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Geometrical and mechanical properties of some remarkable curves

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I.

Problem: Given a perfectly flexible and homogeneous string that nonetheless has unequal thickness and whose elements are subject to the action of central forces that are inversely proportional to the distance, find the law by which the thickness must vary at each point of the curve and the curve that the string must exhibit in the equilibrium state in order for the tension in that state to vary from one point to another in proportion to the thickness or for the string to present a chance of rupture that is everywhere equal.

Solution: Let a be the constant ratio that exists between the thickness ω and the tension T at each point of the string in the equilibrium state. Let $R = f(r)$ denote the intensity of the given central force, which we first suppose to be an arbitrary function of the distance, but we will have:

$$d \cdot \left(T \frac{dx}{ds} \right) = \pm\, a\, TR \frac{x}{r}\, ds,$$

$$d \cdot \left(T \frac{dy}{ds} \right) = \pm\, a\, TR \frac{y}{r}\, ds$$

for any element of the string. The center of the force is taken to be the coordinate origin, and the signs in the right-hand sides are + or – according to whether the force is attractive or repulsive, respectively.

Add the preceding two equations, after having first multiplied then dx / ds and dy / ds, respectively, and then by $-y$ and x, resp. One will get:

$$dT = \pm\, a\, TR\, dr, \qquad d\left[T \left(x \frac{dy}{ds} - y \frac{dx}{ds} \right) \right] = 0,$$

so, upon integrating:

$$(1) \qquad T = C\, e^{\pm a \int R\, dr}, \qquad T\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right) = C'.$$

The penultimate equation already tells us what the tension at each point of the string will be in the equilibrium state. One easily deduces the thickness:

$$(2) \qquad \omega = a\,T = a\,C\,e^{\pm a \int R\, dr}.$$

In order to then get the equation for the equilibrium curve of the string, it will suffice to eliminate T from equations (1). One will then find that:

$$x\frac{dy}{ds} - y\frac{dx}{ds} = \frac{C'}{C} e^{\mp a \int R\, dr},$$

and upon passing to polar coordinates:

$$\frac{r^2 d\theta}{\sqrt{dr^2 + r^2\, d\theta^2}} = \frac{C'}{C} e^{\mp a \int R\, dr},$$

so

$$d\theta = \frac{dr}{r\sqrt{\dfrac{C^2}{C'^2} r^2 e^{\mp 2a \int R\, dr} - 1}};$$

hence, upon integrating:

$$(3) \qquad \theta + \alpha = \int \frac{dr}{r\sqrt{\dfrac{C^2}{C'^2} r^2 e^{\mp 2a \int R\, dr} - 1}}.$$

The constants that were introduced by the integrations are easy to interpret. If the two quadratures that enter into the right-hand side of equation (3) are both taken by starting from the foot of the normal that is drawn through the origin to the curve that is represented by that equation (3) then the constants – α, C' / C, C will be nothing but the values of θ, r, T, respectively, that pertain to that point. As for the constant a, one knows that it expresses the constant ratio that exists between the thickness and the tension at each point of the string in the equilibrium state. It is almost pointless to say that the four constants that one deals with are determined in each special case by expressing the idea that the string passes through two points, that it has a well-defined length between those two points, and finally that it has a known thickness at a point.

II.

We now leave behind the previous generalities in order to occupy ourselves exclusively with the case in which the force is inversely proportional to the distance.

We then set:

$$f(r) = \frac{1}{r},$$

in which the unit of force is the force that acts at a unit distance. Equations (2) and (3) will become:

(4)
$$\omega = aT_0\left(\frac{r}{r_0}\right)^{\pm a}, \qquad \theta - \theta_0 = \int_{r_0}^{r}\frac{dr}{r\sqrt{\left(\frac{r}{r_0}\right)^{2(1\pm a)} - 1}},$$

upon taking the integrals as we did above, starting from the foot of the normal that is drawn through the origin to the curve that the string assumes in the equilibrium state, and calling the values of θ, r, T at that point θ_0, r_0, T_0, respectively.

The integral contained in the right-hand side of the last equation is easily evaluated upon setting:

$$\left(\frac{r}{r_0}\right)^{1\pm a} = \sec\varphi, \qquad \text{so} \qquad \int_{r_0}^{r}\frac{dr}{r\sqrt{\left(\frac{r}{r_0}\right)^{2(1\pm a)} - 1}} = \int_0^{\varphi}\frac{d\varphi}{1\pm a} = \frac{\varphi}{1\pm a},$$

and one will find that this equation is:

(5)
$$r^{1\pm a}\cos(1\pm a)(\theta - \theta_0) = r_0^{1\pm a}.$$

That shows us that the equilibrium curves of the string in the case considered are nothing but the remarkable curves that Serret considered for the first time in Tome VII of this *Journal*, and whose arc lengths represent Eulerian integrals of the second kind in a large number of cases. One knows that these curves include the circle, the equilateral hyperbola, the lemniscate, etc., as special cases. Moreover, the value of a upon which the nature of the equilibrium curve will depend, as well as the values of θ_0 and r_0 that are determined in each special case must be obtained, as was said above, by expressing the ideas that the curve must pass through two points and that it must have a known length between those curves.

III.

Recall the first of equations (4):

(6)
$$\omega = aT_0\left(\frac{r}{r_0}\right)^{\pm a}.$$

That equation tells us the thickness of the string at each point as a function of r, viz., the distance from the origin to the point considered in the equilibrium position. One can once more express the thickness at each point as a function of the volume of the string that is included between that point and a well-defined point. The formula that one obtains to that effect, which has a very remarkable form, has the advantage of being applicable no matter what position that the string occupies, and consequently, when it is stretched into a straight line on a plane.

Take the polar axis to be the normal to the curve (5) that is drawn through the origin, and set:

$$1 \pm a = m, \qquad a\,T_0 = \omega_0\,,$$

to simplify. Equations (5) and (6) become:

$$r^m \cos m\theta = r_0^m\,, \tag{7}$$

$$\omega = \omega_0 \left(\frac{r}{r_0}\right)^{m-1}. \tag{8}$$

We infer the differential of arc length that the string is endowed with from the penultimate equation:

$$ds = r_0 \left(\frac{r}{r_0}\right)^{m+1} d\theta\,,$$

so we will have:

$$V = \omega_0\, r_0 \int_0^\theta \left(\frac{r}{r_0}\right)^{2m} d\theta = \omega_0\, r_0 \int_0^\theta \frac{d\theta}{\cos^2 m\theta} = \frac{\omega_0\, r_0}{m} \tan m\theta$$

for the volume that is counted by starting from the polar axis. If we eliminate θ from that equation and equation (8), which amounts to:

$$\frac{\omega_0}{\omega} = (\cos m\theta)^{\frac{m-1}{m}}\,,$$

then we will find:

$$V^2 = \frac{\omega_0^2\, r_0^2}{m^2} \left[\left(\frac{\omega}{\omega_0}\right)^{\frac{2m}{m-1}} - 1\right] \tag{9}$$

for the desired equation. After the constants ω_0, r_0, and m have been determined, that equation will give the volume of the string as a function of the thickness at the point where one stops, and conversely.

That same equation can serve to determine the form of the string.

We associate the string that has been stretched into a straight line on a plane with the volume that is generated by a circle of variable radius that moves parallel to itself while touching the extremities of the same diameter of a line and a curve, and we propose to determine the equation of that director curve. Let y_0 be the initial diameter of the generating circle, and let $y + y_0$ be the diameter when that circle has traversed the space x along the straight line. We will have:

$$V = \frac{\pi}{4}\int_0^x (y+y_0)^2\,dx, \qquad \omega_0 = \frac{\pi}{4}y_0^2, \qquad \omega = \frac{\pi}{4}(y+y_0)^2.$$

If one substitutes these values in equation (9) then one will get:

$$\int_0^x (y+y_0)^2\,dx = \frac{r_0\,y_0^2}{m}\sqrt{\left(\frac{y+y_0}{y_0}\right)^{\frac{4m}{m-1}} - 1},$$

so upon differentiating this, one will have:

$$dx = \frac{2r_0}{y_0(m-1)} \frac{\left(\frac{y+y_0}{y_0}\right)^{\frac{m+3}{m-1}} dy}{\sqrt{\left(\frac{y+y_0}{y_0}\right)^{\frac{4m}{m-1}} - 1}}, \tag{10}$$

and integrating:

$$x = \frac{2r_0}{y_0(m-1)} \int_0^y \frac{\left(\frac{y+y_0}{y_0}\right)^{\frac{m+3}{m-1}} dy}{\sqrt{\left(\frac{y+y_0}{y_0}\right)^{\frac{4m}{m-1}} - 1}}. \tag{11}$$

The quadrature that is contained in the right-hand side of the last equation is a binomial integral that can be evaluated, from known principles, only when $\frac{m+1}{2m}$ or $\frac{1}{2m}$ is an integer. The first condition is fulfilled for $m = -1$, while the second one is fulfilled for $m = \pm\frac{1}{2}$: For the first of those values for m, equation (7) will represent a circumference; for the other two, it will represent a parabola whose focus is at the origin and an external epicycloid that has modulus $\frac{1}{2}$. If one performs the integration in those three cases then one will find, in succession, that equation (11) is:

$$(y + y_0)\cos\frac{x}{r_0} = y_0,$$

$$\frac{x}{r_0} = \left(\frac{y+y_0}{y_0}\right)^{-2}\left[\left(\frac{y+y_0}{y_0}\right)^{-4} - 1\right]^{1/2} + \log\left\{\left(\frac{y+y_0}{y_0}\right)^{-2} + \left[\left(\frac{y+y_0}{y_0}\right)^{-4} - 1\right]^{1/2}\right\},$$

$$\left(\frac{y+y_0}{y_0}\right)^{-4}\left[1-\left(\frac{x}{2r_0}\right)^2\right]^3 = 1,$$

respectively. However, without stopping to discuss those special cases (which lead, as one knows, to some very complicated results), we remark that one can represent the curve by equation (10) or (11) for any *m* by a geometric construction. Indeed, set:

$$(m + 1)\, x = u, \qquad \left(\frac{y+y_0}{y_0}\right)^{\frac{2(m+1)}{m-1}} = \frac{v}{r_0},$$

so equation (10) will become:

$$du = \frac{dv}{\sqrt{\left(\frac{v}{r_0}\right)^{\frac{2m}{m+1}}-1}}, \tag{12}$$

and it is clear that if we can construct the curve that is represented by the latter equation then we can also have as many points as we desire along the curve that is represented by equation (10) by a geometric construction.

Now, I say that if one rolls the curve that is represented by equation (7) along a straight line then the center of that curve will describe a curve that is included in equation (12).

In order to do that, if we remark that r and θ are polar coordinates of an arbitrary point of the curve, when we take those coordinates with respect to an axis and origin that are arbitrary, but invariably coupled with that curve, and x and y are the rectangular coordinates of the point that serves as the origin when we roll that curve along the x-axis in such a manner that the contact will occur at the point whose coordinates are r and θ then we will have:

$$\frac{dx}{dy} = -\frac{r\,d\theta}{dr}, \qquad y = \frac{r^2 d\theta}{\sqrt{dr^2 + r^2 d\theta^2}}. \tag{13}$$

Now, suppose that the curve that one rolls is the one that is represented by equation (7), so one will then have:

$$r\frac{d\theta}{dr} = \frac{1}{\sqrt{\left(\frac{r}{r_0}\right)^{2m}-1}}, \qquad \frac{r^2 d\theta}{\sqrt{dr^2 + r^2 d\theta^2}} = r_0\left(\frac{r}{r_0}\right)^{4-m},$$

and consequently:

$$\frac{dx}{dy} = -\frac{1}{\sqrt{\left(\frac{r}{r_0}\right)^{2m} - 1}}, \qquad y = r_0 \left(\frac{r}{r_0}\right)^{4-m},$$

so, upon eliminating r / r_0 , one will have:

(14) $$dx = \frac{dy}{\sqrt{\left(\frac{y}{r_0}\right)^{\frac{2m}{1-m}} - 1}},$$

which is an equation that is included in equation (12).

One can deduce some known results from this. If one makes:

$$\frac{2m}{1-m} = -1, \quad \text{or} \quad m = -1$$

then equation (14) will be that of a cycloid and equation (7) will be that of a circle. We then get back to the characteristic property of the cycloid. If:

$$\frac{2m}{1-m} = 2, \quad \text{so} \quad m = \tfrac{1}{2},$$

then equation (14) will be that of a catenary and equation (7) will be that of a parabola when it is referred to its focus, so one can conclude that when one rolls a parabola on a line, the focus of the parabola will describe a catenary. If one sets:

$$\frac{2m}{1-m} = -4, \quad \text{so} \quad m = 2,$$

then equation (14) will be that of a rectangular elastic curve and equation (7) will be that of an equilateral hyperbola when referred to its axes. We can then say that the center of an equilateral hyperbola that rolls on a line will describe a rectangular elastic curve, etc.

IV.

If, instead of rolling the curve that is represented by equation (7) along a line, one rolls a curve that is parallel to the former, then one will get a very simple result that does not differ very much from the one that we obtained by rolling the curve (7).

First, recall that geometers call a curve *parallel* to a given curve when it is the locus of the points that are obtained by taking an arc of the same length a along each normal to that curve that starts from the contact point.

That being the case, one easily sees that if ξ and η are the rectangular coordinates of any point on the curve that is parallel to the curve (7) then one will have:

$$\xi = r \cos \theta + a \cos (m - 1) \theta,$$

$$\eta = r \sin \theta - a \sin (m - 1) \theta,$$

or, upon letting ρ and φ denote the polar coordinates that correspond to the rectangular coordinates ξ and η:

(15) $$\rho e^{\varphi\sqrt{-1}} = r e^{\varphi\sqrt{-1}} + a e^{-(m-1)\theta\sqrt{-1}},$$

(16) $$\rho e^{-\varphi\sqrt{-1}} = r e^{-\varphi\sqrt{-1}} + a e^{(m-1)\theta\sqrt{-1}},$$

with the condition:

(17) $$r^m \cos m\theta = r_0^m .$$

We now recall equations (13). Upon changing r into ρ and θ into φ, they will become:

(18) $$\frac{dx}{dy} = -\frac{\rho\, d\varphi}{d\rho}, \qquad y = \frac{\rho^2\, d\varphi}{\sqrt{d\rho^2 + \rho^2\, d\varphi^2}},$$

and we can eliminate r, θ, ρ, and φ from those two equations and equations (15), (16), and (17).

From the equality (15), we will first infer that:

$$e^{\varphi\sqrt{-1}} d\varphi + \rho e^{\varphi\sqrt{-1}}\sqrt{-1}\, dr = e^{\varphi\sqrt{-1}} dr + r e^{\varphi\sqrt{-1}}\sqrt{-1}\, d\theta - (m-1)\, a\, e^{-(m-1)\theta\sqrt{-1}}\sqrt{-1}\, d\theta,$$

so, upon multiplying this by equation (16) and equating the real and imaginary parts of the two sides, respectively, we will get:

$$\rho\, d\rho = r\, dr - (m - 1)\, dr \sin m\theta\, d\theta + a \cos m\theta\, dr - ar \sin m\theta\, d\theta,$$

$$\rho^2\, d\rho = r^2\, dr - (m - 1)\, dr \cos m\theta\, d\theta + a \sin m\theta\, dr + ar \cos m\theta\, d\theta - (m - 1)\, a^2\, d\theta.$$

If one simplifies this by means of the equation:

$$\cos m\theta\, dr = r \sin \theta\, d\theta,$$

which one deduces from equation (17), then one will get:

$$\rho\, d\rho = dr\, [r - (m - 1)\, a \cos m\theta],$$

$$\rho^2\, d\rho = \frac{(r\cos m\theta + a)[r - (m-1)a\cos m\theta]\, d\theta}{\cos m\theta} = \frac{(r\cos m\theta + a)[r - (m-1)a\cos m\theta]\, dr}{r \sin m\theta}.$$

We infer from this that:

$$\frac{\theta\, d\varphi}{d\rho} = \frac{r\cos m\theta + a}{r\sin m\theta}.$$

Furthermore, equations (15) and (16), when multiplied by each other, will give:

$$\rho^2 = r^2 + a^2 + 2ar\cos m\theta.$$

If one substitutes this in equations (18), while remarking that the second of those equations amounts to:

$$\rho = y\sqrt{1+\frac{dy^2}{dx^2}},$$

then, due to the first, one will find that:

$$r\sin m\theta = y\frac{dy}{dx}, \qquad r\cos m\theta = y - a,$$

or finally, by means of equation (17):

$$dx = \frac{y\,dy}{(y-a)\sqrt{\left(\frac{y-a}{r_0}\right)^{\frac{2m}{1-m}} - 1}}.$$

That equation will become equation (14) when one sets $a = 0$, as it must. One can infer various more or less curious results by varying m. For example, if one sets $m = 2$, in which case the curve (17) will be an equilateral hyperbola, then one will find that the equation of the curve that is generated will be:

$$dx = \frac{\frac{y(y-a)}{r_0^2}\,dy}{\sqrt{1-\left(\frac{y-a}{r_0}\right)^4}}.$$

V.

The curves that are represented by equation (12), which includes the cycloid as a particular case, are obtained by generalizing several properties of that curve.

One knows that for the cycloid, the radius of curvature is twice the normal. We propose to find, more generally, the curve for which the radius of curvature is equal to m times the normal. We will have:

$$\frac{1+p^2}{q} = my, \qquad \text{or} \qquad \frac{q}{1+p^2} = \frac{1}{m}\frac{1}{y}$$

for the differential equation of the desired curve. If one multiplies the two sides by 2 *dy* and observes that:

$$q\,dy = \frac{dp}{dx}dy = p\,dp$$

then one will get:

$$\frac{2p\,dp}{1+p^2} = \frac{2}{m}\frac{dy}{y},$$

so, upon integrating:

$$1 + p^2 = C\,y^{2/m},$$

so

$$dx = \frac{dy}{\sqrt{C\,y^{2/m} - 1}},$$

which is an equation of the same form as equation (12).

Similarly, since the cycloid is the curve of fastest descent in a vacuum, the integral:

$$\int \frac{\sqrt{1+p^2}}{\sqrt{y}}\,dx$$

must be a minimum for the cycloid. Now, the curves that are included in equation (12) will give a minimum for the most general integral:

$$\int y^n \sqrt{1+p^2}\,dx,$$

as one see in (Euler's) *Methodus inveniendi*, etc., page 50.

One will obtain some curves for which the ones that are represented by equations (12) are only special cases upon generalizing some other properties of the cycloid.

Hence, we propose to find the curve for which the radius of curvature ρ has a constant ratio with the n^{th} power of the ordinate. We will first have:

$$\frac{q}{(1+p^2)^{3/2}} = a\,y^{-n},$$

so, upon multiplying by 2 *dy* and integrating:

$$(1 + p^2)^{-1/2} = C + C'y^{1-n},$$

so

(19) $$dx = \frac{(C + C'y^{1-n})\,dy}{\sqrt{1-(C+C'y^{1-n})^2}},$$

which is an equation that will come back to equation (14) when one sets $C = 0$.

One knows that one will find the cycloid upon looking for the curve for which the space that is found between it and its development is smallest; i.e., upon searching for the curve that makes the integral:

$$\int \rho\, ds = \int \frac{(1+p^2)^2}{q}\, dx$$

a minimum.

If, more generally, one proposes to find the curve for which the integral:

$$\int \rho^n\, ds = \int \frac{(1+p^2)^{\frac{3n+1}{2}}}{q^n}\, dx$$

is a minimum then one will find a curve that is included in equation (19).

Euler, who solved the latter problem in his *Methodus inveniendi*, etc., page 66, did not point out that its solution coincides with the curve for which there exists a constant ratio between the radius of curvature and a power of the ordinate.

Here is a simple way of establishing that coincidence:

From the method of variations, the curve whose ordinate makes the integral:

$$\int \frac{(1+p^2)^{\frac{3n+1}{2}}}{q^n}\, dx$$

a minimum will have the differential equation:

$$\frac{(1+p^2)^{\frac{3n+1}{2}}}{q^n} = C + C'p - n\,\frac{(1+p^2)^{\frac{3n+1}{2}}}{q^n}$$

or

$$\frac{(1+p^2)^{\frac{3n+1}{2}}}{q^n} = C + C'p.$$

If ρ is the radius of curvature of the desired curve then one will have:

$$\rho = \frac{(1+p^2)^{3/2}}{q},$$

so

$$\rho^n = \frac{C + C'p}{\sqrt{1+p^2}}$$

can be considered to be the equation of the curve. One infers from this that:

$$n\,\rho^{n-1}\,d\rho = \frac{(C - C'p)}{(1+p^2)^{3/2}}\,dp\,,$$

and since:

$$\rho = \frac{(1+p^2)^{3/2}}{q}\,,$$

one will have:

$$n\,\rho^n\,d\rho = \frac{C - C'p}{q}\,dp = (C - C'p)\,dx = C\,dx - C'dy,$$

so

$$\rho^{n+1} = Cx + C'y + C'',$$

which is an equation that can be put into the form:

$$\rho^{n+1} = ay, \qquad \text{or} \qquad \rho = a^{\frac{1}{n+1}}y^{\frac{1}{n+1}}$$

by suitably changing the position and direction of the axes, and that is what we had proposed to look into.

VI.

We look for the development of the curve that is represented by equation (12), or more simply, by the equation:

(20) $$dx = \frac{dy}{\sqrt{y^m - 1}}\,.$$

Let x, y be the coordinates of an arbitrary point of that curve, let ρ be the radius of curvature at that point, and let α, β be the coordinates of the center of curvature at that point. One will have:

$$\frac{d\beta}{d\alpha} = -\frac{dx}{dy}, \quad d\alpha^2 + d\beta^2 = d\rho^2.$$

Now one deduces from equation (20) that:

$$\frac{dx}{dy} = \frac{1}{\sqrt{y^m - 1}} \text{ and } \qquad \rho = \frac{2}{m}\,y^{\frac{m+2}{2}}\,,$$

so

$$d\rho = \frac{m+2}{m} y^{m/2} dy .$$

One will then have:

$$\frac{d\beta}{d\alpha} = -\frac{1}{\sqrt{y^m - 1}}, \qquad d\alpha^2 + d\beta^2 = \left(\frac{m+2}{m}\right)^2 y^m dy^2 ,$$

so one can infer that:

$$\frac{d\beta}{1} = -\frac{d\alpha}{\sqrt{y^m - 1}} = \sqrt{\frac{d\alpha^2 + d\beta^2}{y^m}} = \frac{m+2}{m} dy,$$

so

$$d\beta = \frac{m+2}{m} dy, \qquad \text{and} \qquad \beta = \frac{m+2}{m} y + C,$$

and

$$d\alpha = \frac{m+2}{m} dy \sqrt{y^m - 1} = d\beta \sqrt{\left(\frac{m}{m+2}\beta - C\right)^m - 1} .$$

If one translates the β-axis parallel to itself in such a manner as to make the constant C disappear, and one replaces α and β with x and y then one will have:

$$dx = dy \sqrt{\left(\frac{m}{m+2}\beta - C\right)^m - 1} .$$

That equation represents a curve that is similar to the one that has the equation:

(21) $$dx = dy \sqrt{y^m - 1} ,$$

and which is obviously the orthogonal trajectory of the curves that are included in equation (14).

Although the curves that were just obtained, and whose equation can always be converted into the form (21), are less remarkable than the ones that are represented by equation (20), they nonetheless enjoy some very curious properties. Like the latter, they include the cycloid, and one will obtain them by generalizing several properties of that curve. Hence, those curves are the ones for which an arbitrary power of the arc length is proportional to the abscissa. Now, one knows that the cycloid is the curve for which the square of the arc length is proportional to the abscissa. The property of tautochronism that the cycloid enjoys shows that for that curve, the integral:

$$\int_0^h \frac{ds}{\sqrt{h - x}}$$

is independent of *h*. More generally, the curves that are represented by equation (21) are the ones for which the integral:

$$\int_0^h (h-x)^n\, ds$$

is independent of *h*. One will recognize that easily by either method that Poisson presented (see his *Mécanique,* Tome premier, page 373) or by the calculus of differentials with fractional indices of Liouville, etc.

VII.

I will conclude by recalling a minimum property that the curves (7) enjoy and which was pointed out by Euler in his *Methodus inveniendi*, etc. page 53. From that property, the curves (7) are the ones for which the integral:

$$\int r^n ds = \int r^n \sqrt{dr^2 + r^2 d\theta^2}$$

is a minimum. Indeed, the method of variation will give us the differential equation of the latter curves as:

$$r^n \sqrt{r^2 + r'^2} = C + \frac{r^n\, r'^2}{\sqrt{r^2 + r'^2}},$$

so

$$r^{n+2} = C\sqrt{r^2 + r'^2}\,;$$

hence:

$$d\theta = \frac{dr}{r\sqrt{\dfrac{r^{2(n+1)}}{C^2} - 1}},$$

and therefore, upon integrating as in § **II**:

$$r^{n+1} \cos (n + 1)(\theta - \theta_0) = C.$$