hat{2}} = i(\varphi_1 + \varphi_3),
\end{cases}$$

and

$$\begin{cases}
\Psi_{1'} = n\Psi_1 - m\Psi_2 = n(\varphi_3 - \varphi_1) - m(\varphi_4 - \varphi_2) = \varphi_{3'} - \varphi_{1'}, \\
\Psi_{2'} = -l\Psi_1 + k\Psi_2 = -l(\varphi_3 - \varphi_1) + k(\varphi_4 - \varphi_2) = \varphi_{4'} - \varphi_{2'}, \\
\Psi_{\hat{1}'} = \bar{n}\Psi_{\hat{1}} - \bar{m}\Psi_{\hat{2}} = \bar{n}i(\varphi_2 + \varphi_4) + \bar{m}i(\varphi_1 + \varphi_3) = i(\varphi_{2'} + \varphi_{4'}), \\
\Psi_{\hat{2}'} = -\bar{l}\,\Psi_{\hat{1}} + \bar{k}\Psi_{\hat{2}} = -\bar{l}i(\varphi_2 + \varphi_4) - \bar{k}i(\varphi_1 + \varphi_3) = i(\varphi_{1'} + \varphi_{3'}).
\end{cases}$$

One then easily sees that the φ's transform by way of the matrix:

$$\Lambda = \tfrac{1}{2}\begin{pmatrix}
\bar{k}+n & \bar{l}-m & \bar{k}-n & \bar{l}+m \\
-l+\bar{m} & k+\bar{n} & l+\bar{m} & -k+\bar{n} \\
\bar{k}-n & \bar{l}+m & \bar{k}+n & \bar{l}-m \\
l+\bar{m} & -k+\bar{n} & l+\bar{m} & k+\bar{n}
\end{pmatrix}$$

The matrices *a*, *b*, *c*, *d*, *e*, *f*, *g* give us expressions for *k*, *l*, *m*, *n* as functions of the angles of rotation. One can then infer expressions for (?).

If one sets:

γ_i = rotation of the i^{th} spatial axis around the time axis,
α_i = Dirac current-matrix,

θ_j = spatial rotation around the j^{th} spatial axis,
σ_j = Dirac spin-matrix

then one will find that for a simple Lorentz transformation:

$$\Lambda_i = \exp\left(\alpha_i \frac{\gamma_i}{2}\right),$$

and for a spatial rotation:

$$\Lambda_j = \exp\left(-i\sigma_j \frac{\theta_j}{2}\right).$$

Hence, one has the general transformation:

$$\Lambda = \prod_{i=1}^{3} \exp\left(\alpha_i \frac{\gamma_i}{2}\right) \cdot \prod_{j=1}^{3} \exp\left(-i\sigma_j \frac{\theta_j}{2}\right).$$

If the transformation is infinitesimal then one will recover the known expression:

$$l = 1 + \boldsymbol{\alpha} \cdot \frac{\delta\boldsymbol{\gamma}}{2} - i\, \boldsymbol{\sigma} \cdot \frac{\delta\boldsymbol{\theta}}{2}.$$

6. Calculating the Cayley-Klein parameters in the case of a general rotation. – Consider the most general Euler rotation. Let $\omega_l = abc$ and $\omega_{\bar{l}} = dfg$. One will then have:

$$L = \omega_l \cdot \omega_{\bar{l}} = \begin{pmatrix} k & l \\ m & n \end{pmatrix},$$

so

$$L = \begin{pmatrix} k & l \\ m & n \end{pmatrix} =$$

$$\begin{pmatrix} \operatorname{ch}\frac{\alpha}{2} & \operatorname{sh}\frac{\alpha}{2} \\ \operatorname{sh}\frac{\alpha}{2} & \operatorname{ch}\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} \operatorname{ch}\frac{\beta}{2} & -i\operatorname{sh}\frac{\beta}{2} \\ i\operatorname{sh}\frac{\beta}{2} & \operatorname{ch}\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{\gamma/2} & 0 \\ 0 & e^{-\gamma/2} \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}$$

so one can identify:

$$\begin{cases} k = \exp\left[\frac{\gamma}{2}+i\frac{\psi+\varphi}{2}\right]\cos\frac{\theta}{2}\left(\text{ch}\frac{\alpha}{2}\text{ch}\frac{\beta}{2}+i\,\text{sh}\frac{\alpha}{2}\text{sh}\frac{\beta}{2}\right)+i\exp-\left[\frac{\gamma}{2}+i\frac{\psi-\varphi}{2}\right]\sin\frac{\theta}{2}\left(\text{sh}\frac{\alpha}{2}\text{ch}\frac{\beta}{2}-i\,\text{sh}\frac{\alpha}{2}\text{ch}\frac{\beta}{2}\right), \\ l = i\exp\left[\frac{\gamma}{2}+i\frac{\psi-\varphi}{2}\right]\sin\frac{\theta}{2}\left(\text{ch}\frac{\alpha}{2}\text{ch}\frac{\beta}{2}+i\,\text{sh}\frac{\alpha}{2}\text{sh}\frac{\beta}{2}\right)+\exp-\left[\frac{\gamma}{2}+i\frac{\psi+\varphi}{2}\right]\cos\frac{\theta}{2}\left(\text{sh}\frac{\alpha}{2}\text{ch}\frac{\beta}{2}-i\,\text{sh}\frac{\alpha}{2}\text{ch}\frac{\beta}{2}\right), \\ m = \exp\left[\frac{\gamma}{2}+i\frac{\psi+\varphi}{2}\right]\cos\frac{\theta}{2}\left(\text{ch}\frac{\alpha}{2}\text{sh}\frac{\beta}{2}+i\,\text{sh}\frac{\alpha}{2}\text{ch}\frac{\beta}{2}\right)+i\exp-\left[-\frac{\gamma}{2}-i\frac{\psi-\varphi}{2}\right]\sin\frac{\theta}{2}\left(\text{ch}\frac{\alpha}{2}\text{ch}\frac{\beta}{2}-i\,\text{sh}\frac{\alpha}{2}\text{bh}\frac{\beta}{2}\right), \\ n = i\exp\left[\frac{\gamma}{2}+i\frac{\psi-\varphi}{2}\right]\sin\frac{\theta}{2}\left(\text{ch}\frac{\alpha}{2}\text{sh}\frac{\beta}{2}+i\,\text{sh}\frac{\alpha}{2}\text{ch}\frac{\beta}{2}\right)+\exp-\left[\frac{\gamma}{2}+i\frac{\psi-\varphi}{2}\right]\cos\frac{\theta}{2}\left(\text{ch}\frac{\alpha}{2}\text{ch}\frac{\beta}{2}-i\,\text{sh}\frac{\alpha}{2}\text{sh}\frac{\beta}{2}\right). \end{cases}$$

7. Expressions in the proper system. – We can suppose that $\Psi_1 = 0$ and $\Psi_{\hat{2}} = 0$, so $\varphi_1 = 0$, $\varphi_2 = a$, $\varphi_3 = 0$, $\varphi_4 = b$. We will then have the invariants:

$$\begin{cases} \Omega_1 = \varphi^*\alpha_4\varphi = a\bar{a} - b\bar{b}, \\ \Omega_2 = \varphi^*\alpha_5\varphi = i(\bar{a}b - a\bar{b}). \end{cases}$$

One can demand that the spinor should be defined up to a multiplicative factor: $\Omega_1 = \cos A$ and $\Omega_2 = \cos A$. One will then find the two solutions:

$$\text{I}\quad \begin{cases} a = \cos\frac{A}{2}, \\ b = -\sin\frac{A}{2}, \end{cases} \qquad \text{II}\quad \begin{cases} a = -i\sin\frac{A}{2}, \\ b = \cos\frac{A}{2}. \end{cases}$$

Take the system I; one has:

$$\begin{pmatrix} 0 \\ \cos\frac{A}{2} \\ 0 \\ -i\sin\frac{A}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{A}{2} & 0 & -i\sin\frac{A}{2} & 0 \\ 0 & \cos\frac{A}{2} & 0 & -i\sin\frac{A}{2} \\ -i\sin\frac{A}{2} & 0 & \cos\frac{A}{2} & 0 \\ 0 & -i\sin\frac{A}{2} & 0 & \cos\frac{A}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \exp\left(-i\sigma_4\frac{A}{2}\right)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Note that σ_4 commutes with α_i and σ_i, so upon setting $A = \theta_4$, one will get the general spinor:

$$\Psi = \prod_{i=1}^{4} \exp\left(-i\sigma_i \frac{\theta_i}{2}\right) \prod_{j=1}^{3} \exp\left(\alpha_i \frac{\gamma_i}{2}\right) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

One has:

$$\varphi = \Lambda \begin{pmatrix} 0 \\ \cos \frac{A}{2} \\ 0 \\ -i \sin \frac{A}{2} \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} \bar{l} \exp\left[-i \frac{A}{2}\right] - m \exp\left[i \frac{A}{2}\right] \\ \bar{n} \exp\left[-i \frac{A}{2}\right] + k \exp\left[i \frac{A}{2}\right] \\ \bar{l} \exp\left[-i \frac{A}{2}\right] + m \exp\left[i \frac{A}{2}\right] \\ \bar{n} \exp\left[-i \frac{A}{2}\right] - k \exp\left[i \frac{A}{2}\right] \end{pmatrix},$$

so

$$\begin{cases} 2\varphi_1 = \bar{l} \exp\left[-i \frac{A}{2}\right] - m \exp\left[i \frac{A}{2}\right] \\ 2\varphi_2 = \bar{n} \exp\left[-i \frac{A}{2}\right] + k \exp\left[i \frac{A}{2}\right] \\ 2\varphi_3 = \bar{l} \exp\left[-i \frac{A}{2}\right] + m \exp\left[i \frac{A}{2}\right] \\ 2\varphi_4 = \bar{n} \exp\left[-i \frac{A}{2}\right] - k \exp\left[i \frac{A}{2}\right] \end{cases} \quad \text{and} \quad \begin{cases} k = (\varphi_2 - \varphi_4) \exp\left[-i \frac{A}{2}\right], \\ l = (\varphi_1^* + \varphi_3^*) \exp\left[-i \frac{A}{2}\right], \\ m = (\varphi_3 - \varphi_1) \exp\left[-i \frac{A}{2}\right], \\ n = (\varphi_2^* + \varphi_4^*) \exp\left[-i \frac{A}{2}\right]. \end{cases}$$

If one takes solution II instead of I then one will find that:

$$\begin{pmatrix} 0 \\ -i \sin A/2 \\ 0 \\ \cos A/2 \end{pmatrix} = \exp\left[-i\sigma_4 \frac{A}{2}\right] \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$
