"Bemerkungen zur Konforminvarianz der Elektrodynamik und der Grundgleichungen der Dynamik," Ann. Phys. (Leipzig) (7) **4** (1959), 96-102.

Remarks on the conformal invariance of electrodynamics and the fundamental equations of dynamics

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Dedicated to Herrn Professor Dr. Walther Gerlach on his 70th birthday

Translated by D. H. Delphenich

Abstract. – The theorem of **Cunningham** and **Bateman** regarding the conformal invariance of electrodynamics will be proved in a new way. With the same method of proof, one can show that the equations of motion will also be conformally-invariant when one transforms the masses like a reciprocal length, as **Heisenberg** suggested.

For quite some time, it has been known that electrodynamics is invariant under not only **Lorentz** transformations, but also the transformations of the 15-parameter conformal group ([1]), which includes the 10-parameter inhomogeneous **Lorentz** group as a subgroup.

In recent years, the properties of wave equations in the theory of elementary particles that emerge from **Lorentz**-invariant symmetries have found a certain interest ([2]); e.g., in its earliest form ([3]), **Heisenberg**'s theory contained five parameters beyond the **Lorentz** group, like the conformal group. In that situation, one must ask whether one can understand conformal invariance from the standpoint to which the theory of elementary particles leads better than before ([4]).

We would not like to pursue such a wide-ranging goal in this brief note. We would only like to show that one can make sense of the conformal invariance of the parameter transformations that **Heisenberg**, *et al.*, have recently introduced ([5]).

([1]) **E. Cunningham**, Proc. London Math. Soc. **8** (1910), 77; **H. Bateman**, *ibidem* **8** (1910), 223; **E. Bessel-Hagen**, Math. Ann. **84** (1921), 258; **A. Einstein**, Berl. Ber., math.-phys. Kl. (1921), pp. 261; **S. A. Bludman**, Phys. Rev. **107** (1957), 1163; **McLennan**, Nuov. Cim. **3** (1956), 1360; **F. Gürsey**, Nuov. Cim. **3** (1956), 988.

([2]) **A. Pais**, Physica **19** (1953), 869; Prog. Theor. Phys. **10** (1953), 457; **A. Pais, M. Gell-Mann**, Proc. of the 1954 Glasgow conference, pp. 342; **M. Gell-Mann**, Suppl. Nuov. Cim. **4** (1956), 848; **K. Nishijima**, Proc. Theor. Phys. **12** (1954), 107; *ibid.* **13** (1955), 285; Fortschr. Phys. **4** (1956), 519; **H. Gell-Mann, A. H. Rosenfeld**, Annual Rev. Nucl. Sci. **7** (1957), 407.

([3]) **H.-P. Dürr, W. Heisenberg, H. Mitter, S. Schlieder, K. Yamazaki**, Z. Naturforschg (in press): "Zur Theorie der Elementarteilchen," The authors must be thanked for furnishing a preprint.

([4]) **F. Klein**, *Vorlesungen über die Entwicklung der Mathematik in 19. Jahrhundert*¸ Berlin, 1927, Bd. II, § 7, pp. 79; - *Gesammelten Abhandlungen I*, Berlin, 1921, XXX "Zum Erlanger Programm," pp. 552. **E. Bessel-Hagen**, *loc. cit.* ([1]).

([5]) Preprint, *loc. cit.* ([3]), pp. 4.

As soon as one goes beyond the transformations of the **Lorentz** group, one is advised to employ the formalism of the general theory of relativity, even when gravitation is not considered, especially since almost all of the equations of electrodynamics, as one knows ([6]), are not only conformally-invariant, but also generally-invariant. It is simpler to look for reductions of the group of all coordinate transformations instead of extensions of the **Lorentz** group.

There are two classes of equations in electrodynamics that are generally-invariant from the outset. The metric tensor first enters them in the equations that couple the quantities of both classes.

The continuity equation:

$$\partial_\mu j^\mu = 0 \tag{I}$$

belongs to first class, along with the **Maxwell** equations:

$$\partial_\varpi \mathfrak{H}^{\mu\nu} = j^\mu \,. \tag{II}$$

Both of them will be generally-invariant when the four-current:

$$(j^\mu) = (j^0, j^1, j^2, j^3) = (c\,\rho\,, \mathfrak{j})$$

is a contravariant tensor density ([7]), and the field [i, k, l = cycl. (1, 2, 3)]:

$$(\mathfrak{H}^{\mu\nu}) = (-\,\mathfrak{H}^{\mu\nu}) = (\boldsymbol{\mathfrak{H}}^{0i}, \boldsymbol{\mathfrak{H}}^{kl}) = (c\,\boldsymbol{\mathfrak{D}}, \boldsymbol{\mathfrak{H}})$$

is a contravariant, antisymmetric tensor density.

The second class of equations can be derived from the **Lorentz** force:

$$K_\mu = Q\, B_{\mu\nu}\, \dot{x}^\nu\,, \qquad \dot{x}^\nu = \frac{dx^\nu}{d\tau}\,, \qquad Q = z\, e. \tag{III}$$

In these expressions, τ is an arbitrary parameter that does not change under coordinate transformations; however, it cannot coincide with the proper time. Equation (III) is generally-invariant only if the field:

$$(B_{\mu\nu}) = (-\,B_{\nu\mu}) = (\mathbf{B}_{0i}, \mathbf{B}_{kl}) = \left(-\frac{1}{c}\boldsymbol{\mathfrak{E}}, \boldsymbol{\mathfrak{B}}\right)$$

is a covariant, antisymmetric tensor, and the **Lorentz** force K_μ is a covariant four-vector. The field tensor is derived from a four-potential:

([6]) **H. Weyl**, *Raum-Zait-Materie,* 5th ed., Berlin, 1923, § **28**; **M. v. Laue**, *Die Relativitätstheorie*, Bd. II, § **36**, 3rd ed., Braunschweig, 1953; see also *loc. cit.* ([7]).

([7]) **H. Weyl**, *loc. cit.* (6), § **14**, pp. 104, 111. Recall that without a metric there will be no transitions between the various kinds of scalars, vectors, and tensors, so it is only in the physical nature of the quantities j^μ and $\mathfrak{H}^{\mu\nu}$ that they should be contravariant densities.

$$(A_\mu) = (A_0\,, A_1\,, A_2\,, A_3) = \left(-\frac{1}{c}\Phi, \mathfrak{A}\right),$$

which is a covariant vector, according to:

$$B_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{IV}$$

such that **Maxwell**'s equations of the second kind will be true:

$$\partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} = 0, \tag{V}$$

and there will exist the invariant generalized potential ([8]):

$$V = -\,Q\,A_\mu\,\dot{x}^\mu\,. \tag{VI}$$

Equations (IV-VI) are generally-invariant ([9]). One does not need the metric tensor for equations (I-VI) when one observes that equations (I) and (II) are determined by only the measurement of charge and (III-VI) are determined by the measurement of forces ([10]).

The metric enters into the picture only in the coupling equations ([11]):

$$\mathfrak{H}^{\mu\nu} = \frac{1}{\mu_0}\sqrt{-g}\;\; g^{\mu\rho}\,g^{\nu\sigma}\,B_{\rho\sigma}\,, \tag{VII}$$

in which the factor $1 / \mu_0$ depends upon the system of units in a known way. Ordinarily, one writes those equations in the form:

$$\mathfrak{H}^{0i} = -\frac{1}{\mu_0}\mathbf{B}_{0i}\,, \qquad \mathfrak{H}^{kl} = +\frac{1}{\mu_0}\mathbf{B}_{kl}\,. \tag{1}$$

One will get a special form for it when one sets $g_{\mu\nu}$ and $g^{\mu\nu}$ equal to the **Minkowski** tensor:

([8]) The four-force that is derived from the generalized potential $V(x, \dot{x})$ reads:

$$K_\mu = -\frac{\partial V}{\partial x^\mu} + \frac{d}{d\tau}\frac{\partial V}{\partial \dot{x}^\mu}\,.$$

If one demands that $V(x, \dot{x})$ must exist then it will follow from equation (III) that equations (IV), (V), and (VI) must be true.

([9]) **H. Weyl**, *loc. cit.* ([6]), § **14**, pp. 107.

([10]) Naturally, the $\mathfrak{H}^{\mu\nu}$ is given by only j^μ on the basis of (II) is still not determined completely. Furthermore, the $\mathfrak{H}^{\mu\nu}$ define a kind of current potential that will first be established by further conditions, namely, the coupling equations (VII). However, the statement that we have an antisymmetric, contravariant tensor density will follow from only (I).

([11]) **H. Weyl**, *loc. cit.* ([6]), § **18**, pp. 132, equation before (66).

$$(g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}. \tag{2}$$

One arrives at **Lorentz** invariance from this when one allows only those coordinate transformations that leave the **Minkowski** tensor invariant.

However, in order to recover equation (1) (which is the only one that is linked with a special coordinate system), one needs only to require that the fourth-rank tensor density:

$$\sqrt{-g}\; g^{\mu\rho}\, g^{\nu\sigma} \tag{3}$$

should not change. That requirement will lead to the conformal group. It implies a new proof for the theorem of **Cunningham** and **Bateman** ([1]) that we would like to briefly outline here due to its potential for generalization.

First, we formulate the condition equations for the invariance of eq. (1). Let:

$$\breve{x}^{\mu} = f^{\mu}(x), \qquad x \equiv (x^0, x^1, x^2, x^3)$$

so we will have:

$$d\,\breve{x}^{\mu} = \alpha^{\mu}{}_{\nu}\, dx^{\nu}, \qquad \alpha^{\mu}{}_{\nu} = \frac{\partial f^{\mu}}{\partial x^{\nu}},$$

and eq. (3) will go to:

$$\sqrt{-\breve{g}}\; \breve{g}^{\mu\rho}\, \breve{g}^{\nu\sigma} = \frac{1}{\alpha}\, \alpha^{\mu}{}_{\mu'}\, \alpha^{\rho}{}_{\rho'}\, \alpha^{\nu}{}_{\nu'}\, \alpha^{\sigma}{}_{\sigma'}\; g^{\mu'\rho'}\, g^{\nu'\sigma'}, \qquad \alpha = \det\left(\alpha^{\mu}_{\nu}\right).$$

The transformed expression will coincide with (3) when one has:

$$\alpha^{\mu}{}_{\mu'}\, \alpha^{\rho}{}_{\rho'}\, \alpha^{\nu}{}_{\nu'}\, \alpha^{\sigma}{}_{\sigma'}\; g^{\mu'\rho'}\, g^{\nu'\sigma'} = \alpha\, g^{\mu\rho}\, g^{\nu\sigma}.$$

$g^{\mu\nu}$ is determined by eq. (2) in this. With the known rules for raising and lowering indices, which we carry out using the **Minkowski** metric ([12]), we can write the latter equation as:

$$\alpha^{\mu\mu'}\, \alpha_{\rho\rho'}\, \alpha^{\nu\nu'}\, \alpha_{\sigma\sigma'} = \alpha\, \delta^{\mu}_{\rho}\, \delta^{\nu}_{\sigma}. \tag{4}$$

When one takes the trace over ν, ρ in this, one will get:

$$\alpha^{\mu\rho'}\, \alpha_{\rho\rho'} = \frac{4\alpha}{\alpha^{\sigma\sigma'}\, \alpha_{\sigma\sigma'}}\, \delta^{\mu}_{\rho},$$

and it will follow that:

([12]) That is an unconventional definition. It does not agree with the usual covariant notation, but it is recommended for the conformal group that is linked closely with the **Lorentz** group by eq. (5).

$$\alpha^{\mu}{}_{\nu} = \gamma l^{\mu}{}_{\nu}, \qquad l^{\mu\rho} l_{\nu\rho} = \delta^{\mu}_{\nu}, \qquad l = \det\,(l^{\mu}{}_{\nu}) = 1. \tag{5}$$

Since the second equation in (5) represents the condition for **Lorentz** transformations, as long as γ and $l^{\mu}{}_{\nu}$ are constant, we will be dealing with **Lorentz** transformations and dilatations. They are the only linear transformations that leave the equations of electrodynamics invariant ([13]). We shall call that group the "extended **Lorentz** group," in order to be able to distinguish it from the complete conformal group.

One will arrive at the conformal group when one observes that γ and $l^{\mu}{}_{\nu}$ can be special space-time functions. It follows from $\alpha^{\mu}{}_{\nu} = \partial_{\nu} f^{\mu}$ that:

$$\partial_{\rho}\left(\gamma l^{\mu}{}_{\nu}\right) = \partial_{\nu}\left(\gamma l^{\mu}{}_{\rho}\right). \tag{6}$$

The infinitesimal transformations:

$$\gamma = 1 + \varepsilon, \qquad l^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}, \qquad \omega_{\mu\nu} + \omega_{\nu\mu} = 0 \tag{7}$$

will imply that:

$$\delta^{\mu}_{\nu}\,\partial_{\rho}\varepsilon - \delta^{\mu}_{\rho}\,\partial_{\nu}\varepsilon = -\,\partial_{\rho}\omega^{\mu}{}_{\nu} - \partial_{\nu}\omega^{\mu}{}_{\rho}. \tag{8}$$

That will imply that:

$$\partial_{\rho}\,\omega_{\mu\nu} + \partial_{\mu}\,\omega_{\nu\rho} + \partial_{\nu}\,\omega_{\rho\mu} = 0,$$

and therefore:

$$\partial_{\mu}\,\omega_{\nu\rho} = g_{\mu\nu}\,\partial_{\rho}\,\varepsilon - g_{\mu\rho}\,\partial_{\nu}\,\varepsilon, \tag{9}$$

so due to the symmetry of $\partial_{\mu\rho}$, differentiating this with ∂_{σ} will yield:

$$-\,g_{\mu\rho}\,\partial_{\nu\rho}\,\varepsilon + g_{\mu\nu}\,\partial_{\rho\sigma}\,\varepsilon = -\,g_{\sigma\rho}\,\partial_{\nu\mu}\,\varepsilon + g_{\sigma\nu}\,\partial_{\rho\mu}\,\varepsilon,$$

and therefore:

$$\varepsilon = \overset{\circ}{\varepsilon} + \varepsilon_{\mu}\,x^{\mu} \tag{10}$$

and

$$\omega_{\nu\rho} = \overset{\circ}{\omega}_{\nu\rho} + \varepsilon_{\rho}\,x_{\nu} - \varepsilon_{\nu}\,x_{\rho}. \tag{11}$$

If we set $f^{\mu} = x^{\mu} + \delta x^{\mu}$ then this equation will follow:

$$\partial_{\nu}\,\delta x^{\mu} = \overset{\circ}{\varepsilon}\,\delta^{\mu}_{\nu} + \varepsilon_{\rho}x^{\rho}\delta^{\mu}_{\nu} + \overset{\circ}{\omega}{}^{\mu}{}_{\nu} + \varepsilon_{\nu}\,x^{\mu} - \varepsilon^{\mu}\,x_{\nu},$$

whose integration will yield the 15-parameter group ([14]):

$$\delta x^{\mu} = \alpha^{\mu} + \overset{\circ}{\varepsilon}\,x^{\mu} + \overset{\circ}{\omega}{}^{\mu}{}_{\nu}\,x^{\nu} + \varepsilon_{\nu}\,x^{\nu}x^{\mu} - \tfrac{1}{2}x^{\nu}x_{\nu}\varepsilon^{\mu}, \tag{12}$$

([13]) **Ph. Frank.**, Ann. Phys. (Leipzig) **35** (1911), 599.

([14]) In integral form, the proper conformal transformations read:

$$\breve{x}^{\mu} = \frac{x^{\mu} - x\cdot x a^{\mu}}{1 - 2a\cdot x + a^{2}x^{2}}.$$

in which four translations, one dilatation, six **Lorentz** transformations, and four proper conformal maps occur in succession ([15]).

So much for the new proof of the well-known theorem! We shall now take one step further. Initially, conformal invariance exists for only the equations of electrodynamics. The equations of motion have a narrower symmetry. They will then change their forms under a transition to conformal coordinates. We ask how the equations of motion will change under conformal transformations.

It is known that the inertial force will follow from the following generally-invariant action integral:

$$-mc\int\sqrt{-\overset{\circ}{g}_{\rho\sigma}\overset{\odot}{x}^{\rho}\overset{\odot}{x}^{\sigma}}\,d\tau,$$

which is obviously parameter-invariant, like the one that leads to the **Lorentz** force:

$$+Q\int\overset{\circ}{A}_{\mu}\overset{\odot}{x}^{\mu}\,d\tau.$$

We also choose the **Minkowski**an proper time here, as well as for arbitrarily-transformed coordinate systems ([16]).

Variation will yield the equations of motion:

$$\frac{d}{d\tau}\left(\frac{mc\,\breve{g}_{\mu\nu}\overset{\odot}{x}^{\nu}}{\sqrt{}}\right)-\frac{mc}{2}\breve{\partial}_{\mu}\breve{g}_{\rho\sigma}\frac{\overset{\odot}{x}^{\rho}\overset{\odot}{x}^{\rho}}{\sqrt{}}=\breve{K}_{\mu}. \tag{13}$$

Since one has:

$$\breve{g}_{\mu\nu}=\frac{1}{\gamma^{2}}g_{\mu\nu}$$

in the coordinate systems that emerge by means of conformal maps of **Minkowski** space, we can write the latter in the form:

$$\frac{d}{d\tau}\frac{m\overset{\odot}{x}_{\mu}}{\gamma}+\breve{\partial}_{\mu}\frac{m}{\gamma}=\frac{1}{c}\breve{K}_{\mu}. \tag{14}$$

Due to the fact that:

$$-g_{\rho\sigma}\,d\breve{x}^{\rho}\,d\breve{x}^{\sigma}=d\tau^{2},$$

that equation will be compatible with the condition that $\breve{K}_{\mu}\overset{\odot}{x}^{\mu}=0$. It is obviously different from the **Minkowski**an form:

([15]) By means of eq. (7), one easily calculates that:

$$\delta(dx_{\mu}\,dx^{\mu})=2\varepsilon\,dx_{\mu}\,dx^{\mu},$$

and conversely the conformal transformations are defined with this.

([16]) Observe that the **Minkowski** proper time is not conformally invariant.

$$\frac{d\,m\,\dot{x}_\mu}{d\tau} = \frac{1}{c} K_\mu . \tag{15}$$

We next consider the case of the extended **Lorentz** group, in which γ = const. If we set:

$$\frac{m}{\gamma} = \breve{m} \tag{16}$$

then (14) will imply the equation:

$$\frac{d\,\breve{m}\,\overset{\odot}{x}{}^{\mu}}{d\tau} = \frac{1}{c} \breve{K}_\mu , \tag{17}$$

which has the same form as (15). The equations of motion will then be invariant under extended **Lorentz** transformations when one transforms the mass parameter like a reciprocal length. The parameter transformation that **Heisenberg** introduced does not occupy a special place in this. *It will come about under dilatation of the coordinate system when one fixes the* ***Minkowski****an metric tensor, despite that transformation.*

It is surprising that the general equation of motion (14) has an invariant form, along with eq. (16). One can write it as:

$$\frac{d\,\breve{m}\,\overset{\odot}{x}{}^{\mu}}{d\tau} + \breve{\partial}_\mu \breve{m} = \frac{1}{c} \breve{K}_\mu . \tag{18}$$

That equation will follow from the action integral that was given before when one brings the mass under the integral, as one must, as long as *m* is *x*-dependent ([17]). Observe that the form of the equation will indeed change in such a way that it will, however, contain the same quantities as eq. (17), such that invariance is not introduced artificially, but is a structural property of the equation. It shows itself in just the fact that one can insert the factor by which the metric tensor differs from the **Minkowski** tensor into the parameter.

The result: We have once more proved the theorem of **Cunningham** and **Bateman** and show that the equations of motion are also conformally-invariant when one transforms the mass like a reciprocal length, as **Heisenberg** has suggested. Conformal invariance will then take on greater significance than it had up to now. If that still does not explain the physical meaning of conformal invariance then in view of the fundamental meaning of eq. (18), it is nonetheless hard to imagine that any equation of physics could violate conformal invariance ([18]).

I would like to express my deepest thanks to Herrn Dr. **Meister** for his help in the formulation of this note and the remarks that he made about it.

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(Submitted to the editor on 23 March 1959)

([17]) Naturally, *m* can be replaced with $\breve{m}$ in the action integral only when we simultaneously replace the actual metric tensor with the **Minkowski**an one.

([18]) Undoubtedly, the scale invariance of the **Dirac** equation for the free electron [see J. M. Jauch, F. Rohrlich, *The theory of photons and electrons*, Cambridge (Mass.), 1955, eq. (5.70)] is the quantum-mechanical counterpart to the extended **Lorentz** invariance of eq. (17).