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Field-mechanical foundations of the Dirac wave equation

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When one derives the inertial force on the electron from a generalized linear electrodynamics, and thus takes the development in the retarded terms one step further than Lorentz, one will obtain an equation of motion that W. Hönl and Papepetrou established for the first time. One must distinguish between the macro-impulse of the particle and its micro-velocity. The center of energy will displace constantly with respect to the center of charge as a result of emission-reabsorption processes, which will lead to a Zitterbewegung that yields a supplementary spin-like moment. In what follows, it will be shown that this supplementary moment is identical to the electron spin with a well-defined modification of Maxwell equations. The quantization of the equation of motion gives a wave equation in that case whose solutions also contain those of the Dirac wave equation. The Dirac wave equation is then closely connected with electron structure.

1. Development of the problem

Dirac ([1]) presented the Maxwellian potential for a moving point-charge in the following form: Let $(x_a) = (\mathfrak{r}, ict)$ be the coordinates of the reference point, and let $[z_a(s)] = [\mathfrak{r}_0(s), ict_0(s)]$, with $\dot{z}_\mu^2 = -c^2$, be those of the path curve of a point charge $+e$. The retarded potential will then read:

$$\varphi_\mu(x) = \frac{2e}{c^2}\int_{\text{ret.}} \dot{z}_\mu(s)\, f(\sigma)\, ds\,. \tag{1}$$

In this, σ means the square of the distance vector $[x_a - z_a(s)]$:

$$\sigma = \frac{1}{c^2}(x_a - z_a(s))^2, \tag{2}$$

and the integration (ret.) is extended over all points of the path curve in the cone through x. In particular, for Maxwell's theory, one will have:

$$f_{\text{M}}(s) = \delta(s). \tag{3}$$

In this, δ is the Dirac point function.

([1]) **P. A. M. Dirac**, Proc. Roy. Soc. (London) **167**A (1938), 148; **M. H. L. Pryce**, Proc. Roy. Soc. (London) **168**A (1938), 389.

It is known that this Ansatz leads to an infinite rest mass for the point charge. The usual modification in the equations of motion that was emphasized especially in the cited paper by Dirac consists of the assumption that one calculate the interaction between charged particles using Maxwell's theory, but one must introduce a finite value for the rest mass. One can describe that by altering the Ansatz in eq. (3) ([2]). The Dirac model follows from:

$$f_{\mathrm{D}}(\sigma) = \frac{m_0 c^3}{e^2}\delta\left(\sqrt{-\sigma}\right) - \frac{1}{2}\delta'\left(\sqrt{-\sigma}\right). \tag{4}$$

The second term alone will yield the basis for Heitler's theory of radiation damping, if one so desires, in which the field self-energy is set to 0. In 1939 ([3]), **Hönl** and **Papepetrou** studied a model that followed from the Ansatz ([4]):

$$f_{\mathrm{HP}}(\sigma) = \frac{m_0 c^3}{e^2}\delta\left(\sqrt{-\sigma}\right) - \frac{1}{2}\delta'\left(\sqrt{-\sigma}\right) + k\delta''\left(\sqrt{-\sigma}\right). \tag{5}$$

The Ansatz:

$$f_{\mathrm{mult.}}(\sigma) = \frac{\gamma}{2}\frac{J_1\left(\gamma\sqrt{-\sigma}\right)}{\sqrt{-\sigma}}, \tag{6}$$

which produces a Yukawa-like modification of the Coulomb potential [$\varphi = \frac{e}{r}(1 - e^{-\kappa r})$], has been examined by numerous authors in the meantime ([5]), ([6]), ([7]), ([8]). **R. P. Feynman** ([9]) recently considered the Ansatz:

$$f_{\mathrm{F}}(\sigma) = \frac{a^2}{2}e^{-a\sqrt{-\sigma}}. \tag{7}$$

All of the modifications of the Maxwell Ansatz $f = \delta(\sigma)$ above guarantee the finitude of the rest mass. The demand of a finite mass does not suffice to determine the potential-generating function $f(\sigma)$ then. A further narrowing of the possibilities can be inferred from the **Wessel** ([10])**-Dirac** ([1]) paradox [cf., **Bopp** ([2, 11])]: If an electron passes through a

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([2]) **F. Bopp**, Ann. Phys. (Leipzig) **43** (1943), 565.

([3]) **H. Hönl** and **A. Papepetrou**, Zeit. Phys. **112** (1939), 512; *ibid.* **114** (1939), 478; *ibid.* **116** (1940), 153.

([4]) **F. Bopp**, Z. Forschg. **1** (1946), 196; cf., Fiat-Review, v. 13, Kernphysik I, Chapter 1.

([5]) **F. Bopp**, Ann. Phys. (Leipzig) (5) **38** (1940), 345.

([6]) **E. C. G. Stueckelberg**, Helv. physica Acta **14** (1941), 51 had previously proposed a related Ansatz.

([7]) **A. Landé** and **L. H. Thomas**, Phys. Rev. **60** (1941), 121; *ibid.* **60** (1941), 514; *ibid.* **65** (1944), 175.

([8]) **B. Podolsky** and **Ch. Kikuchi**, Phys. Rev. **65** (1944), 228.

([9]) **R. P. Feyman**, "A relativistical cut-off for classical electrodynamics," still unpublished. I would like to warmly thank **Bethe** for providing me with the lecture notes, as well as for his discussion of the relevant American papers.

([10]) **W. Wessel**, Zeit. Phys. **92** (1934), 407; *ibid.* **110** (1938), 625.

([11]) **F. Bopp**, Z. Forschg. **1** (1946), 53.

force field then it will accelerate. The deformation of the field that is coupled with it will then lead to an alteration in the particle mass. (The dependency of the acceleration upon the mass has also, in fact, now been insured experimentally by radio spectroscopy.) Once the particle leaves the force field, the change in the field will not return to zero immediately in any case. It does not, by any means, need to seek the old equilibrium, and that is also not the case in the Ansätze in eqs. (4) and (5). In both cases, one obtains the self-acceleration ([12]) of the velocity of light that was discussed by Wessel and Dirac, and also more recently by **Hönl** ([13]). Hönl sought to link that self-acceleration with the bounded lifetime of unstable particles. For stable particles, one must demand that the eigenfield of the particle will revert to the old state of equilibrium when the external force vanishes. From **Bopp** ([2, 11]) and **Landé** ([7]), that will hold true for, e.g., the Ansatz in eq. (6).

Quantitative progress in the determination of the generating function $f(\sigma)$ is to be expected from the requirement of the *stability of the mass*, along with that of finite energy. In Dirac's wave equation (and above all, in the modifications that are to be expected today), however, one finds implicit quantitative structural statements that have still not been exhausted field-theoretically, up to now. When one attempts to derive the electron spin from the properties of the field, that should lead to some restrictions on the function $f(\sigma)$.

Hönl and **Papepetrou** ([2]), in conjunction with **Lubanski** ([14]), have shown what is important to our understanding of the electrons spin. They interpreted the Schrödinger Zitterbewegung as a consequence of the coincidence of the center of charge and the center of mass, and described that formally by the assumption of a mass-dipole that is superimposed with the electron mass. For many years, **Wessel** ([15]) has sought to bound the field interpretation of the spin even more sharply, on the one hand, starting from the radiation force, and on the other, from the Dirac wave equation. In my opinion, it was **Dirac**'s ([1]) sharp separation of the inertial field from the radiation field that first made it possible to formulate Wessel's thoughts free from hypotheses. It would not be the radiation field, but the inertial field that would then determine the spin properties. The fluctuating field energy in the vicinity of the particle – and thus, the quantum-mechanical energy of the emission-absorption processes – ensured the separation of the center of charge and the center of mass that Hönl and Papapetrou assumed. In the first approximation, that gives precisely the same thing as the pole-dipole particle.

2. Derivation of the field-mechanical equation of motion

If one restricts one's consideration to stationary states then one can omit the reaction force that originates in the radiation. It follows from this that for the calculation of the Lorentz force, we must start with the symmetric combination:

([12]) In my opinion, **Dirac**'s final condition for avoiding the self-acceleration gives no solution to the problem, as the example of the potential jump shows (cf., rem. 2, pp. 596).

([13]) **H. Hönl**, Z. Naturforsch. **2a** (1947), 537.

([14]) **Lubansky**, Acta phys. polon. **6** (1937), 356.

([15]) **W. Wessel**, Z. Naturforsch. **1** (1946), 622.

$$\varphi_\mu = \tfrac{1}{2}\left(\varphi_\mu^{\text{ret}} + \varphi_\mu^{\text{adv}}\right) \tag{7a}$$

of the retarded and advanced potentials. It must be emphasized expressly that this restriction is purely methodological and has nothing to do with the main arguments in **Eliezer** ([16]) or **Wheeler** and **Feynman** ([17]). When one convinces oneself of the necessity of a generalized potential law in the sense of eq. (1) – and addressing the question of the finitude of rest mass, as well as the stability of the mass – there will, in my opinion, be no basis for posing the problem of the Sommerfeld condition on emitted radiation. It is obvious that such a presentation would not make contact with the beautiful theorems on the mathematical equivalence of retarded and advanced potential in completely absorbing systems. When one assumes from the outset that there is only absorbed radiation, naturally the force that originates in the emitted radiation will vanish.

In the present case, we shall consider exclusively a single electron in a given external field. With no emitted radiation, the resulting equation of motion can be derived from a variational principle that **Born** ([18]) has given already. It gives the field equations by starting with a Lagrange function when one substitutes the potential in eq. (1) and then no longer varies the field, but the parameters of the motion.

In our case, with $\dot{z}_a = u_a$, one will get ([2]):

$$\delta\int L_0\, ds = 0,$$

$$L_0 = \frac{e^2}{c^3}\int u_a(s)\, u_a(s-\tau)\, f(\sigma)\, d\tau - u_a(s)\, \varphi_a(z). \tag{8}$$

In this, one has:

$$\sigma = \frac{1}{c^2}\,[z_a(s) - z_a(s-\tau)]^2. \tag{9}$$

In the first approximation, that will naturally yield the usual equations of motion.

A precise discussion of the equations of motion that follows from eq. (8) is still lacking. Up to now, two limiting cases have been treated. Since the stability of the mass has been shown already in the non-relativistic approximation, one can ask what happens in the non-relativistic approximation when one considers the retardation completely. One then obtains the condition for the choices ([2, 11]) of the function $f(\sigma)$ that will give stable masses. However, that approximation is certainly unsuitable for the quantitative treatment of practical problems. Hence, due to the large velocity of Zitterbewegung (for the Dirac equation, it is equal to the velocity of light), the relativistic limiting case is actually always present ([18a]). For that reason, it is preferable to calculate in the strongly

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([16]) **C. J. Eliezer**, Rev. Mod. Phys. **19** (1947), 147.

([17]) **J. A. Wheeler** and **R. P. Feynman**, Rev. Mod. Phys. **17** (1945), 157.

([18]) **M. Born**, Proc. Roy. Soc. (London) **143**A (1933/34), 410.

([18a]) Remark added in proof: From recent investigations, one cannot resolve the question of whether $v = c$ or $v \neq c$ by starting with just the Dirac equation (cf., rem. 24).

relativistic limit, in which, however, the development of the retardation goes one step further than usual ([19]). If one considers:

$$u_a(s)\,u_a(s-\tau) = u_a\left(u_a - \tau\dot{u}_a + \frac{\tau^2}{2}\ddot{u}_a\cdots\right) = -\,c^2\left(1+\frac{\tau^2}{2c^2}\dot{u}_a^2\cdots\right)$$

and

$$\sigma = \frac{1}{c^2}\,[z_a(s) - z_a(s-\tau)]^2 = \frac{\tau^2}{2}\left(u_a - \frac{\tau^2}{2}\dot{u}_a + \frac{\tau^2}{6}\ddot{u}_a\cdots\right)^2 = -\,\tau^2 - \frac{\tau^4}{12}\dot{u}_a^2\ldots,$$

and with the dimensionless abbreviation:

$$Q = \frac{l^2}{c^4}\dot{u}_a^2\,, \tag{10}$$

in which l is an arbitrary, suitably-chosen, length factor, one will get the following Lagrange function from eq. (8):

$$L_0 = -\frac{e^2}{c}\int\left(1+\frac{c^2\tau^2}{2l^2}Q\right) f\left(-\tau^2 - \frac{c^2\tau^4}{12l^2}Q\right)d\tau - \frac{e^2}{c}u_a\,\varphi_a\,.$$

The first expression in it is nothing but a function of Q that varies with $f(\sigma)$. We can then set:

$$L_0 = -\,m_0\,c^2\,F(Q) - \frac{e}{c}u_a\,\varphi_a\,. \tag{11}$$

It means:

$$F(Q) = \frac{e^2}{m_0 c^3}\int\left(1+\frac{c^2\tau^2}{2l^2}Q\right) f\left(-\tau^2 - \frac{c^2\tau^4}{12l^2}Q\right)d\tau\,. \tag{12}$$

Instead of studying the influence of $f(\sigma)$, one can immediately examine $F(Q)$. Eq. (11) coincides with the Lagrange function for the Hönl-Papapetrou pole-dipole particle.

In what follows, proper time will hardly be a suitable parameter. Hence, we shall use coordinate time from now on. In that case, the variational principle will read $\delta\int L\,dt = 0$, and the Lagrange function will assume the following form ($\beta = v / c$):

$$L = -\,m_0\,c^2\,F(Q)\,\sqrt{1-\beta^2} + e\varphi - \frac{e}{c}(\mathfrak{v}\,\mathfrak{A}),$$

$$Q = \frac{l^2/c^4}{\left(\sqrt{1-\beta^2}\right)^2}\left\{\dot{\mathfrak{v}}^2 + \left(\frac{\mathfrak{v}\dot{\mathfrak{v}}}{c\sqrt{1-\beta^2}}\right)^2\right\}. \tag{13}$$

([19]) **F. Bopp**, "Massenspektrum der Elementarteilchen," (cf., rem. 4) Zeit. Phys., *in press* (since Dec. 1947).

We will soon see that the velocity $\mathfrak{v}$ corresponds to the velocity of Zitterbewegung for the Dirac wave equation. Up to now, in the examples of all authors, the assumption is still tacitly made that this must also be equal or close to that of light in the corresponding classical theory.

Weyssenhoff and **Raabe** ([20]) have recently proved that it will lead to considerable simplifications if one considers that from the outset. In what follows, it seems that will even yield a definite choice for the function $F(Q)$ in eq. (13). Otherwise, if the mass term in eq. (13) is not to vanish or diverge when $\mathfrak{v} = c$ (i.e., for $\beta = 1$) then one must have:

$$F(Q) = \sqrt[4]{Q}\,. \tag{14}$$

One possible arbitrary factor already comes from the undetermined length factor l in the defining equation (10) of Q. Ultimately, it will be calibrated by the Dirac equation. Naturally, extra terms can arise that vanish when $\beta \to 1$. As long as we consider only the solutions with $\beta = 1$, they will be meaningless. From eq. (14), one will have:

$$L = -\,m_0\,c\,\sqrt{l}\,\sqrt[4]{\dot{\mathfrak{v}}^2 + \left(\frac{\mathfrak{v}\,\dot{\mathfrak{v}}}{\sqrt{1-\beta^2}}\right)^2} + e\,\varphi - \frac{e}{c}(\mathfrak{v}\;\mathfrak{A}). \tag{15}$$

Whether or not retardation effects are contained in the Lagrange function above, at least, in the first approximation (in quantum theory of wave fields, they correspond to emission-reabsorption processes), the field will no longer appear explicitly. That has the consequence that quantization without divergence problems is possible. The classical ones are omitted by the Ansatz in eq. (1). Quantum-mechanically, they do not appear ([21]), even though the usual treatment of emission-reabsorption processes leads to divergent results.

Since the Lagrange function (15) depends upon acceleration, the position $\mathfrak{r}$ and velocity $\mathfrak{v}$ will appear as independent variables in the canonical representation. For the associated canonically-conjugate variables $\mathfrak{p}$ and $\mathfrak{s}$, one will have:

$$\mathfrak{p} = \frac{\partial L}{\partial \mathfrak{v}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathfrak{v}}}\,, \qquad \mathfrak{s} = \frac{\partial L}{\partial \dot{\mathfrak{v}}}\,.$$

One gets the Hamilton function from:

$$H = \mathfrak{v}\;\mathfrak{p} + \dot{\mathfrak{v}}\,\mathfrak{s} - L\,(\mathfrak{r},\,\mathfrak{v},\,\dot{\mathfrak{v}})$$

after eliminating $\dot{\mathfrak{v}}$. If follows from $\mathfrak{s} = \partial L / \partial \dot{\mathfrak{v}}$ that:

([20]) **J. Weyssenhoff** and **A. Raabe**, Acta Phys. Pol. **9** (1947), 7-53.

([21]) That agrees with the results of **R. P. Feynman** in “Relativistic cut-off for quantum electrodynamics,” (lecture notes, rems. 9, 19).

$$\dot{\mathfrak{v}} = -\frac{m_0^2 c^2 l}{4} \frac{\mathfrak{s} - \frac{1}{c^2}(\mathfrak{v}\mathfrak{s})\mathfrak{v}}{\sqrt{\mathfrak{s}^2 - \frac{1}{c^2}(\mathfrak{v}\mathfrak{s})^2}},$$

$$\dot{\mathfrak{v}}\,\mathfrak{s} = \frac{-m_0^2 c^2 l/4}{\sqrt{\mathfrak{s}^2 - \frac{1}{c^2}(\mathfrak{v}\mathfrak{s})^2}},$$

such that one will get the explicit Hamiltonian function:

$$H = -e\,\varphi + \left(\mathfrak{v}, \mathfrak{p} + \frac{e}{c}\mathfrak{A}\right) + \frac{m_0^2 c^2 l}{4\sqrt{\mathfrak{s}^2 - \frac{1}{c^2}(\mathfrak{v}\mathfrak{s})^2}}. \tag{16}$$

It follows directly from the Hamilton equation for $\mathfrak{v}$:

$$\dot{\mathfrak{v}} = \frac{\partial H}{\partial \mathfrak{s}} = -\frac{-m_0^2 c^2 l/4}{\sqrt{\mathfrak{s}^2 - \frac{1}{c^2}(\mathfrak{v}\mathfrak{s})^2}} \left\{\mathfrak{s} - \frac{1}{c^2}(\mathfrak{v}\mathfrak{s})\,\mathfrak{v}\right\} \tag{17}$$

that $|\,\mathfrak{v}\,| = c$ is one integral of the motion. From now on, we would like to consider those solutions, which imply that $\mathfrak{s}^2 - \frac{1}{c^2}(\mathfrak{v}\mathfrak{s})^2 = \frac{1}{c^2}(\mathfrak{v} \times \mathfrak{s})^2$.

The relationship between eq. (16) and the Dirac wave equation has been shown repeatedly. In particular, as with the Dirac wave equation, one must distinguish between the velocity $\mathfrak{v}$ and the impulse $\mathfrak{p}$. In the force-free case, one will have $\mathfrak{p}$ = const.; $\mathfrak{p}$ will then be the macro-impulse. By contrast, from eq. (17), $\mathfrak{v}$ can never be constant, even for particles that are macroscopically at rest. The velocity $\mathfrak{v}$ will then be the micro-velocity that describes the Zitterbewegung. The quantity $\mathfrak{s}$ that is canonically-conjugate to $\mathfrak{v}$ appears as the mass dipole moment in Hönl and Papapetrou. It obviously describes the coincidence of the center of charge and the center of energy.

Having accomplished the transition to the canonical representation, we now revert to the four-dimensional notation. It would be possible – and formally very elegant – to choose a canonical representation in R_4 from the outset. However, that would necessitate a series of remarks that pertain to only the formalism and have no physical meaning. If we set $(\varphi_\mu) = (\varphi_1, \ldots, \varphi_4) = (\mathfrak{A}, i\varphi)$ and $(p_\mu) = (p_1, \ldots, p_4) = (\mathfrak{p}, i\,E/c)$ then it will follow from eq. (16), upon multiplying by $4\,|\,\mathfrak{v} \times \mathfrak{s}\,|\,/\,m_0\,c\,l$, that:

$$K = \left(g_a, p_a + \frac{e}{c}\varphi_a\right) + m_0\,c^2 = 0. \tag{18}$$

In this, one has:

$$(g_a) = \frac{4}{m_0\, c\, l} \,|\, \mathfrak{v} \times \mathfrak{s} \,|\, \cdot (\mathfrak{v}, ic). \tag{19}$$

The g_a in eq. (18) correspond completely to the Dirac matrices. However, they are functions of the canonical variables $\mathfrak{v}$ and $\mathfrak{s}$.

3. The transition to a wave equation

The formal transition to the wave equation is not difficult. We can proceed in a formal way that relates to the variables x_a and p_a throughout. We consider the x_a to be c-numbers and set the p_a equal to the Schrödinger operators:

$$p_a = \frac{\hbar}{i}\frac{\partial}{\partial x_a}. \tag{20}$$

We can also proceed analogously with the ($\mathfrak{v}$, $\mathfrak{s}$), and we have done that on occasion in the past. In that way, we will get a wave equation for a function $\psi(x_a, \mathfrak{v})$. That wave equation will contain internal degrees of freedom for the electron, so to speak. They also remain for force-free ($\varphi_a = 0$) particles that are macroscopically at rest ($\mathfrak{p} = 0$) and give an entire mass spectrum of rest energies, in general. In addition, the states of rest particles that belong to the various energy eigenvalues can be distinguished by a spin-like impulse moment that originates in the Zitterbewegung ([4, 19]); in general, the values of the spins are whole numbers. However, one can show that the Schrödinger operator representation of the canonical variables $\mathfrak{v}$, $\mathfrak{s}$ is not the most general. There are spin-like Ansätze for them that then also make half-integer spin values possible in an obvious way.

Before we do that, let us preface that with yet another remark on the present method of quantization. In the quantum theory of wave fields, one usually starts with a canonical representation of the wave equation in which the Fourier amplitudes of the waves appear as canonical variables. Here, at least the inertial part of the wave fields will be defined by the path parameter of the generating particle; it will then be defined uniquely by it. This represents a double alteration of the usual method. First of all, we have implicitly carried out a transformation of variables. Since the new representation is also canonical, one must also be dealing with a canonical transformation. It should be observed that a remarkable change in the physical results can follow from this, since in fact, the commutation relations of quantum mechanics are canonically-invariant, but not the wave equations. If one represents an ordinary mechanical problem (e.g., the oscillator or rotator) in terms of action and angle variables before quantization, instead of position and impulse coordinates, then one will not get the Schrödinger eigenvalues rigorously, but the older Bohr theory. There is no general principle for distinguishing which of them is the correct choice of variables. The following results might show that, in any event, choosing the particle parameters to be variables will bring the best agreement with experiment.

This preference for the new variables is in contrast to the drawback that up to now we have only been able to treat the radiation to the particle and not the radiation from it. With that, we come to the second point by which the present method differs from the conventional one. The internal radiation will become a quantum-mechanical problem under the transition to the particle parameters. Up to now, the quantization has consisted of a mechanical part and another one that belongs to the theory of radiation. On the other hand, it has already been proved that the separation between particle energy and radiation energy has not occupied a Lorentz-invariant position heretofore. The shift of the boundary between mechanics and radiation theory is closely related to Dirac's considerations on the retarded and advanced potentials. Here, we are formulating the *quantum theory of field mechanics.* It must also include a new form of the theory of radiation, namely, a *quantum theory of emitted radiation.* The following results of the quantum theory of field mechanics might justify a separate investigation of that. One finds programmatic Ansätze for a theory of emitted radiation in **Eliezer** ([16]).

Let us return to the operator representation of $\mathfrak{v}$ and $\mathfrak{s}$! The fact that the Schrödinger representation $\mathfrak{s} = \frac{\hbar}{i}\frac{\partial}{\partial\mathfrak{v}}$ does not subsume all possibilities implies the fact that the g_a in eq. (19) depend upon only $\mathfrak{v}$ and $\mathfrak{v} \times \mathfrak{s}$. The second vector has the form of an angular impulse operator, about which, one knows that the Schrödinger representation will give only integer values, while the matrix representation will also admit half-integer values. For that reason, one should expect that there is a generalized representation of the operators that actually leads to the same commutation relations as the Schrödinger representation, but which also include half-integer solutions. The appropriate starting point for the study of commutation relations is defined by the Poisson brackets ([15]), which will follow from the physical Ansatz here free from arbitrariness.

We calculate the Poisson brackets for the expression from eq. (19). We set $4 / m_0\, c\, l = 2\rho / \hbar$ in this, where ρ is a real number. The splitting off of the factor 2 is convenient. With:

$$g_0 = \frac{2\rho}{\hbar} \,|\, \mathfrak{v} \times \mathfrak{s} \,|, \tag{21}$$

one will have:

$$g_i = g_0\, v_i\,, \qquad g_4 = i\, c\, g_0\,. \tag{22}$$

The Poisson bracket of two functions A and B of the canonical variables $\mathfrak{v}$ and $\mathfrak{s}$ are defined by the equations:

$$\{A, B\} = \frac{\partial A}{\partial\mathfrak{s}}\frac{\partial B}{\partial\mathfrak{v}} - \frac{\partial A}{\partial\mathfrak{v}}\frac{\partial B}{\partial\mathfrak{s}}\,. \tag{23}$$

It will be shown that the Poisson brackets of the functions g_α and:

$$\{g_\alpha, g_\beta\} = g_{\alpha\beta} \tag{24}$$

define a closed system, in such a way that the bracket of any two of those quantities can be represented as a function of g_α and $g_{\alpha\beta}$.

As one will easily verify, the basic formulas for the following calculations read:

$$
\begin{aligned}
\{g_0, v_i\} &= \frac{2\rho}{\hbar}\frac{1}{|\mathfrak{v}\times\mathfrak{s}|}[\mathfrak{v}^2 s_i - (\mathfrak{v}\ \mathfrak{s})\, v_i], \\
\{g_0, s_i\} &= -\frac{2\rho}{\hbar}\frac{1}{|\mathfrak{v}\times\mathfrak{s}|}[\mathfrak{s}^2 v_i - (\mathfrak{v}\ \mathfrak{s})\, s_i].
\end{aligned}
\tag{25}
$$

It follows from this that, on the one hand:

$$\{g_0, \mathfrak{v}^2\} = \{g_0, \mathfrak{v}\mathfrak{s}\} = \{g_0, \mathfrak{s}^2\} = 0; \tag{26}$$

on the other hand, one will have:

$$\{g_i , g_k\} = \{g_0\, v_i , g_0\, v_k\} = \frac{1}{ic}(v_i\, g_{4k} - v_k\, g_{4i}).$$

The remaining Poisson brackets read:

$$
\begin{aligned}
\{g_{\alpha\beta}, g_\gamma\} &= \frac{4\rho^2 c^2}{\hbar^2}(g_\alpha\, \delta_{\beta\gamma} - g_\beta\, \delta_{\alpha\gamma}), \\
\{g_{\alpha\beta}, g_{\mu\nu}\} &= \frac{4\rho^2 c^2}{\hbar^2}(g_{\alpha\nu}\, \delta_{\beta\mu} - g_{\beta\nu}\, \delta_{\alpha\mu} - g_{\alpha\mu}\, \delta_{\beta\nu} + g_{\beta\mu}\, \delta_{\alpha\nu}).
\end{aligned}
\tag{28}
$$

The last equations follow in a purely algebraic way from:

$$\{g_{\alpha\beta}, g_{\mu\nu}\} = \{\{g_{\alpha\beta}, \{g_\mu , g_\nu\}\} = \{\{g_{\alpha\beta}, g_\mu \}, g_\nu\} - \{\{g_{\alpha\beta}, g_\nu \}, g_\mu\}.$$

In the first, we have set $v = c$ ([22]).

In quantum mechanics, the commutation relations appear in place of the Poisson brackets. The following correspondence exists:

$$\{s_i , v_k\} = \delta_{ik} \rightarrow [s_i , v_k] = \frac{\hbar}{i}\,\delta_{ik}\,.$$

In place of an equation with the Poisson bracket:

$$\{A, B\} = C,$$

one will find the commutation relation:

$$[A, B] = \frac{\hbar}{i}\,C.$$

([22]) By the way, eqs. (24) and (28) are also true for $v \neq c$ in the event that $g_0 = \frac{2\rho c}{\hbar}\sqrt{\mathfrak{s}^2 - \frac{1}{c^2}(\mathfrak{v}\mathfrak{s})^2}$.

It then follows from the relations (24) and (28) that:

$$[g_\alpha, g_\beta] = \frac{\hbar}{i} g_{\alpha\beta},$$

$$[g_{\alpha\beta}, g_\gamma] = \frac{4\rho^2c^2}{i\hbar}(g_\alpha\,\delta_{\beta\gamma} - g_\beta\,\delta_{\alpha\gamma}), \tag{29}$$

$$[g_{\alpha\beta}, g_{\mu\nu}] = \frac{4\rho^2c^2}{i\hbar}(g_{\alpha\nu}\,\delta_{\beta\mu} - g_{\beta\nu}\,\delta_{\alpha\mu} - g_{\alpha\mu}\,\delta_{\beta\nu} + g_{\beta\mu}\,\delta_{\alpha\nu}) .$$

One can realize those commutation relations in the following way: Let $\xi = (\xi_1, \ldots, \xi_4)$ and $\xi^* = (\xi_1^*, \ldots, \xi_4^*)$ be spinor variables that appear in place of the velocity $\mathfrak{v}$. We denote the derivatives by:

$$\eta = \frac{\hbar}{i}\frac{\partial}{\partial \xi^*}, \qquad \eta^* = \frac{\hbar}{i}\frac{\partial}{\partial \xi}. \tag{30}$$

We define the operators:

$$\Gamma_\alpha = \tfrac{1}{2}(\xi^*\gamma_4\,\gamma_\alpha\,\gamma_4\,\eta + \eta^*\gamma_\alpha\,\xi) \tag{31}$$

with the Dirac matrices $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, and for $\alpha \neq \beta$:

$$\Gamma_{\alpha\beta} = \tfrac{1}{2}(\xi^*\gamma_4\,\gamma_\alpha\,\gamma_\beta\,\gamma_4\,\eta - \eta^*\gamma_\alpha\,\gamma_\beta\,\xi). \tag{32}$$

They satisfy the following commutation relations:

$$[\Gamma_\alpha, \Gamma_\beta] = \frac{\hbar}{i}\Gamma_{\alpha\beta},$$

$$[\Gamma_{\alpha\beta}, \Gamma_\gamma] = \frac{\hbar}{i}(\Gamma_\alpha\,\delta_{\beta\gamma} - \Gamma_\beta\,\delta_{\alpha\gamma}), \tag{29}$$

$$[\Gamma_{\alpha\beta}, \Gamma_{\mu\nu}] = \frac{\hbar}{i}(\Gamma_{\alpha\nu}\,\delta_{\beta\mu} - \Gamma_{\beta\nu}\,\delta_{\alpha\mu} - \Gamma_{\alpha\mu}\,\delta_{\beta\nu} + \Gamma_{\beta\mu}\,\delta_{\alpha\nu}) .$$

These agree with the ones in eq. (29), up to factors. We will have complete identity when we set:

$$g_\alpha = \frac{2\rho c}{\hbar}\,\Gamma_\alpha, \qquad g_{\alpha\beta} = \frac{2\rho c}{\hbar}\,\Gamma_{\alpha\beta} . \tag{34}$$

The quantities g_α that are defined by eqs. (31) and (34) then coincide with the corresponding functions in eq. (19).

If we introduce those operators and the ones in eq. (20) into eq. (18) then after dividing by c that will give the *wave equation of field mechanics*:

$$\left\{\frac{\rho}{\hbar}(\xi^{*}\gamma_\alpha P_\alpha^{\dagger}\eta-\eta^{*}\gamma_\alpha P_\alpha\xi)-m_0c\right\}\Psi\,(x,\,\xi,\,\xi^{*})=0. \tag{35}$$

In this:

$$P_\alpha=\frac{\hbar}{i}\frac{\partial}{\partial x_\alpha}+\frac{e}{c}\varphi_\alpha, \qquad P_\alpha^{\dagger}=\frac{\hbar}{i}\frac{\partial}{\partial x_\alpha^{*}}+\frac{e}{c}\varphi_\alpha^{*}, \tag{36}$$

and the wave function depends upon the spin variables ξ and ξ^*, in addition to the x_α. In particular, if we assume that they depend upon ξ^* linearly and upon ξ not at all, so we will set:

$$\Psi=\xi^{*}\,\gamma_4\,\psi(x), \tag{37}$$

then if we write the still-undetermined numerical factor $\rho = 1$, it will follow that:

$$\xi^{*}\{i\,\gamma_4\,\gamma_\alpha P_\alpha+m_0\,c\,\gamma_4\}\;\psi(x)=0,$$

or (perhaps by differentiating with respect to ξ^*) the Dirac wave equation:

$$\{\gamma_\alpha P_\alpha - i\,m_0\,c\}\;\psi(x)=0. \tag{38}$$

The Dirac wave equation will then yield special solutions for the general wave equation of field mechanics in eq. (35), which will take the following form when $\rho = 1$:

$$\left\{\xi^{*}\gamma_\alpha P_\alpha^{\dagger}\eta-\eta^{*}\gamma_\alpha P_\alpha\xi-\hbar\,m_0\,c\right\}\Psi\,(x,\,\xi,\,\xi^{*})=0. \tag{39}$$

Since $\xi^*\eta$ and $\eta^*\xi$ commute with the operator of the wave equation, one will get the homogeneous functions ξ and ξ^* as particular solutions of the equations. The special Ansatz in eq. (37) that leads to the Dirac equation is of that type. One should expect that the homogeneous functions of higher degree would lead to wave equations for particles of higher spin. Investigations into that possibility are ongoing ([23]).

It is noteworthy that eq. (14) will yield the Dirac equation ([24]) only for the special Ansatz $F(Q) = \sqrt[4]{Q}$. In general, that means nothing except that the electron spin would follow only from a uniquely-defined structure for the electron. If we assume that eq. (14) is true for all values of Q then a well-defined function $f(\sigma)$ would indeed follow from that. However, that would contradict other experiments. In fact, the Dirac equation only suffices to guarantee eq. (14) for extreme velocities of Zitterbewegung; i.e., only asymptotically when $Q \to \infty$. In that limiting case, it would follow from eq. (12) that:

$$F\,(Q)\to-\frac{e^{2}}{2m_0e\,l^{2}}Q\int f\left(-\frac{c^{2}Q}{12l^{2}}\,\tau^{4}\right)\tau^{2}d\tau\,,$$

([23]) Comment added in proof: That suspicion has been confirmed in the meantime.

([24]) Comment added in proof: The question of the influence of $F(Q)$ will be treated rigorously next. It is very surprising that one also obtains other solutions of the Dirac equation for $v \neq c$.

or, when we set: $-\dfrac{c^2Q}{12l^2}\,\tau^4 = \sigma$, so $\tau = \sqrt[4]{\dfrac{12l^2}{c^2Q}}\sqrt[4]{-\sigma}$:

$$F\,(Q) = \frac{3}{2}\frac{e^2}{m_0c^3}\sqrt[4]{\frac{c^2}{12l^2}}\int\frac{f(\sigma)\,d\sigma}{\sqrt[4]{-\sigma}}\sqrt[4]{Q}\,.$$

We will then come to eq. (14) asymptotically:

$$F\,(Q) = \sqrt[4]{Q}\,,$$

when we choose $f(\sigma)$ in such a way that:

$$\int\frac{f(\sigma)\,d\sigma}{\sqrt[4]{-\sigma}} = \frac{2}{3}\frac{m_0c^3}{e^2}\sqrt[4]{\frac{12l^2}{c^2}}\,. \tag{40}$$

The comparison with previous results (2) will be eased when we set $\sigma = -\xi^2$, $f(-\xi^2) = g(\xi)$. If we then observe that $l = \dfrac{2\hbar}{m_0\,c}$ and choose the units $e = 1$, $c = 1$, $m_0 = 1$, and $\hbar = 137$ then eq. (40) will read:

$$\int g(\xi)\sqrt{\xi}\,d\xi = \frac{4}{3}\sqrt{\hbar\sqrt{3}} = 20.56. \tag{41}$$

Along with that [rem. 2, eq. (2)], one will have $\int f(\sigma)\,d\sigma = 1$; i.e.:

$$\int g(\xi)\,\xi\,d\xi = \tfrac{1}{2}. \tag{42}$$

Eq. (42) is likewise true in the Lorentz approximation. However, in place of eq. (41), one will get [rem. 2, eq. (60)]:

$$\int g(\xi)\,d\xi = 1.$$

One might think that this condition will persist along with eqs. (41) and (42). However, that is physically absurd. Eq. (41) is true for *extreme* velocities of Zitterbewegung, while the Lorentz approximation is true for *negligible* ones. For the electron, one must undoubtedly start with the Hönl-Papapetrou equation, not the Lorentzian one. If one determines g from eqs. (41) and (42) in order to arrive at the inclusion of the Dirac wave equation then that will give the integral:

$$M_0 = \int g(\xi)\,d\xi \neq 1, \tag{43}$$

which will be different from $m_0 = 1$, in general, which is the mass of the particle in the limiting case of small velocity of Zitterbewegung. If we set $g(\xi) = \begin{Bmatrix} g_0 & \text{for} & \xi \leq \xi_0 \\ 0 & \text{for} & \xi > \xi_0 \end{Bmatrix}$ as a first estimate then it will follow from eqs. (41) and (42) that:

$$\frac{2}{3} g_0\, \xi_0^{3/2} = 20.56, \qquad \frac{1}{2} g_0\, \xi^2 = \frac{1}{2}, \quad \text{resp.;}$$

thus:

$$g_0 = A^4, \qquad \xi_0 = \frac{1}{A^2} \qquad (A = 30.84)$$

and

$$M_0 = g_0\, \xi_0 = A^2 = 951.$$

The solutions with small velocities of Zitterbewegung will then yield heavy particles: e.g., mesons or protons. With that qualitative consideration, we then connect with our previous discussion ([3, 4, 19]) of the mass spectrum of elementary particles. The structural statements that are contained in the Dirac equations are subsumed completely by eq. (41). With that, the electron radius can obviously be smaller in order of magnitude than the classical value of Lorentz. Perhaps one can even make M_0 infinite ([25]). How it will behave depends upon the course of $F(Q)$ for small Q. The determination of that function for moderate values of the arguments assumes experiments that would go beyond the scope of the Dirac equation, and which should be treated in another place.

([25]) However, it needs to be pointed out that Dirac's classical theory of the electron will no longer be in harmony with Dirac's wave equation with the Ansatz in eq. (4), since $\int g(\xi)\,\sqrt{\xi}\,d\xi$ will diverge.

It is a special joy for me to be able to dedicate this paper to Hrn. Geheimrat **A. Sommerfeld** on his 80th birthday. Might it be granted that he would delight us with his active participation in the scientific life around the Munich Institute for some time! May the subject of this Jubilee be granted a blessed retirement for many years!