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XXIII

THE PROPAGATION OF WAVES

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FOREWORD

This little book treats the propagation of waves in the strict sense. Propagation in the larger sense will be the subject of a much more extended work that will be prepared for this collection in collaboration with M. Taniuti.

G. B.

INTRODUCTION

A *field* is composed of a set of a certain number N of (real) functions of $n + 1$ variables (n space variables, one time variable) that are solutions of a first order (or reducing to that order) partial differential equations. The givens of a solution may or may not correspond with the givens of physical quantities; what is important is that they fix a *state*.

The notion of a wave is completely as general as that of a field, and since that generality is not without some damage to the comprehension of the term, it is important that it be well defined. A *wave* will be a perturbation that propagates from a perturbed state into an unperturbed one. This definition implies the existence of a boundary between these two states: a *wave surface*, upon the traversing of which there will exist discontinuities in the variables of the field itself (one then starts, more especially, with *shock waves*) or their derivatives. The wave realizes the passage from one state to the other. The second state is still also called a wave, which can lead to confusion, and explains the ultimate acceptance of the abuse of language that makes a wave synonymous with a field. However, if one may not conceive of a wave without a (pre-existing) field then one can just as well imagine a field without a wave (for example, a constant field).

The problem that is posed is the following one: given a certain field that defines the unperturbed state at each place and epoch, a perturbation is created at a certain instant (the initial instant) in a certain region; determine the time evolution of that perturbation. Since that evolution obviously depends upon the field equations, we are thus led to make hypotheses concerning the form of those equations. We therefore suppose that they constitute a *hyperbolic quasi-linear system* of first order partial differential equations, and to discard the possibility of shock waves *a priori* we assume that the discontinuities are of first order when one crosses the wave surface.

The problem of the study of the discontinuity then splits, as Lichnerowicz has remarked on the subject of the Einstein equations ([**14**], *op. cit.*), into two distinct problems: the problem of initial conditions and the problem of evolution.

At the initial instant the discontinuity in the normal (to the wave surface) derivative resolves into a (vectorial) sum of discontinuities; each of them then propagates according to a particular mode and a certain velocity. We specify that one is led to distinguish two types of velocity: the velocity normal to the wave surface and the radial velocity.

Once one has introduced the initial distribution, how do the waves propagate? The discontinuities might never disappear. They might, for revenge, become infinite. This singularity has meaning when a certain product $\theta\Phi$ of two functions of space and time is annulled. The annihilation of Φ is essentially related to the nonlinear character of the field, and one sees, by comparison with the study of one-dimensional fluid flow, that it corresponds to the appearance of shocks. It is then natural to say that the singularity that was described above manifests the birth of shocks. Hence, $\theta = 0$, which is not attributed to the nonlinearity becomes the expression of *linear shocks*, as opposed to the preceding ones that one qualified with the term *nonlinear*.

Nonlinear shock might not exist when the system of field equations *is completely exceptional*, and, similarly, there are linear shocks in this system when the system has *commuting matrices*. This last peculiarity obviously depends on the number of independent variables. When $n = 1$, it is clear that all of the systems have commuting matrices, and one understands why these linear shocks, which are closely connected with the geometry of the wave surface, never accompany plane waves in systems with two independent variables.

With that, one proceeds to account for the phenomena that are produced on the wave front. It is therefore permissible to confirm that a continuous solution of the field equations might not exist beyond the critical instant when one has $\theta\Phi = 0$. If we must give a concrete example, we cite that of a horizontal plane plate that quickly breaks off and falls through the air under the action of its proper weight: the critical time does not exceed 28 seconds.

As an example of a linear shock, we briefly examine only the caustics of optics here, and furthermore, only in general relativity, noting that an analogous phenomenon is produced that is, nevertheless, quite difficult to interpret.

Nonlinear shocks will be apparent in the theory of magnetohydrodynamics and completely exceptional electrodynamical systems that are determined by partial differential equations.

Jean-Louis Destouches was the origin of this work. We wish to acknowledge the interesting discussions and the amity that he afforded us. To André Lichnerowicz, who did us the honor of taking an interest in our research and supporting us, we express our gratitude. We thank Y. Choquet-Bruhat in particular for the information that was provided by some of her work.

We express our amicable respect for professor T. Taniuti, whose cited memoir has served us well.

We would like to recognize professors C. Møller, L. Rosenfeld, and H. Wergeland, for various contributions.

Guy BOILLAT.

BOOK ONE

GENERAL THEORY

1. **Fields, field equations, and waves.-** A field will be represented by a column vector $\mathbf{u}(x^\alpha)$ with N components that are functions of $n + 1$ independent variables x^α and subject to the system of N partial differential equations:

$$A^\alpha(\mathbf{u})\,\mathbf{u}_\alpha = \mathbf{f}(\mathbf{u}, x^\beta) \qquad (\alpha, \beta = 0, 1, 2, \ldots, n). \tag{1.1}$$

In the general case, in which the matrices A^α depend on the field, such a system is called *quasi-linear;* in the contrary case, it is qualified as *semi-linear*, or similarly *linear*, if, moreover, the vector $\mathbf{f}$ that appears in the right-hand side possesses a certain character relative to $\mathbf{u}$. We suppose, to simplify, that the matrices A^α do not depend on the coordinates explicitly.

We make the hypothesis that the function $\mathbf{u}$ is continuous, whereas its first derivatives are continuous on one side and the other of the wave front and tend to two different limits; i.e., $\mathbf{u}$ will be function that is of class piecewise C^1.

We introduce the wave surface by its Cartesian equation:

$$\varphi(x^\alpha) = 0, \tag{1.2}$$

as well as the new variables:

$$\varphi = \varphi(x^\alpha), \quad \xi^i = \xi^i(x^\alpha); \qquad \xi^i, \varphi \in C^2 \qquad (i = 1, 2, \ldots, n). \tag{1.3}$$

We make use of the following symbol for the jump:

$$[\ ] = \underset{\text{perturbed state}}{(\,)_{\varphi=+0}} - \underset{\text{unperturbed state}}{(\,)_{\varphi=-0}}\,, \tag{1.4}$$

which permits us to write, from the classical argument of Hadamard [1]:

$$[\mathbf{u}] = 0, \qquad [\mathbf{u}_{\xi^i}] = 0, \qquad [\mathbf{u}_\varphi] = \boldsymbol{\pi}. \tag{1.5}$$

With these definitions, the problem that one poses is the study of the propagation of waves as the study of $\boldsymbol{\pi}$ as a function of the coordinates. We denote the value of the field in the unperturbed state by $\mathbf{u}_0$:

$$A_0^\alpha(\mathbf{u}_0)\mathbf{u}_{0\alpha} = \mathbf{f}(\mathbf{u}_0, x^\beta); \qquad \mathbf{u}_0 \in C^1. \tag{1.6}$$

If we express the derivatives with the aid of the new variables:

(1.7) $$\mathbf{u}_\alpha = \mathbf{u}_\varphi\,\varphi_\alpha + \mathbf{u}_{\xi^i}\xi^i_\alpha\,,$$

substitute in (1.1), and compute the jump then we obtain:

(1.8) $$A_0^\alpha\,\varphi_\alpha\boldsymbol{\pi} = 0.$$

In order for this linear homogeneous system in the components of $\boldsymbol{\pi}$ to admit a non-null solution it is necessary that the determinant:

(1.9) $$\mathcal{D}(A^\alpha\varphi_\alpha) = 0$$

for the value $\mathbf{u} = \mathbf{u}_0$ of the field.

2. **Normal velocity, radial velocity.** Among the variables x^α, one of them x^0, which we also denote by t, plays the role of time, whereas the other ones x^i (i = 1, 2, …, n) are space variables. In that space, the wave front will be represented by a (hyper) surface S(t) of dimension n – 1 that moves in the course of time, and at each point of which there will be a normal velocity $\lambda\vec{n}$ that is defined at each instant by the formulae:

(2.1) $$\lambda = -\frac{\varphi_t}{|\nabla\varphi|}, \quad \vec{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$$

which is derived from (1.2). The condition (1.9) may be further translated into:

$$\mathcal{D}(A^i n^i - \lambda A^0) = 0.$$

The matrix A^0 is regular; otherwise, there would exist infinite wave velocities, which is unacceptable from a physical point of view. Therefore, nothing prevents us from taking:

(2.2) $$A^0 = I,$$

the identity matrix, which gives:

(2.3) $$\mathcal{D}(A^i n^i - \lambda I) = 0.$$

We are thenceforth assured that (2.2) is certainly realized; if necessary, one multiplies the system (1.1) by the matrix $(A^0)^{-1}$.

To each proper value $\lambda^{(i)}(\mathbf{u}, \vec{n})$, which is a possibly multiple root of the characteristic polynomial (2.3), there correspond the right proper vectors $\mathbf{d}_I^{(i)}(\mathbf{u}, \vec{n})$ and left proper vectors $\mathbf{l}_I^{(i)}(\mathbf{u}, \vec{n})$ of the matrix $A_n = A^k n^k$, which are defined by (*):

(*) and denoted by the initials of the Latin words *dexter* (right), *laevus* (left).

$$(2.4) \qquad \begin{aligned} (\mathrm{A}_n - \lambda^{(i)}\mathrm{I})\mathbf{d}_{\mathrm{I}}^{(i)} &= 0, \\ \mathbf{l}_{\mathrm{I}}^{(i)}(\mathrm{A}_n - \lambda^{(i)}\mathrm{I}) &= 0, \end{aligned}$$

in which the index I can on take as many integer values as there are linearly independent proper values for the value $\lambda^{(i)}$ in question. We say that the system (1.1) is *hyperbolic* if the proper values of A_n are real and the proper vectors of this matrix form a basis for the space of components of **u**, or, in other words, if there exist N linearly independent proper vectors [2]. The hyperbolicity conditions will be assumed to be satisfied in what follows.

We introduce the quantities:

$$(2.5) \qquad \psi^{(i)}(\mathbf{u}, \varphi_\alpha) = \varphi_t + |\nabla\varphi|\, \lambda^{(i)}.$$

By virtue of (2.1), the velocity $\lambda = \lambda^{(i)}$ will satisfy the relation:

$$(2.6) \qquad \psi_0^{(i)}(\mathbf{u}, \varphi_\alpha) = 0,$$

on the wave front, which is the partial differential equation that the characteristic surface (1.2) satisfies.

Classical theory introduced the *characteristic lines* or *rays* $\mathrm{C}^{(i)}$ (of equation (2.6)) for the solution of such equations. They are *bicharacteristics* of the system (1.1), with the differential system:

$$(2.7) \qquad \frac{dx^\alpha}{d\sigma} = \frac{\partial \psi_0^{(i)}}{\partial \varphi_\alpha}, \qquad \frac{d\varphi_\alpha}{d\sigma} = -\frac{\partial \psi_0^{(i)}}{\partial x^\alpha},$$

in which one finds a parameter σ (or, more particularly, denoted by $\sigma^{(i)}$), which is identified with time along the curve $\mathrm{C}^{(i)}$, as well as displaying the equation that was written.

Before proceeding, it is useful to specify the various meanings of the differentiation symbols that we use. We summarize them in the formula:

$$\partial_\alpha \psi(x^\beta, \varphi_\gamma) = \frac{\partial \psi}{\partial \varphi_\gamma}\varphi_{\gamma\alpha} + \frac{\partial \psi}{\partial x^\alpha}.$$

We return to the expression (2.7). Taking into account the remark that was made concerning the parameter s, one sees that the $\dfrac{\partial \psi_0}{\partial \varphi_j}$ form the components of a velocity, the *radial velocity* $\vec{\Lambda}$:

$$(2.8) \qquad \Lambda^{(i),\, j} = \frac{\partial \psi^{(i)}}{\partial \varphi_j},$$

which, from (2.5), is derived directly from $\lambda^{(i)}$:

(2.9) $$\vec{\Lambda}^{(i)}(\mathbf{u},\vec{n}) = \lambda^{(i)}\vec{n} + \frac{\partial\lambda^{(i)}}{\partial\vec{n}} - \left(\vec{n}\cdot\frac{\partial\lambda^{(i)}}{\partial\vec{n}}\right)\vec{n}\,.$$

One will note that:

(2.10) $$\vec{\Lambda}^{(i)}\cdot\vec{n} = \lambda^{(i)}.$$

3. **First expressions.** The following equality results from the defining formulae (2.4):

$$(\mathrm{A}^k n^k - \lambda\mathrm{I})\,\mathbf{d}_{\mathrm{I}}^{(i)} = (\lambda^{(i)} - \lambda)\,\mathbf{d}_{\mathrm{I}}^{(i)},$$

which, when multiplied by the modulus of the gradient of φ, $|\nabla\varphi|$, and considering (2.1, 5), leads to:

(3.1) $$\mathrm{A}^\alpha\varphi_\alpha\mathbf{d}_{\mathrm{I}}^{(i)} = \lambda^{(i)}\,\mathbf{d}_{\mathrm{I}}^{(i)},$$

and an analogous expression:

(3.1′) $$\mathbf{l}_{\mathrm{I}}^{(i)}\,\mathrm{A}^\alpha\varphi_\alpha = \lambda^{(i)}\,\mathbf{l}_{\mathrm{I}}^{(i)},$$

for the left vectors. From this, one deduces the well-known property:

(3.2) $$\mathbf{l}_{\mathrm{J}}^{(j)}\cdot\mathbf{d}_{\mathrm{I}}^{(i)} = 0, \qquad \forall\,(j)\neq(i),$$

which we cite, for the sake of reference.

We multiply the equality (1.8) by the proper vector $\mathbf{l}_{\mathrm{J}^0}^{(j)}$ that corresponds to the proper value $\lambda_0^{(j)}$, and obtain:

$$\mathbf{l}_{\mathrm{J}^0}^{(j)}\mathrm{A}_0^\alpha\,\varphi_\alpha\,\boldsymbol{\pi} = \psi_0^{(j)}\mathbf{l}_{\mathrm{J}^0}^{(j)}\cdot\boldsymbol{\pi} = 0,$$

and then reconsider the surface $\mathrm{S}^{(i)}$ – where (i) is a given value – that satisfies (2.6):

$$\mathbf{l}_{\mathrm{J}^0}^{(j)}\cdot\boldsymbol{\pi}^{(i)} = 0, \qquad\qquad \forall\,(j)\neq(i),$$

namely:

(3.3) $$\boldsymbol{\pi}^{(i)} = \pi^{\mathrm{I}}\,\mathbf{d}_{\mathrm{I}^0}^{(i)},$$

in which the $\pi^{\mathrm{I}}(x^\alpha)$ are the functions to be determined. Therefore, $\boldsymbol{\pi}$ belongs to the vector subspace that is generated by the right vectors that are associated with the mode of propagation that is envisioned. In order to abbreviate the notation, when it will create no ambiguity, we will drop the upper index (i).

In the perturbed neighborhood of S, we may write:

(3.4) $$\mathbf{u} = \mathbf{u}_0 + \varphi\boldsymbol{\pi} + \mathbf{O}(\varphi^2),$$

where $\mathbf{O}(\varphi^2)$ denotes the terms of order greater than or equal to two in the principal infinitesimal φ (Landau's notation). If one substitutes in (1.1) and multiplies by $\mathbf{l}_{\mathrm{I}'}^{(i)}$ then one gets:

(3.5) $$\mathbf{l}_{\mathrm{I}'}\,\mathrm{A}^{\alpha}\mathbf{u}_{0\alpha} + \mathbf{l}_{\mathrm{I}'}\,\mathrm{A}^{\alpha}\varphi_{\alpha}\,\boldsymbol{\pi} + \varphi\,\mathbf{l}_{\mathrm{I}'}\mathrm{A}^{\alpha}\partial_{\alpha}\boldsymbol{\pi} + \mathbf{O}(\varphi^2) = h_{\mathrm{I}'}$$ (*)

upon setting:

(3.6) $$h_{\mathrm{I}'} = \mathbf{l}_{\mathrm{I}'}\cdot\mathbf{f}.$$

We introduce the gradient operator in the space of components of $\mathbf{u}$, namely:

$$\nabla = \left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \cdots, \frac{\partial}{\partial u^{\mathrm{N}}}\right),$$

and calculate – to second order – the various terms that appear in (3.5) by using (3.4):

(3.7) $$\begin{aligned}\mathbf{l}_{\mathrm{I}'}\,\mathrm{A}^{\alpha}\mathbf{u}_{0\alpha} &= \mathbf{l}_{\mathrm{I}'^0}\mathrm{A}_0^{\alpha}\,\mathbf{u}_{0\alpha} + \varphi\boldsymbol{\nabla}(\mathbf{l}_{\mathrm{I}'}\mathrm{A}^{\alpha})_0\,\boldsymbol{\pi}\,\mathbf{u}_{0\alpha} + \cdots,\\ \mathbf{l}_{\mathrm{I}'}\,\mathrm{A}^{\alpha}\varphi_{\alpha}\boldsymbol{\pi} &= \psi^{(i)}\,\mathbf{l}_{\mathrm{I}'}\cdot\boldsymbol{\pi} = \varphi\boldsymbol{\nabla}(\psi_0^{(i)}\cdot\boldsymbol{\pi})\,\mathbf{l}_{\mathrm{I}'^0}\cdot\boldsymbol{\pi} + \cdots,\end{aligned}$$

keeping (2.6) in mind, and:

$$h_{\mathrm{I}'} = h_{\mathrm{I}'^0} + \varphi\boldsymbol{\nabla}h_{\mathrm{I}'^0}\cdot\boldsymbol{\pi} + \cdots$$

If one refers to (3.5), taking (1.6) into account, and one derives with respect to φ then one sees that the following relation will be satisfied on S:

(3.8) $$\nabla(\mathbf{l}_{\mathrm{I}'}\mathrm{A}^{\alpha})_0\,\boldsymbol{\pi}\,\mathbf{u}_{0\alpha} + (\nabla\,\psi_0^{(i)}\cdot\boldsymbol{\pi})\,\mathbf{l}_{\mathrm{I}'^0}\cdot\boldsymbol{\pi} + \mathbf{l}_{\mathrm{I}'^0}\mathrm{A}_0^{\alpha}\,\partial_{\alpha}\boldsymbol{\pi} = \nabla h_{\mathrm{I}'^0}\cdot\boldsymbol{\pi}.$$

If we appeal to (3.3), we will have:

(3.9) $$\mathbf{l}_{\mathrm{I}'^0}\mathrm{A}_0^{\alpha}\partial_{\alpha}\boldsymbol{\pi} = \mathbf{l}_{\mathrm{I}'^0}\mathrm{A}_0^{\alpha}\mathbf{d}_{\mathrm{I}^0}\,\partial_{\alpha}\pi^{\mathrm{I}} + \mathbf{l}_{\mathrm{I}'^0}\mathrm{A}_0^{\alpha}\,\frac{\partial\mathbf{d}_{\mathrm{I}^0}}{\partial\varphi_{\beta}}\,\varphi_{\alpha\beta}\,\pi^{\mathrm{I}} + \mathbf{l}_{\mathrm{I}'^0}\mathrm{A}_0^{\alpha}\boldsymbol{\nabla}\mathbf{d}_{\mathrm{I}^0}\mathbf{u}_{0\alpha}\,\pi^{\mathrm{I}}.$$

First of all, consider the first term of the right-hand side. Derive (3.1) with respect to φ_{α}:

(3.10) $$\mathrm{A}^{\alpha}\mathbf{d}_{\mathrm{I}} + \mathrm{A}^{\beta}\varphi_{\beta}\,\frac{\partial\mathbf{d}_{\mathrm{I}}}{\partial\varphi_{\alpha}} = \frac{\partial\psi^{(i)}}{\partial\varphi_{\alpha}}\,\mathbf{d}_{\mathrm{I}} + \psi^{(i)}\,\frac{\partial\mathbf{d}_{\mathrm{I}}}{\partial\varphi_{\alpha}}.$$

Hence:

(*) Upon considering (3.1′) and (2.6), one sees why the terms of order two contribute nothing in (3.4).

(3.11) $$\mathbf{l}_{\mathrm{I}'}\,\mathrm{A}^{\alpha}\mathbf{d}_{\mathrm{I}} = \frac{\partial\psi^{(i)}}{\partial\varphi_{\alpha}}\,\mathbf{l}_{\mathrm{I}'}\cdot\mathbf{d}_{\mathrm{I}},$$

and:

(3.12) $$\mathbf{l}_{\mathrm{I}'^0}\,\mathrm{A}_0^{\alpha}\mathbf{d}_{\mathrm{I}^0}\,\partial_{\alpha}\pi^{\mathrm{I}} = \frac{d\pi^{\mathrm{I}}}{d\sigma}\mathbf{l}_{\mathrm{I}'^0}\cdot\mathbf{d}_{\mathrm{I}^0}\,,$$

because, from (2.7):

(3.13) $$\frac{d}{d\sigma} = \frac{\partial\psi_0^{(i)}}{\partial\varphi_{\alpha}}\partial_{\alpha}$$

denotes the derivative along the characteristic line $\mathrm{C}^{(i)}$.

We pass on to the second term of (3.9). Deriving (3.10) with respect to φ_β gives:

(3.14) $$\mathrm{A}^{\alpha}\frac{\partial\mathbf{d}_{\mathrm{I}}}{\partial\varphi_{\beta}} + \mathrm{A}^{\beta}\frac{\partial\mathbf{d}_{\mathrm{I}}}{\partial\varphi_{\alpha}} + \mathrm{A}^{\gamma}\varphi_{\gamma}\frac{\partial^2\mathbf{d}_{\mathrm{I}}}{\partial\varphi_{\alpha}\partial\varphi_{\beta}} = \frac{\partial^2\psi^{(i)}}{\partial\varphi_{\alpha}\partial\varphi_{\beta}}\mathbf{d}_{\mathrm{I}} + \frac{\partial\psi^{(i)}}{\partial\varphi_{\alpha}}\frac{\partial\mathbf{d}_{\mathrm{I}}}{\partial\varphi_{\beta}} + \frac{\partial\psi^{(i)}}{\partial\varphi_{\beta}}\frac{\partial\mathbf{d}_{\mathrm{I}}}{\partial\varphi_{\alpha}}.$$

As a consequence:

(3.15) $$\mathbf{l}_{\mathrm{I}'^0}\mathrm{A}_0^{\alpha}\frac{\partial\mathbf{d}_{\mathrm{I}^0}}{\partial\varphi_{\beta}}\varphi_{\alpha\beta}\pi^{\mathrm{I}} = \frac{1}{2}\frac{\partial^2\psi^{(i)}}{\partial\varphi_{\alpha}\partial\varphi_{\beta}}\varphi_{\alpha\beta}\,\mathbf{l}_{\mathrm{I}'^0}\cdot\boldsymbol{\pi} + \pi^{\mathrm{I}}\mathbf{l}_{\mathrm{I}'^0}\frac{\partial\mathbf{d}_{\mathrm{I}^0}}{\partial\varphi_{\alpha}}\frac{d\varphi_{\alpha}}{d\sigma}.$$

As for the last term of (3.9) and the first term of (3.8), they are not, in general, susceptible to being given simpler expressions.

These various results permit us to write (3.8) as follows:

(3.16) $$\frac{d\pi^{\mathrm{I}}}{d\sigma}\mathbf{l}_{\mathrm{I}'^0}\mathbf{d}_{\mathrm{I}^0} + \left(\frac{1}{2}\frac{\partial^2\psi_0^{(i)}}{\partial\varphi_{\alpha}\partial\varphi_{\beta}} + \nabla\psi_0^{(i)}\cdot\boldsymbol{\pi}\right)\mathbf{l}_{\mathrm{I}'^0}\cdot\boldsymbol{\pi} + \pi^{\mathrm{I}}\mathbf{l}_{\mathrm{I}'^0}\frac{\partial\mathbf{d}_{\mathrm{I}^0}}{\partial\varphi_{\alpha}}\frac{d\varphi_{\alpha}}{d\sigma}$$
$$+\ \pi^{\mathrm{I}}\{\nabla(\mathbf{l}_{\mathrm{I}'}\mathrm{A}^{\alpha})_0\,\mathbf{d}_{\mathrm{I}^0} + \mathbf{l}_{\mathrm{I}'}\mathrm{A}_0^{\alpha}\nabla\mathbf{d}_{\mathrm{I}^0}\}\mathbf{u}_{0\alpha} = \nabla h_{\mathrm{I}'^0}\cdot\boldsymbol{\pi}.$$

4. **The differential system at the discontinuities.** Consider a point $\mathrm{M}(x^{\alpha})$ on a curve $\mathrm{C}^{(i)}$ at the instant σ, which was found at $\mathrm{M}_0(x_0^{\alpha})$ at the initial instant, and let $q^i(x^{\alpha})$ be n quantities that are constant along that curve. We then have:

$$\frac{dq^i}{d\sigma} = \frac{\partial\psi_0^{(i)}}{\partial\varphi_{\alpha}}q_{\alpha}^i = 0.$$

We now differentiate with respect to x^j:

$$\left(\frac{\partial^2\psi_0}{\partial x^j\partial\varphi_\alpha}+\frac{\partial^2\psi_0}{\partial\varphi_\beta\partial\varphi_\alpha}\varphi_{\beta j}\right)q^i_\alpha+\frac{\partial\psi_0}{\partial\varphi_\alpha}q^i_{\alpha j}=0,$$

i.e.:

(4.1)
$$\frac{d}{d\sigma}q^i_j=-\left(\frac{\partial^2\psi_0}{\partial x^j\partial\varphi_k}+\frac{\partial^2\psi_0}{\partial\varphi_l\partial\varphi_k}\varphi_{lj}\right)q^i_k.$$

We recall that if a matrix Q(σ) satisfies the relation:

$$\frac{dQ}{d\sigma}=\mathrm{MQ},$$

then its determinant $\mathcal{D}$(Q) satisfies:

$$\frac{d}{d\sigma}\mathcal{D}(\mathrm{Q})=(\mathrm{Tr\ M})\mathcal{D}(\mathrm{Q}),$$

where Tr M denotes the trace of the matrix M. As a consequence, if one introduces:

$$\theta=\sqrt{\left|\frac{\mathcal{D}(q^i_j)_0}{\mathcal{D}(q^i_j)}\right|},$$

in such a way that $\theta = 1$ when $\sigma = 0$:

(4.2)
$$\frac{\partial^2\psi_0}{\partial\varphi_i\partial\varphi_j}\varphi_{ij}+\frac{\partial^2\psi_0}{\partial x^i\partial\varphi_i}=2\frac{d}{d\sigma}\mathrm{Log}\ \theta.$$

From the constancy of q^i ($q^i = (q^i)_0$) and the properties of the functional determinant it then follows that:

$$\frac{\mathcal{D}(q^i_j)_0}{\mathcal{D}(q^i_j)}=\frac{\mathrm{D}(q^i)}{\mathrm{D}(x^i)}\frac{\mathrm{D}(x^i)}{\mathrm{D}(q^i)}=\frac{\mathrm{D}(x^i)}{\mathrm{D}(x^i_0)},$$

namely:

$$\theta=\sqrt{\left|\frac{\mathrm{D}(x^i)}{\mathrm{D}(x^i_0)}\right|}.$$

Finally, one may rewrite (3.16) in the form:

(4.4)
$$\mathbf{l}_{\mathrm{I}'^0}\left\{\frac{d\boldsymbol{\pi}}{d\sigma}+\left(\frac{d}{d\sigma}\mathrm{Log}\,\theta-\frac{1}{2}\frac{\partial\Lambda_0^{(i),k}}{\partial x^k}+|\nabla\varphi|\,\boldsymbol{\nabla}\lambda_0^{(i)}\cdot\boldsymbol{\pi}\right)\boldsymbol{\pi}\right\}$$
$$+\boldsymbol{\nabla}(\mathbf{l}_{\mathrm{I}'}\mathrm{A}^\alpha)_0\,\boldsymbol{\pi}\,\mathbf{u}_{0\alpha}-\pi^{\mathrm{I}}\frac{\partial\mathbf{l}_{\mathrm{I}'^0}}{\partial\varphi_\alpha}\mathrm{A}^\beta\varphi_\beta\frac{\partial\mathbf{d}_{\mathrm{I}^0}}{\partial x^\alpha}=\boldsymbol{\nabla}h_{\mathrm{I}'^0}\cdot\boldsymbol{\pi}.$$

Since the proper vectors are defined only up to a multiplicative factor, it happens that this equation is invariant under the replacement of the left vector $\mathbf{l}_{I'0}$ with a collinear vector; the penultimate term of the right-hand side assures this invariance.

The discontinuities displace with the radial velocity along the characteristic rays (*). The systems (2.7) and (4.4), to which we shall add the equality (3.3), permit us to determine their values. When equations (2.7) are integrated once, they give:

$$x^0 = \sigma, \qquad x^i = x^i\left(x_0^i, \sigma\right), \tag{4.5}$$

provided that one is given $(\nabla\varphi)_0$ as a function of the x_0^i, i.e., the point of the surface wave S_0 that was M_0 at the initial instant $\sigma = 0$ (**):

$$\varphi^0(x_0^i) = 0. \tag{4.6}$$

Once one has substituted the expressions (4.5) into (4.4), all that remains is to solve this differential system in order to obtain $\boldsymbol{\pi}$. We immediately remark that this system is not, in general, linear, and that it may, on the other hand, become singular, while any singularity of $\mathbf{u}_0$ is then isolated. We shall examine these circumstances at length in the next sections.

5. **The propagation of waves in a constant state.** – We direct our attention to the waves that propagate in a region where the field is constant. We mark the values the field takes in this region with an asterisk, values that, from (1.6), must be such that:

$$\mathbf{f}(\mathbf{u}_*, x^a) \equiv 0, \quad \mathbf{u}_* = \text{const.}, \tag{5.1}$$

in which we have obviously assumed that a solution exists. Important simplifications then ensue. First of all, since $\psi_*^{(i)}$ no longer depends upon the coordinates explicitly, (2.7) shows that the φ_α are constant along the curves $C^{(i)}$. As a result, the relations (4.5) immediately give:

$$x^0 = \sigma, \qquad x^j = x_0^j + \Lambda_{*0}^{(i),j}(\mathbf{u}_*, \vec{n}_0)\sigma, \tag{5.2}$$

in which the normal vector $\vec{n}_0(x_0^i)$ to S_0 is calculated by means of (4.6). From this, one deduces:

$$\theta = \sqrt{D(\sigma\partial_{i_0}\Lambda_{*0}^{(i),j} + \delta_i^j)}, \tag{5.3}$$

with:

(*) To use the language of the physicist, we shall say that the characteristic rays are the "guiding waves."

(**) Upon solving (4.5) for the x_0^i that one substitutes in (4.6) one will find the equation (1.2) for $S(t)$.

$$\delta_i^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

On the other hand, the system (4.4) becomes:

(5.4) $$\left\{\frac{d\pi^{\mathrm{I}}}{d\sigma} + \left(\frac{d}{d\sigma}\log\theta + |\nabla\varphi|\,\nabla\lambda_* \cdot \boldsymbol{\pi}\right)\pi^{\mathrm{I}}\right\}\mathbf{l}_{\mathrm{I}^*} \cdot \mathbf{d}_{\mathrm{I}^*} = \nabla h_{\mathrm{I}^*} \cdot \boldsymbol{\pi}\,.$$

Starting with the determinant:

(5.5) $$\Delta^{(i)} = \mathcal{D}(\mathbf{l}_{\mathrm{I}}, \mathbf{d}_{\mathrm{I}'})\,,$$

one introduces the quantities $f^{\mathrm{I}}_{\mathrm{I}''}$: the product $\Delta^{(i)} \cdot f^{\mathrm{I}}_{\mathrm{I}''}$ equals the determinant that is obtained by replacing the I^{th} column in $\Delta^{(i)}$ by that of the elements:

$$\nabla h_{\mathrm{I}''}\mathbf{d}_{\mathrm{I}'}\,,$$

in which I'' takes on all of the same values as I. It is essential to notice that $\Delta^{(i)}$ is non-null. Indeed, if L and D denote the matrices that are formed from the left and right proper vectors of Λ_n then, by virtue of (3.2):

(5.6) $$\mathcal{D}(\mathrm{LD}) = \prod_{(i)} \Delta^{(i)} = \mathcal{D}(\mathrm{L})\mathcal{D}(\mathrm{R}) \neq 0,$$

since the system (1.1) is hyperbolic (cf. sec. 2). We may therefore recall (5.4), which is solved for the term in braces:

(5.7) $$\frac{d\pi^{\mathrm{I}}}{d\sigma} + \left(\frac{d}{d\sigma}\log\theta + |\nabla\varphi|\,\nabla\lambda_* \cdot \boldsymbol{\pi}\right)\pi^{\mathrm{I}} = f^{\mathrm{I}}_{\mathrm{I}'*} \cdot \pi^{\mathrm{I}'}\,,$$

and, after integration, gives the law of propagation of the discontinuities [3]:

(5.8) $$\boldsymbol{\pi} = \frac{\boldsymbol{\eta}}{\theta\Phi}\,,$$

with:

(5.9) $$\Phi = 1 + |\nabla\varphi| \int_0^\sigma \nabla\lambda_*^{(i)} \cdot \boldsymbol{\eta}(\tau)\,\frac{d\tau}{\theta(\tau)}\,,$$

(5.10) $$\boldsymbol{\eta} = \eta^{\mathrm{I}}\,\mathbf{d}_{\mathrm{I}*}\,;$$

in which the η^{I} are (continuous, as well as $f^{\mathrm{I}}_{\mathrm{I}'*}$) solutions of:

(5.11) $$\frac{d\eta^{\mathrm{I}}}{d\sigma} = f^{\mathrm{I}}_{\mathrm{I}'*} \cdot \eta^{\mathrm{I}'}\,, \qquad \eta^{\mathrm{I}}_0 = \pi^{\mathrm{I}}_0(x^i_0)\,.$$

The given of the perturbing field $\mathbf{u}^0(x_0^{\mathrm{I}})$ determines the π_0^{I}. At the initial instant:

$$\mathrm{S}_0^{(i)} = \mathrm{S}_0^{(j)} = \mathrm{S}_0, \qquad \forall (i), (j),$$

and in the perturbed neighborhood of S_0:

$$\mathbf{u}^0 = \mathbf{u}_* + \varphi^0 \boldsymbol{\pi}^0 + \ldots \tag{5.12}$$

$\boldsymbol{\pi}^0$ is solved for its components in the basis of proper vectors:

$$\boldsymbol{\pi}^0(x_0^i) = \sum_{(k)} \sum_{\mathrm{K}} \pi_0^{\mathrm{K}} \mathbf{d}_{\mathrm{K}*}^{(k)} = \sum_{(k)} \boldsymbol{\pi}_0^{(k)} . \tag{5.13}$$

From this, one deduces:

$$\mathbf{l}_{\mathrm{I}'*} \cdot \boldsymbol{\pi}^0 = \pi_0^{\mathrm{I}} \mathbf{l}_{\mathrm{I}'*} \cdot \mathbf{d}_{\mathrm{I}*}, \tag{5.14}$$

and from that, the π_0^{I} are obtained by solving the latter system.

We make two remarks: The degree of the polynomial $\theta^2(\sigma)$ is less than or equal to $n - 1$. There exists one relation between the components of the radial velocity $\vec{\Lambda}_{*0}^{(i)}(x_0^i)$ since there exists one relation between the variables x_0^i (equation (4.6) for the surface S_0) and the $\mathcal{D}(\partial_{i_0} \Lambda_{*0}^{(i),j})$ is null *ipso facto*.

When the function $\mathbf{f}$ in the right-hand side of (1.1) is identically null, the integral:

$$\int_0^\sigma \frac{d\tau}{\theta(\tau)},$$

which is found in Φ, involves only elementary functions in the usual case, where $n = 3$. For more numerous variables, one must appeal to the elliptic functions ($n = 4, 5$) and the hypoelliptic ones ($n > 5$).

6. **Plane waves, translating waves, and parallel waves.** In this section we treat three particularly important cases. When the surface S_0 is a plane:

$$\vec{n}_0 = \text{const.}, \qquad \vec{\Lambda}_{*0}^{(i)} = \text{const.}, \qquad \theta = 1, \tag{6.1}$$

and the formulas (5.2) show that S is likewise a plane surface. We briefly state: the plane waves remain planes. These various properties confer their importance on such waves.

Likewise, the translating waves possess a radial velocity that is constant, but for a different reason: it does not depend on the normal vector:

(6.2) $$\vec{\Lambda}_*^{(i)} = \vec{\Lambda}_*^{(i)}(\mathbf{u}_*) = \text{const.}, \qquad \theta = 1.$$

From this, it results that S is derived from S_0 by a simple translation; therefore, the latter does not suffer any deformation in the course of time.

The parallel waves are characterized by having a constant scalar normal velocity:

(6.3) $$\lambda_*^{(i)} = \lambda_*^{(i)}(\mathbf{u}_*) = \text{const.},$$

from which, and (2.9), it ensues that:

(6.4) $$\vec{\Lambda}_*^{(i)} = \lambda_*^{(i)}\vec{n}$$

and (cf. (5.3)):

(6.5) $$\theta^2 = \mathcal{D}(\lambda_*^{(i)}\sigma\partial_{i_0} n_0^j + \delta_i^j).$$

Now, by virtue of the Rodriguez relation:

$$d\vec{\mathrm{M}} + \mathrm{R}_0 d\vec{n}_0 = 0,$$

the principle radii of curvature R_0 of the surface S_0 at the point M_0 are solutions of:

$$\mathcal{D}(\mathrm{R}_0\partial_{i_0} n_0^j + \delta_i^j) = 0,$$

and, as a result, the polynomial θ^2 admits them as roots. Thus, in ordinary physical space (n = 3):

(6.6) $$\theta^2 = \frac{(\lambda_*^{(i)}\sigma - \mathrm{R}_0^1)(\lambda_*^{(i)}\sigma - \mathrm{R}_0^2)}{\mathrm{R}_0^1\mathrm{R}_0^2},$$

or furthermore [4]:

(6.7) $$\theta = \sqrt{\lambda_*^2\mathrm{K}_0\sigma^2 - 2\lambda_*\Omega_0\sigma + 1},$$

in which we employ the mean curvature Ω_0 and the total curvature K_0.

More especially, if the waves are spherical and of radius r:

$$\theta = \frac{r}{r_0},$$

and if they are circular cylindrical:

$$\theta = \sqrt{\frac{r}{r_0}}.$$

7. **Shocks.** – We say that a *shock* is produced when $\boldsymbol{\pi}$ takes on an infinite value; the field may then be discontinuous in its own right. In reality, one is presented with a very

rapid variation in intensity by "steps," by "bumps," or by "fingers of a glove" in a physical region of negligible thickness. In any event, equations (1.1) cease to be valid in this region, and must be replaced with other ones. One already knows how to treat conservative systems:

$$\partial_\alpha \mathbf{f}^\alpha(\mathbf{u}) = 0, \tag{7.1}$$

by writing the shock conditions as:

$$\tilde{\lambda}\,|\,\mathbf{f}^0\,| = [\mathbf{f}^i]\, n^i, \tag{7.2}$$

which, on the one hand, the field values, and on the other, the wave surface of the shock must obey [5].

From the mathematical point of view, the problem amounts to considering the discontinuous solutions to (1.1) – the *weak solutions* – and a question of uniqueness is posed immediately: there might exist an infinitude of such solutions that correspond to the same initial data [2][6]. Remedies (or should we say, palliatives?) such as introducing a viscosity term [7][8] or taking microscopic phenomena into consideration have been proposed.

8. **Nonlinear shocks.** – These are due to the annulling of Φ, and are absent in linear fields. They do not exist if the conditions:

$$\nabla\lambda^{(i)}\cdot\mathbf{d}_{\mathrm{I}} \neq 0, \tag{8.1}$$

which assures the continuity of the first derivatives of $\lambda^{(i)}$:

$$[\partial_\varphi \lambda^{(i)}] = 0, \tag{8.2}$$

are verified. In this case, we say, with Lax, that the system (1.1) is *exceptional* for the wave in question [2]. When it possesses this property for all of the proper values it is *completely exceptional*. From all evidence, such is notably the case for semi-linear systems, as well as the following one [9]:

$$\mathrm{A}_n = \begin{pmatrix} \mathrm{B} & 0 \\ 0 & \mathrm{C} \end{pmatrix}, \tag{8.3}$$

in which B and C are square matrices of dimensions k and $\mathrm{N} - k$, respectively, B is independent of the first k components of $\mathbf{u}$, and C is independent of the other ones:

$$\mathrm{B} = \mathrm{B}(u^{k+1}, \ldots, u^{\mathrm{N}}, \vec{n}), \qquad \mathrm{C} = \mathrm{C}(u^1, \ldots, u^k, \vec{n}). \tag{8.4}$$

On the contrary, a system of the type:

$$\mathrm{A}_n = \begin{pmatrix} \mathrm{B} & 0 \\ \mathrm{D} & \mathrm{C} \end{pmatrix}, \tag{8.5}$$

in which C is the matrix that one must define and B and D are arbitrary, is exceptional only with respect to the waves that are determined by C.

Shocks will certainly appear if:

$$\nabla\lambda_*^{(i)} \cdot \boldsymbol{\pi} < 0. \tag{8.6}$$

What does this condition represent? We have:

$$\left[\frac{\partial}{\partial t}\log|\,\lambda^{(i)}\,|\right] = \frac{1}{\lambda_*^{(i)}}\left[\frac{\partial\lambda^{(i)}}{\partial\varphi}\right]\varphi_t = -\,|=\varphi\,|\;\nabla\lambda_*^{(i)} \cdot \boldsymbol{\pi}, \tag{8.7}$$

and (8.6) translates into:

$$\left[\frac{\partial\,|\,\lambda^{(i)}\,|}{\partial t}\right] > 0. \tag{8.8}$$

The left-hand side is nothing but the relative acceleration of the perturbed posterior face, as one calls it, of the wave surface with respect to the anterior face; when the accelerated one catches up with the other one, a shock is produced. The latter are the local manifestation here of a global phenomenon of envelope formation that has been studied in detail in aerodynamics for one-dimensional flows [5].

9. **Linear shocks.** – These are due to the annulling of θ, and inherit nothing from the nonlinear character of the field; hence, they are not foreign to the case of linear fields. In general, θ is annulled on a surface Σ, which is called the *focal surface* and consists of n – 1 sheets. Its points are called *foci* or *focal points*, and Σ further constitutes the envelope of characteristic lines $C^{(i)}$.

As far as the formal origin of these shocks is concerned, we shall establish the following proposition [10]:

Linear shocks are due to the non-commutation of the matrices A^i. Suppose that these matrices commute between themselves, i.e., that one has, for the value $\mathbf{u} = \mathbf{u}_*$:

$$A^iA^j = A^jA^i, \quad \forall i, j = 1, 2, \ldots, n. \tag{9.1}$$

From (3.1) (recall (2.2)), one may then write:

$$\mathbf{l}_J A^\alpha A^\beta \varphi_\beta \mathbf{d}_I = \psi^{(i)}\mathbf{l}_J A^\alpha \mathbf{d}_I = \mathbf{l}_J A^\beta \varphi_\beta A^\alpha \mathbf{d}_I = \psi^{(j)}\mathbf{l}_J A^\alpha \mathbf{d}_I,$$

namely:

$$\mathbf{l}_J^{(j)} A^\alpha \mathbf{d}_I^{(i)} = 0, \qquad \forall\ (j) \neq (i).$$

Upon multiplying (3.10) on the left by $\mathbf{l}_J$, by reason of the orthogonality of that vector with $\mathbf{d}_I$, it follows that:

$$\mathbf{l}_J \frac{\partial \mathbf{d}_J}{\partial \varphi_\alpha} = 0,$$

which translates into:

$$\frac{\partial \mathbf{d}_{\mathrm{I}}^{(i)}}{\partial \varphi_\alpha} = c_{\mathrm{I}}^{\alpha,\mathrm{I}'} \mathbf{d}_{\mathrm{I}}^{(i)} .$$

When this latter expression is substituted into (3.10), one finally has:

(9.2) $$\mathrm{A}^\alpha \mathbf{d}_{\mathrm{J}} = \frac{\partial \psi^{(i)}}{\partial \varphi_\alpha} \mathbf{d}_{\mathrm{I}},$$

which is equivalent to (cf. (2.8)):

(9.3) $$(\mathrm{A}^j - \Lambda^{(i),j}\mathrm{I})\, \mathbf{d}_{\mathrm{I}}^{(i)} = 0.$$

Thus, the radial velocity, just like the proper vectors, does not depend on $\vec{n}$. All of the associated waves of the system are translating waves (sec. 6) and no linear shock may be produced.

Conversely, let there be a system (1.1) for which all of the radial velocities and associated proper vectors depend only upon the field, let $\mathrm{K}^i(\mathbf{u})$ be the diagonal matrix that is formed from the i^{th} components of these velocities, and let $\mathrm{D}(\mathbf{u})$ the matrix of right proper vectors. Thanks to (2.10), we have:

$$\mathrm{A}^i n_i = \mathrm{D}\mathrm{K}^i n_i \mathrm{D}^{-1},$$

i.e.:

(9.4) $$\mathrm{A}^i = \mathrm{D}\mathrm{K}^i\mathrm{D}^{-1};$$

the matrices then commute. We note that one may nevertheless encounter translating waves in systems with non-commuting matrices; the associated proper vectors will generally depend on $\vec{n}$ (cf. Alfvén's waves, book II, sec 3). In summary, we say that in order for linear shocks to be produced it is necessary that the equalities (9.1) are not satisfied. One will recall here that there is very general principle of physics that asserts that "for such a phenomenon to be produced, it is necessary that certain symmetry elements do not exist [11]."

10. **Second order discontinuities.** – This amounts to determining the terms of order two in the development (3.4). We suppose, to simplify, that the function **f** in (1.1) is identically null, and we consider the propagation in a constant state. In the perturbed neighborhood of S, we thus write:

(10.1) $$\mathbf{u} = \mathbf{u}_* + \varphi\,\boldsymbol{\pi} + \frac{\varphi^2}{2}\boldsymbol{\chi} + \mathbf{O}(\varphi^3),$$

in which, obviously:

(10.2) $$\boldsymbol{\chi} = [\mathbf{u}_{\varphi\varphi}].$$

From the equality:

(10.3) $$\mathbf{l}_{\mathrm{J}}^{(j)}\mathrm{A}^{\alpha}\mathbf{u}_{\alpha} = 0, \qquad (j) \neq (i),$$

we have, to first order:

$$\mathbf{l}_{\mathrm{J}}\,\mathrm{A}^{\alpha}(\varphi\,\boldsymbol{\pi} + \varphi_{\alpha}\boldsymbol{\pi} + \varphi\,\varphi_{\alpha}\boldsymbol{\chi}) + \mathbf{O}(\varphi^{2}) = 0,$$

so that furthermore, upon using (3.1, 3):

$$\varphi\mathbf{l}_{\mathrm{J}*}\mathrm{A}_{*}^{\alpha}(\pi_{\alpha}^{\mathrm{I}}\mathbf{d}_{\mathrm{I}*} + \pi^{\mathrm{I}}\mathbf{d}_{\mathrm{I}*\alpha}) + \varphi\psi_{*}^{(j)}\mathbf{l}_{\mathrm{J}*}\boldsymbol{\chi} + \psi^{(j)}\mathbf{l}_{\mathrm{J}}\cdot\mathbf{d}_{\mathrm{I}*}\pi^{\mathrm{I}} + \mathbf{O}(\varphi^{2}) = 0.$$

However:

(10.4) $$\mathbf{d}_{\mathrm{I}} = \mathbf{d}_{\mathrm{I}*} + \nabla\mathbf{d}_{\mathrm{I}*}(\mathbf{u} - \mathbf{u}_{*}) + \ldots = \mathbf{d}_{\mathrm{I}*} + \varphi\,\nabla\mathbf{d}_{\mathrm{I}*}\,\boldsymbol{\pi} + \mathbf{O}(\varphi^{2}),$$

and, making use of (3.10), one obtains:

$$\varphi\psi_{*}^{(j)}\mathbf{l}_{\mathrm{J}*}\left\{\boldsymbol{\chi} - \pi^{\mathrm{I}}\nabla\mathbf{d}_{\mathrm{I}*}\boldsymbol{\pi} - \pi_{\alpha}^{\mathrm{I}}\frac{\partial\mathbf{d}_{\mathrm{I}*}}{\partial\varphi_{\alpha}} - \frac{1}{2}\pi^{\mathrm{I}}\partial_{\alpha}\left(\frac{\partial\mathbf{d}_{\mathrm{I}*}}{\partial\varphi_{\alpha}}\right)\right\} + \mathbf{O}(\varphi^{2}) = 0.$$

It finally ensues that:

(10.5) $$\boldsymbol{\chi} = \pi^{\mathrm{I}}\nabla\mathbf{d}_{\mathrm{I}*}\boldsymbol{\pi} + \pi_{\alpha}^{\mathrm{I}}\frac{\partial\mathbf{d}_{\mathrm{I}*}}{\partial\varphi_{\alpha}} + \frac{1}{2}\pi^{\mathrm{I}}\partial_{\alpha}\left(\frac{\partial\mathbf{d}_{\mathrm{I}*}}{\partial\varphi_{\alpha}}\right) + \chi^{\mathrm{I}}\mathbf{d}_{\mathrm{I}*}.$$

Because the proper vectors generally depend on the field and the normal vector $\vec{n}$, the second order discontinuities depend on the first order ones. It remains for us to determine the functions χ^{I}. To do this, one proceeds as in section 3: one writes:

(10.6) $$\mathbf{l}_{\mathrm{I}'}^{(i)}\mathrm{A}^{\alpha}\mathbf{u}_{\alpha} = 0,$$

upon neglecting only the terms of order greater than two. One has:

(10.7) $$\varphi\mathbf{l}_{\mathrm{I}'}\mathrm{A}^{\alpha}\partial_{\alpha}\boldsymbol{\pi} + \psi^{(i)}\mathbf{l}_{\mathrm{I}'}\cdot\boldsymbol{\pi} + \frac{1}{2}\varphi^{2}\,\mathbf{l}_{\mathrm{I}'}\,\mathrm{A}^{\alpha}\partial_{\alpha}\boldsymbol{\chi} + \varphi\psi^{(i)}\,\mathbf{l}_{\mathrm{I}'}\,\boldsymbol{\chi} + \mathbf{O}(\varphi^{3}) = 0.$$

We give the calculation of these various terms without comment:

a) $\mathbf{l}_{\mathrm{I}'}\mathrm{A}^{\alpha}\partial_{\alpha}\boldsymbol{\pi}$

$$= \mathbf{l}_{\mathrm{I}'}\mathrm{A}^{\alpha}(\partial_{\alpha}\pi^{\mathrm{I}}\mathbf{d}_{\mathrm{I}*} + \pi^{\mathrm{I}}\partial_{\alpha}\mathbf{d}_{\mathrm{I}*})$$

$$= \mathbf{l}_{I'} A^{\alpha} \pi^{I}_{\alpha} (\mathbf{d}_{I} - \varphi \nabla \mathbf{d}_{I*} \boldsymbol{\pi}) + \pi^{I} \left\{ \mathbf{l}_{I'} A^{\alpha} \frac{\partial \mathbf{d}_{I}}{\partial \varphi_{\beta}} \varphi_{\alpha\beta} - \varphi \mathbf{l}_{I'*} A^{\alpha}_{*} \partial_{\alpha} (\nabla \mathbf{d}_{I*}) \pi \right\} + \mathbf{O}(\varphi^{3})$$

(cf. (10.4))

$$\mathbf{l}_{I'} A^{\alpha} \partial_{\alpha} \boldsymbol{\pi}$$

$$= \left(\frac{d\pi^{I}}{d\sigma} + \frac{1}{2} \partial_{k} \Lambda^{k}_{*} \pi^{I} \right) \mathbf{l}_{I'*} \cdot \mathbf{d}_{I*} + \varphi \left\{ \left(\frac{d\pi^{I}}{d\sigma} + \frac{1}{2} \partial_{k} \Lambda^{k}_{*} \pi^{I} \right) \nabla (\mathbf{l}_{I'} \cdot \mathbf{d}_{I})_{*} \boldsymbol{\pi} + (\nabla \Lambda^{k}_{*} \cdot \boldsymbol{\pi}) \mathbf{l}_{I'*} \cdot \partial_{k} \boldsymbol{\pi} \right.$$

$$\left. + \frac{1}{2} \pi^{I} \partial_{k} (\nabla \Lambda^{k}_{*}) \boldsymbol{\pi} (\mathbf{l}_{I'*} \cdot \mathbf{d}_{I*}) - \frac{1}{2} \pi^{I} (\nabla \psi^{*} \cdot \boldsymbol{\pi}) \mathbf{l}_{I'*} \cdot \partial_{\alpha} \left(\frac{\partial \mathbf{d}_{I*}}{\partial \varphi_{\alpha}} \right) - \mathbf{l}_{I'*} A^{\alpha}_{*} \partial_{\alpha} (\pi^{I} \nabla \mathbf{d}_{I*}) \boldsymbol{\pi} \right\} + \mathbf{O}(\varphi^{2})$$

(cf. (3.14))

b) $\psi^{(i)} \mathbf{l}_{I'} \cdot \boldsymbol{\pi}$

$$= \pi^{I} \{ \varphi \nabla \psi^{*} \cdot \boldsymbol{\pi} + \frac{1}{2} \varphi^{2} \nabla \psi^{*} \nabla \boldsymbol{\chi} + \varphi^{2} \boldsymbol{\pi}^{T} \nabla (\nabla \psi^{T})^{*} \mathbf{p} + \ldots \}$$

$$\times \left\{ \mathbf{l}_{I'*} \cdot \mathbf{d}_{I*} + \varphi \nabla (\mathbf{l}_{I'} \cdot \mathbf{d}_{I})^{*} \boldsymbol{\pi} - \varphi \mathbf{l}_{I'*} \nabla \mathbf{d}_{I*} \boldsymbol{\pi} + \cdots \right\}$$

$$= \varphi (\nabla \psi^{*} \cdot \boldsymbol{\pi}) \mathbf{l}_{I'*} \nabla \boldsymbol{\pi} + \varphi^{2} \Big(\pi^{I} (\nabla \psi^{*} \cdot \boldsymbol{\pi}) \left\{ \nabla (\mathbf{l}_{I'} \cdot \mathbf{d}_{I})^{*} \boldsymbol{\pi} - \mathbf{l}_{I'*} \nabla \mathbf{d}_{I*} \boldsymbol{\pi} \right\}$$

$$+ \{ \frac{1}{2} \nabla \psi^{*} \cdot \boldsymbol{\pi} + \boldsymbol{\pi}^{T} \nabla (\nabla \psi^{T})^{*} \boldsymbol{\pi} \} \mathbf{l}_{I'*} \cdot \boldsymbol{\pi} \Big) + \mathbf{O}(\varphi^{3})$$

(cf. (10.4); ^{T} denotes the transpose)

c) $$\frac{1}{2} \varphi^{2} \mathbf{l}_{I'} A^{\alpha} \partial_{\alpha} \boldsymbol{\chi} = \frac{1}{2} \varphi^{2} \mathbf{l}_{I'*} A^{\alpha}_{*} \partial_{\alpha} \boldsymbol{\chi} + \mathbf{O}(\varphi^{3})$$

d) $$\varphi \psi^{(i)} \mathbf{l}_{I'} \boldsymbol{\chi} = \varphi^{2} (\nabla \psi^{*} \cdot \boldsymbol{\pi}) \mathbf{l}_{I'*} \cdot \boldsymbol{\chi} + \mathbf{O}(\varphi^{3}).$$

By definition, the second order conditions translate into:

(10.8) $$\frac{1}{2} \mathbf{l}_{I'} A^{\alpha} \partial_{\alpha} \boldsymbol{\chi} + (\nabla \psi^{*} \cdot \boldsymbol{\pi}) \mathbf{l}_{I'*} \cdot \boldsymbol{\chi} + \{ \frac{1}{2} \nabla \psi^{*} \cdot \boldsymbol{\pi} + \boldsymbol{\pi}^{T} \nabla (\nabla \psi^{T})^{*} \boldsymbol{\pi} + \frac{1}{2} \partial_{k} (\nabla \Lambda^{k}_{*}) \boldsymbol{\pi} \} \mathbf{l}_{I'*} \cdot \boldsymbol{\chi}$$

$$- \mathbf{l}_{I'*} A^{\alpha}_{*} \partial_{\alpha} (\pi^{I} \nabla \mathbf{d}_{I*}) \boldsymbol{\pi} + (\nabla \Lambda^{k}_{*} \cdot \boldsymbol{\pi}) \mathbf{l}_{I'*} \partial_{k} \boldsymbol{\pi}$$

$$- \pi^{I} (\nabla \psi^{*} \cdot \boldsymbol{\pi}) \left\{ \frac{1}{2} \mathbf{l}_{I'*} \partial_{\alpha} \left(\frac{\partial \mathbf{d}_{I*}}{\partial \varphi_{\alpha}} \right) + \mathbf{l}_{I'*} \nabla \mathbf{d}_{I'*} \boldsymbol{\pi} \right\}$$

$$= 0.$$

One sees that this system of differential equations for the functions χ^{I} is linear. Precisely, it is of the form:

(10.9) $$\frac{1}{2} \left\{ \frac{d\chi^{I}}{d\sigma} + \left(\frac{d}{d\sigma} \log \theta + 2 \nabla \psi^{*} \cdot \boldsymbol{\pi} \right) \chi^{I} + (\nabla \psi^{*} \cdot \mathbf{d}_{I''*}) \pi^{I} \chi^{I''} \right\} \mathbf{l}_{I'*} \cdot \mathbf{d}_{I*}$$

$$= \mathbf{l}_{I'*} \cdot \mathbf{b}_{*} (x^{i}_{0}, \sigma) ,$$

in which $\mathbf{b}_*(x_0^i, \sigma)$ may be determined by means of the results of section 5.

11. **The case of plane waves. –** We now put ourselves in the case of plane waves (see sec. 6) with simple proper values. Then:

$$\boldsymbol{\pi} = \pi \mathbf{d}_*, \qquad \pi = \frac{\pi_0}{\Phi} \tag{11.1}$$

$$\Phi = 1 + |\nabla\varphi|\, \nabla\lambda^* \cdot \boldsymbol{\pi}_0\, \sigma, \tag{11.2}$$

$$\boldsymbol{\chi} = \pi^2 \nabla \mathbf{d}_* \mathbf{d}_* + \frac{\partial \mathbf{d}_*}{\partial \varphi_k} \partial_k \pi + \chi \mathbf{d}_* . \tag{11.3}$$

We then note that, from (3.14):

$$\mathbf{l}_* \mathrm{A}_*^\alpha \frac{\partial \mathbf{d}_*}{\partial \varphi_\beta} \pi_{\alpha\beta} = \frac{1}{2} \frac{\partial^2 \psi_*}{\partial \varphi_\alpha \partial \varphi_\beta} \pi_{\alpha\beta} \mathbf{l}_* \cdot \mathbf{d}_* + \frac{\partial \psi_*}{\partial \varphi_\alpha} \pi_{\alpha\beta} \mathbf{l}_* \frac{\partial \mathbf{d}_*}{\partial \varphi_\beta},$$

and, from the law of propagation:

$$\frac{\partial \psi_*}{\partial \varphi_\alpha} \pi_\alpha + (\nabla \psi_* \cdot \mathbf{d}_*) \pi^2 = 0,$$

from which, one derives:

$$\frac{\partial \psi_*}{\partial \varphi_\alpha} \pi_{\alpha\beta} + 2\pi \pi_\beta (\nabla \psi_* \cdot \mathbf{d}_*) = 0,$$

and one easily obtains, by starting with (10.8):

$$\frac{d\chi}{d\sigma} + 3(\nabla\psi \cdot \mathbf{d})_* \pi\chi + \{\nabla\psi \nabla \mathbf{d}\mathbf{d} + 2\mathbf{d}^{\mathrm{T}} \nabla(\nabla\psi^{\mathrm{T}})\mathbf{d}\}_* \pi^3 + \left(2\nabla\Lambda^k \cdot \mathbf{d} + \Delta\psi \cdot \frac{\partial \mathbf{d}}{\partial \varphi_k}\right)_* \pi \partial_k \pi + \frac{1}{2} \frac{\partial \Lambda_*^k}{\partial \varphi_l} \partial_{kl} \pi = 0. \tag{11.4}$$

Since:

$$\begin{aligned} \partial_k \pi &= \Phi^{-2} \partial_{k_0} \pi_0, \\ \partial_{kl} \pi &= \Phi^{-3} (\Phi \partial_{k_0 l_0} \pi_0 - 2\partial_{k_0} \pi_0 \partial_{l_0} \pi_0 \nabla \psi_* \cdot \mathbf{d}_* \sigma), \end{aligned} \tag{11.5}$$

integration leads to:

$$\chi\Phi^3 - \chi_0 + |\nabla\varphi|\, \pi_0^3 \sigma \{\nabla\lambda\, \nabla \mathbf{d}\, \mathbf{d} + 2\mathbf{d}^{\mathrm{T}} \nabla(\nabla\lambda^{\mathrm{T}})\, \mathbf{d}\}_*$$

$$+\sigma\left\{\left(2\nabla\Lambda^k\cdot\mathbf{d}+|\nabla\varphi|\,\nabla\lambda\cdot\frac{\partial\mathbf{d}}{\partial\varphi_k}\right)_*\pi_0\partial_{k_0}\pi_0+\frac{1}{2}\frac{\partial\Lambda_*^k}{\partial\varphi_l}\partial_{k_0l_0}\pi_0\right\}$$
$$+\frac{1}{4}\sigma^2|\nabla\varphi|\,\nabla\lambda_*\cdot\mathbf{d}^*\frac{\partial\Lambda_*^k}{\partial\varphi_l}(\pi_0\partial_{k_0l_0}\pi_0-2\partial_{k_0}\pi_0\partial_{l_0}\pi_0)=0. \tag{11.6}$$

If the waves present an exceptional character (cf. 8.1):

$$\nabla(\nabla\lambda\cdot\mathbf{d}_{\mathrm{I}})\,\mathbf{d}_{\mathrm{I}'}=\mathbf{d}_{\mathrm{I}}^{\mathrm{T}}\nabla(\nabla\lambda^{\mathrm{T}})\mathbf{d}_{\mathrm{I}'}+\nabla\lambda\nabla\mathbf{d}_{\mathrm{I}}\mathbf{d}_{\mathrm{I}'}\equiv 0, \tag{11.7}$$

and equation (11.6) shows that:

$$\chi=\chi_0(x_0^i)\,, \tag{11.8}$$

when one adds the initial restriction:

$$\pi_0=\text{const.} \tag{11.9}$$

then one has the condition:

$$\mathbf{d}^{\mathrm{T}}\nabla(\nabla\lambda^{\mathrm{T}})\mathbf{d}_*=0. \tag{11.10}$$

From (8.1), or rather, from:

$$\nabla\psi\cdot\mathbf{d}\equiv 0\,,$$

which is equivalent to it, one further deduces upon differentiating with respect to φ_k:

$$|\nabla\varphi|\,\nabla\lambda\cdot\frac{\partial\mathbf{d}}{\partial\varphi_k}=-\nabla\Lambda^k\cdot\mathbf{d}\,, \tag{11.11}$$

and (11.6) may be simply written:

$$\chi-\chi_0+\sigma\left\{|\nabla\varphi|\,\mathbf{d}^{\mathrm{T}}\nabla(\nabla\lambda^{\mathrm{T}})\mathbf{d}^*\pi_0^3+\nabla\Lambda_*^k\cdot\mathbf{d}_*\pi_0\partial_{k_0}\pi_0+\frac{1}{2}\frac{\partial\Lambda_*^k}{\partial\varphi_l}\partial_{k_0l_0}\pi_0\right\}=0. \tag{11.12}$$

BOOK TWO

APPLICATIONS. EXAMPLES

I. – Classical electromagnetism.

1. **The caustics of optics. –** We make no pretense of great originality in briefly summarizing the now classical study of luminous waves here, but we have to find an example that illustrates some of the simplest phenomena of linear shocks. Therefore, starting with the Maxwell equations:

$$\dot{\vec{\mathrm{H}}} + \mathrm{rot}\,\vec{\mathrm{E}} = 0, \qquad \dot{\vec{\mathrm{E}}} - \mathrm{rot}\,\vec{\mathrm{H}} = 0, \tag{1.1}$$

$$\mathrm{div}\,\vec{\mathrm{H}} = 0, \qquad \mathrm{div}\,\vec{\mathrm{E}} = 0, \tag{1.2}$$

and introduce delta symbol to denote the first order discontinuities, the set of which constitutes the vector **p**, we then find:

$$-\lambda\delta\vec{\mathrm{H}} + \vec{n}\times\delta\vec{\mathrm{E}} = 0, \qquad \lambda\delta\vec{\mathrm{E}} + \vec{n}\times\delta\vec{\mathrm{H}} = 0, \tag{1.3}$$

$$\vec{n}\cdot\delta\vec{\mathrm{H}} = 0, \qquad \vec{n}\cdot\delta\vec{\mathrm{E}} = 0. \tag{1.4}$$

These latter constraints prohibit the existence of longitudinal stationary waves, which permits only the equations (1.3):

$$\delta\vec{\mathrm{H}} = \pi^{(1)}\vec{n}, \qquad \delta\vec{\mathrm{E}} = \pi^{(2)}\vec{n}.$$

Therefore, if one desires that λ not be null, one obtains:

$$(\lambda^2 - 1)\delta\vec{\mathrm{E}} + (\vec{n}\cdot\delta\vec{\mathrm{E}})\vec{n} = 0,$$

namely:

$$\lambda = \pm 1. \tag{1.5}$$

Equations (1.3) show further that:

$$\delta\vec{\mathrm{H}} = \pm\pi\vec{h}_0,$$
$$\delta\vec{\mathrm{E}} = \pi\vec{e}_0,$$

in which $\vec{h}_0$ and $\vec{e}_0$ are unit vectors that are functions of the point M_0 on S_0 and are such that they form a directed orthonormal frame $(\vec{e}_0, \vec{h}_0, \vec{n})$ with $\vec{n}$. By reason of (1.5), the surfaces S(t) are parallel, and:

(1.6) $$\pi = \frac{\pi_0}{\theta},$$

in which θ has the expression (6.7), which is annulled on a two-sheeted surface, the geometric locus of the centers of curvature of the wave surface S_0. We have already recognized the caustic, whose brilliance manifests the appearance of linear shocks. One knows the importance that optics attaches to the determination of that surface. Given the experimental evidence, it permits one to calculate, *a posteriori*, the form of the wave surfaces [12]. On the other hand, the luminous intensity, which is proportional to $\sqrt{\mathrm{E}^2 + \mathrm{H}^2}$, is known from its variation on that of the field. The latter will present a maximum on the focal surface.

II. – **General relativity.**

2. **Gravitational waves. –** When one uses isothermal coordinates that satisfy the de Donder conditions:

(2.1) $$g^{\alpha\beta}\partial_\beta g_{\gamma\alpha} - \frac{1}{2} g^{\alpha\beta}\partial_\gamma g_{\alpha\beta} = 0,$$

the Einstein equations for the external case may be written [13]:

(2.2) $$\mathrm{R}_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta}\partial_{\alpha\beta} g_{\mu\nu} + \mathrm{F}_{\mu\nu} = 0,$$

in which the $\mathrm{F}_{\mu\nu}$ are quadratic forms of the first derivatives of the gravitational potentials $g_{\alpha\beta}$. The latter are continuous, along with their first derivatives, whereas discontinuities $\pi_{\alpha\beta} = [\partial_{\varphi\varphi} g_{\alpha\beta}]$ may exist in the second derivatives [14] upon crossing the characteristic hypersurfaces that satisfy:

(2.3) $$g^{\alpha\beta}\varphi_\alpha\varphi_\beta = 0.$$

The exception relation (I, 8.2) is verified since λ depends only upon the $g_{\alpha\beta}$, which possess continuous first derivatives. (The Einstein system of equations then corresponds to the type (I, 8.3).) In pseudo-euclidian space:

$$g_{\alpha\beta*} = \eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1), \tag{2.4}$$

the waves are parallel, and their velocity of propagation is equal to that of light in vacuo:

$$\lambda_* = \pm 1. \tag{2.5}$$

The system (2.2) is obviously hyperbolic for the values (2.4). If we set:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} ,$$

and make the weak field approximation (see, for example [15]) then one deduces:

$$\Box h_{\alpha\beta} = 0.$$

As a consequence, the discontinuities obey a law that is identical to (1.6), namely [16]:

$$\pi_{\alpha\beta} = \frac{\pi^0_{\alpha\beta}}{\theta} . \tag{2.6}$$

In addition, Lichnerowicz has shown (*op. cit.*) that the $\pi^0_{0\alpha}$ are stripped of all physical significance and that one may always assume that they are null. Finally, when one accounts for the restrictions that are expressed by (2.1), one notes that:

$$\begin{aligned} \pi_{0\alpha} &= 0 \\ \delta^{ij}\pi_{ij} &= 0 \\ n^j\pi_{ij} &= 0, \end{aligned} \tag{2.7}$$

in such a way that the initial discontinuities depend only upon two arbitrary functions of the point M_0.

Physically, the π_{ij} have the effect of creating a discontinuity in the relative acceleration of the two particles, which are situated on either side of the wave front. That discontinuity then varies with their positions; it is null if they are aligned along the normal $\vec{n}$, which shows the transverse nature of gravitational waves [17].

III. – **Magnetohydrodynamics.**

3. **The Lundquist equations, Alfvén waves.** – The Lundquist equations relate the magnetic field $\vec{b}$, the fluid velocity $\vec{u}$, its density ρ, and the entropy S, by introducing the sound velocity $c(\rho, \mathrm{S})$ and a constant μ (the magnetic permeability), according to [18]:

(3.1)
$$\begin{cases} \dot{\vec{b}} + \mathrm{rot}(\vec{b}\times\vec{u}) = 0 \\ \dot{\vec{u}} + (\vec{u}\cdot\nabla)\vec{u} + \dfrac{c^2}{\rho}\nabla\rho + \dfrac{\vec{b}}{\mu\rho}\times\mathrm{rot}\,\vec{b} = 0 \\ \dot{\rho} + \mathrm{div}(\rho u) = 0 \\ \dot{\mathrm{S}} + (\vec{u}\cdot\nabla)\mathrm{S} = 0. \end{cases}$$

Outside of the contact surfaces that displace with the fluid, there exist three possible modes of propagation, which are associated with Alfvén waves, fast (supersonic) waves, and slow (subsonic) waves, respectively [19]. (In general relativity, the corresponding velocities have been determined by Choquet-Bruhat [20].)

The Alfvén waves with the velocities:

(3.2)
$$\lambda = u_n \pm v_n,$$

or:

(3.3)
$$\vec{v} = \frac{\vec{b}}{\sqrt{\mu\rho}}, \qquad u_n = \vec{u}\cdot\vec{n}, \qquad \text{etc.},$$

are transverse waves:

(3.4)
$$\mathbf{d} = \begin{pmatrix} \vec{n}\times\vec{b} \\ \pm\vec{v}\times\vec{n} \\ 0 \\ 0 \end{pmatrix}.$$

They are exceptional:

(3.5)
$$\boldsymbol{\nabla}\lambda = \left(\pm\frac{\vec{n}}{\sqrt{\rho\mu}}, \quad \vec{n}, \quad -\frac{1}{2}\frac{v_n}{\rho}, \quad 0 \right),$$

and (I, 8.1) is verified, and since:

(3.6)
$$\vec{\Lambda} = \vec{u} \pm \vec{v},$$

they are displaced by translation (see book I, sec 6),

(3.7)
$$\boldsymbol{\pi} = \boldsymbol{\pi}_0.$$

As far as second order discontinuities are concerned:

(3.8)
$$\boldsymbol{\nabla}\mathbf{d}\;\mathbf{d} = \begin{pmatrix} b_n\vec{n} - \vec{b} \\ \pm(\vec{v} - v_n\vec{n}) \\ 0 \\ 0 \end{pmatrix},$$

from which:

$$\nabla\lambda\, \nabla\mathbf{d}\ \mathbf{d} \equiv 0.$$

We likewise confirm that one has:

$$\nabla\vec{\Lambda}\cdot\mathbf{d} \equiv 0,$$

and that the plane waves lead to (cf. I, 11.7, 12):

(3.9) $$\chi = \chi_0.$$

4. **Fast and slow waves. –** They are defined by the velocities:

(4.1) $$\lambda = u_n + w, \qquad w^2 = \frac{1}{2}(c^2 + v^2 \pm \sqrt{\Delta}\,),$$

(4.2) $$\Delta = (c^2 + v^2)^2 - 4c^2 v_n^2 = (c^2 + v^2)^2 + 4c^2(v^2 - v_n^2) \geq 0,$$

(4.3) $$\vec{\Lambda} = \vec{u} + w\vec{n} \pm \frac{c^2 v_n}{w\sqrt{\Delta}}(\vec{n}v_n - \vec{v}),$$

and the proper vectors:

(4.4) $$\mathbf{d} = \begin{pmatrix} \vec{b} - b_n\vec{n} \\ \pm\left(w\vec{n} - \dfrac{v_n}{w}\vec{v}\right) \\ \rho\left(1 - \dfrac{v_n^2}{w^2}\right) \\ 0 \end{pmatrix}.$$

A calculation then gives:

$$\frac{\partial\lambda}{\partial\vec{b}} = \pm\frac{1}{\sqrt{\mu\rho}\sqrt{\Delta}}\left(w\vec{v} - c^2\frac{v_n}{w}\vec{n}\right)$$

$$\frac{\partial\lambda}{\partial\vec{u}} = \vec{n}$$

$$\frac{\partial\lambda}{\partial\rho} = \pm\frac{1}{2\sqrt{\Delta}}\left\{\left(2cc' - \frac{v^2}{\rho}\right)w - \left(2cc' - \frac{c^2}{\rho}\right)\frac{v_n^2}{w}\right\}$$

with:

$$c > 0, \qquad c' = \frac{\partial c}{\partial\rho} > 0.$$

If is convenient to introduce the quantity:

(4.5) $$q(\mathbf{u}, \vec{n}) = \frac{v_n^2}{w^2},$$

by means of which one easily expresses that [21]:

(4.6) $$\nabla\lambda \cdot \mathbf{d} = -w(q-1)\mathrm{Q}(q),$$

with:

(4.7) $$\mathrm{Q}(q) = \frac{1}{2}\left(\frac{c'}{c}\rho + \frac{5}{2}\right)\frac{q - q_1}{q - q_2},$$

(4.8) $$q_1 = 1 + \frac{3(v^2 - c^2)}{2cc'\rho + 5c^2}, \qquad q_2 = \frac{1}{2}\left(1 + \frac{v^2}{c^2}\right).$$

Thanks to (4.4), it then follows that:

(4.9) $$[\dot{\rho}] = |{=}\varphi|\,\lambda\pi\rho(q-1),$$

in such a way that if one assumes that one is propagating in a fluid at rest $(\vec{u} = 0)$ in which there is a constant magnetic field $\vec{b}$ then:

(4.10) $$|\nabla\varphi|\,\nabla\lambda \cdot \boldsymbol{\pi} = -\frac{[\dot{\rho}]}{\rho}\mathrm{Q}(q).$$

In the case where $v < c$, $\frac{c'}{c}\rho < \frac{1}{2}$:

$$\frac{v^2}{c^2} < q_1 < q_2 < 1,$$

the inequalities:

$$0 \,[\, v_n^2 \,[\, v^2,$$

impose the following domain of variation upon q:

For fast waves:

$$0 \,[\, q \,[\, \frac{v^2}{c^2},$$

For slow waves:

$$0 \,[\, q \,[\, 1 + \frac{v^2}{c^2}.$$

As a result, for the set of shock waves one has:

(4.11) $$\frac{c'}{c}\rho + 1 \,[\, \mathrm{Q} \,[\, \frac{3}{2},$$

and the latter inequalities will change their sense if $\frac{c'}{c}\rho$ is greater than $\frac{1}{2}$.

Therefore, just as in aerodynamics (where w = !c, Q = 1 + $\frac{c'}{c}\rho$) shocks will be produced if the fluid experiences a compression (see I, 8.6):

$$[\dot{\rho}] > 0 \tag{4.12}$$

during the passage of a fast or slow wave. It will suffice that this condition is initially satisfied.

IV. – **Nonlinear electrodynamics**

5. **The field equations**. – The components of the electric field vector $\vec{\mathrm{E}}$ and the magnetic field vector $\vec{\mathrm{H}}$ on the axes of an orthonormal frame may be expressed in terms of a world vector (q^α) by the formulae [22]:

$$\begin{aligned} &\mathrm{H}^1 = q_2^3 - q_3^2, && \mathrm{E}^1 = q_1^0 - q_0^1 \\ &\mathrm{H}^2 = q_3^1 - q_1^3, && \mathrm{E}^2 = q_2^0 - q_0^2 \\ &\mathrm{H}^3 = q_1^2 - q_2^1, && \mathrm{E}^3 = q_3^0 - q_0^3 \end{aligned} \tag{5.1}$$

in which one has set:

$$q_\beta^\alpha = \partial_\beta q^\alpha \qquad (x^0 = t,\ c = 1),$$

and which have the immediate consequence that:

$$\dot{\vec{\mathrm{H}}} + \mathrm{rot}\,\vec{\mathrm{E}} = 0, \tag{5.2}$$

$$\mathrm{div}\,\vec{\mathrm{H}} = 0. \tag{5.3}$$

On the other hand, one knows that the electromagnetic field tensor:

$$\mathrm{F}_{\alpha\beta} = \partial_\alpha q_\beta - \partial_\beta q_\alpha \tag{5.4}$$

permits us to construct two invariants [13][14]:

$$\mathrm{Q} = \frac{1}{2}(\vec{\mathrm{H}}^2 - \vec{\mathrm{E}}^2), \qquad \mathrm{R} = \vec{\mathrm{E}} \cdot \vec{\mathrm{H}}, \tag{5.5}$$

which will be conveniently expressed in the following fashion in the rest of this section:

$$\begin{aligned} 2\mathrm{Q} &= q_i^j q_i^j - q_j^i q_i^j - q_i^0 q_i^0 - q_0^i q_0^i + 2 q_i^0 q_0^i, \\ \mathrm{R} &= \varepsilon_{ijk}(q_j^0 q_k^i - q_0^j q_k^i), \end{aligned} \tag{5.6}$$

in which we have used the permutation indicator ε_{ijk}, which equals +1 if *ijk* constitutes an even permutation of the natural sequence 123, –1 if it constitutes an odd permutation, and 0 in all other cases.

If one does not impose the condition of linearity on the field equations then one may take the Lagrangian density to be a function of the two quantities Q, R [9]:

$$\mathcal{L} = \mathcal{L}(\mathrm{Q}, \mathrm{R}). \tag{5.7}$$

The variational principle:

(5.8) $$\delta \int \mathcal{L}\, d\mathcal{V} = 0$$

translates into the Euler equations [15]:

(5.9) $$\partial_\beta \left(\frac{\partial \mathrm{L}}{\partial q^\alpha_\beta} \right) = \frac{\partial \mathrm{L}}{\partial q^\alpha}\,.$$

One has:

$$\frac{\partial \mathrm{L}}{\partial q^\alpha_\beta} = \mathrm{L}_\mathrm{Q}\frac{\partial \mathrm{Q}}{\partial q^\alpha_\beta} + \mathrm{L}_\mathrm{R}\frac{\partial \mathrm{R}}{\partial q^\alpha_\beta}\,, \qquad\qquad \frac{\partial \mathrm{L}}{\partial q^\alpha} \equiv 0,$$

since one has, upon using the expressions (5.6):

(5.10) $$\begin{aligned} &\frac{\partial \mathrm{Q}}{\partial q^i_0} = -\frac{\partial \mathrm{Q}}{\partial q^0_i} = \mathrm{E}^i\,, \quad \frac{\partial \mathrm{R}}{\partial q^i_0} = -\frac{\partial \mathrm{R}}{\partial q^0_i} = -\mathrm{H}^i\,, \\ &\frac{\partial \mathrm{Q}}{\partial q^i_j} = -\varepsilon_{ijk}\mathrm{H}^k\,, \qquad \frac{\partial \mathrm{R}}{\partial q^i_j} = -\varepsilon_{ijk}\mathrm{E}^k\,. \end{aligned}$$

Finally, equations (5.9) may be written:

(5.11) $$\begin{aligned} &(\mathcal{L}_{\mathrm{QQ}}\dot{\mathrm{Q}} + \mathcal{L}_{\mathrm{QR}}\dot{\mathrm{R}})\,\vec{\mathrm{E}} + \vec{\mathrm{H}}\times(\mathcal{L}_{\mathrm{QQ}}\nabla\mathrm{Q} + \mathcal{L}_{\mathrm{QR}}\nabla\mathrm{R}) + \mathcal{L}_{\mathrm{Q}}(\dot{\vec{\mathrm{E}}} - \mathrm{rot}\,\vec{\mathrm{H}}) \\ &\quad - (\mathcal{L}_{\mathrm{QR}}\dot{\mathrm{Q}} + \mathcal{L}_{\mathrm{RR}}\dot{\mathrm{R}})\,\vec{\mathrm{H}} + \vec{\mathrm{E}}\times(\mathcal{L}_{\mathrm{QR}}\nabla\mathrm{Q} + \mathcal{L}_{\mathrm{RR}}\nabla\mathrm{R}) + \mathcal{L}_{\mathrm{R}}(\dot{\vec{\mathrm{H}}} - \mathrm{rot}\,\vec{\mathrm{E}}) = 0, \end{aligned}$$

(5.12) $$(\mathcal{L}_{\mathrm{QR}}\vec{\mathrm{H}} - \mathcal{L}_{\mathrm{RR}}\vec{\mathrm{E}})\nabla\mathrm{Q} + (\mathcal{L}_{\mathrm{RR}}\vec{\mathrm{H}} - \mathcal{L}_{\mathrm{QR}}\vec{\mathrm{E}})\nabla\mathrm{R} + \mathcal{L}_{\mathrm{R}}\mathrm{div}\,\dot{\vec{\mathrm{H}}} - \mathcal{L}_{\mathrm{Q}}\mathrm{div}\,\vec{\mathrm{E}} = 0.$$

(5.2) and (5.11), along with (5.5), constitute a system of six partial differential equations for the components of the vectors $\vec{\mathrm{E}}, \vec{\mathrm{H}}$, which are subject to the constraints (5.3, 12), in addition.

6. **The characteristic equation.** – We once more denote the first order discontinuities by $\delta\vec{\mathrm{E}}, \delta\vec{\mathrm{H}}$, as in the first section. Equation (5.2) gives:

(6.1) $$\lambda\delta\vec{\mathrm{H}} = \vec{n}\times\delta\vec{\mathrm{E}}\,,$$

from which, if we suppose that $\lambda \neq 0$:

$$\delta Q = \left(\frac{\vec{H}\times\vec{n}}{\lambda} - \vec{E}\right)\delta\vec{E}, \tag{6.2}$$
$$\delta R = \left(\frac{\vec{E}\times\vec{n}}{\lambda} + \vec{H}\right)\delta\vec{E}.$$

If one multiplies (5.11) by λ then one obtains:

$$\begin{aligned}
&(\{\mathcal{L}_{QQ}(\vec{H}\times\vec{n} - \lambda\vec{E}) + \mathcal{L}_{QR}(\vec{E}\times\vec{n} + \lambda\vec{H})\}\cdot\delta\vec{E})(\vec{H}\times\vec{n} - \lambda\vec{E}) \\
&+ (\{\mathcal{L}_{QR}(\vec{H}\times\vec{n} - \lambda\vec{E}) + \mathcal{L}_{RR}(\vec{E}\times\vec{n} + \lambda\vec{H})\}\cdot\delta\vec{E})(\vec{E}\times\vec{n} + \lambda\vec{H}) \\
&- \mathcal{L}_Q\{(\lambda^2 - 1)\,\delta\vec{E} + (\vec{n}\cdot\delta\vec{E})\vec{n}\} = 0.
\end{aligned} \tag{6.3}$$

As for (5.3) and (5.12), they simply project (6.1) and (6.3) onto the normal to the wave surface. In revenge, when λ is null they provide conditions that are independent of the ones that one obtains by starting with (5.2, 11) (cf. sec. 7).

If we suppose that the vectors $\vec{E}, \vec{H}$ are not collinear then we may write:

$$\delta\vec{E} = a^1\vec{H} + a^2\vec{E} + a^3\vec{S}, \tag{6.4}$$

in which we have introduced the Poynting vector:

$$\vec{S} = \vec{E}\times\vec{H}. \tag{6.5}$$

When the decomposition (6.4) that was introduced in (6.3) is successively multiplied by $\vec{H}, \vec{E}, \vec{S}$, one obtains three scalar equations that one may transcribe in the form [23]:

$$(\lambda^2\alpha_{ij} + \lambda\beta_{ij} + \gamma_{ij})\,a^j = 0, \qquad (i, j = 1, 2, 3). \tag{6.6}$$

The various coefficients are symmetric in their indices, and calculation gives:

$$\begin{aligned}
\alpha_{11} &= R^2\mathcal{L}_{QQ} - 2\,H^2\,R\mathcal{L}_{QR} + H^4\mathcal{L}_{RR} - H^2\,\mathcal{L}_Q \\
\alpha_{12} &= E^2R\,\mathcal{L}_{QQ} - (R^2 + E^2H^2)\,\mathcal{L}_{QR} + H^2\,R\,\mathcal{L}_{RR} - R\,\mathcal{L}_Q \\
\alpha_{13} &= 0 \\
\alpha_{22} &= E^4\mathcal{L}_{QQ} - 2E^2\,R\,\mathcal{L}_{QR} + R^2\mathcal{L}_{RR} - E^2\mathcal{L}_Q \\
\alpha_{23} &= 0 \\
\alpha_{33} &= -\,S^2\mathcal{L}_Q.
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
\beta_{11} &= 2\,S_n(R\,\mathcal{L}_{QR} - H^2\mathcal{L}_{PP}) \\
\beta_{12} &= -\,S_n\{R\,\mathcal{L}_{QQ} - (E^2 + H^2)\,\mathcal{L}_{QR} + R\,\mathcal{L}_{RR}\}
\end{aligned}$$

$$\begin{aligned}
&\beta_{13} = (\mathrm{H}^2\mathrm{E}_n - \mathrm{RH}_n)(\mathrm{R}\mathcal{L}_{\mathrm{QQ}} - \mathrm{H}^2\mathcal{L}_{\mathrm{QR}}) - (\mathrm{E}^2\mathrm{H}_n - \mathrm{RE}_n)(\mathrm{R}\mathcal{L}_{\mathrm{QR}} - \mathrm{H}^2\mathcal{L}_{\mathrm{RR}})\\
(6.8)\qquad &\beta_{22} = 2\mathrm{S}_n(\mathrm{R}\mathcal{L}_{\mathrm{QR}} - \mathrm{E}^2\mathcal{L}_{\mathrm{QQ}})\\
&\beta_{23} = (\mathrm{H}^2\mathrm{E}_n - \mathrm{RH}_n)(\mathrm{E}^2\,\mathcal{L}_{\mathrm{QQ}} - \mathrm{R}\,\mathcal{L}_{\mathrm{QR}}) - (\mathrm{E}^2\mathrm{H}_n - \mathrm{RE}_n)(\mathrm{E}^2\mathcal{L}_{\mathrm{QR}} - \mathrm{R}\,\mathcal{L}_{\mathrm{RR}})\\
&\beta_{23} = 0.
\end{aligned}$$

$$\begin{aligned}
&\gamma_{11} = \mathrm{S}_n^2\mathcal{L}_{\mathrm{RR}} + (\mathrm{H}^2 - \mathrm{H}_n^2)\mathcal{L}_{\mathrm{Q}}\\
&\gamma_{12} = -\mathrm{S}_n^2\mathcal{L}_{\mathrm{QR}} + (\mathrm{R} - \mathrm{E}_n\mathrm{H}_n)\mathcal{L}_{\mathrm{Q}}\\
&\gamma_{13} = \mathrm{S}_n\{(\mathrm{H}^2\mathrm{E}_n - \mathrm{RH}_n)\mathcal{L}_{\mathrm{QR}} - (\mathrm{E}^2\mathrm{H}_n - \mathrm{RE}_n)\mathcal{L}_{\mathrm{RR}} - \mathrm{H}_n\mathcal{L}_{\mathrm{Q}}\}\\
(6.9)\qquad &\gamma_{22} = \mathrm{S}_n^2\mathcal{L}_{\mathrm{QQ}} + (\mathrm{E}^2 - \mathrm{E}_n^2)\mathcal{L}_{\mathrm{Q}}\\
&\gamma_{23} = \mathrm{S}_n\{(\mathrm{E}^2\,\mathrm{H}_n - \mathrm{RE}_n)\,\mathcal{L}_{\mathrm{QR}} - (\mathrm{H}^2\mathrm{E}_n - \mathrm{RH}_n)\,\mathcal{L}_{\mathrm{QQ}} - \mathrm{E}_n\,\mathcal{L}_{\mathrm{Q}}\}\\
&\gamma_{33} = (\mathrm{H}^2\mathrm{E}_n - \mathrm{RH}_n)^2\,\mathcal{L}_{\mathrm{QQ}} - 2(\mathrm{H}^2\mathrm{E}_n - \mathrm{RH}_n)(\mathrm{E}^2\mathrm{H}_n - \mathrm{RE}_n)\,\mathcal{L}_{\mathrm{QR}}\\
&\qquad\qquad + (\mathrm{E}^2\mathrm{H}_n - \mathrm{RE}_n)^2\,\mathcal{L}_{\mathrm{RR}} + (\mathrm{S}^2 - \mathrm{S}_n^2)\mathcal{L}_{\mathrm{Q}}
\end{aligned}$$

with:

$$\mathrm{S}_n = \vec{\mathrm{S}}\cdot\vec{n}\,,\qquad \text{etc.} \tag{6.10}$$

The proper values, which are solutions of the characteristic equation:

$$\mathcal{D}(\lambda^2\alpha_{ij} + \lambda\beta_{ij} - \gamma_{ij}) = 0, \tag{6.11}$$

are roots of a sixth degree polynomial, provided that:

$$\mathcal{D}(\alpha_{ij}) = -\,\mathrm{S}^4\mathcal{L}_{\mathrm{Q}}\{\mathrm{S}^2(\mathcal{L}_{\mathrm{QQ}}\,\mathcal{L}_{\mathrm{RR}} - \mathcal{L}_{\mathrm{QR}}^2) - (\mathrm{E}^2\mathcal{L}_{\mathrm{QQ}} + \mathrm{H}^2\mathcal{L}_{\mathrm{RR}} - 2\mathrm{R}\,\mathcal{L}_{\mathrm{QR}} - \mathcal{L}_{\mathrm{Q}})\,\mathcal{L}_{\mathrm{Q}}\} \tag{6.12}$$

are different from zero. This condition (which expresses the regularity of the matrix A^0: cf. book I, sec. 2) must be verified in order for all of the propagation velocities to be finite. One thus supposes that:

$$\mathcal{L}_{\mathrm{Q}} \neq 0. \tag{6.13}$$

One confirms that the expression in braces is always strictly positive when the following inequalities are, moreover, simultaneously satisfied:

$$\mathcal{L}_{\mathrm{QQ}}\,\mathcal{L}_{\mathrm{RR}} - \mathcal{L}_{\mathrm{QR}}^2 \geq 0, \qquad \mathcal{L}_{\mathrm{Q}}\,(\mathcal{L}_{\mathrm{RR}} + \mathcal{L}_{\mathrm{RR}}) \leq 0. \tag{6.14}$$

The characteristic polynomial involves terms of odd degree that, under the hypothesis that the three second derivatives of the Lagrangian are not simultaneously null, disappear only with S_n; one verifies, for example, that the coefficient of λ^5 is equal to:

$$2\,\mathrm{S}^4\,\mathrm{S}_n\,\mathcal{L}_{\mathrm{Q}}\{(\mathrm{E}^2 + \mathrm{H}^2)(\mathcal{L}_{\mathrm{QQ}}\,\mathcal{L}_{\mathrm{RR}} - \mathcal{L}_{\mathrm{QR}}^2) - \mathcal{L}_{\mathrm{Q}}(\mathcal{L}_{\mathrm{RR}} + \mathcal{L}_{\mathrm{RR}})\}. \tag{6.15}$$

As a consequence, there exist certain waves that propagate with various velocities in the two senses of a direction that is not perpendicular to the Poynting vector. In this fashion, the privileged role that this vector plays appears already. It is confirmed when one verifies that a symmetry with respect to $\vec{\mathrm{S}}$ leaves the spectrum invariant. In effect, considering a vector that is symmetric about $\vec{n}$ amounts to performing the substitution:

$$\mathrm{H}_n \to -\mathrm{H}_n, \qquad \mathrm{E}_n \to -\mathrm{E}_n,$$

which changes only the quantities β_{13}, β_{23}, γ_{13}, γ_{23} into their opposites. If one develops the determinant (6.11) in one's mind then it is easy to confirm its invariance.

Taking into account the identity:

$$\mathrm{S}^2 - \mathrm{S}_n^2 - \mathrm{E}^2\mathrm{H}_n^2 - \mathrm{H}^2\mathrm{E}_n^2 + 2\mathrm{R}\,\mathrm{E}_n\,\mathrm{H}_n \equiv 0, \tag{6.16}$$

which results from:

$$\vec{\mathrm{S}} = -\mathrm{E}_n\vec{\mathrm{H}}\times\vec{n} + \mathrm{H}_n\vec{\mathrm{E}}\times\vec{n} + \mathrm{S}_n\vec{n},$$

as well as from:

$$\mathrm{S}^2 \equiv \mathrm{E}^2\,\mathrm{H}^2 - \mathrm{R}^2, \tag{6.17}$$

which follows immediately from the definition (6.5), then the determinant $\mathcal{D}$ may be developed according to:

$$\mathcal{D} = -\,\mathrm{S}^4\,\mathcal{L}_{\mathrm{Q}}^2\,\lambda^2\,\mathfrak{P}(\mathbf{u}, \lambda),$$

with:

$$\mathfrak{P}(\mathbf{u}, \lambda) = \omega\,\mathfrak{P}_1 - \mathfrak{P}_2, \tag{6.18}$$

$$\begin{aligned}\mathfrak{P}_1 = {} & \mathrm{S}^2\lambda^2 - 2\mathrm{S}_n(\mathrm{E}^2 + \mathrm{H}^2)\lambda^3 \\ & + \{(\mathrm{E}^2 + \mathrm{H}^2)(\mathrm{E}^2 + \mathrm{H}^2 - \mathrm{E}_n^2 - \mathrm{H}_n^2) + 2(2\mathrm{S}_n^2 - \mathrm{S}^2)\}\lambda^2 \\ & - 2\mathrm{S}_n(\mathrm{E}^2 + \mathrm{H}^2 - 2(\mathrm{E}_n^2 + \mathrm{H}_n^2))\}\lambda \\ & + \mathrm{S}^2 - (\mathrm{E}^2 + \mathrm{H}^2 - \mathrm{E}_n^2 - \mathrm{H}_n^2)(\mathrm{E}_n^2 + \mathrm{H}_n^2),\end{aligned} \tag{6.19}$$

$$\begin{aligned}\mathfrak{P}_2 = {} & (\lambda^2 - 1)\{(\mathrm{E}^2\mathcal{L}_{\mathrm{QQ}} + \mathrm{H}^2\,\mathcal{L}_{\mathrm{RR}} - 2\mathrm{R}\,\mathcal{L}_{\mathrm{QR}} - \mathcal{L}_{\mathrm{Q}})\lambda^2 - 2\mathrm{S}_n(\mathcal{L}_{\mathrm{QQ}} + \mathcal{L}_{\mathrm{RR}})\lambda \\ & + \mathrm{H}^2\,\mathcal{L}_{\mathrm{QQ}} + \mathrm{E}^2\,\mathcal{L}_{\mathrm{RR}} + 2\mathrm{R}\,\mathcal{L}_{\mathrm{QR}} + \mathcal{L}_{\mathrm{Q}} - (\mathcal{L}_{\mathrm{QQ}} + \mathcal{L}_{\mathrm{RR}})(\mathrm{E}_n^2 + \mathrm{H}_n^2)\}\end{aligned} \tag{6.20}$$

and:

$$\omega = \frac{\mathcal{L}_{\mathrm{QQ}}\mathcal{L}_{\mathrm{RR}} - \mathcal{L}_{\mathrm{QR}}^2}{\mathcal{L}_{\mathrm{Q}}}. \tag{6.21}$$

7. **Stationary waves.** – Up till now, we have assumed that λ is non-null. Now, the root $\lambda = 0$ exists, and we shall show that it is a double root. First of all, (6.1) implies that:

$$\delta\vec{\mathrm{E}} = \pi\vec{n} \tag{7.1}$$

whereas, referring to (5.11), one obtains:

(7.2) $$(\mathcal{L}_{QQ}\delta Q + \mathcal{L}_{QR}\delta R)\,\vec{H}\times\vec{n} + (\mathcal{L}_{QR}\delta Q + \mathcal{L}_{RR}\delta R)\,\vec{E}\times\vec{n} + \mathcal{L}_Q\vec{n}\times\delta\vec{H} = 0.$$

From (5.3, 12), it then follows that:

(7.3) $$\vec{n}\cdot\delta\vec{H} = 0,$$

(7.4) $$(\mathcal{L}_{QQ}\delta Q + \mathcal{L}_{QR}\delta R)E_n - (\mathcal{L}_{QR}\delta Q + \mathcal{L}_{RR}\delta R)H_n + \pi\,\mathcal{L}_Q = 0.$$

In all of the cases that one may pose, one has, in view of (7.2):

(7.5) $$\delta\vec{H} = a^{1'}H + a^{2'}\vec{E} + a^{3'}\vec{n},$$

from which it results that:

(7.6) $$\delta Q = a^{1'}H^2 + a^{2'}R + a^{3'}H_n - \pi E_n,$$
$$\delta R = a^{1'}R + a^{2'}E^2 + a^{3'}E_n + \pi H_n.$$

Consider two cases:

1. $S_n = 0$. The vectors $\vec{H}\times\vec{n}$, $\vec{E}\times\vec{n}$ are collinear with $\vec{S}$: $a^{3'} = 0$, one of the remaining coefficients is arbitrary, and the other one may be deduced from equation (7.2). The set $\delta\vec{E}, \delta\vec{H}$ depends on two parameters; the proper value $\lambda = 0$ is double.

2. $S_n \neq 0$. One derives from (7.2) that:

(7.7) $$\mathcal{L}_{QQ}\delta Q + \mathcal{L}_{QR}\delta R = a^{1'}\mathcal{L}_Q,$$
$$\mathcal{L}_{QR}\delta Q + \mathcal{L}_{RR}\delta R = a^{2'}\mathcal{L}_Q.$$

π and $a^{3'}$ are arbitrary, $a^{1'}$ and $a^{2'}$ are determined by (7.6, 7). Indeed, one verifies that the system that is composed of these equations is regular; its determinant:

(7.8) $$S^2(\mathcal{L}_{QQ}\mathcal{L}_{RR} - L_{QR}^2) - \mathcal{L}_Q(H^2\mathcal{L}_{QQ} + E^2\mathcal{L}_{RR} + 2R\mathcal{L}_{QR} - \mathcal{L}_Q)$$

is strictly positive if the inequalities (6.14) are true.

It remains for us to include the constraints (7.3, 4). With (7.6, 7), they form a system of four homogeneous linear equations for the four unknowns $a^{1'}, a^{2'}, a^{3'}, \pi$, and are such that the determinant:

$$(\mathcal{L}_{QQ}\mathcal{L}_{RR} - \mathcal{L}_{QR}^2)\{S^2 - 2Q(E_n^2 - H_n^2) + (E_n^2 + H_n^2)^2 + 4R\,E_n H_n\}$$
$$- \mathcal{L}_Q\{(H^2 - H_n^2 + E_n^2)\mathcal{L}_{QQ} + 2(R - 2E_n H_n)\,\mathcal{L}_{QR} + (E^2 - E_n^2 + H_n^2)\,\mathcal{L}_{RR} - \mathcal{L}_Q\};$$

which, thanks to (6.16), is also:

$$(\mathcal{L}_{QQ}\,\mathcal{L}_{RR} - \mathcal{L}_{QR}^2)(S_n^2 + \{E_n(\vec{E} - E_n\vec{n}) + H_n(\vec{H} - H_n\vec{n})\}^2)$$
$$- \frac{\mathcal{L}_Q}{\mathcal{L}_{QQ} + \mathcal{L}_{RR}}(\{(\vec{H} - H_n\vec{n})\,\mathcal{L}_{QQ} + (\vec{E} - E_n\vec{n})\,\mathcal{L}_{QR}\}^2$$
$$+ (\vec{E} - E_n\vec{n})\,\mathcal{L}_{RR} + (\vec{H} - H_n\vec{n})\,\mathcal{L}_{QR}\}^2 + (E_n\,\mathcal{L}_{QQ} - H_n\,\mathcal{L}_{QR})^2$$
$$+ (H_n\,\mathcal{L}_{RR} - E_n\,\mathcal{L}_{QR})^2 + (\mathcal{L}_{QQ}\,\mathcal{L}_{RR} - \mathcal{L}_{QR}^2)(E^2 + H^2)) + \mathcal{L}_Q^2$$

is strictly positive if one accepts (6.14). As a consequence:

$$a^{1'} = a^{2'} = a^{3'} = \pi = 0;$$

there exist no stationary waves.

8. **Propagation at the fundamental velocity.** – Do there exist waves that propagate with the fundamental velocity, or, in other words, can one find non-null solutions $\delta\vec{E}$ of equation (6.3) for the values:

(8.1) $$\lambda = \pm 1?$$

When:

(8.2) $$\mathcal{L}_{QQ}\mathcal{L}_{RR} - \mathcal{L}_{QR}^2 = 0$$

the first two vectors between braces in the right-hand side of that equation are collinear; it suffices to choose $\delta\vec{E}$ to be perpendicular to the plane that it determines with $\vec{n}$:

(8.3) $$\begin{aligned}\delta\vec{E} &= \pi\{\mathcal{L}_{QQ}(H_n\vec{n} - \vec{H} \mp \vec{E}\times\vec{n}) + \mathcal{L}_{QR}(E_n\vec{n} - \vec{E} \pm \vec{H}\times\vec{n})\},\\ \delta\vec{H} &= \pi\{\mathcal{L}_{QQ}(E_n\vec{n} - \vec{E} \pm \vec{H}\times\vec{n}) - \mathcal{L}_{QR}(H_n\vec{n} - \vec{H} \mp \vec{E}\times\vec{n})\}.\end{aligned}$$

Now suppose that (8.2) is not satisfied. It is then necessary and sufficient that the three vectors $\vec{H}\times\vec{n} \mp \vec{E}$, $\vec{E}\times\vec{n} \pm \vec{H}$, $\vec{n}$ are coplanar, i.e., that the mixed product:

(8.4) $$((\vec{H}\times\vec{n} \mp \vec{E}, \vec{n}, \vec{E}\times\vec{n} \pm \vec{H})) = 2S_n \mp (H^2 - H_n^2 + E^2 - E_n^2) = 0.$$

From this, one deduces (see 6.16, 17):

(8.5) $$\begin{aligned}&(H^2 - H_n^2 + E^2 - E_n^2)^2 - 4S_n^2\\ &\quad = \{(H^2 - E^2) - (H_n^2 - E_n^2)\}^2 + 4(R - E_n H_n)^2 = 0.\end{aligned}$$

It is thus necessary that the following are true [23]:

(8.6) $$H_n^2 - E_n^2 = 2Q, \qquad E_n H_n = R$$

or:

(8.7) $$\mathrm{H}_n^2 = \mathrm{Q} + \sqrt{\mathrm{Q}^2 + \mathrm{R}^2}\,, \qquad \mathrm{E}_n^2 = -\,\mathrm{Q} + \sqrt{\mathrm{Q}^2 + \mathrm{R}^2}\,.$$

Therefore, there exist two directions of propagation that are symmetric with respect to the Poynting vector, which are found in the acute angle of the dihedral that is formed from the planes $(\vec{\mathrm{S}}, \vec{\mathrm{H}})$, $(\vec{\mathrm{S}}, \vec{\mathrm{E}})$, in which the waves attain the velocity of light. (It is remarkable that these directions do not depend on the nonlinear form of the Lagrangian.) It then results that the absolute value of a velocity that is less than (or greater than) 1 is everywhere (except possibly in the two indicated directions) strictly less than (or greater than) 1. With regard to (8.4), the sense of the displacement in the two distinguished directions is the one that makes $\vec{\mathrm{S}}$ an acute angle. In particular, if R = 0, Q > 0 then the two directions are situated in the plane $(\vec{\mathrm{S}}, \vec{\mathrm{H}})$, whereas they belong to the plane $(\vec{\mathrm{S}}, \vec{\mathrm{E}})$ if Q < 0. If Q = 0 then they make angles with $\vec{\mathrm{E}}$ and $\vec{\mathrm{H}}$ that are equal to the ones that these vectors make between them. They both agree with the support of $\vec{\mathrm{S}}$ when Q = R = 0.

9. **In a direction perpendicular to the Poynting vector.** – In such a direction the characteristic polynomial is quartic; one may then study the values of the velocity relative to the velocity of light. One has:

(9.1) $$\mathrm{S}_n = 0$$

and (6.18-20) gives us the right to write $\mathcal{L}_{\mathrm{Q}}\,\mathfrak{P} = 0$ in the form:

(9.2) $$\begin{aligned}(\mathcal{L}_{\mathrm{QQ}}\mathcal{L}_{\mathrm{RR}} - \mathcal{L}_{\mathrm{QR}}^2)\,\{\mathrm{S}^2(\lambda^2 - 1)^2 + (\mathrm{E}^2 + \mathrm{H}^2 - \mathrm{E}_n^2 - \mathrm{H}_n^2)^2\}&\\ - \mathcal{L}_{\mathrm{Q}}(\lambda^2 - 1)^2(\mathrm{E}^2\mathcal{L}_{\mathrm{QQ}} + \mathrm{H}^2\mathcal{L}_{\mathrm{RR}} - 2\mathrm{R}\,\mathcal{L}_{\mathrm{QR}} - \mathcal{L}_{\mathrm{Q}})&\\ + (\lambda^2 - 1)\,(\mathrm{E}^2 + \mathrm{H}^2 - \mathrm{E}_n^2 - \mathrm{H}_n^2)\,\{(\mathrm{E}^2 + \mathrm{H}^2)\,(\mathcal{L}_{\mathrm{QQ}}\mathcal{L}_{\mathrm{RR}} - \mathcal{L}_{\mathrm{QR}}^2)&\\ - \mathcal{L}_{\mathrm{Q}}(\mathcal{L}_{\mathrm{QQ}} + \mathcal{L}_{\mathrm{RR}})\} = 0.&\end{aligned}$$

If the inequalities (6.14) – which are strict inequalities – are true then the coefficient of $\lambda^2 - 1$ is positive and the same is true for the sum of the other terms. (We remark that:

$$\begin{aligned}&- \mathcal{L}_{\mathrm{Q}}(\mathrm{E}^2\mathcal{L}_{\mathrm{QQ}} + \mathrm{H}^2\mathcal{L}_{\mathrm{RR}} - 2\mathrm{R}\,\mathcal{L}_{\mathrm{QR}})\\ &\quad = \frac{-\mathcal{L}_{\mathrm{Q}}}{\mathcal{L}_{\mathrm{QQ}} + \mathcal{L}_{\mathrm{RR}}}\{(\vec{\mathrm{E}}\mathcal{L}_{\mathrm{QQ}} - \vec{\mathrm{H}}\mathcal{L}_{\mathrm{QR}})^2 + (\vec{\mathrm{H}}\mathcal{L}_{\mathrm{RR}} - \vec{\mathrm{E}}\mathcal{L}_{\mathrm{QR}})^2 + (\mathrm{E}^2 + \mathrm{H}^2)(\mathcal{L}_{\mathrm{QQ}}\mathcal{L}_{\mathrm{RR}} - \mathcal{L}_{\mathrm{QR}}^2)\}\end{aligned}$$

is positive.) As a consequence:

$$\lambda^2 < 1,$$

in all spatial directions (except for the ones for which equality applies) provided that λ exists (see the preceding section). Now, in the present case the discriminant of the quartic polynomial $\mathfrak{P}$, namely:

(9.3) $$(E^2 + H^2 - E_n^2 - H_n^2)^2 \{(\mathcal{L}_{QQ} - \mathcal{L}_{RR} - 2Q\omega)^2 + 4(\mathcal{L}_{QR} - R\omega)^2\}$$

is never negative. Furthermore, this does not suffice to assure the reality of the roots; it is also necessary that the coefficients of λ^4, $-\lambda^2$, and the independent term have the same sign, and that the inequalities (6.14) do not suffice to satisfy these latter conditions for all $\vec{E}$, $\vec{H}$, $\vec{n}$.

Assume (6.14). The conditions translate into:

(9.4) $$Y > 0,$$
(9.5) $$Z > 0,$$

with:

(9.6) $$Y = \{2S^2 - (E^2 + H^2)(E^2 + H^2 - E_n^2 - H_n^2)^2\} \mathcal{L}_Q \omega$$
$$+ \{2Q(\mathcal{L}_{QQ} - \mathcal{L}_{RR}) + 4R\mathcal{L}_{QR} + 2\mathcal{L}_Q - (\mathcal{L}_{QQ} + \mathcal{L}_{RR})(E_n^2 - H_n^2)\} \mathcal{L}_Q,$$

(9.7) $$Z = \{2S^2 - (E^2 + H^2)(E^2 + H^2 - E_n^2 - H_n^2)^2\} \mathcal{L}_Q \omega$$
$$+ \{H^2\mathcal{L}_{QQ} + E^2\mathcal{L}_{RR}) + 2R\mathcal{L}_{QR} + \mathcal{L}_Q - (\mathcal{L}_{QQ} + \mathcal{L}_{RR})(E_n^2 - H_n^2)\} \mathcal{L}_Q.$$

These inequalities must be satisfied for all directions in the plane ($\vec{E}$, $\vec{H}$). One is then led to determine the minima Y_m, Z_m of the functions Y(X), Z(X), which are linear and quadratic, respectively, in the variable:

(9.8) $$X = (E_n^2 - H_n^2)$$

over its domain of variation. This domain is calculated by searching for the extrema of X when considered as a function of E_n (for example), on account of the relation:

$$S^2 - E^2 H_n^2 - H^2 E_n^2 + 2R\, E_n H_n = 0.$$

(Cf. 6.16; 9.1.) One thus finds that X varies over the segment:

(9.9) $$\frac{1}{2}(E^2 + H^2) - \sqrt{Q^2 + R^2}, \qquad \frac{1}{2}(E^2 + H^2) + \sqrt{Q^2 + R^2}.$$

From this, one deduces:

(9.10) $$Y_m = -2(Q^2 + R^2)\mathcal{L}_Q\omega$$
$$+ \mathcal{L}_Q\{\mathcal{L}_{QQ}(2Q + \sqrt{Q^2 + R^2}) + \mathcal{L}_{RR}(-2Q + \sqrt{Q^2 + R^2}) + 4R\mathcal{L}_{QR} + 2\mathcal{L}_Q\}$$
$$- \mathcal{L}_Q\{\mathcal{L}_{QQ} + \mathcal{L}_{RR})(E^2 + H^2)\left(\sqrt{Q^2 + R^2}\,\varpi + \frac{1}{2}\right),$$

where:

(9.11) $$\varpi = \frac{\omega}{\mathcal{L}_{QQ} + \mathcal{L}_{RR}}.$$

For given Q, R the quantity $E^2 + H^2$ may takes values that are as large as one desires. One thus assumes:

(9.12) $$\sqrt{Q^2 + R^2}\,\varpi + \frac{1}{2} > 0,$$

namely:

(9.13) $$0 \leq -\varpi < \frac{1}{2\sqrt{Q^2 + R^2}}.$$

On the other hand, since:

(9.14) $$E^2 + H^2 > 2\sqrt{Q^2 + R^2},$$

one will have:

(9.15) $$Y_m > 2Y_1,$$

(9.16) $$Y_1 = -2(Q^2 + R^2)\mathcal{L}_Q\omega + \mathcal{L}_Q\{Q(\mathcal{L}_{QQ} - \mathcal{L}_{RR}) + 2R\mathcal{L}_{QR} + \mathcal{L}_Q\}.$$

On thus obtains:

(9.17) $$Z_m = \mathcal{L}_Q\{(Q + \sqrt{Q^2 + R^2})\mathcal{L}_{QQ} + (-Q + \sqrt{Q^2 + R^2})\mathcal{L}_{RR} + 2R\mathcal{L}_{QR} + \mathcal{L}_Q\}$$

and one sees that:

$$Y_1 = Z_m - 2\mathcal{L}_Q(\mathcal{L}_{QQ} + \mathcal{L}_{RR})\sqrt{Q^2 + R^2}\left(\sqrt{Q^2 + R^2}\,\varpi + \frac{1}{2}\right).$$

Note the sequence of implications:

$$Z_m > 0 \rightarrow Y_1 > 0 \rightarrow Y_m > 0.$$

In summary, the existence of roots is assured in any direction that is perpendicular to the Poynting vector when one assumes that the inequalities (6.14) are subject to the condition, besides (9.12), that the following inequality is satisfied:

(9.18) $$Z_m > 0.$$

10. **The determination of the velocities for the values** Q = R = 0**.** – In this eventuality:

(10.1) $$E^2 = H^2 = |\vec{S}| = S$$

and, from (6.16):

(10.2) $$S(E_n^2 + H_n^2) = S^2 - S_n^2,$$

which justifies the notation:

$$\mathfrak{P}_1 = \frac{1}{S^2}(S\lambda - S_n)^4,$$
$$\mathfrak{P}_2 = (\lambda^2 - 1)\left\{\frac{1}{S}(S\lambda - S_n)^2(\mathcal{L}_{QQ} + \mathcal{L}_{RR}) + \mathcal{L}_Q(\lambda^2 - 1)\right\}.$$

One must find solutions to:

(10.3) $$\omega(S\lambda - S_n)^4 - S(\lambda^2 - 1)(S\lambda - S_n)^2(\mathcal{L}_{QQ} + \mathcal{L}_{RR}) + S^2\mathcal{L}_Q(\lambda^2 - 1)^2 = 0.$$

If one sets:

(10.4) $$\frac{(S\lambda - S_n)^2}{1 - \lambda^2} = \mu$$

then this comes down to solving the equation:

(10.5) $$\omega\mu^2 + S(\mathcal{L}_{QQ} + \mathcal{L}_{RR})\mu + S^2\mathcal{L}_Q = 0,$$

a solution that is obtained without undue effort:

(10.6) $$\mu = \frac{-S\mathcal{L}_Q(\mathcal{L}_{QQ} + \mathcal{L}_{RR}) \pm \sqrt{\Delta}}{2(\mathcal{L}_{QQ}\mathcal{L}_{RR} - \mathcal{L}_{QR}^2)} \qquad (\omega \neq 0)$$

with:

(10.7) $$\begin{aligned}\Delta &= S^2\{(\mathcal{L}_{QQ} + \mathcal{L}_{RR})^2 - 4(\mathcal{L}_{QQ}\,\mathcal{L}_{RR} + \mathcal{L}_{QR}^2)\} \\ &= S^2\{(\mathcal{L}_{QQ} - \mathcal{L}_{RR})^2 + 4\,\mathcal{L}_{QR}^2\}.\end{aligned}$$

In turn, (10.4) shows that λ is a root of the trinomial:

(10.8) $$(S^2 + \mu)\lambda^2 - 2S\,S_n\lambda + S_n^2 - \mu = 0,$$

in which μ takes the values (10.6). In order for the reduced discriminant:

(10.9) $$\mu(S^2 - S_n^2 + \mu)$$

to be non-negative for any magnitude of S it is necessary and sufficient that the same be true for μ. This restriction (or rather, these restrictions) is equivalent to (6.14).

Finally:

(10.10) $$\lambda = \frac{\mathrm{SS}_n \pm \sqrt{\mu(\mathrm{S}^2 - \mathrm{S}_n^2 + \mu)}}{\mathrm{S}^2 + \mu}.$$

When $\omega = 0$ this formula is still valid when one makes:

(10.11) $$\mu = \frac{-\mathrm{S}\mathcal{L}_{\mathrm{Q}}}{\mathcal{L}_{\mathrm{QQ}} + \mathcal{L}_{\mathrm{RR}}}, \qquad (\omega = 0)$$

while one finds both of the velocities to be:

$$\lambda = \pm 1.$$

Of course, this case is susceptible to a general treatment when one starts with the expression (6.20) for $\mathfrak{P}_2$.

11. **Completely exceptional systems.** The system of equations (5.9), in which the right-hand side is identically null, is conservative. Thanks to the expression (5.10), when shocks are present, one may then write the conditions:

$$\tilde{\lambda}\left[\frac{\partial \mathrm{L}}{\partial q_0^\alpha}\right] = \left[\frac{\partial \mathrm{L}}{\partial q_j^\alpha}\right] n^j,$$

as:

(11.1) $$\tilde{\lambda}[\mathcal{L}_{\mathrm{Q}}\vec{\mathrm{E}} - \mathcal{L}_{\mathrm{R}}\vec{\mathrm{H}}] = [\mathcal{L}_{\mathrm{Q}}\vec{\mathrm{H}} \times \vec{n} + \mathcal{L}_{\mathrm{R}}\vec{\mathrm{E}} \times \vec{n}],$$
$$[\mathcal{L}_{\mathrm{Q}}\mathrm{E}_n - \mathcal{L}_{\mathrm{R}}\mathrm{H}_n] = 0.$$

This poses the question: What sort of system is completely exceptional? Since it is not impossible to essentially determine the roots of the characteristic polynomial, i.e., the values of λ that are solutions of:

(11.2) $$\mathfrak{P} = 0,$$

we proceed in the following fashion: From the equation above, we deduce that:

(11.3) $$\frac{\partial \mathfrak{P}}{\partial \lambda}\delta\lambda + \delta\mathfrak{P} = 0$$

where:

(11.4) $$\delta\mathfrak{P} = \nabla\mathfrak{P} \cdot \boldsymbol{\delta}\mathbf{u} = \lambda^\kappa \delta c_\kappa(\mathbf{u})$$

upon specifying that:

(11.5) $$\mathfrak{P} = c_\kappa(\mathbf{u})\lambda^\kappa,$$

and that the values of λ are all of the values that satisfy (11.2). Therefore (see 6.18):

(11.6) $$\frac{\partial\mathfrak{P}}{\partial\lambda}\delta\lambda = \delta\mathfrak{P}_1 - \omega\delta\mathfrak{P}_2 - \mathfrak{P}_1\delta\omega.$$

Starting with (6.1, 4), one obtains, in full generality ($\lambda \neq 0$):

(11.7)
$$\begin{aligned}
\lambda\delta\vec{\mathrm{H}} &= -a^1\vec{\mathrm{H}}\times\vec{n} - a^2\vec{\mathrm{E}}\times\vec{n} + a^3(\mathrm{H}_n\vec{\mathrm{E}} - \mathrm{E}_n\vec{\mathrm{H}})\\
\frac{1}{2}\delta(\mathrm{E}^2) &= a^1\mathrm{R} + a^2\mathrm{E}^2\\
\frac{1}{2}\lambda\delta(\mathrm{H}^2) &= a^2\mathrm{S}_n + a^3(\mathrm{H}_n\mathrm{R} - \mathrm{E}_n\mathrm{H}^2)\\
\lambda\delta\mathrm{Q} &= -a^1\lambda\mathrm{R} + a^2(\mathrm{S}_n - \lambda\mathrm{E}^2) + a^3(\mathrm{H}_n\mathrm{R} - \mathrm{E}_n\mathrm{H}^2)\\
\lambda\delta\mathrm{R} &= a^1(\lambda\mathrm{H}^2 - \mathrm{S}_n) + a^2\lambda\mathrm{R} + a^3(\mathrm{H}_n\mathrm{E}^2 - \mathrm{E}_n\mathrm{R})\\
\delta(\mathrm{E}_n) &= a^1\mathrm{H}_n + a^1\mathrm{E}_n + a^3\mathrm{S}_n\\
\delta(\mathrm{H}_n) &= 0\\
\lambda\delta(\mathrm{S}_n) &= a^1(\mathrm{R} - \mathrm{E}_n\mathrm{H}_n) + a2(\mathrm{E}^2 - \mathrm{E}_n^2 + \lambda\mathrm{S}_n) + a^3\{\lambda(\mathrm{R}\mathrm{H}_n - \mathrm{H}^2\mathrm{E}_n) - \mathrm{E}\mathrm{S}_n\}.
\end{aligned}$$

It then follows that:

(11.8)
$$\begin{aligned}
\frac{\delta\mathfrak{P}_2}{\lambda^2-1} &= \mathcal{L}_{\mathrm{QQ}}\{\lambda^2(\delta(\mathrm{E}^2) - \delta\mathrm{Q}) - 2\lambda\delta(\mathrm{S}_n) + \delta(\mathrm{H}^2) + \delta\mathrm{Q} - \delta(\mathrm{E}_n^2 + \mathrm{H}_n^2)\}\\
&+ \mathcal{L}_{\mathrm{RR}}\{\lambda^2\delta(\mathrm{H}^2) - 2\lambda\delta(\mathrm{S}_n) + \delta(\mathrm{E}^2) - \delta(\mathrm{E}_n^2 + \mathrm{H}_n^2)\}\\
&+ \{a\mathcal{L}_{\mathrm{QQQ}} - 2\mathrm{R}(\lambda^2-1)\mathcal{L}_{\mathrm{QQR}} + b\mathcal{L}_{\mathrm{QRR}}\}\delta\mathrm{Q}\\
&+ \{a\mathcal{L}_{\mathrm{QQR}} - 2\mathrm{R}(\lambda^2-1)\mathcal{L}_{\mathrm{QRR}} + b\mathcal{L}_{\mathrm{RRR}} - 3(\lambda^2-1)\mathcal{L}_{\mathrm{QR}}\}\delta\mathrm{R},
\end{aligned}$$

with:

(11.9)
$$\begin{aligned}
a &= \mathrm{E}^2\lambda^2 - 2\mathrm{S}_n\lambda + \mathrm{H}^2 - \mathrm{E}_n^2 - \mathrm{H}_n^2,\\
b &= \mathrm{H}^2\lambda^2 - 2\mathrm{S}_n\lambda + \mathrm{E}^2 - \mathrm{E}_n^2 - \mathrm{H}_n^2.
\end{aligned}$$

The expressions (11.7) provide:

(11.10) $$\lambda^2(\delta(\mathrm{E}^2) - \delta\mathrm{Q}) - 2\lambda\delta(\mathrm{S}_n) + \delta(\mathrm{H}^2) + \delta\mathrm{Q} - \delta(\mathrm{E}_n^2 + \mathrm{H}_n^2) = -3(\lambda^2-1)\delta\mathrm{R}$$

(11.11) $$\lambda^2\delta(\mathrm{H}^2) - 2\lambda\delta(\mathrm{S}_n) + \delta(\mathrm{E}^2) - \delta(\mathrm{E}_n^2 + \mathrm{H}_n^2) = 0.$$

The product $2\mathrm{S}_n\lambda$ that appears in $\mathfrak{P}_2$ is deduced from (11.2) and substituted into (11.9), which gives:

(11.12) $$(\lambda^2-1)a = (\lambda^2-1)^2\beta + \varpi\,\mathfrak{P}_1,$$

(11.13) $$(\lambda^2 - 1)b = (\lambda^2 - 1)^2 \alpha + \varpi\,\mathfrak{P}_1,$$

where:

(11.14) $$\alpha = \frac{\mathcal{L}_{\mathrm{Q}} + 2\mathrm{Q}\mathcal{L}_{\mathrm{QQ}} + 2\mathrm{R}\mathcal{L}_{\mathrm{QR}}}{\mathcal{L}_{\mathrm{QQ}} + \mathcal{L}_{\mathrm{RR}}}, \qquad \beta = \frac{\mathcal{L}_{\mathrm{Q}} + 2\mathrm{Q}\mathcal{L}_{\mathrm{QR}} - 2\mathrm{Q}\mathcal{L}_{\mathrm{RR}}}{\mathcal{L}_{\mathrm{QQ}} + \mathcal{L}_{\mathrm{RR}}},$$

and we recall that:

(11.15) $$\varpi = \frac{\omega}{\mathcal{L}_{\mathrm{QQ}} + \mathcal{L}_{\mathrm{RR}}}.$$

(11.9-15) then permit us to write (11.8):

(11.16) $$\delta\mathfrak{P}_2 = -(\lambda^2 - 1)^2(\mathcal{L}_{\mathrm{QQ}} + \mathcal{L}_{\mathrm{RR}})\left(\frac{\partial\alpha}{\partial\mathrm{Q}}\delta\mathrm{Q} + \frac{\partial\beta}{\partial\mathrm{R}}\delta\mathrm{R}\right) + \varpi\,\mathfrak{P}_1\delta(\mathcal{L}_{\mathrm{QQ}} + \mathcal{L}_{\mathrm{RR}}).$$

We now calculate $\delta\mathfrak{P}_1$. By adding the identities (11.10, 11) term-by-term, one obtains:

(11.17) $$\delta(\mathrm{E}_n^2 + \mathrm{H}_n^2) = \frac{1}{2}(\lambda^2 - 1)\delta(\mathrm{E}^2 + \mathrm{H}^2) - 2\lambda\delta(\mathrm{S}_n) + (\lambda^2 - 1)\delta\mathrm{Q}.$$

Using that equality, one finds that:

(11.18) $$\delta\mathfrak{P}_1 = (\lambda^2 - 1)^2\left\{\delta(\mathrm{S}^2) - \frac{1}{2}(\mathrm{E}^2 + \mathrm{H}^2)\delta(\mathrm{E}^2 + \mathrm{H}^2)\right\} - (\lambda^2 - 1)(a + b)\,\delta\mathrm{Q},$$

while it is easy to show, using only (6.17), that:

(11.19) $$\frac{1}{2}(\mathrm{E}^2 + \mathrm{H}^2)\delta(\mathrm{E}^2 + \mathrm{H}^2) - \delta(\mathrm{S}^2) = 2\mathrm{R}\delta\mathrm{R} + 2\mathrm{Q}\delta\mathrm{Q}.$$

Finally, (11.6) becomes:

(11.20) $$\frac{\partial\mathfrak{P}}{\partial\lambda}\delta\lambda = \mathfrak{P}_1\{\varpi\delta(\mathcal{L}_{\mathrm{QQ}} + \mathcal{L}_{\mathrm{RR}}) + 2\omega\varpi\,\delta\mathrm{Q} - \delta\omega\}$$
$$-(\lambda^2 - 1)^2\left\{(\mathcal{L}_{\mathrm{QQ}} + \mathcal{L}_{\mathrm{RR}})\left(\frac{\partial\alpha}{\partial\mathrm{Q}}\delta\mathrm{Q} + \frac{\partial\beta}{\partial\mathrm{R}}\delta\mathrm{R}\right) - 2\omega(\mathrm{Q}\delta\mathrm{Q} + \mathrm{R}\delta\mathrm{R}) - \omega(\alpha + \beta)\delta\mathrm{Q}\right\}.$$

When the system of equations (5.2, 11) is completely exceptional, the left-hand – and, as a result, the right-hand – side of (11.20) is identically null for the four characteristic values that satisfy (11.2). One must then annul the coefficients of $\delta\mathrm{Q}$ and $\delta\mathrm{R}$ inside each pair of braces in (11.20). One thus arrives at the system of partial differential equations:

(11.21) $$\frac{\partial \varpi}{\partial \mathrm{Q}} - 2\varpi^2 = 0,$$

(11.22) $$\frac{\partial \varpi}{\partial \mathrm{R}} = 0,$$

(11.23) $$\frac{\partial \alpha}{\partial \mathrm{Q}} - 2\varpi\alpha = 0,$$

(11.24) $$\frac{\partial \alpha}{\partial \mathrm{R}} - 2\varpi \mathrm{R} = 0.$$

To these, one adds:

(11.25) $$\alpha - \beta = 2\mathrm{Q}.$$

There are two cases to consider:

1. $\omega = 0$. The last two equations may be integrated to:

$$\alpha = \text{const.}$$

One is then reduced to taking solutions of the system:

(11.26) $$\begin{cases} \omega = 0, \\ \alpha = \text{const.} = -k. \end{cases}$$

We parenthetically note that μ (10.11) is equal to $\mathrm{S}k_1$.

2. $\omega \neq 0$. (11.21, 22) give:

(11.27) $$2(\mathrm{Q} + k_1)\varpi = -1$$

and (11.24) gives:

(11.28) $$\alpha = \frac{-\mathrm{R}^2}{2(\mathrm{Q}+k_1)} + f(\mathrm{Q}).$$

The value of α that is given by (11.28) must satisfy (11.23). From this, the following differential equation results:

(11.29) $$\frac{df}{d\mathrm{Q}} + \frac{f}{\mathrm{Q}+k_1} = 0,$$

whose solution:

$$f = \frac{-k_2}{2(\mathrm{Q}+k_1)}$$

leads to the system:

(11.30)
$$\begin{cases} 2(\mathrm{Q}+k_1)\varpi = -1, \\ 2(\mathrm{Q}+k_1)\alpha = -(\mathrm{R}^2+k_2). \end{cases}$$

(k_1, k_2 are constants.)

Here, one will have (see (10.6)):

$$\mu = \mathrm{S}(k_1 \pm \sqrt{k_1^2 - k_2}\,).$$

The Born-Infeld theory uses the Lagrangian [24]:

(11.31)
$$\mathrm{L} = (1 + 2\mathrm{Q} - \mathrm{R}^2)^{\frac{1}{2}}$$

which produces an exceptional system $\left(\varpi = \dfrac{-1}{2(\mathrm{Q}+1)}, \alpha = -\dfrac{\mathrm{R}^2+1}{2(\mathrm{Q}+1)}\right)$. Likewise, if one considers the solutions for which $\mathrm{R} \equiv 0$, starting with the Lagrangian:

(11.31′)
$$\mathrm{L} = (1 + 2\mathrm{Q})^{\frac{1}{2}},$$

then the system is completely exceptional ($\omega = 0$, $\alpha = -1$). This fact has already been confirmed in the one-dimensional study of that theory; the characteristic curves form two families of isoclines [9][25].

One likewise verifies that the Heisenberg-Euler theory, which makes use of the following Lagrange function [26]:

(11.32)
$$\mathcal{L} = -\mathrm{Q} + k\mathrm{Q}^2 + \frac{7}{4}k\mathrm{R}^2,$$

in which k is a certain positive constant, does not lead to a completely exceptional system. One may refer to [27] for the study of discontinuities in the latter theory and to [28] for the study of shocks.

In conclusion, we note that the Lagrangian (11.31′) is the only one (up to a choice of constant that one obtains by integrating (11.26)) depends uniquely on Q in a nonlinear fashion that leads to a completely exceptional system [25].

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