"Interpretation de l'équation de Dirac…" Séminaire L. de Broglie. Théories physiques **25** (1955-1956), rep. no. 15, 1-21.

Interpretation of the Dirac equation as a linear approximation to the equation of a wave that propagates in a chaotically-agitated vorticial fluid of Dirac ether type

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PART TWO

1. Introduction. – In the preceding report, which was presented at this same seminar (rep. no. 8, 10 January 1956), we proposed a Lagrangian that was expressed in terms of spinors and permitted one to describe the behavior of a relativistic rotating body.

Here, we shall address that question once more and show that a perfect fluid without pressure (viz., the pure matter schema) whose molecules are "tops" will obey the Mathisson equations.

2. Relativistic body in rotation. – We start with a Lagrangian that has the form:

$$L = \tfrac{1}{2} K^{[\alpha\beta]} \, \omega_{[\alpha\beta]} \, ,$$

as we had proposed before.

One has:

$\omega_{\alpha\beta}$: angular velocity tensor,
$K_{\alpha\beta}$: kinetic moment.

We express that in terms of spinors.

One knows that if one defines an infinitesimal Lorentz transformation by:

$$\delta x_\alpha = \delta_{\alpha\beta} \, x^\beta \qquad (\delta_{\alpha\beta} = - \delta_{\beta\alpha})$$

then the transformation of a spinor can be written ([1]):

$$\delta\Psi = \tfrac{1}{4} \delta_{\alpha\beta} \gamma^\alpha \gamma^\beta \Psi, \qquad \text{with } \alpha \neq \beta$$
$$\delta\Psi^+ = -\tfrac{1}{4} \Psi^+ \delta_{\alpha\beta} \gamma^\alpha \gamma^\beta .$$

If one divides the two sides by the proper time interval $\delta\tau$ and denotes derivation with respect to proper time by a dot then one will have:

([1]) We take the world-variables $x_1, x_2, x_3, x_4 = ict$. The γ are the von Neumann matrices.

$$\dot{\Psi} = \tfrac{1}{4}\,\omega_{\alpha\beta}\,\gamma^{\alpha}\,\gamma^{\beta}\,\Psi \qquad \left(\omega_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{\delta\tau}\right)$$
$$\dot{\Psi}^{+} = -\tfrac{1}{4}\,\Psi^{+}\,\omega_{\alpha\beta}\,\gamma^{\alpha}\,\gamma^{\beta}\,\Psi.$$

Introduce the tensor:

$$A_{[\mu\nu]} = i\,(\Psi^{+}\gamma_{\mu}\,\gamma_{\nu}\,\dot{\Psi} - \dot{\Psi}^{+}\,\gamma_{\mu}\,\gamma_{\nu}\,\Psi),$$

in which one sets $\mu \neq \nu$ explicitly.

Upon specifying $\dot{\Psi}$ and $\dot{\Psi}^{+}$, one will get:

$$A_{\mu\nu} = \frac{i}{4}\,\omega_{\alpha\beta}\,\Psi^{+}\,(\gamma_{\mu}\,\gamma_{\nu}\,\gamma^{\alpha}\,\gamma^{\beta} - \gamma^{\alpha}\,\gamma^{\beta}\,\gamma_{\mu}\,\gamma_{\nu})\,\Psi.$$

All of the terms in that sum for which one has $\mu = \nu$ or $\alpha = \beta$ are zero, and similarly, the ones with $\mu \neq \alpha$ and $\nu = \beta$.

One will then have:

$$A_{\mu\nu} = \frac{i}{2}\,\omega_{\mu\nu}\,\Psi^{+}\,(\gamma_{\mu}\,\gamma_{\nu}\,\gamma_{\mu}\,\gamma_{\nu} + \gamma_{\mu}\,\gamma_{\nu}\,\gamma_{\mu}\,\gamma_{\nu})\,\Psi.$$
$$+\ \frac{i}{4}\sum{}'\,\omega_{\alpha\beta}\,\Psi^{+}\,(\gamma_{\mu}\,\gamma_{\nu}\,\gamma^{\alpha}\,\gamma^{\beta} + \gamma^{\alpha}\,\gamma^{\beta}\,\gamma_{\mu}\,\gamma_{\nu})\,\Psi,$$

upon setting $\mu \neq \nu \neq \alpha \neq \beta$.

Further, let:

$$A_{\mu\nu} = -\,i\,(\Psi^{+}\,\Psi\ \omega_{\mu\nu} + \frac{i}{2}\,\Psi^{+}\gamma_5\,\Psi\ \delta_{\mu\nu\alpha\beta}\,\omega^{\alpha\beta}),$$

$$\boxed{A_{\mu\nu} = -i(\Psi^{+}\Psi\,\omega_{\mu\nu} + i\Psi^{+}\gamma_5\Psi\,\overline{\omega_{\mu\nu}}),}$$

in which one has:

$$\overline{\omega_{\mu\nu}} = \tfrac{1}{2}\,\delta_{\mu\nu\alpha\beta}\,\omega^{\alpha\beta} = \frac{i}{2}\,\varepsilon_{\mu\nu\alpha\beta}\,\omega^{\alpha\beta},$$

with $\varepsilon_{\mu\nu\alpha\beta} = \pm 1$ according to the parity of the permutation $(\mu, \nu, \alpha, \beta)$.

One similarly has:

$$\overline{A_{\mu\nu}} = -\,i\,(\Psi^{+}\,\Psi\ \overline{\omega_{\mu\nu}} - i\,\Psi^{+}\gamma_5\Psi\,\omega_{\mu\nu}),$$

so:

$$(1) \qquad \boxed{\omega_{\mu\nu} = i\,\frac{\Psi^{+}\Psi A_{\mu\nu} - i\Psi^{+}\gamma_5\Psi\,\overline{A_{\mu\nu}}}{(\Psi^{+}\Psi)^2 + (i\Psi^{+}\gamma_5\Psi)^2}.}$$

We suppose that the spinor Ψ is normalized, so we will then have (see the Appendix to Part One in report 8):

(2) $$\Psi = \exp\left(-i\gamma_5 \frac{A}{2}\right)\prod_{i=1}^{3}\exp\left(-i\sigma_i \frac{\theta_i}{2}\right)\prod_{i=1}^{3}\exp\left(\alpha_i \frac{\varphi_i}{2}\right)\begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$

so

$$\Psi^+ \Psi = i \cos A,$$
$$\Psi^+\gamma_5\Psi = \sin A,$$

and

$$(\Psi^+ \Psi)^2 + (i\, \Psi^+\gamma_5\Psi)^2 = -1.$$

The Lagrangian will then be written:

$$L = \frac{1}{2} K^{\alpha\beta} [\Psi^+ \Psi\, (\Psi^+\gamma_\alpha \gamma_\beta \dot{\Psi} - \dot{\Psi}^+ \gamma_\alpha \gamma_\beta \Psi) - \frac{i}{2}\, \Psi^+\gamma_5\Psi\, \delta_{\alpha\beta\gamma\delta} (\Psi^+\gamma^\gamma\gamma^\delta \dot{\Psi} - \dot{\Psi}^+ \gamma^\gamma\gamma^\delta \Psi)].$$

Now, one knows that:

$$\tfrac{1}{2}\, \delta_{\alpha\beta\gamma\delta}\, \gamma^\gamma\gamma^\delta = \gamma_5\, \gamma^\alpha\gamma^\beta,$$

(3) $$L = \frac{1}{2} K^{\alpha\beta} [\Psi^+(\Psi^+ \Psi - \Psi^+ \gamma_5 \Psi \gamma_5)\, \gamma_\alpha \gamma_\beta \dot{\Psi} - \dot{\Psi}^+ \gamma_\alpha \gamma_\beta (\Psi^+ \Psi - \Psi^+ \gamma_5 \Psi \gamma_5)\Psi)].$$

The canonically-conjugate spinors to Ψ and Ψ^* will be:

$$\Pi^+ = \frac{\partial L}{\partial \dot{\Psi}} \qquad \text{and} \qquad \Pi = \frac{\partial L}{\partial \dot{\Psi}^+}.$$

If one takes into account the fact that:

$$\Psi^+ \Psi - \Psi^+\gamma_5\Psi\gamma_5 = i \cos A - \sin A\, \gamma_5 = i \exp(i\, \gamma_5 A)$$

then one will find that:

(4) $$\boxed{\begin{aligned}\Pi^+ &= \frac{i}{2}\Psi^+ K_{\alpha\beta}\gamma^\alpha\gamma^\beta \exp(i\gamma_5 A),\\ \Pi &= -\frac{i}{2} K_{\alpha\beta}\gamma^\alpha\gamma^\beta \exp(i\gamma_5 A)\Psi,\end{aligned}}$$

Now consider the tensor:

$$M_{\mu\nu} = [\Pi^+\gamma_\mu \gamma_\nu\Psi - \Psi^+ \gamma_\mu \gamma_\nu \Pi].$$

Upon specifying Π, one can write it as:

$$M_{\mu\nu} = \frac{i}{4} \Psi^+ (\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta + \gamma_\alpha \gamma_\beta \gamma_\mu \gamma_\nu) \exp(i\, \gamma_5 A)\, \Psi \cdot K^{\alpha\beta}.$$

Calculation will then give:

$$M_{\mu\nu} = -\,i\,[\Psi^{+}\exp(i\,\gamma_5\,A)\,\Psi\,K_{\mu\nu} + \mathrm{i}\,\Psi^{+}\gamma_5\exp(i\,\gamma_5\,A)\,\Psi\,\overline{K_{\mu\nu}}\,].$$

Taking into account the expression for the spinor that was given in (2) and the commutation relations between matrices, one will have:

$$\Psi^{+}\exp(i\,\gamma_5\,A)\,\Psi = (\Psi^{+}\Psi)_{A=0} = i,$$
$$\Psi^{+}\,\gamma_5\exp(i\,\gamma_5\,A)\,\Psi = (\Psi^{+}\gamma_5\Psi)_{A=0} = 0.$$

One will then have:

(5) $$\boxed{M_{\mu\nu} = K_{\mu\nu} = \tfrac{1}{2}[\Pi^{+}\gamma_\mu\gamma_\nu\Psi - \Psi^{+}\gamma_\mu\gamma_\nu\Pi].}$$

We now make two hypotheses:

1) Suppose that $K_{\mu\nu}$ represents the *proper kinetic moment* of the body – i.e., the moment that is taken with respect to the center-of-gravity, in the Möller sense ([2]).

The proper system will be defined by $\varphi_j = 0$ here [see the expression (2) for the spinor], and we then suppose that:

$$\varphi_j = 0 \qquad \text{implies} \qquad K_{i4} = 0.$$

2) The spatial components K_{ij}^{0} of the kinetic moment in the proper system form a spatial vector. We can orient the spatial axes of the proper system in such a way that the *z*-axis coincides with the kinetic moment. That system, which is linked with the body, will then be such that:

$$\theta_j = \varphi_j = 0.$$
$$K_{14}^{0} = K_{23}^{0} = K_{31}^{0} = 0,$$
$$K_{12}^{0} = k,$$

and

$$\Psi^{0} = \exp\left(-i\gamma_5\,\frac{A}{2}\right)\begin{pmatrix}0\\1\\0\\0\end{pmatrix}.$$

The system thus-defined is different from the one that is chosen in classical mechanics, in which one takes the axes to be the axes of the ellipsoid of inertia. One sees

([2]) The center-of-gravity is the point that is the center-of-mass in the proper system of the body – i.e., in the system where the spatial components of the *total impulse* are annulled. [MÖLLER, Annales de l'Institut Henri Poincaré, **11** (1949)]. The temporal components of the proper kinetic moment will be zero in that system.

that our system simplifies the calculations in terms of spinors and is the one that is chosen implicitly in Dirac's theory ([3]).

In that system, we will have [if we take (4) into account]:

$$\Pi_0 = -i\ K_0^{12} \exp\,(i\,\gamma_5 A)\ \gamma_1\ \gamma_2\ \exp\left(-i\gamma_5 \frac{A}{2}\right)\begin{pmatrix}0\\1\\0\\0\end{pmatrix}.$$

Now let:

$$\Pi_0 = -k \exp\,(i\,\gamma_5 A)\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\exp\left(-i\gamma_5 \frac{A}{2}\right).$$

$$= +k \exp\,(i\,\gamma_5 A)\cdot \exp\left(-i\gamma_5 \frac{A}{2}\right)\begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$

$$\Pi_0 = k \exp\,(i\,\gamma_5 A)\,\Psi_0\,,$$

and then, in an invariant fashion:

(6) $$\boxed{\Pi = k\exp(i\gamma_5 A)\Psi}$$

and likewise:

$$\boxed{\Pi^+ = -k\Psi^+\exp(i\gamma_5 A).}$$

If we introduce the expressions (6) into (5) then we will get:

$$K_{[\mu\nu]} = -k\,\Psi^+ \exp\,(i\,\gamma_5 A)\ \gamma_\mu\ \gamma_\nu\,\Psi,$$

Further, let:

$$\begin{aligned} K_{[\mu\nu]} &= -k\,(\cos A\cdot \Psi^+\gamma_\mu\,\gamma_\nu\,\Psi + i\sin A\ \Psi^+\,\gamma_5\,\gamma_\mu\,\gamma_\nu\,\Psi) \\ &= ik\,(\Psi^+\Psi\cdot\Psi^+\gamma_\mu\,\gamma_\nu\,\Psi - i\,\Psi^+\gamma_5\Psi,\ \overline{\Psi^+\gamma_\mu\,\gamma_\nu\,\Psi}) \\ &= \frac{k}{\sqrt{-(\Psi^+\Psi)^2-(i\Psi^+\gamma_5\Psi)^2}}\,\overline{(\Psi^+\gamma_5\,\gamma_\mu\Psi\cdot\Psi^+\,\gamma_\nu\Psi - \Psi^+\gamma_5\,\gamma_\nu\Psi\cdot\Psi^+\,\gamma_\mu\Psi)}\,, \end{aligned}$$

from a formula of Kofink ([4]).

([3]) One sees that we are dealing with an adaptation of the mathematical formalism here, and not a physical hypothesis. Contrary to what we said in our first seminar talk, we have *not at all restricted the generality of the problem* in that way.

([4]) Cf., O. COSTA de BEAUREGARD, *Thése*, 1943.

One then sets:

$$\sigma_\mu = \Psi^+ \gamma_5 \gamma_\mu \Psi, \qquad u_\mu = \frac{\Psi^+ \gamma_\mu \Psi}{\sqrt{-(\Psi^+\Psi)^2 - (i\Psi^+\gamma_5\Psi)^2}},$$

so:

(7) $$\boxed{K_{\mu\nu} = k(\overline{\sigma_\mu u_\nu - \sigma_\nu} u_\mu).}$$

One easily deduces that u_μ is *the velocity unit vector of the particle* from the expression (2) for the spinor.

As for the Lagrangian, it can now be written:

$$L = \Pi^+\dot{\Psi} + \dot{\Psi}^+\Pi = -k\,[\Psi^+ \exp(i\,\gamma_5 A)\,\dot{\Psi} - \dot{\Psi}^+ \exp(i\,\gamma_5 A)\,\Psi],$$

or further:

(8) $$\boxed{L = ik[\Psi^+\Psi(\Psi^+\dot{\Psi} - \dot{\Psi}^+\Psi) - \Psi^+\gamma_5\Psi(\Psi^+\gamma_5\dot{\Psi} - \dot{\Psi}^+\gamma_5\Psi)].}$$

One then infers from the equations of motion for a relativistic rotating body, namely:

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{\Psi}} - \frac{\partial L}{\partial \Psi} = 0,$$

that:

$$\boxed{(\Psi^+\Psi - \Psi^+\gamma_5\Psi\cdot\gamma_5)\dot{\Psi} + (\Psi^+\dot{\Psi} - \Psi^+\gamma_5\dot{\Psi}\cdot\gamma_5)\Psi = 0.}$$

However, the study of those equations is not the goal of the present work.

3. Equations of a continuous field of rotating bodies (fluid with spin). – We first remark that the expression (8) for the Lagrangian is lacking two factors:

The factor $\dfrac{1}{(\Psi^+\Psi)^2 + (i\,\Psi^+\gamma_5\Psi)^2}$, which yields the expression for the $\omega_{\mu\nu}$,

The factor $\sqrt{-(\Psi^+\Psi)^2 - (i\,\Psi^+\gamma_5\Psi)^2}$, which is found implicitly in the expression for the kinetic moment.

These two factors, which are equal to unity, do not appear for an isolated body.

Likewise, we must add the term m ([5]), which is the rest energy of the body, to the Lagrangian. That constant term will not enter into the equation for the isolated body either, but it will play a role in the equation of the fluid.

We then write the Lagrangian (8) in the equivalent form:

G. PETIAU, J. Math. pures et app. **112** (1947).

([5]) We take the speed of light to be equal to unity.

$$(9) \qquad L = \frac{-ik}{\sqrt{-(\Psi^+\Psi)^2 - (i\,\Psi^+\gamma_5\Psi)^2}}[\Psi^+\Psi(\Psi^+\dot{\Psi} - \dot{\Psi}^+\Psi) - \Psi^+\gamma_5\Psi(\Psi^+\gamma_5\dot{\Psi} - \dot{\Psi}^+\gamma_5\Psi)] + m\sqrt{-(\Psi^+\Psi)^2 - (i\,\Psi^+\gamma_5\Psi)^2}\,.$$

We would like to describe a fluid whose molecules are bodies in motion that all have the same mass m and spin k, and they do not interact with each other. That would be a fluid that generalizes the "pure matter" schema ([6]).

In order to write the Lagrangian of our fluid, it will suffice to multiply the Lagrangian (9) by an invariant density D. The form of the Lagrangian will then show that it would be equivalent to multiply the spinor Ψ by $\sqrt{D}$. The fluid is then described by the Lagrangian (9), in which one takes the spinor to not be normalized.

The derivative with respect to proper time will now be the Lagrangian derivative along the streamlines. We write:

$$\dot{\Psi} = u^\mu\,\partial_\mu\,\Psi,$$

in which u_μ is the unit velocity quadri-vector, namely:

$$(10) \qquad u_\mu = \frac{\Psi^+\gamma_\mu\Psi}{\sqrt{-(\Psi^+\Psi)^2 - (i\,\Psi^+\gamma_5\Psi)^2}}\,.$$

If one introduces (10) into (9) then one will obtain the fluid Lagrangian:

$$(11) \qquad \boxed{L = \frac{ik\Psi^+\gamma_\mu\Psi}{(\Psi^+\Psi)^2 + (i\Psi^+\gamma_5\Psi)^2}[\Psi^+\Psi\cdot\Psi^+[\partial_\mu]\Psi - \Psi^+\gamma_5\Psi\cdot\Psi^+\gamma_5[\partial_\mu]\Psi + m\sqrt{-(\Psi^+\Psi)^2 - (i\Psi^+\gamma_5\Psi)^2}\,.}$$

The Lagrangian is obviously gauge-invariant, and we apply the general results from the classical theory of fields ([7]) to it, while refraining from writing down the equations of motion in spinorial form.

We first write the expression for the current:

$$j_\mu = \frac{i}{2k}\left[\frac{\partial L}{\partial\Psi_\mu}\Psi - \Psi^+\frac{\partial L}{\partial\Psi_\mu^+}\right].$$

A simple calculation gives:

$$j_\mu = \Psi^+\,\gamma_\mu\,\Psi,$$

([6]) One knows that the energy tensor of such a fluid is written $T_{\mu\nu} = \rho\,u_\mu\,u_\nu$. It is a perfect fluid without pressures. Cf., A. LICHNEROWICZ, *Théories relativistes de la Gravitation et de l'électromagnétisme*, Paris, Masson, 1954.

([7]) PAULI, Rev. Mod. Phys. **13** (1931), 203.

as one must have, from (10).

One then knows that by virtue of the field equations:

$$\partial_\mu \frac{\partial L}{\partial \Psi_\mu} - \frac{\partial L}{\partial \Psi} = 0,$$

one will have conservation of current, namely:

$$\partial_\mu j^\mu = 0,$$

which can also be written:

(12) $$\boxed{\partial_\mu (D u^\mu) = \dot{D} = 0.}$$

The homogeneity of the Lagrangian also implies, after a somewhat long calculation:

$$\Psi^+ \left(\partial_\mu \frac{\partial L}{\partial \Psi^+_\mu} - \frac{\partial L}{\partial \Psi^+} \right) + \left(\partial_\mu \frac{\partial L}{\partial \Psi_\mu} - \frac{\partial L}{\partial \Psi} \right) \Psi = -2L = 0.$$

The Lagrangian is then zero in the course of motion.

We shall now write the energy-impulse tensor.

One has:

$$T_{\mu\nu} = \frac{\partial L}{\partial \Psi_\nu} \Psi_\mu + \Psi^+_\mu \frac{\partial L}{\partial \Psi^+_\nu} - L\, g_{\mu\nu}\,.$$

One has $L = 0$ here, and a simple calculation will give us:

$$T_{\mu\nu} = \frac{ik\,\Psi^+ \gamma_\nu \Psi}{(\Psi^+\Psi)^2 + (i\Psi^+\gamma_5\Psi)^2} (\Psi^+\Psi \cdot \Psi^+[\partial_\mu]\, \Psi - \Psi^+\gamma_5\, \Psi \; \mathrm{Y} \; \Psi^+\gamma_5\, [\partial_\mu]\, \Psi).$$

One sees that this tensorial density is the product of two factors:

(13) $$\boxed{T_{\mu\nu} = k_\mu u_\nu\,,}$$

namely, the unitary velocity vector:

$$u_\nu = \frac{\Psi^+ \gamma_\nu \Psi}{\sqrt{-(\Psi^+\Psi)^2 - (i\Psi^+\gamma_5\Psi)^2}}\,,$$

and the vectorial density of energy-impulse:

(14) $$\boxed{k_\mu = \frac{-ik}{\sqrt{-(\Psi^+\Psi)^2-(i\Psi^+\gamma_5\Psi)^2}}(\Psi^+\Psi\cdot\Psi^+[\partial_\mu]\Psi-\Psi^+\gamma_5\Psi\cdot\Psi^+\gamma_5[\partial_\mu]\Psi).}$$

k_μ is not collinear with u_μ, and $T_{\mu\nu}$ has exactly the form that was proposed by Weyssenhoff ([8]).

We shall return to the vector k_μ at the end of this discussion.

The antisymmetric part of $T_{\mu\nu}$ will be provided to us by the Belinfante-Rosenfeld relation ([9]):

$$T_{[\mu\nu]} = \frac{i}{2}\partial^\lambda\left[\frac{\partial L}{\partial\Psi_\lambda}\gamma_\mu\gamma_\nu\Psi-\Psi^+\gamma_\mu\gamma_\nu\frac{\partial L}{\partial\Psi^+_\lambda}\right].$$

It becomes:

$$T_{[\mu\nu]} = -k\,\partial^\lambda\left[\frac{\Psi^+\gamma_\lambda\Psi}{(\Psi^+\Psi)^2+(i\Psi^+\gamma_5\Psi)^2}(\Psi^+\Psi\cdot\Psi^+\gamma_\mu\gamma_\nu\Psi-i\Psi^+\gamma_5\Psi\cdot\overline{\Psi^+\gamma_\mu\gamma_\nu\Psi})\right],$$

$$T_{[\mu\nu]} = k\partial^\lambda[u_\lambda(\overline{\sigma_\mu u_\nu-\sigma_\nu u_\mu})],$$

(15) $$\boxed{T_{[\mu\nu]} = \partial^\lambda(u_\lambda K_{[\mu\nu]}).}$$

Since $K_{\mu\nu}$ is a density, that will simply be the derivative of $K_{\mu\nu}$ along the streamline, so one has:

(16) $$T_{[\mu\nu]} = k_\mu\,u_\nu - k_\nu\,u_\mu = \dot{K}_{\mu\nu}.$$

However, by virtue of the equations of motion, one will also have:

(17) $$\partial^\nu\,T_{\mu\nu} = \partial^\nu\,(k_\mu\,u_\nu) = \dot{k}_\mu = 0.$$

Equation (16) can then be written:

(18) $$\boxed{\partial^\lambda[u_\lambda(x_\mu k_\nu - x_\nu k_\mu)]+\partial^\lambda(u_\lambda K_{\mu\nu}) = 0.}$$

It thus emerges from the equations of motion that the total kinetic moment (orbital + spin) is conserved, which Weyssenhoff postulated in order to deduce (15).

If we then contract (16) by u_ν, while taking into account the fact that $u_\mu\,u^\mu = -1$ then we will get:

(19) $$k_\mu = -k_\lambda\,u^\lambda\cdot u_\mu - \dot{K}_{\mu\nu}\,u^\nu,$$

However, if one takes (14) into account then the Lagrangian (11) can be written:

([8]) WEYSSENHOFF, Acta Physica Polonica **9** (1947).
([9]) Cf., PAULI, *loc. cit.*

$$L = k_\lambda u^\lambda + m D,$$

in which $D = \sqrt{-(\Psi^+\Psi)^2 - (i\Psi^+\gamma_5\Psi)^2}$ is the invariant density of the fluid that was introduced at the outset.

Since the Lagrangian is zero in the course of motion, we will have:

$$-k_\lambda u^\lambda = mD = \mu_0, \tag{20}$$

which is the proper mass density, as Weyssenhoff has postulated.

From (12), we will have, moreover:

$$\dot{\mu}_0 = m\,\partial^\mu(u_\mu D) = 0; \tag{21}$$

i.e., the conservation of the proper mass density of the fluid.

On the other hand:

$$K_{\mu\nu}\, u^\nu = 0,$$

which is a relation that expresses simply that $K_{i4} = 0$ in the proper system in an invariant fashion ([10]).

One deduces from this that:

$$-\dot{K}_{\mu\nu} u^\nu = K_{\mu\nu} \dot{u}^\nu .$$

If one takes (20) into account then one can write (19) as:

$$k_\mu = \mu_0 u_\mu + K_{\mu\nu}\dot{u}^\nu . \tag{22}$$

That is the Weyssenhoff decomposition.

If we introduce (22) into (16) then we will have:

$$\dot{K}_{\mu\nu} = K_{\mu\lambda}\dot{u}^\lambda u_\nu - K_{\nu\lambda}\dot{u}^\lambda u_\mu . \tag{23}$$

If we introduce (22) into (17) then we will have:

$$\dot{\mu}_0\, u_\mu + \mu_0\, \dot{u}_\mu + \dot{K}_{\mu\nu}\dot{u}^\nu + K_{\mu\nu}\dot{u}^\nu = 0.$$

However, from (21), $\dot{\mu}_0 = 0$, and from (23), $\dot{K}_{\mu\nu}\dot{u}^\nu = 0$ ([11]).

One will then have:

([10]) It is verified in an obvious way by taking into account the fact that:

$$K_{\mu\nu} = k\ \overline{\sigma_\mu u_\nu - \sigma_\nu u_\mu} .$$

([11]) Because $u_\mu u^\mu = -1$ and $K_{\mu\nu} = -K_{\nu\mu}$.

(24) $$\boxed{\mu_0 \dot{u}_\mu + K_{\mu\nu} \ddot{u}^\nu = 0.}$$

Equations (23) and (24) are the Mathisson equations ([12]).

4. – It is interesting to compare these results with the expressions for the dynamical quantities that characterize the hydrodynamical representation for the Dirac equation ([13]). From the concept that inspired the present work, they represent, in effect, waves of weak amplitude that propagate in the ether, which is considered to be a fluid of tops. If one represents those waves by a fictitious fluid then one can easily exhibit a close analogy between the laws of a real fluid that supports waves and the laws of a fictitious fluid that represents those waves.

Start with the Dirac Lagrangian in terms of spinors:

$$L = -\frac{ih}{2}(\Psi^+ [\partial_\mu]\, \gamma_\mu \Psi - 2\, K\, \Psi^+ \Psi).$$

One infers the wave equations $\partial_\mu \gamma_\mu \Psi + K \Psi = 0$ and $\partial_\mu \Psi^+ \gamma_\mu + K \Psi^+ = 0$ from this, which gives:

$$L = 0.$$

The current is given by:

$$J_\mu = i\left(\frac{\partial L}{\partial \Psi_\mu}\Psi - \Psi^+ \frac{\partial L}{\partial \Psi^+_\mu}\right),$$

namely:

$$\boxed{J_\mu = \hbar\, \Psi^+ \gamma_\mu \Psi,}$$

and the canonical energy-impulse tensor is given by:

$$T_{\mu\nu} = \frac{\partial L}{\partial \Psi_\nu}\Psi_\mu + \Psi^+_\mu \frac{\partial L}{\partial \Psi^+_\nu} - \delta_{\mu\nu} L = \frac{\partial L}{\partial \Psi_\nu}\Psi_\mu + \Psi^+_\mu \frac{\partial L}{\partial \Psi^+_\nu},$$

namely:

$$\boxed{T_{\mu\nu} = -\frac{i\hbar}{2}\Psi^+[\partial_\mu]\gamma_\nu \Psi,} \qquad \text{with} \qquad \partial_\nu T_{\mu\nu} = 0.$$

One then writes the current $J_\mu = \rho\, u_\mu$ upon introducing the unit velocity u_μ of the fluid ($u_\mu u_\mu = -1$) and its density ρ :

([12]) MATHISSON, Acta Physica Polonica **6** (1937).
WEYSSONHOFF, *loc. cit.*
Louis de BROGLIE, *Th. des Part. de spin 1/2* .
MÖLLER, *loc. cit.*

([13]) Report no. 6 to this seminar, which was presented on 20 December 1955 by J. P. VIGIER, from the note of HALBWACHS, LOCHAK, and VIGIER, C. R. Acad. Sc. Paris, **241** (1955), pp. 692.

$$J_\mu J_\mu = -\rho^2 = \hbar^2 \Psi^+ \gamma_\mu \Psi \cdot \Psi^+ \gamma_\mu \Psi,$$

$$\boxed{\rho = \hbar\sqrt{-\Psi^+\gamma_\mu\Psi\cdot\Psi^+\gamma_\mu\Psi}.}$$

One knows that:

$$\Psi^+ \gamma_\mu \Psi \cdot \Psi^+ \gamma_\mu \Psi = (\Psi^+ \Psi)^2 + (i\,\Psi^+\gamma_5\Psi)^2.$$

Hence, the density will be:

$$\rho = \hbar\sqrt{-(\Psi^+\Psi)^2 - (i\Psi^+\gamma_5\Psi)^2}\ .$$

One naturally has:

$$u_\mu = \frac{\Psi^+\gamma_\mu\Psi}{\sqrt{-(\Psi^+\Psi)^2 - (i\Psi^+\gamma_5\Psi)^2}}\,,$$

and likewise $\partial_\mu(\rho\, u_\mu) = \dot{\rho} = 0$ along a streamline.

Those are the *same* expressions as the ones that we found for the fluid of tops [viz., equations (10) and (12)].

We then perform a decomposition of the energy-impulse tensor $T_{\mu\nu}$ (which is not symmetric) in such a fashion as to make an impulse k_μ appear that is not collinear with the velocity, which conforms to the ideas of Möller and Weyssenhoff. One must introduce a tensor $\Theta_{\mu\nu}$ of internal stresses that we subject to the condition $\Theta_{\mu\nu}\, u_\nu = 0$, which signifies that all of those components are in proper space, in accord with the usual concepts of hydrodynamics:

$$T_{\mu\nu} = k_\mu\, u_\nu + \Theta_{\mu\nu}.$$

If we contract this with u_ν then we will get:

$$T_{\mu\nu}\, u_\nu = -\, k_\mu\,,$$

since $u_\nu\, u_\nu = -1$ and that $\Theta_{\mu\nu}\, u_\nu = 0$.

We can then calculate k_μ :

$$\boxed{-k_\mu = -\frac{i\hbar}{2}\Psi^+[\partial_\mu]\gamma_\nu\Psi\cdot\frac{\Psi^+\gamma_\nu\Psi}{\sqrt{-(\Psi^+\Psi)^2 + (\Psi^+\gamma_5\Psi)^2}}.}$$

We use the Kofink relation:

$$\Psi^+[\partial_\mu]\,\gamma_\nu\Psi\cdot\Psi^+\gamma_\nu\Psi = \Psi^+\Psi\cdot\Psi^+[\partial_\mu]\,\Psi - \Psi^+\gamma_5\,\Psi\cdot\Psi^+[\partial_\mu]\,\gamma_5\Psi$$

and get:

$$k_\mu = \frac{i\hbar}{2}\frac{\Psi^+\Psi\cdot\Psi^+[\partial_\mu]\Psi - \Psi^+\gamma_5\Psi\cdot\Psi^+[\partial_\mu]\gamma_5\Psi}{\sqrt{-(\Psi^+\Psi)^2 + (\Psi^+\gamma_5\Psi)^2}}\,.$$

Here again, we recover *the same expression* for the impulse vector as for the fluid of tops [viz., equation (14)]. By contrast, whereas the fluid of tops is a Weyssenhoff fluid with no internal stresses, the representative fluid of the Dirac equation does involve internal stresses:

$$k_\mu\, u_\nu = \frac{i\hbar^3}{2\rho^2}\Psi^+ [\partial_\mu]\ \gamma_\lambda \Psi \cdot \Psi^+\gamma_\lambda \Psi \cdot \Psi^+\gamma_\nu \Psi \neq T_{\mu\nu}\,,$$

$$T_{\mu\nu} - k_\mu\, u_\nu = \Theta_{\mu\nu} = -\frac{i\hbar}{2}\Psi^+ [\partial_\mu]\ \gamma_\nu \Psi - \frac{i\hbar^3}{2\rho^2}\Psi^+ [\partial_\mu]\ \gamma_\lambda \Psi \cdot \Psi^+\gamma_\lambda \Psi \cdot \Psi^+\gamma_\nu \Psi,$$

$$\boxed{\Theta_{\mu\nu} = -\frac{i\hbar^3}{2\rho^2}\Psi^+[\partial_\mu]\gamma_\lambda\Psi\left\{\delta_{\mu\nu}\frac{\rho^2}{\hbar^2} + \Psi^+\gamma_\lambda\Psi\Psi^+\gamma_\nu\Psi\right\},}$$

and one has, in fact, $\Theta_{\mu\nu}\, u_\nu = 0$.

On the other hand, the angular momentum is given by $x_\nu\, T_{\mu\lambda} - x_\mu\, T_{\mu\lambda}$. If we take its divergence then we will get:

$$\partial_\lambda\, (x_\nu\, T_{\mu\lambda} - x_\mu\, T_{\mu\lambda}) = x_\nu\, \partial_\lambda\, T_{\mu\lambda} - x_\mu\, \partial_\lambda\, T_{\nu\lambda} + \delta_{\lambda\nu}\, T_{\mu\lambda} - \delta_{\lambda\mu}\, T_{\nu\lambda}\,.$$

The first two terms are zero, since $T_{\mu\lambda}$ is conservative. The last two give $T_{\mu\nu} - T_{\nu\mu} = 2\, T_{[\mu\nu]}$, which is not zero, since $T_{\mu\nu}$ is not symmetric.

The angular momentum is not conservative, since it takes into account only the orbital rotation, while the Dirac fluid also involves a proper rotation.

The proper angular momentum (viz. angular spin) is given by the Belinfante formula:

$$f_{[\lambda\mu]\nu} = \mathrm{Re}\left\{\frac{\partial L}{\partial \Psi_\nu} I_{[\lambda\mu]}\mathrm{op.}\Psi\right\}.$$

Here:

$$I_{[\lambda\mu]} = \tfrac{1}{2}(\gamma_\lambda\, \gamma_\mu - \gamma_\lambda\, \gamma_\lambda),$$

so:

$$f_{[\lambda\mu]\nu} = -\frac{i\hbar}{4}\Psi^+\ \gamma_\nu\, (\gamma_\lambda\, \gamma_\mu - \gamma_\lambda\, \gamma_\lambda)\ \Psi = -\frac{i\hbar}{2}\Psi^+\ \gamma_\nu\, (\gamma_\lambda\, \gamma_\mu - \delta_{\lambda\mu})\ \Psi.$$

It is easy to see that the only non-zero components are the ones for which $\lambda \neq \mu \neq \nu$, and that the tensor is completely antisymmetric:

$$f_{[\lambda\mu\nu]} = -\frac{i\hbar}{2}\Psi^+\ \gamma_\nu\, \gamma_\lambda\, \gamma_\mu\, \Psi.$$

That is indeed the form that was given above for the spin. From the Belinfante method, one must form the tensor:

$$\begin{aligned} S_{[\lambda\mu]\nu} = \tfrac{1}{2}(f_{\lambda\mu\nu} + f_{\nu\lambda\mu} - f_{\mu\nu\lambda}) &= \tfrac{1}{2}(-i\hbar)\,\Psi^+\ (\gamma_\lambda\, \gamma_\mu\, \gamma_\nu + \gamma_\nu\, \gamma_\lambda\, \gamma_\mu - \gamma_\mu\, \gamma_\nu\, \gamma_\lambda)\ \Psi, \\ &= \tfrac{1}{2}(-i\hbar)\,\Psi^+\ \gamma_\lambda\, \gamma_\mu\, \gamma_\nu\, \Psi, \end{aligned}$$

which gives $\frac{1}{2} f_{\lambda\mu\nu}$, so we can write:

$$\sigma_{\lambda\mu\nu} = -\frac{i\hbar}{2} \Psi^{+} \gamma_\lambda \gamma_\mu \gamma_\nu \Psi,$$

or, upon taking the dual:

$$\boxed{\sigma_\mu = \frac{\hbar}{2} \Psi^{+} \gamma_5 \gamma_\rho \Psi.}$$

The divergence of that tensor $\partial_\nu \sigma_{\lambda\mu\nu}$ provides the energy-impulse tensor of proper rotation $t_{\lambda\mu}$, which $T_{\lambda\mu} + t_{\lambda\mu}$ is both conservative and symmetric. $t_{\lambda\mu}$, which is antisymmetric in the case of the Dirac equation ($t_{\mu\nu} = -T_{[\mu\nu]}$), is easily transformed into:

$$t_{\lambda\mu} = \partial_\nu \sigma_{[\lambda\mu\nu]} = \partial_\nu \delta_{\lambda\mu\nu\rho} \sigma_\rho ,$$

if one lets σ_ρ denote the dual of $\sigma_{[\lambda\mu\nu]}$, which is the spin, properly speaking.

$$t_{\lambda\mu} = \tfrac{1}{2}(\partial_\nu \delta_{\lambda\mu\nu\rho} \sigma_\rho + \partial_\rho \delta_{\lambda\mu\nu\rho} \sigma_\nu) = \tfrac{1}{2} \delta_{\lambda\mu\nu\rho} (\partial_\nu \sigma_\rho - \partial_\rho \sigma_\nu),$$

so one has the well-known result that relates to the Dirac wave equation:

$$t_{\lambda\mu} = \tfrac{1}{2}\overline{(\partial_\lambda \sigma_\mu - \partial_\mu \sigma_\lambda)} = -T_{[\mu\nu]} ,$$

and the energy-impulse of proper rotation is represented by the dual of the rotation of the spin.

We shall now justify that hydrodynamical representation of the Dirac waves [which is equivalent to the one that was proposed by Takabayasi ([14])], by studying the dynamics of a fluid droplet that is contained in a spacelike volume. One shows that the droplet obeys three dynamical laws: conservation of mass, quantity of motion, and kinetic moment, which justifies completely the interpretation of the four relativistic quantities ρ, u_μ, K_μ, and σ_μ as the mass, velocity, impulse density, and internal rotation density, respectively, as well as the interpretation of the proper space tensor $\Theta_{\mu\nu}$ as the internal stress tensor. A

([14]) Takabayasi decomposed the energy-impulse tensor into:

$$T_{\mu\nu} = k_\mu u_\nu + \frac{c\hbar}{2} P \,(s_\nu \partial_\mu A + i\, \delta_{\nu\alpha\beta\gamma} u^\alpha s^\beta \partial_\mu u^\gamma) \qquad (*),$$

in which s^μ is the unit vector that is collinear with the spin ($s_\mu u_\mu = 0$, $s_\mu s_\mu = -1$). It then represents $\Theta_{\mu\nu}$, and that decomposition is equivalent to ours, so Takabayasi's k_μ is the same as ours (**). One then obtains another expression for the internal stresses:

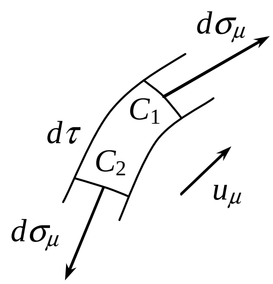
$$\Theta_{\mu\nu} = \frac{c\hbar}{2} \frac{P}{\sqrt{-\sigma_\lambda \sigma_\lambda}} [\sigma_\nu \partial_\mu A + i\, \delta_{\nu\alpha\beta\gamma} u^\alpha \sigma^\beta \partial_\mu u^\gamma] .$$

The second term, which one can write as $i\, k_{\mu\gamma} \partial_\mu u^\gamma$ or $-i\, u^\gamma \partial_\mu k_{\mu\gamma}$, is classical, and characterized all spinning fluids. The first one, in which A is defined by starting with the two invariants $\tan A = \frac{i\Psi^{+}\gamma_5 \Psi}{\Psi^{+}\Psi}$, is proper to the Dirac fluid, and seems difficult to interpret.

(*) TAKABAYASI, Nuovo Cimento III, 2 (February 1956).

(**) HALBWACHS, LOCHAK, VIGIER, C. R. Acad. Sc. Paris **241** (1955), pp. 744.

moving element of the droplet will be represented by a relativistic domain that is bounded by an infinitely-thin current hyper-tube and two proper space hyper-endcaps C_1, C_2 that are orthogonal to the current and infinitely close to each other.



1) Integrate the equation for conservation of current in that domain:

$$\iiint\int \partial_\mu(\rho u_\mu)\,d\omega = 0 = \iiint_S \rho u_\mu\, d\sigma_\mu \qquad \text{(Gauss's theorem)}$$

in which $d\sigma_\mu$ is the hypersurface element.

Decompose the hypersurface S into three parts. $d\sigma_\mu$ is perpendicular to *um* on the hyper-wall of the tube, $u_\mu\, d\sigma_\mu = 0$, so the integral is zero.

One the hyper-endcaps, one has:

$$C_1: \quad d\sigma_\mu = u_\mu\, dV_0\,, \qquad u_\mu\, d\sigma_\mu = u_\mu\, u_\mu\, dV_0 = -\,dV_0\,,$$

$$C_2: \quad d\sigma_\mu = -\,u_\mu\, dV_0\,, \qquad u_\mu\, d\sigma_\mu = -\,u_\mu\, u_\mu\, dV_0 = dV_0\,,$$

in which dV_0 is the *proper* volume element.

$$\iiint\int \partial_\mu(\rho u_\mu)\,d\omega = \iiint_{C_2} \rho\, dV_0 - \iiint_{C_1} \rho\, dV_0 = 0.$$

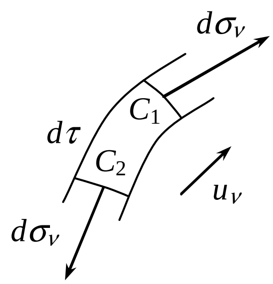
The two integrals represent the masses m_2 and m_1 of the droplet at the two instants considered. That mass is indeed constant.

2) Now integrate the conservation equation for the canonical tensor, namely, $\partial_\nu T_{\mu\nu} = 0$:

$$\iiint\int \partial_\nu T_{\mu\nu}\, d\omega = \iiint\int \partial_\nu(k_\mu u_\nu)\,d\omega + \iiint\int \partial_\nu \Theta_{\mu\nu}\, d\omega = 0.$$

Apply Gauss's theorem to the first term; one will get:

$$\iiint\int \partial_\nu(k_\mu u_\nu)\,d\omega = \iiint_S k_\mu u_\nu\, d\sigma_\nu\,.$$



One can decompose the hypersurface S as before on the hyper-wall of the tube and get:

$$u_\nu \, d\sigma_\nu = 0,$$

so the integral will be zero. All that will remain are the hyper-endcap terms, namely:

$$\iiint_{C_1} k_\mu u_\nu \, d\sigma_\nu + \iiint_{C_2} k_\mu u_\nu \, d\sigma_\nu \ ,$$

with

$$(u_\nu d\sigma_\nu)_{C_1} = - dV_0 \ , \qquad (u_\nu d\sigma_\nu)_{C_2} = dV_0 \ ,$$

so:

$$\iiint_S k_\mu u_\nu \, d\sigma_\nu = - \iiint_{C_1} k_\mu u_\nu \, d\sigma_\nu + \iiint_{C_2} k_\mu u_\nu \, d\sigma_\nu = \frac{d}{d\tau}\left[\iiint_{V_0} k_\mu dV_0\right] d\tau$$
$$= \left[\iiint_{V_0} \dot{k}_\mu dV_0\right] d\tau \ ,$$

in which, $\dot{k}_\mu$ denotes the derivative of the impulse density along the streamline.

One then has, by definition:

$$\iiiint \partial_\nu T_{\mu\nu} \, d\omega = \iiiint \partial_\nu \Theta_{\mu\nu} \, d\omega + \left[\iiint_{V_0} \dot{k}_\mu dV_0\right] d\tau = 0.$$

The interpretation is immediate: The first term can be written:

$$\int \left[\iiint \partial_\nu \Theta_{\mu\nu} \, dV_0\right] d\tau$$

or even $\left[\iiint \partial_\nu \Theta_{\mu\nu} \, dV_0\right] d\tau$, if the two hyper-endcaps are sufficiently close. $-\partial_\mu \Theta_{\mu\nu}$ is classically a force density of internal stresses f_μ, and one will finally have:

$$\iiint_{V_0} \dot{k}_\mu dV_0 = \iiint_{V_0} f_\mu dV_0 \ ,$$

or upon integrating:

$$\boxed{\dot{K}_\mu = F_\mu .}$$

That is the quantity of motion theorem for the droplet, taken as a whole.

3) Integrate the equation for the conservation of the moment of the *total* energy-impulse tensor $T_{\mu\nu} + t_{\mu\nu}$, while using the same method:

$$\iiiint \partial_\nu [k_\nu (T_{\mu\nu} + t_{\mu\nu}) - k_\mu (T_{\nu\lambda} + t_{\nu\lambda})] d\omega = 0,$$

$$2 \iiiint T_{[\mu\nu]}\, d\omega + 2 \iiiint t_{[\mu\nu]}\, d\omega = 0,$$

$$2 \iiiint T_{[\mu\nu]}\, d\omega = \iiiint (k_\mu u_\nu - k_\nu u_\mu)\, d\omega + 2 \iiiint \Theta_{[\mu\nu]}\, d\omega,$$

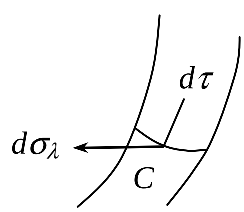
$$2 \iiiint t_{\mu\nu}\, d\omega = 2 \iiiint \partial_\lambda \sigma_{\mu\nu\lambda}\, d\omega = 2 \iiint_S \sigma_{\mu\nu\lambda}\, d\sigma_\lambda$$

$$= 2 \iiint_P \sigma_{\mu\nu\lambda}\, d\sigma_\lambda - 2 \frac{d}{d\tau} \left[\iiint_{V_0} \sigma_{\mu\nu\lambda}\, u_\lambda\, dV_0 \right] d\tau .$$

One can identify the tensor 2 $\sigma_{[\mu\nu\lambda]}\, u_\lambda$ with the Weyssenhoff spin $k_{[\mu\nu]}$. It is a proper space tensor because one obviously has:

$$k_{\mu\nu}\, u_\nu = 0.$$

On the other hand, one can specify the vector $d\sigma_\lambda$ on the hyper-wall by $d\sigma_\lambda = ds_{0\lambda}\, d\tau$ ([15]), in which $d\sigma_\lambda$ is in proper space, as well as $ds_{0\lambda}$, and it then represents the infinitesimal proper space vector that is normal to the proper space hyper-contour C, which is nothing but the surface Σ_0 of the droplet in proper space.



The product $\sigma_{\mu\nu\lambda}\, ds_{0\lambda}$ is then the element of spin flux that crosses the surface of the droplet in proper space. One can, moreover, write $\delta_{\mu\nu\lambda\rho}\, \sigma_\rho\, ds_{0\lambda}$ or $\overline{\sigma_\mu ds_{0\mu}}$.

([15]) We have $d\sigma_\lambda = \overline{[dx^\mu\, dx^\nu\, dx^\rho]} = \delta_{\lambda\mu\nu\rho}\, [dx^\mu\, dx^\nu\, dx^\rho]$. If we take the proper axes then $d\sigma_\lambda = \delta_{\lambda\mu\nu\rho}$ $[dx_1^\mu dx_2^\nu u^\rho d\tau]$, with $u^4 = ic$, $u^1 = u^2 = u^3 = 0$, $dx_1^4 = dx_2^4 = 0$, so:

$$d\sigma_\lambda = \delta_{\lambda\mu\nu}\, [dx_1^\mu dx_2^\nu d\tau] = ic\, d\tau\, \delta_{\lambda\mu\nu}\, [dx_1^\mu dx_2^\nu], \text{ with } \lambda, \mu, \nu \neq 4.$$

which one can write:

$$d\sigma_i = d\tau\, \varepsilon_{ijk}\, [dx_1^J dx_2^K] = d\tau\, \overline{[dx^J dx^K]},$$

in which the dual is a dual with three indices in proper space: $ic\ \delta_{iJK} = \varepsilon_{iJK}$. In fact, the proper surface element is $dS_{0i} = \overline{[dx^J dx^K]}$, which indeed gives $d\sigma_\lambda = ds_{0i} \cdot d\tau\, (\lambda \neq K)$.

One then ultimately has:

$$2 \iiiint T_{[\mu\nu]}\, d\omega + 2\iiiint t_{\mu\nu}\, d\omega = 0$$
$$= \int d\tau \iiint_{V_0} (k_\mu u_\nu - k_\nu u_\mu)\, dV_0 + 2\int d\tau \iiint_{V_0} \Theta_{\mu\nu}\, dV_0$$
$$+ 2\int d\tau \iiint_{\Sigma_0} \overline{\sigma_\nu\, ds_{0\mu}} - \frac{d}{d\tau}\left[\iiint_{V_0} k_{\mu\nu}\, dV_0\right] d\tau = 0.$$

If the time interval between C and C' is sufficiently small then one can simply it by $d\tau$, and what will remain (upon setting $dk_{\mu\nu} / d\tau = \dot{k}_{\mu\nu}$) will be:

$$\iiint_{V_0} (k_\mu u_\nu - k_\nu u_\mu)\, dV_0 + 2\iiint_{V_0} \Theta_{[\mu\nu]}\, dV_0 + 2\iiint_{\Sigma_0} \overline{\sigma_\nu\, ds_{0\mu}} - \iiint_{V_0} \dot{k}_{\mu\nu}\, dV_0 = 0,$$

or finally, upon using the Weyssenhoff notations for the proper-volume integrals over the droplet:

$$\boxed{K_\mu u_\nu - K_\nu u_\mu = \dot{K}_{\mu\nu} - 2\iiint_{V_0} \Theta_{[\mu\nu]} dV_0 + 2\iint_{\Sigma_0} \overline{\sigma_\mu ds_{0\nu}}.}$$

This formula generalizes the classical Weyssenhoff formula to the case of a fluid that is endowed with internal stresses.

The last two terms characterize what one can call the *torsion* of the fluid. The existence of internal stresses and their asymmetric character must necessarily contribute to the dynamics of the rotating droplet, on the one hand, in the volume integral of the asymmetric stresses that follow the collective rotation of the droplet, step-by-step, and on the other hand, in the surface effects that are coupled to the *proper* rotation of all the external fluid elements that touch the droplet immediately and which compel it to rotate while rolling on them.

One can conclude from the preceding calculations that the Dirac equation can be *represented* by a coherent hydrodynamical model that is the same as the one that Takabayasi proposed. The fluid must be considered as possessing internal stresses and an infinitesimal vorticial motion. A droplet obeys the three fundamental laws of classical dynamics: conservation of mass, the quantity of motion theorem, and the kinetic moment theorem.

By contrast, the representative fluid differs from the real fluid by which we are forced to represent the ether, which is a "pure matter" fluid that is devoid of internal stresses, conforming to the Weyssenhoff model.

The comparison between the two fluids is expressed by:

$T_{\mu\nu} = k_\mu\, u_\nu$ for Weyssenhoff,

$T_{\mu\nu} = k_\mu\, u_\nu + \Theta_{\mu\nu}$ for Dirac-Takabayasi,

$T_{[\mu\nu]} = \dot{k}_{\mu\nu}$ for Weyssenhoff,

$$T_{[\mu\nu]} = \dot{k}_{\mu\nu} + \lim_{\Sigma_0 \to 0}\left[\frac{1}{V_0}\iint_{\Sigma_0} \overline{\sigma_\nu \, ds_{0\mu}}\right] = \tfrac{1}{2}(\overline{\partial_\nu \sigma_\mu - \partial_\mu \sigma_\nu}) \qquad \text{for Dirac-Takabayasi.}$$

5. Remarks on the preceding results:

1) We have shown that a spinning fluid is nothing but an ordinary fluid whose molecules are endowed with a *classical* motion of proper rotation. The introduction of a spin density is justified when one goes to a sufficiently large scale that the fluid can be considered to be continuous. That density obviously cannot be reduced to a rotation, since it is a macroscopic quantity that accounts for the vorticial motions that take place in the domains whose dimensions are considered to be negligible at the scale that one considers.

In the ether model that we propose, we have introduced some stable vorticial structures, which are considered to be rigid in the first approximation and are assumed to be *small in comparison to the dimensions of the electron.* We then have the right to treat the ether as a spinning fluid *at the quantum level*.

2) The expression (14) of the impulse vector k_μ is *the same as that of the impulse for the Dirac fluid* in Takabayasi's hydrodynamical representation.

That confirms the idea that was published already ([16]) that the Dirac fluid is a Weyssenhoff fluid with no internal stresses.

Our calculations show that the impulse of the Dirac fluid is simply that of a classical spinning fluid. Its expression has nothing to do with quantum theory, except to the extent that spin is a phenomenon that is at the basis for quantum theory.

One likewise sees that the differential operator that appears linearly in the expression for the impulse is the operator ∂_μ ; i.e., the one in quantum theory.

Indeed, if Π and Π^+ are conjugate to Ψ and Ψ^+, respectively, with respect to proper time then the impulse density will be given by:

$$\Pi^+ \underset{\rightarrow}{\partial}_\mu \Psi - \Psi^+ \underset{\leftarrow}{\partial}_\mu \Pi \, .$$

The operator ∂_μ then seems to stand out quite simply from the fact that the equations describe spinning particles ([17]) and the fact that one is dealing with spinors.

Similarly, one sees:

$$P_{\mu\nu} = (x_\mu \, \partial_\nu - x_\nu \, \partial_\mu)$$

appear in equation (18), which is the orbital kinetic moment operator, as in quantum theory.

([16]) HALBWACHS, LOCHAK, VIGIER, C. R. Acad. Sc. Paris **241** (1955), pp. 744.

([17]) *All* of the equations, including the Schrödinger equation, which is the non-relativistic approximation to the Dirac equation, in which the spin is assumed to be parallel to a fixed direction. Schrödinger's Ψ is, in reality, one component of a spinor.

One has:

$$x_\mu k_\nu - x_\nu k_\mu = \Pi^+ \underset{\longrightarrow}{P_{\mu\nu}} \Psi - \Psi^+ \underset{\longleftarrow}{P_{\mu\nu}} \Pi.$$

The latter remarks are, of course, quite fragmentary, but they do give one hope that one can give a simple interpretation to the correspondence that is commonly assumed to exist between physical quantities and the operators that are wrongfully attributed to quantum theory alone.

Finally, as we have seen, the theory of spinning fluids can be deduced from the Lagrangian:

$$L = \tfrac{1}{2} K^{\alpha\beta}\, \omega_{\alpha\beta} + D\, m_0\, c^2,$$

which expresses, in relativistic form ([18]), the idea that the energy of a body is provided essentially from its relativistic rotational motion in space-time, which is comparable to the hypothesis that Louis de Broglie put at the basis for wave mechanics, and which is expressed as follows:

"Whenever a material element, in the most general sense, in a reference system possesses an energy W, there will exist a periodic phenomenon in that system whose frequency ν is defined by the relation $W = h\nu$, in which h denotes Planck's constant."

([18]) Indeed, in the proper system of a vortex, and due to the equations of motion, one has:

$$L = -\tfrac{1}{2} k \cdot \mathbf{S} \frac{d\boldsymbol{\theta}}{dt} + m_0 c^2 = 0$$

(in which $d\boldsymbol{\theta} / dt$ denotes the angle of rotation around the kinetic moment $k\,\mathbf{s}$), which is analogous to the classical de Broglie relation:

$$E = h\nu_0 = m_0\, c^2.$$