"Zur zweiten Variation bei isoperimetrischen Problemen," Math. Ann. **57** (1903), 44-47.

On the second variation for isoperimetric problems

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In volume 55 of these Annalen, **Kneser** gave a rigorous proof of *the necessity of the Jacobi condition for the simplest class of isoperimetric problems,* although the question remained undecided in a certain exceptional case (*).

The goal of the following note is to show that this *exceptional case* can also be resolved quite simply with the help of a method that **H. A. Schwarz** had developed in his lectures (**) for the analogous proof in the case of the simplest problem without auxiliary conditions.

The problem that will be treated can be formulated as follows:

Let H_1, H_2, U be three functions of t that are regular in the interval (t_0, t_1). Moreover, let $H_1 > 0$, while U is not zero identically in that interval. Furthermore, let:

$$\Psi(w) \equiv H_2\, w - \frac{d}{dt}\left(H_1 \frac{dw}{dt}\right) ,$$

and let u, v be solutions of the two differential equations:

$$(1) \qquad \Psi(u) = 0 , \qquad \Psi(v) = U$$

that both vanish at t_0 (***):

$$(2) \qquad u(t_0) = 0 , \qquad v(t_0) = 0 ,$$

(*) Cf., also the dissertation of **Hormann**, *Untersuchungen über die Grenzen, zwischen welchen Unduloide und Nodoide, etc.*, Göttingen 1887, which was directed towards the corresponding investigations of **Weierstrass**, and the same exceptional case was still left unresolved.

(**) The method was made known to me from a postscript by Herrn Dr. **J. C. Fields** to a lecture on the calculus of variations in the Winter semester 1898/99. It is the same method that **Sommerfeld** extended to double integrals in Jahresberichte der Deutschen Mathematiker-Vereinigung, v. VIII, pp. 188.

(***) It is known how one can exhibit such solutions as soon as the general integral of Euler's differential equation is known. Cf., **Hormann**, *loc. cit.*, and **Kneser**, *loc. cit.* The functions u, v are linear combinations of the functions that **Kneser** denoted by A, B.

and finally let:

$$m = \int_{t_0}^{t} uU\,dt\,, \quad n = \int_{t_0}^{t} vU\,dt\,,$$

$$\Delta\,(t) = m\,v - n\,u\,.$$

One then has $\Delta\,(t_0) = 0$. Let t_0' be the next zero of $\Delta\,(t)$ after t_0 (the *conjugate point* to t_0), and assume that:

(3) $$t_0' < t_1\,.$$

It will then be shown that one can always find functions w of t that vanish at t_0 and t_1 :

(4) $$w\,(t_0) = 0\,, \qquad w\,(t_1) = 0\,,$$

for which:

(5) $$\int_{t_0}^{t_1} wU\,dt = 0\,,$$

and for which the integral (*):

$$J_2 = \int_{t_0}^{t_1} \left[H_1 \left(\frac{dw}{dt} \right)^2 + H_2\,w^2 \right] dt$$

assumes a negative value.

In that way, w itself will be continuous in the entire interval (t_0, t_1), dw / dt will exist and be continuous, except for a finite number of points, and the forward and backward derivatives will exist and be finite at the exceptional points, as well.

For the case in which u and v do not both vanish at t_0', **Kneser** carried out the proof in the cited treatise by showing that $\Delta\,(t)$ will then vanish to odd order at t_0', which will imply the desired result from an argument that goes back to **Weierstrass**.

It still remains for us to examine the exceptional case:

(6) $$u\,(t_0') = 0\,, \qquad v\,(t_0') = 0\,.$$

Since:

(7) $$v\,\Psi\,(u) - u\,\Psi\,(v) = \frac{d}{dt} H_1 (u\,v' - u'\,v)\,,$$

it will follow (**) from (1) and (2) that:

(8) $$H_1 (u\,v' - u'\,v) = -\,m\,.$$

(*) It is known from **Weierstrass** that the second variation can be transformed into that form in the present case.
(**) Cf., **Kneser**, *loc. cit.*, equation (22). The prime denotes derivation with respect to t.

However, it will follow that under the present assumption (6), one will also have:

$$m(t_0') = \int_{t_0}^{t_0'} uU\,dt = 0 . \tag{9}$$

If one then chooses:

$$w = u \quad \text{in} \quad (t_0, t_0'),$$
$$w = 0 \quad \text{in} \quad (t_0', t_1),$$

w will satisfy the conditions (4) and (5) and make $J_2 = 0$. That is because if $\tau_1, \tau_2, \ldots, \tau_n$ generally denote the places where dw / dt is discontinuous then J_2 can be put into the form:

$$J_2 = \sum_{\nu=1}^{n} \left[H_1 \, w \frac{dw}{dt} \right]_{\tau_\nu + 0}^{\tau_\nu - 0} + \int_{t_0}^{t_1} w \Psi(w)\, dt \tag{10}$$

by partial integration, which will yield $J_2 = 0$ in the present case.

Now, in order to obtain a function that makes negative J_2, we follow **Schwarz**'s procedure and choose w to be a function that deviates from the one above only slightly, namely:

$$\begin{aligned} w &= u + \kappa\omega \quad \text{in} \quad (t_0, t_0'), \\ w &= \phantom{u+{}} \kappa\omega \quad \text{in} \quad (t_0', t_1), \end{aligned} \tag{11}$$

in which κ is a small constant, and ω is a function of t that satisfies the following conditions:

1) ω is continuous with continuous first and second derivatives in (t_0, t_1) .

2) $\omega(t_0) = 0$, $\omega(t_1) = 0$.

3) $\omega(t_0') \neq 0$.

4) $\int_{t_0}^{t_1} \omega U\,dt = 0$.

The function w that is defined in that way will fulfill the conditions (4) and (5). It is itself continuous, but its first derivative suffers a jump at the location t_0' . One will then have to consider the term that arises from the discontinuity (*) by an application of the formula (10) and after a simple calculation, in which one makes use of the identity:

(*) In so doing, one should observe that a discontinuity of the type considered will have no effect on the first variation and the conversion of the second variation into **Weierstrass** form, due to the continuity of w.

$$u\,\Psi(\omega) - \omega\,\Psi(u) = \frac{d}{dt} H_1(\omega u' - \omega' u)\,,$$

that will yield the result that:

(12) $$J_2 = 2\kappa\, H_1\, \omega \frac{du}{dt}\bigg|_{t_0'} + \kappa^2 V\,,$$

in which V is a finite quantity, just as in the case the **Schwarz** treated.

However, by assumption, H_1 and w are non-zero at t_0', just like du / dt, since $u(t_0') = 0$, and t_0' is a non-singular location for the differential equation $\Psi(u) = 0$. However, it will follow from this that *one can make the integral* J_2 *negative* by a suitable choice of κ.

It only remains for us to show that we can always determine a function ω that satisfies one of the four conditions above. Let ω_1 be any function that satisfies one of the first three conditions, e.g., $\omega_1 = (t - t_0)(t - t_1)$. If it should, by chance, also satisfy the fourth one then $\omega = \omega_1$ would be a useful function. However, the integral:

$$\int_{t_0}^{t_1} \omega_1 U\, dt = C_1$$

will be non-zero, in general. In that case, one chooses a second function ω_2 as follows: From the assumption that was made about U, one can always find a subinterval (τ', τ'') of (t_0, t_1) in which $U \neq 0$. One then sets:

$$\omega_2 = (t - \tau')^3 (\tau'' - t)^3 (t - t_0')^2$$

inside of (τ', τ'') and $\omega_2 \equiv 0$ outside of (τ', τ''). The integral:

$$\int_{t_0}^{t_1} \omega_2 U\, dt = C_2\,,$$

is certainly non-zero then. However, it will follow from this that the function:

$$\omega = C_2\,\omega_1 - C_1\,\omega_2$$

satisfies all of the conditions that were posed above.

Therefore, a minimum cannot exist beyond the conjugate point to t_0 *in the exceptional case that was consider, either.*

University of Chicago, 27 February 1902.