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Solving some problems of mechanics

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I.

Consider a perfectly-flexible and homogeneous string that has the same density everywhere. Suppose that its extremities are fixed in an invariable manner at the two points *A* and *B*, and each of its elements are subject to a given force that is subject to only the condition that it must vary in a continuous manner in magnitude, as well as in direction, when one passes from one element to the following element. We will have the three known equations for the equilibrium of an arbitrary element:

$$(1)\qquad \begin{cases} d\cdot\left(T\dfrac{dx}{ds}\right)+X\,ds=0, \\ d\cdot\left(T\dfrac{dy}{ds}\right)+Y\,ds=0, \\ d\cdot\left(T\dfrac{dz}{ds}\right)+Z\,ds=0. \end{cases}$$

x, y, z represent the rectangular coordinates of a point of the element considered, s is the arc length of the string between a fixed point and the variable point (x, y, z), T is the tension at the latter point, X, Y, Z are the forces per unit length and parallel to the axes that are found at that point, and finally, all of the differentials refer to the same infinitely-small displacement that is performed on the string, in a sense that can be arbitrary, but which we suppose to be always the sense in which the positive values of s are counted in order to fix ideas and make the differential ds positive.

If the equilibrium curve of the string is planar by virtue of the nature of the forces X, Y, Z then one can refer the equilibrium to two axes that are situated in the plane of that curve, and it will then suffice to consider the first two of equations (1).

That is the case that we shall examine first.

If one develops the two equations of equilibrium then upon taking s to be the independent variable:

$$T\frac{d^2x}{ds^2}+\frac{dT}{ds}\frac{dx}{ds}+X=0,$$

$$T\frac{d^2y}{ds^2}+\frac{dT}{ds}\frac{dy}{ds}+Y=0,$$

so

$$T=\frac{Y\dfrac{dx}{ds}-X\dfrac{dy}{ds}}{\dfrac{d^2x}{ds^2}\dfrac{dy}{ds}-\dfrac{d^2y}{ds^2}\dfrac{dx}{ds}},$$

$$\frac{dT}{ds}=\frac{X\dfrac{d^2y}{ds^2}-Y\dfrac{d^2x}{ds^2}}{\dfrac{d^2x}{ds^2}\dfrac{dy}{ds}-\dfrac{d^2y}{ds^2}\dfrac{dx}{ds}},$$

and upon eliminating T :

$$d\cdot\left(\frac{Y\dfrac{dx}{ds}-X\dfrac{dy}{ds}}{\dfrac{d^2x}{ds^2}\dfrac{dy}{ds}-\dfrac{d^2y}{ds^2}\dfrac{dx}{ds}}\right)-\frac{X\dfrac{dy}{ds}-Y\dfrac{dx}{ds}}{\dfrac{d^2x}{ds^2}\dfrac{dy}{ds}-\dfrac{d^2y}{ds^2}\dfrac{dx}{ds}}ds=0. \tag{2}$$

That equation represents the curve that the string will affect in the equilibrium position.

One can put it into a simpler form. If s is the independent variable then one will have:

$$\frac{dx}{ds}\frac{d^2x}{ds^2}+\frac{dy}{ds}\frac{d^2y}{ds^2}=0,$$

so

$$\frac{d^2x/ds}{dy/ds}=-\frac{d^2y/ds}{dx/ds}=\frac{d^2x}{ds^2}\frac{dy}{ds}-\frac{d^2y}{ds^2}\frac{dx}{ds}=\pm\frac{1}{\rho},$$

upon calling the radius of curvature ρ and observing that:

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2=1 \quad\text{and}\quad \sqrt{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2}=\frac{1}{\rho}.$$

We infer from this that:

$$\frac{d^2x}{ds^2}=\pm\frac{dy/ds}{\rho},\qquad \frac{d^2y}{ds^2}=\mp\frac{dx/ds}{\rho},\qquad \frac{d^2x}{ds^2}\frac{dy}{ds}-\frac{d^2y}{ds^2}\frac{dx}{ds}=\pm\frac{1}{\rho},$$

in which the upper or lower sign must be taken in the last relation according to whether the derivative of dy / dx with respect to s is negative or positive, resp.; i.e., according to whether tangent of the angle, and consequently, the angle that the tangent to the equilibrium curve makes with the positive part of the x-axis, diminishes or increases, resp., for an infinitely-small displacement of the contact point that is performed on the string in the sense where one counts the positive values of the arc length *s*.

If one substitutes that in equation (2) then one will get:

$$d \cdot \left[\rho \left(Y \frac{dx}{ds} - X \frac{dy}{ds} \right) \right] \pm \left(X \frac{dx}{ds} + Y \frac{dy}{ds} \right) ds = 0,$$

in which the sign ± is determined in the way that was just described.

Now let R denote the intensity of the force that acts upon the string at the point x, y, z, and let α denote the positive angle that the direction of that force makes with the positive part of the x-axis. At the same time, let φ be the positive angle that the portion of the tangent to the string that is obtained by prolonging the element of arc length of that string in the positive sense makes with the same part of the x-axis as before. One will have:

$$X = R \cos \alpha, \quad Y = R \sin \alpha,$$

$$\frac{dx}{ds} = \cos \varphi, \quad \frac{dy}{ds} = \sin \varphi,$$

and the preceding equality will become:

$$d \cdot [R \rho \sin (\alpha - \varphi)] \pm R \cos (\alpha - \varphi)\, ds = 0. \tag{3}$$

If θ is the angle (less than 180°) that the direction of the force R makes with the portion of the tangent to the string that is obtained by prolonging the element of arc length in the positive sense then one will have:

$$\sin (\alpha - \varphi) = \pm \sin \theta \qquad \text{and} \qquad \cos (\alpha - \varphi) = \cos \theta,$$

in which the sign in front of $\sin \theta$ is + or – according to whether the angle φ diminishes or increases, resp., for an infinitely-small displacement that is performed in the sense of positive s, which is easy to assure oneself by examining all of the positions that can be presented and remarking that the force R must always act towards the convexity of the string.

That will finally give the equation of the equilibrium curve of the string:

$$d \cdot (R \rho \sin \theta) + R \cos \theta\, ds = 0. \tag{4}$$

In order to bring more uniformity to our calculations, we shall always employ the preceding equation when we would like to know the figure of equilibrium of a string when the force is given, or conversely, to determine the force when we know the figure of equilibrium. We must point out only that it is sometimes more advantageous to proceed directly.

II.

If we suppose, first of all, that the force R is normal to the equilibrium curve of the string then we will have:

$$\theta = \frac{\pi}{2}, \qquad \text{so} \qquad \sin\theta = 0, \qquad \cos\theta = 0,$$

and equation (4) will become:

$$R\rho = c,$$

which will show that the equilibrium curve in the case considered has its radius of curvature inversely proportional to the force at each of its points, so one deduces that if the force is constant then the curve will be a circle, etc. All of those results are known.

III.

Suppose, secondly, that the force R is always parallel to the same direction. Take the positive part of the x-axis to be a parallel to the force R and the positive part of the y-axis to be a perpendicular that is drawn through the side along which are situated the portions of the tangent that are obtained by prolonging the elements of arc length s in the sense of the positive values of those arc lengths. It is easy to see that since the force R always points towards the convexity of the string, the angle θ will increase then for a displacement of the string in the sense of positive values of s, and that one will have the differentials:

$$ds = \rho\, d\theta,$$

if they refer to a displacement in the same sense, as above. That being the case, equation (4) will then become:

(5) $$d \cdot (R\,\rho \sin\theta) + R\rho\cos\theta\, d\theta = 0.$$

It was only to fix ideas and arrive more rapidly at the preceding equation that one has supposed that the positive part of the y-axis points in a certain sense. However, it is easy to see that equation (5) will still be true when the positive part of the y-axis is taken in the opposite sense. Furthermore, one can establish that equation in another manner that will exhibit what we just proposed.

If one returns to equation (3) and sets $\alpha = 0$ in that equation then one will get:

$$d \cdot (R\rho \sin\varphi) \mp R\rho\cos\varphi\, ds = 0,$$

in which the upper or lower sign will be taken according to whether φ diminishes or increases, resp., for a displacement that is carried out on the string in the sense of positive s; however, one will obviously have:

$$ds = \mp\, \rho\, d\varphi,$$

in which the sign is determined in the same manner. Hence, the equation above will come down to:

$$d \cdot R\rho \sin \varphi + R\rho \cos \varphi \, d\varphi = 0;$$

moreover:

$$\varphi = \theta \quad \text{or} \qquad \varphi = 2\pi - \theta.$$

When one substitutes this, one will find equation (5), which is then found to be established in every case.

If one develops equations (5) then it will become:

$$\sin \theta \, d \cdot R\rho + 2R\rho \cos \theta \, d\theta = 0,$$

so upon integrating this:

(6) $$R\rho \sin^2 \theta = C.$$

Let us now make some hypotheses on R. In the first place, set (*):

$$R = \alpha \sin^m \theta,$$

so equation (6) will become:

(7) $$\rho \sin^{m+2} \theta = \frac{C}{a} = C';$$

upon remarking that:

$$y' = \tan \theta,$$

so

$$\frac{dx}{C'} = \sin^{-(m+2)} \theta \cos \theta \, d\theta,$$

and upon integrating, that will give:

$$-\frac{(m+1)\,x}{C'} = \frac{x}{x_0} = \sin^{-(m+1)} \theta.$$

One does not include the constant, since it is will always be possible to make it disappear by displacing the y-axis parallel to itself.

Finally, replace $\sin \theta$ with its value $\frac{y'}{\sqrt{1+y'^2}}$, so one will get:

$$\frac{x}{x_0} = \left(\frac{y'}{\sqrt{1+y'^2}}\right)^{-(m+1)},$$

so

(*) One will get this expression for the force when one considers a flexible sheet of rectangular form with two opposite sides that are fixed and supposes that the pressure of the moving air on a fixed surface element is proportional to the area of that surface and to the m^{th} power of the air velocity, when estimated in the sense of the normal to that element.

(8) $$dy = \frac{dx}{\sqrt{\left(\frac{x}{x_0}\right)^{\frac{2}{m+1}} - 1}}.$$

We then obtain the curves that we considered in another article (*), which enjoy several remarkable properties.

One can deduce some known results from that. If one sets $m = 0$ then the force R will be constant and equation (8) will be that of a catenary. If $m = 1$ then the force R will be directly proportional to the sine of the angle between the x-axis and the tangent to the curve at the point where that force is applied. In other words, the forces that act upon the elements of the string are proportional to the projections of those elements onto the y-axis, and in that case, equation (8) will represent a parabola that is, in effect, the curve of the suspension bridge, etc.

Equation (8) will take on an illusory form when $m = -1$. In that case, which is the one in which the component of the given force that is normal to the equilibrium curve of the string is constant, one must revert to equation (7); one will then get:

$$\rho \sin \theta = C',$$

where, upon replacing ρ with its value as a function of θ:

$$\frac{dx}{C'} = \frac{\cos\theta \, d\theta}{\sin\theta},$$

so if one integrates twice without introducing constants (since it is always possible to make them disappear by displacing the axes parallel to themselves) then:

$$e^{x/C'} = \cos \frac{y}{C'}.$$

If one changes the axes in such a way that the positive part of the x-axis and the positive part of the y-axis become negative part of the y-axis and the positive part of x-axis, respectively, then one will have:

$$e^{\frac{y}{C'}} \cos \frac{x}{C'} = 1.$$

That is the result to which one will arrive when one looks for the equation of the catenary of equal resistance, as Coriolis did on page 57 of tome I of this journal.

If one supposes that the force R is proportional to a power of θ and a power of ρ, instead of supposing that it is simply proportional to a power of θ, then one will once more find the curves (8). Indeed, one will then have:

(*) See Journal de Mathématiques pures et appliquées, tome IX, page 97.

$$R = a \sin^m \theta \rho^n,$$

and equation (6) will become:

$$\rho^{n+1} \sin^{m+2} \theta = \frac{C}{a} = C'$$

or

$$\rho \sin^{\frac{m+2}{m+1}} \theta = C'^{\frac{1}{n+1}} = C'',$$

so, upon integrating as above:

$$dy = \frac{dx}{\sqrt{\left(\frac{x}{x_0}\right)^{\frac{2(n+1)}{m-n+1}} - 1}}.$$

Suppose, moreover, that the force R is proportional to an arbitrary power of $\cos \theta$, in such a way that:

$$R = a \cos^m \theta,$$

so equation (6) will become:

$$\rho \sin^2 \theta \cos^m \theta = \frac{C}{a} = C',$$

from which one easily infers that:

$$\frac{dx}{C'} = \frac{d\theta}{\sin^2 \theta \cos^{m-1} \theta}.$$

In particular, if $m = 1$, in which case, the force R will have an intensity that is easy to interpret, then the preceding equation will give:

$$x \tan \theta = x \frac{dy}{dx} = - C',$$

when one drops the constant, so, upon integrating:

$$y = - C' lx,$$

which is quite remarkable.

IV.

Without stopping to make more applications of the case in which the force R is always parallel to the same direction [which is a case that we must point out is almost always simpler to deal with directly, without passing through equation (5)], we pass on to some other hypotheses about the direction of the force.

Suppose that the angle θ that the force R makes with the tangent to the equilibrium curve is constant. Equation (5) will then become:

$$d \cdot R\rho + R \cot \theta \, ds = 0.$$

However, if φ is always the angle that the tangent to the curve makes with the x-axis then one will have:

$$ds = \pm \rho \, d\varphi,$$

in which the sign is easy to determine. The preceding equation will then amount to:

$$d \cdot R\rho \pm R\rho \cot \theta \, d\varphi = 0,$$

so, upon integrating:

(9) $$R\rho = C \, e^{m\varphi},$$

when one sets:

$$\mp \cos \theta = m,$$

to simplify.

We now make some hypotheses on R. Suppose, first of all, that R is constant; equation (9) will then reduce to:

(10) $$\rho = C' e^{m\varphi}.$$

All of the curves that are included in the latter equation have developments that are similar to it. Indeed, one can infer from that equation that:

$$d\rho = m \, C' e^{m\varphi} \, d\varphi = m\rho \, d\varphi,$$

so

$$\frac{d\rho}{d\varphi} = m\rho.$$

Now, $\pm \, d\rho \, / \, d\varphi$ is the radius of curvature of the development of the curve, and the preceding equation will show that this quantity is proportional to the radius of curvature.

One knows that the logarithmic spirals enjoy that property. I say, moreover, that the logarithmic spirals are the only curves that are included in equation (10).

Indeed, upon replacing ρ with its value, that equation will give:

$$dx = \pm \, C' \cos \varphi \, e^{m\varphi} \, d\varphi,$$

and consequently:

$$dy = \pm \, C' \sin \varphi \, e^{m\varphi} \, d\varphi;$$

hence, integrating by parts will give:

$$x = \pm \, C' \frac{e^{m\varphi}(\sin \varphi + m \cos \varphi)}{1+m^2},$$

$$y = \pm \, C' \frac{e^{m\varphi}(m \sin \varphi - \cos \varphi)}{1+m^2},$$

while leaving aside the constants, since it is always possible to make them disappear.

In order to eliminate φ from the preceding two equations, we divide the one by the other and get:

$$\frac{y}{x} = \frac{m\sin\varphi - \cos\varphi}{\sin\varphi + m\cos\varphi} = \frac{m\tan\varphi - 1}{\tan\varphi + m},$$

so

$$\tan\,\varphi = \frac{x+my}{mx-y} = \frac{\frac{1}{m}+\frac{y}{x}}{1-\frac{1}{m}\cdot\frac{y}{x}}.$$

Dividing by the value of $x^2 + y^2$, while recalling that:

$$m = \mp\cot\,\theta,$$

and setting:

$$\sqrt{x^2+y^2} = r, \qquad \arctan\frac{y}{x} = \omega,$$

one will find, after all reductions have been made:

$$r = A e^{m(\omega\mp\theta)}.$$

One will arrive at an analogous conclusion if one sets:

$$R = a\,e^{n\varphi}.$$

One can make some other hypotheses on R that will lead to results that are more or less curious.

V.

Now suppose that the force R constantly passes through the same point, which we assume to be the coordinate origin, so the angle that the force R makes with the positive part of the x-axis, and which we represented by α in § I, will be equal to the azimuth of the point where the force is applied in this case or to that azimuth plus 180°. Hence, if we represent the azimuth of an arbitrary point by ω then we will have:

$$\alpha = \omega \qquad \text{or} \qquad \alpha = 180^{\circ} + \omega,$$

so in both cases:

$$d\alpha = d\omega.$$

Now recall equation (4) and replace ds with its value $\mp\rho\,d\rho$ in that equation. It will become:

$$d\,(R\rho\sin\,\theta) \mp R\rho\cos\,\theta\,d\varphi = 0,$$

where the upper or lower sign must be taken according to whether the angle φ diminishes or increases for an infinitely-small displacement in the sense of positive *s*. However, as we saw in § I, we have:

$$\sin(\alpha - \varphi) = \pm \sin\theta, \qquad \cos(\alpha - \varphi) = \cos\theta,$$

so

$$d\alpha - d\varphi = d\omega - d\varphi = \pm\, d\theta,$$

in which the sign in the right-hand side is determined in the same manner. We will then have:

$$d(R\rho \sin\theta) + R\rho\cos\theta\, d\theta \mp R\rho\cos\theta\, d\omega = 0,$$

in which the sign $\mp$ is always determined as was described above. We deduce from this that:

$$\sin\theta\; d\cdot R\rho + 2R\rho\cos\theta\, d\theta \mp R\rho\cos\theta\, d\omega = 0,$$

so

(11) $$\frac{d\cdot R\rho}{R\rho} = \frac{-2\cos\theta\, d\theta \pm \cos\theta\, d\omega}{\sin\theta}.$$

That will equation will be mostly useful for telling us the force when the equilibrium curve of the string is given. Let us make some applications of it.

Suppose that the equilibrium curve is such that θ is constant and first look for the equation of the curve. In order to do that, we remark that since we have:

$$\sin(\alpha - \varphi) = \pm \sin\theta, \qquad \cos(\alpha - \varphi) = \cos\theta,$$

as we said in § I, we will also have:

$$\tan(\alpha - \varphi) = \pm \tan\theta,$$

and consequently:

$$\tan(\omega - \varphi) = \pm \tan\theta,$$

since $\alpha = \omega$ or $= 180^\circ + \omega$. However, since *r* is the radius vector that corresponds to the azimuth ω, we will have:

$$\tan(\omega - \varphi) = -\frac{r\, d\omega}{dr};$$

hence:

$$\frac{r\, d\omega}{dr} = \mp \tan\theta,$$

in which the sign is always determined in the same way; upon integrating this, we infer that:

(12) $$r = C\, e^{\mp\cot\theta\,\omega}$$

is the equation of the desired curve.

Now, since θ is constant, equation (11) will become:

$$\frac{d\,R\rho}{R\rho} = \pm\cos\,\theta\,d\theta,$$

so, upon integrating:

$$R\rho = C' e^{\pm\cot\theta\omega}.$$

Furthermore, one infers the radius of curvature from equation (12):

$$r = \frac{r}{\sin\theta} = \frac{C}{\sin\theta} e^{\mp\cot\theta\omega};$$

one will then have:

$$R = C'\sin\theta\, e^{\pm 2\cot\theta\omega} = \frac{C''}{r^2}$$

for the desired force.

Suppose, secondly, that the equilibrium curve is such that:

$$\mp\, d\theta = m\, d\omega$$

or

$$\frac{\pi}{2} \mp \theta = m\omega,$$

upon taking the polar axis suitably, and the sign is determined as it was above. If one recalls that one has:

$$r\frac{d\omega}{dr} = \mp\,\tan\,\theta,$$

in general, the polar equation of the curve considered will be:

$$r\frac{d\omega}{dr} = -\cot\, m\omega,$$

so, upon integrating:

(13) $$r^m = a^m \cos\, m\omega.$$

Now, under the present hypothesis, equation (11) will become:

$$\frac{d\cdot R\rho}{R\rho} = \pm\frac{(2m+1)\cos\theta\, d\omega}{\sin\theta} = \frac{(2m+1)\sin m\omega\, d\omega}{\cos m\omega},$$

so upon integrating:

$$R\rho\cos^{\frac{2m+1}{m}} m\omega = C.$$

Moreover, one infers the radius of curvature from equation (13):

$$\rho = \frac{a}{m+1}(\cos m\omega)^{\frac{1}{m}-1}.$$

One will then have:

$$R = C' \cos^{-\frac{m+2}{m}} m\omega = C'' r^{-(m+2)}$$

for the desired force.

VI.

Up to now, we have always supposed that the equilibrium figure of the string is a planar curve. If the equilibrium curve is a curve of double curvature, by virtue of the nature of the force *R* and the position of the fixed points *A* and *B* then one must remark that the projection of that curve of double curvature onto an arbitrary plane can be considered to be the equilibrium figure of a string whose elements are acted upon by the components of the forces that act upon the first string that are parallel to the plane of projection, as one will easily recognize, either by means of equations (1) or by immediately employing the elementary theorems on the composition of forces. That being the case, one can always put the equation of the projections of the equilibrium curve of an arbitrary string onto the coordinate planes into the form of equation (4), and then conclude that in several cases, either the projections of the equilibrium figure of the string when the force is given or conversely, the force when one knows the projections of the equilibrium figure.

VII.

Consider the motion in space of an arbitrary material point. Let *x*, *y*, *z* be the coordinates of that point after a time *t*, let *R* be the intensity of the force that acts upon it, and finally, let α, β, γ be the angles that the direction of the force *R* makes with the positive parts of the coordinate axes. As one knows, one will have the three equations:

$$(14) \qquad \left\{ \begin{aligned} \frac{d^2x}{dt^2} &= R\cos\alpha, \\ \frac{d^2y}{dt^2} &= R\cos\beta, \\ \frac{d^2z}{dt^2} &= R\cos\gamma, \end{aligned} \right.$$

If the motion takes place in a plane then one can take the coordinate axes in that plane, and it will then suffice to consider the first two of equations (14). That is the case that we shall examine first.

Let *v* denote the velocity of the moving body after time *t*, let θ denote the angle between the direction of the force *R* and the tangent to the point *x*, *y*, *z*, when prolonged in the sense of motion, and finally, let ρ denote the radius of curvature of the trajectory at that point. As one knows, one can replace the equations of motion with the following:

$$\frac{v^2}{\rho} = R \sin \theta, \qquad \frac{dv}{dt} = R \cos \theta.$$

We deduce from the first equation that:

$$d\,(R\rho \sin \theta) = 2v\,dv,$$

where the differentials in the last equation refer, as before, to an infinitely-small displacement that is performed in the sense of motion. However, since s is the arc length of the trajectory, when measured from a fixed point in the same sense, one will have:

$$v = \frac{ds}{dt}.$$

The preceding equation then amounts to:

$$d\,(R\rho \sin \theta) - 2\frac{dv}{dt}\,ds = 0,$$

or, from one of the equations that were written above:

$$d\,(R\rho \sin \theta) - 2\,R \cos \theta\,ds = 0, \tag{15}$$

which is an equation that represents the trajectory of the moving body.

If we compare equation (15) to equation (4) then we will notice a certain analogy: The first terms are the same in the two equations, and the second term in the first equation is equal to twice the second term in the second one, with its sign changed. We deduce a remarkable consequence from that: Imagine that once we have decomposed the force R into a normal force and a tangential one, we take each of those components in its opposite sense, while reducing the normal component to one-half its value. Call the resultant of those two new forces R', and let θ' be the angle that the force R' makes with the tangent to the trajectory, when it is prolonged in the sense of motion. We will then have:

$$R \sin \theta = 2R' \sin \theta', \quad R' \cos \theta' = -R \cos \theta;$$

hence, upon substituting that in equation (15):

$$d\,(R' \sin \theta') + R' \cos \theta'\,ds = 0.$$

That equation coincides with equation (4). We can then say:

The trajectory that describes a moving body under the action of the force R is the equilibrium curve for a string whose elements are each acted upon by a force that is deduced from R by taking the normal and tangential components of that force and reducing the normal component by one-half.

Of course, it is intended that the string must be, in addition, constrained to pass through two points of the trajectory and to have its length between those two points equal to the length of the arc length of the curve that is included between those points.

After taking the normal and tangential components of the force P in the opposite sense, one can double the tangential component, instead of reducing the normal component to one-half, and one will once more arrive at the same consequence.

One can present the preceding result in its converse form and say:

If an arbitrary plane curve is the equilibrium figure of a string whose elements are each acted upon by a force R then the same curve will be the trajectory of a moving body that is acts upon by a force that one deduces from R by taking its normal and tangential components to the curve and doubling the normal component or reducing the tangential component by one-half.

Of course, it is intended here that the initial situation of the motion of the body must be chosen suitably. Hence, as one will easily recognize, it is necessary that the starting point of the moving body must be the extremity of the chain, that its initial velocity must be directed along the tangent to the string at that extremity, and finally that the square of its initial velocity, divided by the normal component of the force that acts at the start of the motion, must be equal to the radius of curvature of the string at that extremity.

VIII.

The remark that we just made in the preceding paragraph establishes a certain relation between equilibrium and motion can be useful in several situation. For example, we can use it to prove the theorem that we presented on page 113 of this volume very easily, and which was stated as:

If a moving body is subject to the action of forces $F, F', F'', \ldots$ in succession and it always starts from the point A with the velocities v_0, v_0', v_0'', …, resp., which have the same directions, but different intensities, and it describes the same curve AMB then if that same moving body started from the point A with the velocity V_0 and were subject to the action of the resultant of the forces $F, F', F'', \ldots$ then it would again describe the curve AMB, provided that the velocity V_0 has the same direction as the velocities v_0, v_0', v_0'', …, and that its intensity is such that:

$$V_0^2 = v_0^2 + v_0'^2 + v_0''^2 + \ldots$$

Indeed, let T and N be the components of the force F that are tangential and normal to the curve *AMB*, resp. Let T' and N' be the analogous components of the force F', and so on. If one applies a force T and a force $N / 2$ to the various points of a string in the senses that are opposite to the those of the components of the force F then that string will have the curve *AMB* for its equilibrium curve. The same thing will be true if one applies the force T' and the force $N' / 2$ in the senses that are opposite to those of the respective components of the force F', and so on. Hence, the same thing will also be true if one applies the resultant of the forces T, T', T'', … to each element of the string, along with the resultant of the forces $N / 2$, $N' / 2$, $N'' / 2$, …, when taken in the manner that was described. It will result from this that if one subjects a moving body to the action of the resultant of the tangential components T, T', T'', … and the resultant of the normal components N, N', N'', … of the forces F, F', F'', …, or what amounts to the same thing, to the action of the resultant of the forces F, F', F'', …, then that moving body will describe the curve *AMB*, provided that its starting point is nonetheless at A and its initial velocity V_0 is directed along the tangent to A along the curve *AMB*, or in other words, like the velocities v_0, v_0', v_0'', …, and finally that the square of the velocity V_0 must be equal to the sum of the normal components N, N', N'', …, multiplied by the radius of curvature of *AMB* at the A, or what amounts to the same thing, to the sum of the squares of the velocities v_0, v_0', v_0'', …

One can make the preceding proof independent of the remark in § VII. Indeed, preserving our notations, we see that by virtue of the hypothesis, the equations:

$$d\,(N\rho) - 2T\,ds = 0, \quad d\,(N'\rho) - 2T'ds = 0, \quad d\,(N''\rho) - 2T''ds = 0, \ \ldots$$

are all verified for the curve *AMB*. The same thing will then be true for their sum; i.e., for:

$$d \cdot [(N + N' + N'' + \ldots)\,\rho] - 2\,(T + T' + T'' + \ldots)\,ds = 0.$$

However, the latter equation includes that of the trajectory of the moving body under the composed motion. Hence, if *AMB* satisfies the initial conditions of motion that determine the constants, as well, then that curve will be the trajectory that one is dealing with. Now, as one saw in the preceding proof, the initial conditions are found to be fulfilled when the relations between the initial velocities that are required by its statement are themselves fulfilled.

The two proofs that we just gave suppose that the trajectory is planar. However, we can easily pass on to the general case, as we will see below.

IX.

In many cases, equation (15) can serve to determine the trajectory that is described by a moving body very simply when one knows that force that acts upon it, or conversely, to determine that force when the trajectory is known. Let us make some applications.

Suppose that the force R that acts upon the moving body is always parallel to the same direction – for example, to the x-axis. One will then have:

$$ds = -\rho\,d\theta,$$

as one will easily see upon examining all of the cases that can present themselves, and equation (15) will become:

$$d\,(R\rho \sin \theta) + 2R\rho \cos \theta\, d\theta = 0$$

or

$$\sin \theta\, d \cdot R\rho = -\, 3R\rho \cos \theta\, d\theta.$$

Hence, upon integrating:

$$R\rho \sin^3 \theta = C.$$

Now, let:

$$R = a \sin^m \theta,$$

so that will become:

$$\rho \sin^{m+3}\theta = \frac{C}{a} = C',$$

so upon integrating, as in § III:

$$dy = \frac{dx}{\sqrt{\left(\dfrac{x}{x_0}\right)^{\frac{2}{m+2}} - 1}}.$$

If one suppose that $m = 0$ (i.e., that the force R is constant) then the preceding equation will become that of a parabola, as it must be. If $m = -\, 1$ (i.e., if the normal component of the force R is constant) then the equation will represent a catenary, etc. For $m = -\, 2$, the preceding equation will take on an illusory form, but in that case, as we saw in § III, one must take the equation:

$$e^{x/C'} = \cos \frac{y}{C'}.$$

One can suppose that the force R is proportional to a power of ρ and a power of $\sin \theta$ or to a power of $\cos \theta$; one will find results for all of those cases that are analogous to the ones in § III.

In the second place, assume that the force R always makes the same angle with the tangent to the trajectory so θ will then be constant, and equation (15) will become:

$$d \cdot R\rho = 2R \cot \theta\, ds,$$

or, upon remarking that:

$$ds = \pm\, \rho\, d\varphi,$$

in which φ represents, as above, the angle that the positive part of the x-axis makes with the tangent to the trajectory:

$$d \cdot R\rho = \pm\, 2R \cot \theta\, d\varphi,$$

so, upon integrating:

$$R\rho = C\, e^{\pm 2\varphi \cot \theta}.$$

If P is constant then one will find a logarithmic spiral, as in § III; the same thing will be true if one sets:

$$R = C\, e^{n\varphi}, \text{ etc.}$$

One can finally suppose that R is a central force. Equation (15) will then be mainly useful for giving one the force when the trajectory is given. However, the equation that one obtains is much less convenient than the one that one possesses for that same purpose. Moreover, that equation will lead to some results that are analogous to the ones in § V. It is then shown that the curves:

$$r^m = a^m \cos m\theta$$

are described by central forces that are proportional to the power – $(2m + 3)$ of the distance, etc.

X.

Up to now, we have supposed that the trajectory that is described by the moving body is planar. If that trajectory has double curvature then one will remark that, since that results from equations (14) of motion of an arbitrary point, the trajectory of double curvature that is described by a moving body under the action of a force R will always have the trajectory that a moving body would describe under the action of the component of R that is parallel to each of the coordinate planes as its projections onto those planes. Of course, if the coordinate that is perpendicular to the plane of projection enters into the expression for that component then one will replace it by its value as a function of one of the other two coordinates, which is a value that must be known, moreover. That being the case, one can put the equations of the projections of the trajectory into the form (15) and then employ, as one has seen, those equations to determine the trajectory when one knows the force or above all, conversely, to find the force when one knows the trajectory. From that same remark, one will also see how one can extend the theorem that was proved in § VIII to the case where the trajectory has double curvature. Indeed, if the forces $F, F', F'', \ldots$, acting separately, make a moving body describe the same curve of double curvature AMB then the components of those forces parallel to any of the coordinate planes and expressed as functions of the coordinates that relate to that plane, acting separately, will also make it describe the plane curve $A'M'B'$ that is the projection of AMB onto the plane considered. Hence, if the hypothetical relationship between the initial velocities is satisfied then the resultant of the components of the forces $F, F', \ldots$, when expressed as a function of the two coordinates, will also make it describe the plane curve $A'M'B'$. That will be true for the three coordinate planes, so one sees, by a very simple argument, that the curve of double curvature AMB is described by the moving body under the action of the resultant of the forces $F, F', F'', \ldots$

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SUPPLEMENT

At the end of § III in the preceding note, I showed that the figure of equilibrium of the catenary of equal resistance is nothing but the curve that a string will affect in the equilibrium state when its various elements are subject to forces whose directions are constant and whose intensities make the component that is normal to the string constant. That result admits a very remarkable extension:

The equilibrium figure of a string of equal resistance whose elements are subject to the action of a central force that is proportional to the distance is nothing but the curve that a string will affect in its equilibrium state when its various elements are subject to a central force whose intensity is such that the component normal to the string is constant.

Indeed, if one adopts the usual notions and takes the origin to be the center of the forces that act upon the elements of the string then one will easily find that the equations of the latter curve are:

$$T\frac{r^2 d\theta}{ds} = C, \qquad dT + R\,dr = 0,$$

with the condition that:

$$\frac{Rr\,d\theta}{ds} = a.$$

If one eliminates R and T from those three equations then one will get:

$$d\cdot\frac{ds}{r^2 d\theta} + \frac{a}{C}\frac{ds}{d\theta}\frac{dr}{r} = 0,$$

so upon developing this while taking the independent variable to be θ:

$$\frac{d^2 s}{d\theta^2} - 2\frac{ds}{d\theta}\frac{ds}{r\,d\theta} + \frac{a}{C}\frac{ds}{d\theta}\frac{dr}{d\theta} = 0$$

or

$$\frac{\dfrac{d^2 s}{d\theta^2}}{\dfrac{ds}{d\theta}} = \frac{2\,dr}{r\,d\theta} - \frac{a}{C}\frac{r\,dr}{d\theta},$$

and integrating:

$$\frac{ds}{r^2\,d\theta} = C' e^{-\frac{a}{2C}r^2},$$

so one finally has:

$$\theta + \alpha = \int \frac{dr}{\sqrt{C'^2 r^2 e^{-\frac{a}{C}r^2} - 1}}.$$

It is easy to see now that this equation coincides with the one that one obtains by setting $R = r$ in equation (3) of page 98 of this volume. Now, the latter equation represents the curve of equilibrium of a string of equal resistance that is subject to the action of the central force R. That will prove what we have proposed.