"Mémoire sur la théorie des surfaces applicables sur une surface donnée," J. l'École poly. **41** (1860), 209-230.

# Memoir on the theory of surfaces that can be mapped to a given surface (*)

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## INTRODUCTION

The idea of analytically defining a surface by three equations that serve to express the ordinary coordinates as functions of two arbitrary independent variables *u* and *v* had to come to mind in the early research in analytic geometry, and it is found to have been pointed out explicitly in the writings of Euler and Lagrange, moreover; however, it is Gauss that deserves credit for having shown its importance. Indeed, in his celebrated *Disquisitiones generales circa superficies curvas*, that famous author established a set of formulas and theorems that constitute a true school of thought that is entirely new upon adopting the general viewpoint that we just mentioned. Among those theorems, one of the more remarkable ones relates to the surfaces that can be mapped to each other. Gauss obtained it by chance, in effect: He proposed to evaluate what he called the measure of the curvature of the surface (i.e., the inverse of the product of the radii of principal curvature) as a function of the variables *u* and *v*, and found that the measure depended only upon some functions *E*, *F*, *G* that appear in the expression for the line element of the surface. He concluded from this that if two surfaces can be mapped to each other (i.e., one can make each point of the first one correspond to a point of the second one, in such a fashion that the distance between two arbitrary infinitely-close points on the first one will be constantly equal to the distance between the two corresponding points on the second one, in which case the functions *E*, *F*, *G* can be considered to have the same values on the two surfaces) then the measures of the curvature will also be the same for the corresponding points.

That beautiful theorem of Gauss has led geometers to found a theory of surfaces that can be mapped to an arbitrary given surface that is analogous to the older theory of developable surfaces, namely, ones that be mapped to a plane. Minding, in tome XIX of the *Crelle's Journal*, and after him, several geometers (**), have already indicated the means of recognizing when two given surfaces can or cannot be mapped to each other.

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(*) This paper is unaltered from the one that was presented in competition for the mathematics prize that was proposed by the Académie des Sciences de Paris (year 1860).

(**) *See* Liouville's notes in Monge's *Analyse appliquée* and my paper "Sur la théorie générales des surfaces" (Journal de l'École Polytechnique, Cahier **32**, pp. 80).

For his own part, an English geometer Jellett has proved several remarkable properties in a very beautiful paper. Nonetheless, the subject has only been touched upon. Among the questions that remain to be solved, one of the more important ones has the goal of determining all of the surfaces that can be mapped to a given one. That is the one that the Academy has chosen to be the subject of the competition and that we have tried to solve in the present paper. Our article is composed of two parts. The first one contains a new proof of Gauss's theorem. One knows that the numerous known proofs leave much to be desired. Those of Bertrand, Puiseux, are Diguet are direct, but do not lead to a precisely expression for the measure of curvature; those of Gauss and Liouville tell one what the measure of curvature is, but they are indirect and suppose that the theorem is known in advance. Our proof presents none of those inconveniences. We take our starting point to be the relation that expresses the equality of the corresponding elements on the two surfaces, and by some simple transformations, we will be led to Gauss's theorem and an expression for the measure of curvature of the surfaces considered in a natural and, so to speak, inevitable manner. The second part of the paper is dedicated to the determination of the surfaces that be mapped to a given surface. Upon employing variables whose importance Gauss and Liouville showed a long time ago, we will easily reduce the question to the integration of a second-order partial differential equation. That integration is unapproachable in the general case, but some simple cases for which the result is easy to predict will permit one to verify the method and, at the same time, show how the calculations can be achieved.

In addition, we shall give several formulas that relate to the lines of curvature, asymptotic lines, and the radii of principal curvature that are useful in the case where one wants the desired surfaces to fulfill certain geometric condition. Those formulas lead to the following theorem:

*Two surfaces that can be mapped to each other and for which the asymptotic lines of one and the other system are corresponding lines must necessarily coincide.*

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## PART ONE

*Proof of Gauss's theorem. – Method of recognizing whether two given surfaces can or cannot be mapped to each other.*

**1. –** Suppose that we have two surfaces, which we represent by $S$ and $S'$, for brevity. Suppose that the rectangular coordinates $\xi, \eta, \zeta$ of the various points of the first one are expressed as functions of two arbitrary independent variables $u$ and $v$. The element $ds$ of that surface will be given by the equality:

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

in which one has:

$$E = \left(\frac{d\xi}{du}\right)^2 + \left(\frac{d\eta}{du}\right)^2 + \left(\frac{d\zeta}{du}\right)^2,$$

$$F = \frac{d\xi}{du}\frac{d\xi}{dv} + \frac{d\eta}{du}\frac{d\eta}{dv} + \frac{d\zeta}{du}\frac{d\zeta}{dv},$$

$$G = \left(\frac{d\xi}{dv}\right)^2 + \left(\frac{d\eta}{dv}\right)^2 + \left(\frac{d\zeta}{dv}\right)^2.$$

Similarly, if we suppose that the rectangular coordinates $\xi', \eta', \zeta'$ of the various points of the surface $S'$ are expressed as functions of the two variables $u'$ and $v'$ then we can determine the element $ds'$ of the second surface by the equality:

$$ds'^2 = E'du'^2 + 2F'du'\,dv' + G'dv'^2,$$

where:

$$E' = \left(\frac{d\xi'}{du'}\right)^2 + \left(\frac{d\eta'}{du'}\right)^2 + \left(\frac{d\zeta'}{du'}\right)^2,$$

$$F' = \frac{d\xi'}{du}\frac{d\xi'}{dv} + \frac{d\eta'}{du}\frac{d\eta'}{dv} + \frac{d\zeta'}{du}\frac{d\zeta'}{dv},$$

$$G' = \left(\frac{d\xi'}{dv'}\right)^2 + \left(\frac{d\eta'}{dv'}\right)^2 + \left(\frac{d\zeta'}{dv'}\right)^2,$$

and if the surfaces can be mapped to each other (i.e., in such a way that each point of the first one corresponds to a point of the second one in such a fashion that the distance between two infinitely-close points of the first one will be constantly equal to the distance between the two corresponding points on the second one) then there must exist values of $u'$ and $v'$ that are functions of one and the other of $u$ and $v$ and that verify the relation:

$$E\,du^2 + 2F\,du\,dv + G\,dv^2 = E\,du'^2 + 2F\,du\,dv + G\,dv'^2, \tag{1}$$

no matter what $u$, $v$, $du$, $dv$ are.

**2.** – In order to deduce the equality (1) of the various conditions that this equality includes, it is first important to simplify the variables $u$, $v$, $u'$, $v'$ by specializing them. Now, Gauss showed in his "Mémoire sur les cartes géographiques, couronné par l'Académie de Copenhague" that there exist variables $u$ and $v$ for any surface for which one has:

$$E = G, \qquad F = 0.$$

If one denotes those variables by $\alpha$ and $\beta$ when one is dealing with the first surface and by $\alpha'$ and $\beta'$ when one is dealing with the second one then the equality (1) can be written thus:

$$\lambda\,(d\alpha^2 + d\beta^2) = \lambda'(d\alpha'^2 + d\beta'^2),$$

Upon setting:

$$\alpha + i\,\beta = x, \qquad \alpha - i\,\beta = y,$$

$$\alpha' + i\,\beta' = x', \qquad \alpha' - i\,\beta' = y',$$

one will have, even more simply:

$$\varphi^2\,dx\,dy = \varphi'^2\,dx'\,dy', \tag{2}$$

in which $\varphi^2$ represents the value of $\lambda$ when it is expressed in terms of $x$ and $y$, and $\varphi'^2$ is the value of $\lambda'$ when it is expressed in terms of $x'$ and $y'$.

**3.** – The equality (2) shows immediately that $x'$ depends upon only one of the variables $x$ and $y$, and that $y'$ will depend upon the second of those variables. Indeed, if one sets:

$$dx' = \frac{dx'}{dx}dx + \frac{dx'}{dy}dy\,,$$

$$dy' = \frac{dy'}{dx}dx + \frac{dy'}{dy}dy$$

then the equality (2) will give:

$$\varphi^2\,dx\,dy = \varphi'^2\left(\frac{dx'}{dx}dx + \frac{dx'}{dy}dy\right)\left(\frac{dy'}{dx}dx + \frac{dy'}{dy}dy\right).$$

Now, since $dx$ and $dy$ are arbitrary, the coefficients of $dx^2$ and $dy^2$ in the right-hand side must be zero; one will then have:

$$\frac{dx'}{dx}\frac{dy'}{dx} = 0, \qquad \frac{dx'}{dy}\frac{dy'}{dy} = 0;$$

so, since $x$ and $y$ are independent variables:

$$x' = f(x), \qquad y' = f_1(y),$$

or rather:

$$x' = f_1(y), \qquad y' = f(x).$$

**4. –** The first result reduces the equality (2) to:

$$\varphi^2 = \varphi'^2\,f(x)\;f_1'(y)\,; \tag{3}$$

hence, upon taking logarithms:

$$\log \varphi^2 = \log \varphi'^2 + \log f'(x) + f_1'(y) .$$

Upon differentiating this once with respect to $x$ and once with respect to $y$, one will get:

$$\frac{d^2 \log \varphi^2}{dx\,dy} = \frac{d^2 \log \varphi'^2}{dx\,dy},$$

or rather, when one recalls the values of $x'$ and $y'$:

$$\frac{d^2 \log \varphi^2}{dx\,dy} = \frac{d^2 \log \varphi'^2}{dx'\,dy'} f'(x)\,f_1'(y),$$

and due to the relation (3):

$$\frac{1}{\varphi}\frac{d^2 \log \varphi^2}{dx\,dy} = \frac{1}{\varphi'^2}\frac{d^2 \log \varphi'^2}{dx'\,dy'} .$$

We then obtain the remarkable consequence that if the two surfaces $S$ and $S'$ can be mapped to each other then the function:

$$k = \frac{1}{\varphi}\frac{d^2 \log \varphi^2}{dx\,dy}$$

will have the same value for the two surfaces at the corresponding points.

**5.** – It now remains for us to understand the geometric significance of $k$. In order to do that, we shall determine the form that this function takes when one employs the arbitrary variables $u$ and $v$, in place of the particular values $x$ and $y$.

I first return to the variables $\alpha$ and $\beta$, which have the advantage of being real, so I will have:

$$4k = \frac{1}{\lambda}\left(\frac{d^2 \log \lambda}{d\alpha^2} + \frac{d^2 \log \lambda}{d\beta^2}\right).$$

I then multiply this by the surface element $dS$ on the surface, which is an element that will have the value $\lambda\, d\alpha\, d\beta$ here, and integrate while taking the boundary to be an arbitrary closed contour that is traced on the surface considered; I obtain:

(4) $$4\iint k\, dS = \iint \frac{d^2 \log \lambda}{d\alpha^2} d\alpha\, d\beta + \iint \frac{d^2 \log \lambda}{d\beta^2} d\alpha d\beta .$$

The right-hand side can be simplified. Consider the first term:

$$\iint \frac{d^2 \log \lambda}{d\alpha^2} d\alpha\, d\beta .$$

Upon performing the integration over $\alpha$ and omitting the second integration, for the moment, one will get:

$$d\beta \left[ -\left(\frac{d \log \lambda}{d\alpha}\right)_1 + \left(\frac{d \log \lambda}{d\alpha}\right)_2 - \left(\frac{d \log \lambda}{d\alpha}\right)_3 + \left(\frac{d \log \lambda}{d\alpha}\right)_4 + \cdots - \left(\frac{d \log \lambda}{d\alpha}\right)_{2m-1} + \left(\frac{d \log \lambda}{d\alpha}\right)_{2m} \right].$$

1, 2, 3, …, $2m$ are the successive points (which are always even in number), where the coordinate line $\beta$ = const. (prolonged in the positive sense) meets the boundary contour, and represents the value of $d \log \lambda / d\alpha$ at the point $p$, in general. However, if one supposes that the boundary contour is traversed in the sense that the points that immediately follow the points 3, 5, …, $2m - 1$ are likewise on the side to which one counts $\alpha$ as negative with respect to the line $\beta$ = const. then one will have:

$$\sqrt{\lambda}\, d\beta = \sin i \, ds$$

for the points of odd rank 1, 3, 5, …, $2m - 1$, in which one generally lets $i$ denote the positive angle that the contour thus-traversed will form with the positive $\beta$ and lets $ds$ denote the element of the contour (which is taken positively):

$$\sqrt{\lambda}\, d\beta = -\sin i \, ds$$

for the points of even rank 2, 4, …, $2m$. Having said that, the expression above will amount to:

$$-\sum \frac{d \log \lambda}{d\alpha} \sin i \frac{ds}{\sqrt{\lambda}} ,$$

in which the sum is taken over all point 1, 2, 3, …, $2m$ where the boundary contour is met by the line $\beta$ = const.; consequently, upon performing the integration over $\beta$ that has been temporarily omitted, one will find that for the reduced value of the term $\iint \frac{d^2 \log \lambda}{d\alpha^2} d\alpha\, d\beta$, the simple integral:

$$-\int \frac{d \log \lambda}{d\alpha} \sin i \frac{ds}{\sqrt{\lambda}} = -\int \frac{d\sqrt{\lambda}}{d\alpha} \sin i \frac{ds}{\lambda} ,$$

when it is taken over all points on the perimeter of the contour.

When an analogous transformation is applied to the term $\iint \frac{d^2 \log \lambda}{d\beta^2} d\alpha\, d\beta$, that will permit one to replace that term by the simple integral:

$$2\int \frac{d\sqrt{\lambda}}{d\beta}\cos i \frac{ds}{\lambda},$$

which is also extended over all points on the perimeter of the contour, and the equality (4) will then reduce to:

$$2\iint k\,dS = -\int \left( \frac{d\sqrt{\lambda}}{d\alpha}\sin i - \frac{d\sqrt{\lambda}}{d\beta}\cos i \right)\frac{ds}{\lambda},$$

or rather to:

$$(5) \qquad 2\iint k\,dS = \int \left( \frac{d(\sin i/\sqrt{\lambda})}{d\alpha} - \frac{d(\cos i/\sqrt{\lambda})}{d\beta} \right) ds - \int \left( \frac{\cos i}{\sqrt{\lambda}}\frac{di}{d\alpha} - \frac{\sin i}{\sqrt{\lambda}}\frac{di}{d\beta} \right) ds\,.$$

Now let $d\alpha$ and $d\beta$ be the positive or negative increments that $\alpha$ and $\beta$ will receive when one passes from the first extremity of the element $ds$ to the other, and let $\delta\alpha$, $\delta\beta$, be the increments in the same variables for a displacement that is equal to $\delta n$, perpendicular to the element $ds$, and external to the boundary contour: No matter what point of the contour that one considers, one will have:

$$(6) \qquad \begin{aligned} \sqrt{\lambda}\,d\alpha &= \cos i\;ds, & \sqrt{\lambda}\,d\beta &= \sin i\;ds, \\ \sqrt{\lambda}\,\delta\alpha &= -\sin i\;\delta n\,, & \sqrt{\lambda}\,\delta\beta &= \cos i\;\delta n, \end{aligned}$$

so

$$(7) \qquad d\beta = -\frac{ds}{\delta n}\,\delta\alpha, \qquad d\alpha = \frac{ds}{\delta n}\,\delta\beta.$$

The relation (6) permits one to put the equality (5) into the form:

$$2\iint k\,dS = \int \left( \frac{d(1/ds)}{d\alpha}d\beta - \frac{d(1/ds)}{d\beta}d\alpha \right) ds - \int \left( \frac{di}{d\alpha}\frac{d\alpha}{ds} + \frac{di}{d\beta}\frac{d\beta}{ds} \right) ds\,,$$

and the relations (7) will then give:

$$2\iint k\,dS = \int \left( \frac{d\cdot ds}{d\alpha}\frac{\delta\alpha}{ds} + \frac{d\cdot ds}{d\beta}\frac{\delta\beta}{ds} \right) ds - \int \left( \frac{di}{d\alpha}\frac{d\alpha}{ds} + \frac{di}{d\beta}\frac{d\beta}{ds} \right) ds\,;$$

i.e.:

$$2\iint k\,dS = \int \frac{\delta\cdot ds}{\delta n}ds - \int di\,,$$

in which $di$ is the positive or negative increment that $i$ will receive when one passes from the first extremity of the element $ds$ to the second one, and $\delta ds$ is the increment that $ds$ receives under a displacement that is equal to $\delta n$ and normal to $ds$, and external for the boundary contour. Furthermore:

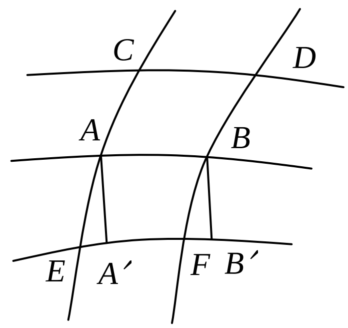
$$\int di = A + B + C + \ldots - (n-2)\,\pi,$$

where $A$, $B$, $C$, … are the interior angles of the contour, and $n$ is the number of those angles; one finally has:

$$2\iint k\,dS = \int \frac{\delta \cdot ds}{\delta n\,ds}\,ds - A - B - C - \ldots + (n-2)\,\pi \tag{8}$$

then.

**6.** – The preceding relation implicitly includes all of the formulas of the "Disquisitiones generales circa superficies curvas," those of my paper on the general theory of surfaces, and the somewhat-more-general ones that Liouville gave in his course at the Collége de France. Here, we shall confine ourselves to deducing the value of $k$ as a function of the variables $u$ and $v$.




Suppose that the boundary contour to which the integrals that are contained in the right-hand sides of the equality (8) are referred is the infinitely-small parallelogram *BACD* that is defined by the coordinate lines $u$, $v$, $u + du$, $v + dv$ ($du$ and $dv$ are positive). The integral $\iint k\,dS$ will reduce to just one element, and since $dS = \sqrt{EG - F^2}\;du\,dv$, in which $E$, $F$, $G$ are always the functions that figure in the expression of the line of the surface, one will have simply:

$$2\sqrt{EG - F^2}\cdot k\,du\,dv$$

for the left-hand side of the equality (8). As for the right-hand side, one will first have $n = 4$, and then, upon letting the angle $\omega$ be between the lines $u$ and $v$ :

$$A = \omega,$$

$$B = \pi - \left(\omega + \frac{d\omega}{dv}\,dv\right),$$

$$C = \pi - \left(\omega + \frac{d\omega}{du}\,du\right),$$

$$D = \omega + \frac{d\omega}{du}du + \frac{d\omega}{dv}dv + \frac{d^2\omega}{du\,dv}du\,dv\,,$$

and consequently:

$$-A - B - C - \ldots + (n-2)\,\pi = -\frac{d^2\omega}{du\,dv}du\,dv\,,$$

i.e.:

$$-\frac{d}{dv}\left[\frac{1}{2\sqrt{EG-F^2}}\left(\frac{F}{E}\frac{dE}{du} + \frac{F}{G}\frac{dG}{du} - 2\frac{dF}{du}\right)\right]du\,dv\,,$$

upon observing that:

$$\cos\,\omega = \frac{F}{\sqrt{EG}}\,.$$

Finally, the integral $\int \frac{\delta\,ds}{\delta n\,ds}ds$ reduces to four elements that correspond to the four sides of the parallelogram *BACD*. The element that relates to the side *AB* has the value:

$$\frac{A'B' - AB}{AA'}\,,$$

*AA′* and *BB′* are normal to *AB*, and *EAF′B′* is the line $(u + \delta u)$; however:

$$AA' = AE\,\sin\,\omega = -\frac{\sqrt{EG-F^2}}{\sqrt{G}}\,\delta u,$$

$$A'B' - AB = EF - AB + FB' - EA' = \frac{d\sqrt{G}}{du}dv\;\delta u - \frac{d(F/\sqrt{G})}{dv}dv\;\delta u\,;$$

one will then have:

$$-\frac{1}{\sqrt{EG-F^2}}\left(\frac{dG}{du} + \frac{F}{G}\frac{dG}{dv} - 2\frac{dF}{dv}\right)dv$$

for that element. One immediately deduces from this that the element that relates to the side *CD* is:

$$\frac{1}{2\sqrt{EG-F^2}}\left(\frac{dG}{du} + \frac{F}{G}\frac{dG}{dv} - 2\frac{dF}{dv}\right)dv + \frac{d}{du}\left[\frac{1}{2\sqrt{EG-F^2}}\left(\frac{dG}{du} + \frac{F}{G}\frac{dG}{dv} - 2\frac{dF}{dv}\right)\right]du\,dv.$$

One will likewise find:

$$\frac{d}{du}\left[\frac{1}{2\sqrt{EG-F^2}}\left(\frac{dG}{du}+\frac{F}{G}\frac{dG}{dv}-2\frac{dF}{dv}\right)\right]du\,dv$$

for the sum of the two elements that relate to *AC* and *BD*; hence, the integral $\int\frac{\delta\,ds}{\delta n\,ds}$ will have the value:

$$\frac{d}{du}\left[\frac{1}{2\sqrt{EG-F^2}}\left(\frac{dG}{du}+\frac{F}{G}\frac{dG}{dv}-2\frac{dF}{dv}\right)\right]du\,dv+\frac{d}{dv}\left[\frac{1}{2\sqrt{EG-F^2}}\left(\frac{dE}{dv}+\frac{F}{E}\frac{dE}{du}-2\frac{dF}{du}\right)\right]du\,dv;$$

consequently, the equality (8) will reduce to:

$$(9)\qquad 4\sqrt{EG-F^2}\,k = \frac{d}{du}\left[\frac{1}{\sqrt{EG-F^2}}\left(\frac{dG}{du}+\frac{F}{G}\frac{dG}{dv}-2\frac{dF}{dv}\right)\right]-\frac{d}{dv}\left[\frac{1}{\sqrt{EG-F^2}}\left(\frac{dE}{dv}-\frac{F}{G}\frac{dG}{du}\right)\right].$$

That is the formula that give the value of *k* as a function of the arbitrary variables *u* and *v*.

**7. –** It is now quite easy to find the geometric significance of *k* and to prove that two times that function will express the measure of curvature, up to sign. Indeed, suppose that *u* and *v* are the rectangular coordinates $\xi$, $\eta$, so one will have:

$$E = 1+\left(\frac{d\zeta}{d\xi}\right)^2,\qquad F=\frac{d\zeta}{d\xi}\frac{d\zeta}{d\eta},\qquad G = 1+\left(\frac{d\zeta}{d\eta}\right)^2,$$

in which $\zeta$ is the third coordinate, and then:

$$\frac{1}{\sqrt{EG-F^2}}\left(\frac{dG}{du}+\frac{F}{G}\frac{dG}{dv}-2\frac{dF}{dv}\right)=\frac{-2\dfrac{d\zeta}{d\xi}\dfrac{d^2\zeta}{d\eta^2}}{\left[1+\left(\dfrac{d\zeta}{d\eta}\right)^2\right]\sqrt{1+\left(\dfrac{d\zeta}{d\xi}\right)^2+\left(\dfrac{d\zeta}{d\eta}\right)^2}},$$

$$\frac{1}{\sqrt{EG-F^2}}\left(\frac{dE}{dv}-\frac{F}{G}\frac{dG}{du}\right)=\frac{2\dfrac{d\zeta}{d\xi}\dfrac{d^2\zeta}{d\xi\,d\eta}}{\left[1+\left(\dfrac{d\zeta}{d\eta}\right)^2\right]\sqrt{1+\left(\dfrac{d\zeta}{d\xi}\right)^2+\left(\dfrac{d\zeta}{d\eta}\right)^2}};$$

consequently:

$$2k = \frac{1}{\sqrt{1+\left(\frac{d\zeta}{d\xi}\right)^2+\left(\frac{d\zeta}{d\eta}\right)^2}}\left\{\frac{d}{d\xi}\frac{-\frac{d\zeta}{d\xi}\frac{d^2\zeta}{d\eta^2}}{\left[1+\left(\frac{d\zeta}{d\eta}\right)^2\right]\sqrt{1+\left(\frac{d\zeta}{d\xi}\right)^2+\left(\frac{d\zeta}{d\eta}\right)^2}}\right.$$

$$\left.+\frac{d}{d\eta}\frac{\frac{d\zeta}{d\xi}\frac{d^2\zeta}{d\xi\,d\eta}}{\left[1+\left(\frac{d\zeta}{d\eta}\right)^2\right]\sqrt{1+\left(\frac{d\zeta}{d\xi}\right)^2+\left(\frac{d\zeta}{d\eta}\right)^2}}\right\},$$

$$= \frac{1}{\sqrt{1+\left(\frac{d\zeta}{d\xi}\right)^2+\left(\frac{d\zeta}{d\eta}\right)^2}}\left\{-\frac{d^2\zeta}{d\eta^2}\frac{d}{d\xi}\frac{-\frac{d\zeta}{d\xi}}{\left[1+\left(\frac{d\zeta}{d\eta}\right)^2\right]\sqrt{1+\left(\frac{d\zeta}{d\xi}\right)^2+\left(\frac{d\zeta}{d\eta}\right)^2}}\right.$$

$$\left.+\frac{d^2\zeta}{d\xi\,d\eta}\frac{d}{d\eta}\frac{\frac{d\zeta}{d\xi}}{\left[1+\left(\frac{d\zeta}{d\eta}\right)^2\right]\sqrt{1+\left(\frac{d\zeta}{d\xi}\right)^2+\left(\frac{d\zeta}{d\eta}\right)^2}}\right\},$$

or, upon developing this and reducing it:

$$2k = \frac{\left(\frac{d^2\zeta}{d\xi\,d\eta}\right)^2-\frac{d^2\zeta}{d\xi^2}\frac{d^2\zeta}{d\eta^2}}{\left[1+\left(\frac{d\zeta}{d\xi}\right)^2+\left(\frac{d\zeta}{d\eta}\right)^2\right]^{3/2}},$$

which is indeed the expression for the inverse of the product of the radii of principal curvature in terms of $\xi$, $\eta$, $\zeta$, up to sign.

**8.** – It results from all of the preceding that when two surfaces $S$ and $S'$ can be mapped to each other, the measures of the curvatures will be equal for the two surfaces at the corresponding points. One will also get that beautiful theorem of Gauss as a consequence of the equality (1). Moreover, formula (9), which one met up with in the course of the proof, will provide the value of the measure of curvature for the general case and in the elegant form that Liouville had pointed out. One knows that none of the known proofs present that double advantage.

**9.** – Gauss's theorem constitutes a necessary, but not sufficient, condition; one must therefore always revert to the equality (1) in order to know whether two given surfaces can or cannot be mapped to each other. Nevertheless, observe that when one obtains a first relation between the corresponding points of the two surfaces, it is easy to deduce a second one. One can then calculate the values of $u$ and $v$ that are the only one admissible ones, and one will no longer have to substitute those values in the equality (1) in order to see whether the surfaces are or are not truly mappable to each other. Let us go into that topic in some detail.

**10.** – Let $k$ be a function of $u$ and $v$, and let $k'$ be a function of $u'$ and $v'$. Suppose that when the two surfaces $S$ and $S$ can be mapped to each other, one has:

$$k = k' \tag{10}$$

for the corresponding points. (One knows from Gauss's theorem that this will be true when one takes $k$ and $k'$ to be the measures of curvature.) Upon differentiation, that will give:

$$\frac{dk}{du}du + \frac{dk}{dv}dv = \frac{dk'}{du'}du' + \frac{dk'}{dv'}dv',$$

or, to simplify the writing:

$$m\,du + n\,dv = m'\,du' + n'\,dv'. \tag{11}$$

Equation (1) and equation (11) determine $du'$ and $dv'$ as functions of $du$ and $dv$. However, the values of those differentials must be linear in $du$ and $dv$, since $u'$ and $v'$ are supposed to be functions of $u$ and $v$; due to the form of equations (1) and (11), that obviously cannot be true, unless there exists a certain relation between $m$, $n$, $m'$, $n'$, $E$, $F$, $G$, $E'$, $F'$, $G'$.

In order to get that relation, I take the square of equation (11) and add the corresponding sides of that with equation (1), which has been previously multiplied by an indeterminate factor $\lambda$; we will have:

$$(m^2 + \lambda E)\,du^2 + 2\,(mn + \lambda F)\,du\,dv + (n^2 + \lambda G)\,dv^2$$

$$= (m'^2 + \lambda E')\,du'^2 + 2\,(m'n' + \lambda F')\,du'\,dv' + (n'^2 + \lambda G')\,dv'^2.$$

If one now determines $\lambda$ in such a fashion that the left-hand side is the square of a first-degree binomial in $du$ and $dv$ then it will be necessary that the right-hand side becomes the square of a first-degree binomial in $du'$ and $dv'$. In other words, the values of $\lambda$ that make both sides into squares must be equal. One will then have:

$$\frac{En^2 - 2Fmn + Gm^2}{EG - F^2} = \frac{E'n'^2 - 2F'm'n' + G'm'^2}{E'G' - F'^2}.$$

That is the desired relation. We then write it as:

(12) $$H = H',$$

and upon adjoining equation (1) to it, along with the two corresponding differential equations:

(13) $$\begin{cases} m\,du + n\,dv = m'\,du' + n'\,dv', \\ \dfrac{dH}{du}du + \dfrac{dH}{dv}dv = \dfrac{dH'}{du'}du' + \dfrac{dH'}{dv'}dv', \end{cases}$$

we will get four equations that permit us to determine $u'$, $v'$, $du'$, $dv'$ as functions of $u$, $v$, $du$, $dv$, in such a way that it will suffice to require those values to verify the equality (1), for any $u$, $v$, $du$, $dv$, in order to get the conditions that express the idea that the two surfaces can be mapped to each other.

**11.** – One can perform some of the calculations and obtain a simpler result. I first observe that if one subtract the product of corresponding sides of the equalities (1) and (12) from the square of the equality (11) then one will get:

$$\frac{[(En - Fm)\,du + (Fu - Gm)\,dv]^2}{EG - F^2} = \frac{[(E'n' - F'm')\,du' + (F'u' - G'm')\,dv']^2}{E'G' - F'^2},$$

i.e.:

(14) $$(en - fm)\,du + (fn - gm)\,dv = (e'n' - f'm')\,du' + (f'n' - g'm')\,dv',$$

upon setting:

$$\frac{E}{\sqrt{EG - F^2}} = e, \qquad \frac{F}{\sqrt{EG - F^2}} = f, \qquad \frac{G}{\sqrt{EG - F^2}} = g,$$

$$\frac{E'}{\sqrt{E'G' - F'^2}} = e', \qquad \frac{F'}{\sqrt{E'G' - F'^2}} = f', \qquad \frac{G'}{\sqrt{E'G' - F'^2}} = g',$$

which is a relation that can replace the equality (1).

Upon now eliminating $du'$ and $dv'$ from equations (11), (13), and (14), and expressing the idea that the final equation is satisfied for any $du$ and $dv$, one will have:

$$\frac{\dfrac{dH}{du}[(e'n' - f'm')n - (fn - gm)m'] + \dfrac{dH}{dv}[-(e'n' - f'm')m - (en - fm)m']}{en^2 - 2fmn + gm^2} = \frac{dH'}{du'},$$

$$\frac{\dfrac{dH}{du}[(f'n'-g'm')n-(fn-gm)n']+\dfrac{dH}{dv}[-(f'n'-g'm')m-(en-fm)n']}{en^2-2fmn+gm^2}=\frac{dH'}{dv'},$$

from which, one easily deduces that:

$$\frac{n\dfrac{dH}{du}-m\dfrac{dH}{dv}}{en^2-2fmn+gm^2}=\frac{n'\dfrac{dH'}{du'}-m'\dfrac{dH'}{dv'}}{e'n'^2-2f'm'n'+g'm'^2},$$

$$\frac{(fn-gm)\dfrac{dH}{du}-(en-fm)\dfrac{dH}{dv}}{en^2-2fmn+gm^2}=\frac{(f'n'-g'm')\dfrac{dH'}{du'}-(e'n'-f'm')\dfrac{dH'}{dv'}}{e'n'^2-2f'm'n'+g'm'^2},$$

or rather, due to the equality (12):

(15) $$\frac{n\dfrac{dH}{du}-m\dfrac{dH}{dv}}{\sqrt{EG-F^2}}=\frac{n'\dfrac{dH'}{du'}-m'\dfrac{dH'}{dv'}}{\sqrt{EG-F^2}},$$

(16) $$\frac{(Fn-Gm)\dfrac{dH}{du}-(En-Fm)\dfrac{dH}{dv}}{EG-F^2}=\frac{(F'n'-G'm')\dfrac{dH'}{dn'}-(E'n'-F'm')\dfrac{dH'}{dv'}}{E'G'-F'^2}.$$

It will then suffice that the values of $u'$ and $v'$ that are both inferred from the four equations (10), (12), (15), (16) will verify the other two, not matter what $u$ and $v$ are.

**12.** – Before going on, it would not be pointless to indicate the geometric significance of the equalities (12), (15), (16). Now, in the first place, the equality (12) expresses the idea that the quotient that is obtained upon dividing by $\delta s$ the increment $\delta k$ that $k$ takes on for a displacement that is equal to $\delta s$ that is performed on the surface $S$ normally to the curve $k$ = const. is equal to the quotient that is obtained upon dividing by $\delta s'$ the increment $\delta k'$ that $k'$ takes on for a displacement that is equal to $\delta s'$ that is performed on the surface $S'$ normally to the curve $k'$ = const. Indeed, if we let $\delta u$ and $\delta v$ denote the increments that $u$ and $v$ take on when one passes from the first extremity of $\delta s$ to the other then we will have the three equations:

$$m\,\delta u+n\,\delta v=\delta k,$$

$$(En-Fm)\,\delta u+(Fn-Gm)\,\delta v=0,$$

$$E\,\delta u^2+2F\,\delta u\,\delta v+G\,\delta v^2=\delta s^2.$$

Squaring the first and adding corresponding sides to the third one, when it was previously multiplied by $\dfrac{En^2 - 2Fmn + Gm^2}{F^2 - EG}$, one will get:

$$\frac{[(En - Fm)\,\delta u + (Fn - Gm)\,\delta v]^2}{EG - F^2} = \delta k^2 + \frac{En^2 - 2Fmn + Gm^2}{F^2 - EG}\,ds^2,$$

or simply, due to the second:

$$\frac{\delta k^2}{ds^2} = \frac{En^2 - 2Fmn + Gm^2}{EG - F^2}.$$

One will likewise find that:

$$\frac{\delta k'^2}{ds'^2} = \frac{E'n'^2 - 2F'm'n' + G'm'^2}{E'G' - F'^2};$$

hence, the equality indeed comes down to:

$$\frac{\delta k}{ds} = \frac{\delta k'}{ds'}.$$

As for the relations (15) and (16) – or rather, the ones that one gets upon multiplying corresponding sides of the latter with $\sqrt{\dfrac{1}{H}} = \sqrt{\dfrac{1}{H'}}$ – they express the idea (one easily verifies this, so it should not be necessary to give a proof) that the two quotients that are obtained upon dividing by $ds$ the increments that $H$ takes on under displacements that are equal to $ds$ and are performed on the surface $S$ along the curve $k$ = const. and normally to that curve are respectively equal to the two quotients that are obtained upon dividing by $ds'$ the increments that $H'$ takes on under displacements that are equal to $ds'$ and are performed on the surface $S'$ along the curve $k'$ = const. and normally to that curve.

**13.** – We have excluded two cases from the preceding:

1. The one in which equations (10) and (12) are incompatible.

2. The one in which those two equations imply each other.

In the former case, the surfaces cannot be mapped to each other. In the latter case, which will occur only when $H$ is a function of $k$ and $H'$ is the same function of $k'$, the surfaces can be mappable to each other, but one must employ some special considerations in order to insure that they really are.

First observe that equations (10) and (12) will be satisfied by that very reason itself, so one can always replace the equality (1) with the equality (14):

$$(14) \qquad (en - fm)\,du + (fn - gm)\,dv = (e'n' - f'm')\,du' + (f'n' - g'm')\,dv',$$

in such a way that the question comes down to expressing the idea that there exist values of $u'$ and $v'$, which are functions of $u$ and $v$, resp., that verify equations (10) and (14) for any $u$, $v$, $du$, $dv$.

If we substitute the values of $v'$ and $dv'$ in the equality (14) that are inferred from equation (10) and the corresponding differential equation:

$$m\,du + n\,dv = m'\,du' + n'\,dv'$$

then we will have:

$$\begin{aligned}\left[\overline{n}'(en-fm)-m\overline{(f'n'-g'm')}\right]du+\left[\overline{n}'(fn-gm)-n\overline{(f'n'-g'm')}\right]dv&\\ +\left[\overline{\overline{m}'(f'n'-g'm')}-\overline{n(e'n'-f'm')}\right]du'&=0\end{aligned}$$

(the line that is placed over an expression indicates that one has replaced $v'$ with its value that is inferred from setting $k = k'$), and it will suffice to demand that the latter equation is satisfied by a value of $u'$ that is a function of $u$ and $v$ for any $u$, $v$, $du$, and $dv$. In order to do this, it is necessary that the integrability condition must be fulfilled, which is, as one knows:

$$\begin{aligned}&\left[\overline{n}'(en-fm)-m\overline{(f'n'-g'm')}\right]\\
&\times\left\{(fn-gm)\left(\overline{\frac{dn'}{du'}}-\overline{\frac{m'}{n'}\frac{dn'}{dv'}}\right)-n\left[\overline{\frac{d(f'n'-g'm')}{du'}}-\overline{\frac{m'}{n'}\frac{d(f'n'-g'm')}{dv'}}\right]\right.\\
&\left.+\ \overline{(e'n'-f'm')\frac{n}{n'}\frac{dn'}{dv'}}+n\overline{\frac{d(e'n'-f'm')}{dv'}}-\overline{(f'n'-g'm')\frac{n}{n'}\frac{dm'}{dv'}}-\overline{\frac{nm'}{n'}\frac{d(f'n'-g'm')}{dv'}}\right\}\\
&+\left[\overline{n}'(fn-gm)-m\overline{(f'n'-g'm')}\right]\\
&\times\left\{-(en-fm)\left(\overline{\frac{dn'}{du'}}-\overline{\frac{m'}{n'}\frac{dn'}{dv'}}\right)-m\left[\overline{\frac{d(f'n'-g'm')}{du'}}-\overline{\frac{m'}{n'}\frac{d(f'n'-g'm')}{dv'}}\right]\right.\\
&\left.-\ \overline{(e'n'-f'm')\frac{m}{n'}\frac{dn'}{dv'}}-m\overline{\frac{d(e'n'-f'm')}{dv'}}+\overline{(f'n'-g'm')\frac{n}{n'}\frac{dm'}{dv'}}-\overline{\frac{nm'}{n'}\frac{d(f'n'-g'm')}{dv'}}\right\}\\
&+\left[\overline{m'(f'n'-g'm')}-\overline{n}'(e'n'-fm')\right]\\
&\times\left[\ \overline{n}'\frac{d(en-fm)}{dv}+(en-fm)\overline{\frac{n}{n'}\frac{dn'}{dv'}}-\overline{(f'n'-g'm')}\frac{dm}{dv}\right.\end{aligned}$$

$$- m \, \overline{\frac{d(f'n'-g'm')}{dv'}\frac{n}{n'}} - \overline{n'}\frac{d(fn-gm)}{du} - (fn-gm)\,\overline{\frac{m}{n'}\frac{dn'}{dv'}} + \overline{(f'n'-g'm')}\frac{dn}{du} + n\,\overline{\frac{m}{n'}\frac{d(f'n'-g'm')}{dv'}}\Bigg]$$
$$= 0,$$

or, upon simplifying this by means of the relations $\frac{dm}{dv} = \frac{dn}{du}$, $\frac{dm'}{dv'} = \frac{dn'}{du'}$:

$$\frac{\dfrac{d(en-fm)}{dv} - \dfrac{d(fn-gm)}{du}}{en^2 - 2fmn + gm^2} = \frac{\dfrac{d(e'n'-f'm')}{dv'} - \dfrac{d(f'n'-g'm')}{du}}{e'n'^2 - 2f'm'n' + g'm'^2}.$$

That condition, which one can then write:

$$\frac{\dfrac{d\cdot\dfrac{En-Fm}{\sqrt{En^2-2Fmn+Gm^2}}\sqrt{H}}{dv} - \dfrac{d\cdot\dfrac{Fn-Gm}{\sqrt{En^2-2Fmn+Gm^2}}\sqrt{H}}{du}}{H\sqrt{EG-F^2}}$$

$$= \frac{\dfrac{d\cdot\dfrac{E'n'-F'm'}{\sqrt{E'n'^2-2F'm'n'+G'm'^2}}\sqrt{H'}}{dv'} - \dfrac{d\cdot\dfrac{F'n'-G'm'}{\sqrt{E'n'^2-2F'm'n'+G'm'^2}}\sqrt{H'}}{du'}}{H'\sqrt{E'G'-F'^2}},$$

upon recalling the values of $e, f, g, e', f', g'$, and which will then reduce to the following:

$$\frac{\dfrac{d\cdot\dfrac{En-Fm}{\sqrt{En^2-2Fmn+Gm^2}}}{dv} - \dfrac{d\cdot\dfrac{Fn-Gm}{\sqrt{En^2-2Fmn+Gm^2}}}{du}}{\sqrt{EG-F^2}}$$

$$= \frac{\dfrac{d\cdot\dfrac{E'n'-F'm'}{\sqrt{E'n'^2-2F'm'n'+G'm'^2}}}{dv'} - \dfrac{d\cdot\dfrac{F'n'-G'm'}{\sqrt{E'n'^2-2F'm'n'+G'm'^2}}}{du'}}{\sqrt{E'G'-F'^2}},$$

since $k = k'$, $H = f(k)$, $H' = f(k')$, will replace equation (12) in the present case. If we set:

(17) $$L = L',$$

for brevity, then upon operating as in the general case, we will see that the necessary and sufficient condition for the two surfaces to be mapped to each other is that the values of $u$ and $v$ that are deduced from the two equation (10) and (17) must verify the following equations identically:

$$(18)\quad \begin{cases} \dfrac{n\dfrac{dL}{du}-m\dfrac{dL}{dv}}{\sqrt{EG-F^2}} = \dfrac{n'\dfrac{dL'}{du'}-m'\dfrac{dL'}{dv'}}{\sqrt{E'G'-F'^2}}, \\[2ex] \dfrac{(Fn-Gm)\dfrac{dL}{du}-(En-Fm)\dfrac{dL}{dv}}{EG-F^2} = \dfrac{(F'n'-G'n')\dfrac{dL'}{du'}-(E'n'-F'm')\dfrac{dL'}{dv'}}{E'G'-F'^2}. \end{cases}$$

**14.** – One sees effortlessly that the second of the preceding equations is a consequence of the two equations:

$$k = k', \qquad L = L',$$

and can consequently be omitted. Indeed, from the formula that I gave in the volume **42** of the *Comptes rendus*, and much later in my paper on the use of a new system of variables in the study of the properties of curved surfaces, which was included in volume V, second series, of the *Journal de Liouville*, $L$ represents the geodesic curvature of the curves $k$ = const.; hence, the expression:

$$\frac{(Fn-Gm)\dfrac{dL}{du}-(En-Fm)\dfrac{dL}{dv}}{EG-F^2}$$

is, up to the factor $\sqrt{1/H}$, the quotient that is obtained upon dividing by $ds$ the increment that this geodesic curvature will take on under a displacement that is equal to $ds$ and situated on the surface $S$ and the normal to the curve $k$ = const. On the other hand, since $H$ is a function of $k$, the curves $k$ = const. are equidistant and have geodesic lines for their trajectories; consequently [*see* my paper on the general theory of surfaces (Journal de l'École Polytechnique, t. **32**, p. 52)], one has:

$$\frac{(Fn-Gm)\dfrac{dL}{du}-(En-Fm)\dfrac{dL}{dv}}{EG-F^2}\sqrt{\frac{1}{H}} + L^2 = k.$$

One similarly has:

$$\frac{(F'n'-G'm')\dfrac{dL'}{du'}-(E'n'-F'm')\dfrac{dL'}{dv'}}{E'G'-F'^2}\sqrt{\frac{1}{H'}} + L'^2 = k';$$

having said that, the two conditions:

$$k = k', \qquad L = L',$$

the first of which already includes $H = H'$, will necessarily imply the second of equations (18).

**15.** – It can happen that the relation $L = L'$ will lead back to the equality (10), or in other words, that $L$ is a function of $k$ and $L'$ is the same function of $k'$. In that case, upon multiplying corresponding sides of equation (14) and this one:

$$\frac{1}{\sqrt{H}} e^{-\int \frac{L}{\sqrt{H}} dk} = \frac{1}{\sqrt{H'}} e^{-\int \frac{L'}{\sqrt{H'}} dk'},$$

one will obtain a new equality whose left-hand side is the exact differential of a function of $u$ and $v$ and whose right-hand side is an exact differential of a function of $u'$ and $v'$. Indeed, one has:

$$\frac{d \cdot \frac{en - fm}{\sqrt{H}} e^{-\int \frac{L}{\sqrt{H}} dk}}{dv} - \frac{d \cdot \frac{fn - gm}{\sqrt{H}} e^{-\int \frac{L}{\sqrt{H}} dk}}{du}$$

$$= e^{-\int \frac{L}{\sqrt{H}} dk} \left\{ \frac{d \cdot \frac{En - Fm}{\sqrt{En^2 - 2Fmn + Gm^2}}}{dv} - \frac{d \cdot \frac{Fn - Gm}{\sqrt{En^2 - 2Fmn + Gm^2}}}{du} \right.$$

$$\left. + \frac{L}{H} \left[ (fn - gm) \frac{dk}{du} - (en - fm) \frac{dk}{dv} \right] \right\}$$

$$= e^{-\int \frac{L}{\sqrt{H}} dk} \left[ L\sqrt{EG - F^2} - \frac{L}{H}(en^2 - 2\,fmn + gm^2) \right] = 0,$$

and also:

$$\frac{d \cdot \frac{e'n' - f'm'}{\sqrt{H}} e^{-\int \frac{L'}{\sqrt{H'}} dk'}}{dv'} - \frac{d \cdot \frac{f'n' - g'm'}{\sqrt{H'}} e^{-\int \frac{L'}{\sqrt{H'}} dk'}}{du'} = 0 ;$$

hence, one can reduce equations (14) to another one of the form:

(19) $$F(u, v) = F'(u', v') + \text{const.}$$

by quadratures.

In the last case, the surfaces will always be mappable to each other, and the values of $u'$ and $v'$ will be given by equations (10) and (19). Due to the arbitrary constant that is contained in equation (19), one will see that there is an infinitude of systems of corresponding points.

**16.** – It still remains for us to speak of the interesting special case in which the measure of curvature $k$ of the surface $S$ is equal to a constant. In that case, one will immediately see whether the surface $S'$ is or is not mappable to the surface $S$. In order for the two surfaces to be mappable to each other, it is necessary and sufficient that the measure of the curvature $k'$ must be the curvature of $S'$, reduced by the same constant as the measure $k$ of the surface $S$. However, the search for the corresponding points on the two surfaces $S$ and $S'$ presents some very great difficulties. That search demands that one must know the geodesic lines on the surface $S'$. Now, although the first integral of the general equation for the geodesic lines on surfaces of constant curvature has a very simple, well-known form, as far as I know, it has not been further obtained when the surface is given in a general manner by three equations that give the coordinates $x$, $h$, $z$ as functions of two *arbitrary* independent variables $u$ and $v$.

To conclude this first part, I would like to remark that the analysis that I made use of in order to recognize whether two surfaces are or are not mappable to each other offers some great analogies with the one that Minding presented in volume **19** of *Crelle's Journal*. Meanwhile, I have added some developments to the solution of that German scholar of geometry that seem to me to be worthy of attention.

(*End of Part One*)