"Die Theorie des starren Elektrons in der Kinematik des Relativitätsprinzips," Ann. Phys. (Leipzig) **30** (1909), 1-56.

The theory of the rigid electron in the kinematics of the principle of relativity

By **Max Born**

Dedicated to the memory of Hermann Minkowski

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Contents

Introduction.

The great significance that the concepts of rigid bodies and rigid coupling take on in **Newton**ian mechanics is linked most intimately with one's fundamental intuitions about space and time upon which that discipline is constructed. The demand that lengths at different times should be comparable to each other then leads directly to the definition of the concepts of yardsticks whose lengths are independent of time and motion – i.e., they are *rigid*. Later, that concept of a rigid body also proves to be fruitful for the construction of dynamics itself. The rigid body is then not just a continuous system of masses with only six degrees of freedom that has the utmost simplicity not just kinematically, but also dynamically, since a combination of the forces that act at its points will admit just as many "resultant" forces and moments, whose knowledge will suffice for the description of motion. In the final analysis, all of those possibilities are based upon the **Galilei-Newton** coupling of space and time into a four-dimensional manifold [which I, with **Minkowski** ([1]) would like to call the "world"], which is a coupling that is essentially contained in the theorem that the laws of nature should be independent of not only the choice of zero-point and time unit, as well as the position of the spatial reference system and unit of length, but also of a uniform translation that the reference system is endowed with while preserving the measurement of time.

It is precisely those foundations of kinematics that one must surrender if the electrodynamical relativity principle, as it was presented by **Lorentz**, **Einstein**, **Minkowski**, and others, is to be valid. The coupling of space and time into "world" will then be a different one in that case: The independence of the laws of nature of the uniform translation of spatial reference systems will be true only when the time parameter also experiences an alteration that does not just arise from a shift of the zero-point and a choice of a different unit. That is most closely linked with the fact that a yardstick that preserves its length in a comoving coordinate system under uniform translation will suffer a change in length when it is considered in a rest system, namely, a shortening in the direction of its velocity. With that, the concept of a rigid body breaks down, at least in the conceptualization of it that is adapted to **Newton**ian kinematics.

Nonetheless, a corresponding concept is also by no means lacking from the new kinematics, since otherwise the comparison of lengths of moving bodies at different times would be illusory. No problem will arise in the definition of that concept for systems that move uniformly relative to each other either, and the aforementioned authors of the fundamental papers on that theory appealed to that fact without giving a special definition of rigidity.

The problem first arises when accelerations are present. Only one attempt was made along those lines, namely, by **Einstein** ([2]), but without entirely clarifying the state of affairs. I have therefore undertaken the task of working out *the kinematics of rigid bodies when it is based upon the relativity postulate.* The possibility of doing that is obvious from the outset, since in every respect **Newton**ian kinematics represents a limiting case of the new one, namely, the case in which the speed of light c is regarded as infinitely large. The method that I propose consists of defining rigidity by a differential law, instead of an integral one.

In fact, in that way, one arrives at the general rigidity conditions in differential form, which are very analogous to the corresponding considerations in the old kinematics and goes over to them when $c = \infty$. I have carried out the integration of those conditions, which is very simple to do in the old kinematics and leads to the constancy of the

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([1]) **H. Minkowski**, "Raum and Zeit," Phys. Zeit. **10** (1909), pp. 104, and Jahresber. d. deutsch. Mathematiker-Vereinigung **18**. (Which also appears as a reprint.) Some knowledge of that paper will be assumed in my own presentation.

([2]) **A. Einstein**, Jahrb. der Radioakt. und Electronik **4.** Heft **4** (1907), § 18.

distances between rigidly-coupled points, but for the case of uniformly-accelerated translation here. The result is hardly inferior to the older kinematics in terms of simplicity and intuitiveness, and it is closely-related to one's speculation about what arbitrary curvilinear and rotatory motions might yield; however, I shall not go into that. The main result is that under uniform motion, once more, the motion of an individual point of the rigid body will be determined along with all of the other ones by a very simple law, namely, that the body has *only one degree of freedom* then.

That now raises the question of whether the rigid mechanics has simple properties in regard to its dynamical behavior in the new mechanics, just as it did in the old one, and naturally, we will be dealing with electromagnetic forces here.

The practical value of the new definition of rigidity must then make itself known in the dynamics of electrons. To a certain degree, the degree of transparency of the result thus-obtained will then also tend to support or contradict the assumption of the relativity principle itself, since experiments have probably given no unique direction in that regard, and probably never will.

Abraham's theory, which studies the motion of rigid electrons in their usual sense in the force field that they generate, has indeed led to not just a qualitatively-satisfying explanation for the inertial phenomena of free electrons, based upon on a purely electrical foundation, but also for the dependence of the electromagnetic mass upon the velocity for small acceleration as a quantitative rule that one must probably regard as having still not been disproved by experiments. However, that theory, which is grafted onto the electrodynamics of rigid bodies when it is adapted to the old mechanics, does not satisfy the principle of relativity, and therefore it happens that its further development, in which **Sommerfeld** ([1]), **P. Hertz** ([2]), **Herglotz** ([3]), **Schwarzschild** ([4]), *et al.*, have been engaged, will lead to extraordinary mathematical complications. Now, **Lorentz** has attempted to adapt **Abraham**'s theory to the principle of relativity, and to that end, he constructed his "deformable" electron. That electron can be called precisely rigid according to the definition that I will give. The fact that, despite that agreement, **Lorentz**'s theory has shown to give rise to contradictions with **Abraham**'s theory ([5]) is based upon the fact that one carries over the law of the composition of forces on rigid bodies into results without criticizing the old mechanics. How one must modify that law will become self-evident in the representation that is chosen here. **Lorentz**'s formula for the dependency of mass on velocity, according to which the experiments can be represented just as well by the **Abraham**'s formula, proves to be applicable in the rigorous theory, as well. As **Einstein** has remarked already, and I have worked out for arbitrary currents in a paper ([6]) on "die träge Masse und das Relativitätsprinzip," that law follows directly from kinematics and is not at all connected with the proper electrodynamical mass, namely, the "rest mass."

However, my theory rigorously yields the dependency of the rest mass upon the acceleration for a class of motions that correspond to the simplest-possible accelerated

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([1]) **A. Sommerfeld**, Nachr. d. k. Ges. d. Wissensch. zu Göttingen, math.-physik Kl. Heft 2 and 5 (1905).

([2]) **P. Hertz**, Math. Ann. **65** (1907), pp. 1.

([3]) **G. Herglotz**, Nachr. d. k. Ges. d. Wissensch. zu Göttingen, math.-physik Kl. Heft 6 (1903).

([4]) **K. Schwarzschild**, Nachr. d. k. Ges. d. Wissensch. zu Göttingen, math.-physik Kl. Heft 6 (1903).

([5]) Cf., **M. Abraham**, *Theorie der Elektrizität*, 2nd ed., v. 2, § 22.

([6]) **M. Born**, Ann. Phys. (Leipzig) **28** (1909), pp. 571.

motions – namely, the uniformly-accelerated motions of the old mechanics – which I call "hyperbolic motions," and in fact the rest mass proves to be constant, up to enormous accelerations. Equations of motion are true for those motions that take the form of the fundamental mechanical equations, when they are adapted to the principle of relativity ([1]). However, since any accelerated motion can be approximated by such hyperbolic motions when its acceleration does not vary too suddenly, one will arrive at an electrodynamical basis for the fundamental equations of mechanics in that way. That theory will break down only for very rapidly-varying accelerations; along with the inertial damping, radiation damping will also appear. *It is remarkable that no matter how large its acceleration might be, an electron in hyperbolic motion will emit no actual radiation, but its field will move with it,* which was known only for uniformly-moving electrons, up to now. *Radiation and the radiation reaction first appear for deviations from hyperbolic motion.*

My definition of rigidity proves to be every bit as reasonable in the system of **Maxwell**ian electrodynamics as the definition of rigidity in the system of **Galilei-Newton** mechanics proves to be. The electron that is rigid in my sense represents the simplest electron motion dynamically. One can even go so far as to state that the theory yields a clear proof of the atomistic structure of electricity, which is in no way the case for **Abraham**'s theory. My theory is then in agreement with the atomistic instincts of so many experimenters for whom the interesting attempt of **Levi-Civita** ([2]) to describe the motion of electricity as a freely-moving fluid that is bound by no kinematical constraints under the action of its own field will hardly provoke applause.

Since the simplicity of the dynamics of the new rigid bodies is not accordingly less than the simplicity of their kinematics, one will have to attribute the same fundamental significance to this concept of rigidity in the system of the electromagnetic world-view that the usual rigid bodies have in the system of the mechanical world-view.

([1]) Cf., **A. Einstein**, Ann. Phys. (Leipzig) **17** (1905), pp. 891. **M. Planck**, Verh. d. deutsch. Phys. Ges. **8** (1906), pp. 136; **H. Minkowski**, Nachr. d. k. Ges. d. Wissensch. zu Göttingen, math.-phys. Kl. (1908), pp. 54; cf., **M. Born**, *loc. cit.*

([2]) **T. Levi-Civita**, "Sui campi elettromagnetici puri," by C. Ferrari, Venice, 1908; "Sulle azione meccaniche, etc." Rendiconti d. R. Acad. dei Lincei 18.5a. This theory also seems to lead to contradictions with experiments when it is applied to cathode rays.

CHAPTER ONE

The kinematics of rigid bodies

§ 1. Rigid bodies in the older mechanics

For the sake of the electrodynamical applications in the second and third chapter, we will not concern ourselves with rigid systems of discrete particles, but with continuous rigid bodies. A continuous flow of matter can be represented in the manner that is called the **Lagrangian** picture in such a way that one gives the spatial coordinates x, y, z as functions of time t and three parameters ζ, η, ζ – perhaps the values of x, y, z at time t = 0:

$$(1)\qquad \begin{cases} x = x(\xi, \eta, \zeta, t), \\ y = y(\xi, \eta, \zeta, t), \\ z = z(\xi, \eta, \zeta, t). \end{cases}$$

The mass system is rigid when the distance between any two of its points:

$$(2)\qquad r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

is independent of time, and will thus be equal to:

$$\sqrt{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2}\ .$$

It will then follow from this that equations (1) will have the form:

$$(3)\qquad \begin{cases} x = a_1 + a_{11}\xi + a_{12}\eta + a_{13}\zeta, \\ y = a_2 + a_{21}\xi + a_{22}\eta + a_{23}\zeta, \\ z = a_3 + a_{31}\xi + a_{32}\eta + a_{33}\zeta, \end{cases}$$

in which the quantities a_α, $a_{\alpha\beta}$ are functions of time t, and the matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (a_{\alpha\beta})$$

is orthogonal ([1]); i.e., if $\tilde{A}$ means the transpose of the matrix A and 1 is the identity matrix then:

([1]) In order to avoid becoming verbose, I shall appeal to the matrix calculus, which is most appropriate to these arguments. One can find a very simple-to-understand presentation of it that assumes no prior

(4) $$\tilde{A}A = 1.$$

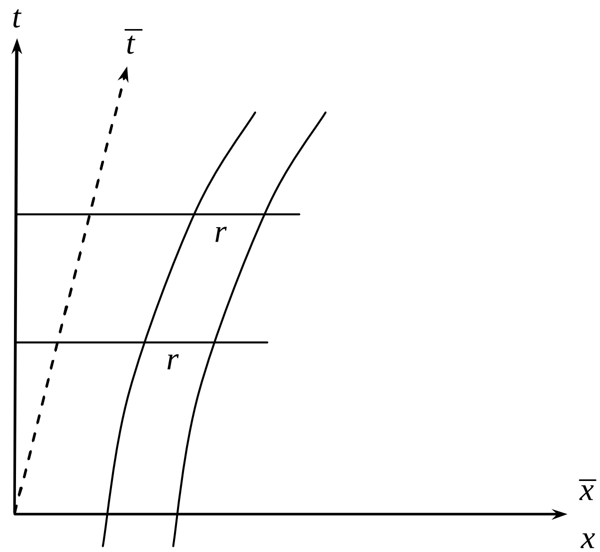

Figure 1.

In order to see clearly the possibility of generalizing that condition to the kinematics of the principle of relativity, it is advantageous to appeal to the interpretation of the variables x, y, z, t as parallel coordinates in a four-dimensional space that is called the "world," which **Minkowski** employed in the aforementioned paper. In what follows, the figures will always mean the plane section $y = 0$, $z = 0$ through that four-dimensional space; in them, the x-axis will be horizontal and the t-axis will point upwards. The path of a point will be represented in the $xyzt$-manifold (i.e., the world) as a curve – namely, the "world-line" – and the motion of a body will be represented by a family of world-lines. Now, the condition dr / dt above means that the line that connects the intersection points of any two world-lines with a three-dimensional structure t = const. will have the same length for all of those structures. That will refer to the spaces t = const. that are "parallel" to the space $t = 0$.

The meaning of that rigidity condition for **Newton**ian mechanics is based upon the fact that it is invariant under transformations that take **Newton**'s equations of motion into themselves. When those transformations preserve the zero-point, they will have the form:

(5) $$\begin{cases} x = k_{11}\,\bar{x} + k_{12}\,\bar{y} + k_{13}\,\bar{z} + k_1\,t, \\ y = k_{21}\,\bar{x} + k_{22}\,\bar{y} + k_{23}\,\bar{z} + k_2\,t, \\ z = k_{31}\,\bar{x} + k_{32}\,\bar{y} + k_{33}\,\bar{z} + k_3\,t, \end{cases}$$

in which $k_{\alpha\beta}$, k_α are constants, and the matrix:

knowledge in § 11 of the paper of **Minkowski** on "die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körper," (cited in rem. 1, pp. 4).

$$K = (k_{\alpha\beta})$$

will be orthogonal:

(6) $$\tilde{K}\,K = 1.$$

That orthogonal part of the transformation means only the transition from the original coordinate system to one that is rotated about the zero point; however, the second part means uniform translation in time. In our four-dimensional world, that represents the transition from the original t-axis to an inclined $\bar{t}$-axis. One sees immediately (Fig. 1) that the quantity r will, in fact, remain unchanged in that way.

The principle of relativity in electrodynamics expresses the invariance of natural laws under *other* linear substitutions, and therefore the meaning of the quantity will disappear. Those "Lorentz transformations" couple the four quantities x, y, z, t with four new ones $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$ by linear transformations:

(7) $$\left\{\begin{aligned} x &= k_{11}\,\bar{x} + k_{12}\,\bar{y} + k_{13}\,\bar{z} + k_{14}\,\bar{t}, \\ y &= k_{21}\,\bar{x} + k_{22}\,\bar{y} + k_{23}\,\bar{z} + k_{24}\,\bar{t}, \\ z &= k_{31}\,\bar{x} + k_{32}\,\bar{y} + k_{33}\,\bar{z} + k_{34}\,\bar{t}, \\ t &= k_{41}\,\bar{x} + k_{42}\,\bar{y} + k_{43}\,\bar{z} + k_{44}\,\bar{t}, \end{aligned}\right.$$

that transform the expression:

(8) $$x^2 + y^2 + z^2 - c^2\,t^2$$

into itself, in which c means the speed of light.

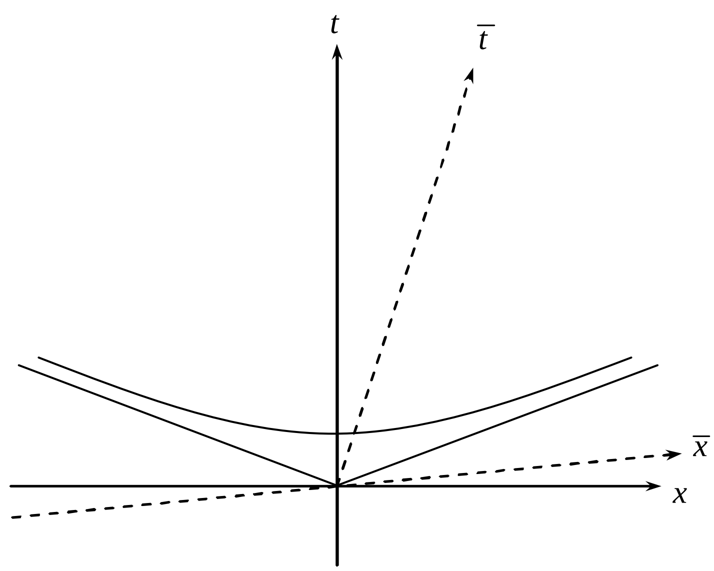

Figure 2.

Thus, the time (or rather the quantity $ct\,\sqrt{-1}$) will be transformed symmetrically with the coordinates, and not only will the t-axis be inclined by the transformation, but the space $t = 0$ will also take on a different place in the four-dimensional world ([1]). Since the spaces t = const. will not go to the spaces $\bar{t}$ = const. then, neither the quantity r, nor the condition $dr / dt = 0$ will be invariant now.

At first, it also does not seem possible for one to give an analogous condition between two world-lines, since there are no three-dimensional spaces that are distinguished under the transformation (7), (8) in the same way that the spaces t = const. were distinguished under (5).

Therefore, for the sake of generality, one must look for another definition of rigidity in the old mechanics. For that, one can employ the fact that one can replace the condition r = const. that exists between two finitely-separated world-lines with a differential condition between infinitely-close world-lines in such a way that when the differential condition is fulfilled in all of space, it will have the equation r = const. as a consequence.

To that end, we consider the distance between two infinitely-close world-lines at time t; i.e., the arc length:

$$ds = \sqrt{dx^2 + dy^2 + dz^2} .$$

If one sets that equal to a constant ε then the equation:

$$ds^2 = \varepsilon^2$$

will represent an infinitely-small sphere. It comes about during the motion that is represented by (1) by way of an infinitely-small ellipsoid that one will get when one represents the quantity ds^2 as a quadratic form in $d\xi$, $d\eta$, $d\zeta$ by means of the equations:

$$(9) \qquad \begin{cases} dx = \dfrac{\partial x}{\partial \xi} d\xi + \dfrac{\partial x}{\partial \eta} d\eta + \dfrac{\partial x}{\partial \zeta} d\zeta , \\[2ex] dy = \dfrac{\partial y}{\partial \xi} d\xi + \dfrac{\partial y}{\partial \eta} d\eta + \dfrac{\partial y}{\partial \zeta} d\zeta , \\[2ex] dz = \dfrac{\partial z}{\partial \xi} d\xi + \dfrac{\partial z}{\partial \eta} d\eta + \dfrac{\partial z}{\partial \zeta} d\zeta . \end{cases}$$

Let that form be:

$$(10) \qquad ds^2 = p_{11}\, d\xi^2 + p_{22}\, d\eta^2 + p_{33}\, d\zeta^2 + 2\, p_{12}\, d\xi\, d\eta + 2\, p_{13}\, d\xi\, d\zeta + 2\, p_{23}\, d\eta\, d\zeta .$$

Hence, the matrix of "deformation quantities" $p_{\alpha\beta}$:

$$P = (p_{\alpha\beta})$$

is composed from the matrix:

([1]) For a more detailed geometric description of Lorentz transformations, cf., **H. Minkowski**, "Raum und Zeit," *loc. cit.* (rem. 1, pp. 2).

(11) $$A = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{pmatrix}$$

in the following way:

(12) $$P = \tilde{A} A .$$

We now call the motion *rigid in the smallest part* when an infinitely-small structure does not change under the motion, so when all of the $p_{\alpha\beta}$ are independent of time. We will then have the *infinitesimal rigidity condition:*

(13) $$\frac{dp_{\alpha\beta}}{dt} = 0.$$

If ξ, η, ζ are the initial values of x, y, z then the matrix A will be equal to the identity matrix 1 for $t = 0$, so (12) will then read:

$$P = \tilde{A} A = 1.$$

Now, it is an elementary theorem in infinitesimal geometry ([1]) that when this condition is fulfilled everywhere, the flow will be represented by equations of the form (3), so one will be dealing with the motion of a rigid body.

This infinitesimal rigidity condition (13) can now be easily carried over to the kinematics of the relativity principle.

§ 2. The differential condition for rigidity

In what follows, only those quantities that are invariant under Lorentz transformations (7), (8) will be physically meaningful.

We now consider a flow that we represent, not by equations of the form (1), but by the following equations, which better exhibit the symmetry in the quantities x, y, z, t that the principle of relativity demands:

(14) $$\begin{cases} x = x(\xi, \eta, \zeta, \tau), \\ y = y(\xi, \eta, \zeta, \tau), \\ z = z(\xi, \eta, \zeta, \tau), \\ t = t(\xi, \eta, \zeta, \tau). \end{cases}$$

([1]) The theorem is seldom formulated explicitly, but it is an immediate consequence of the simplest mapping theorems.

Hence, let τ be the *proper time*; i.e., the identity exists:

(15) $$\left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial y}{\partial \tau}\right)^2 + \left(\frac{\partial z}{\partial \tau}\right)^2 - c^2\left(\frac{\partial t}{\partial \tau}\right)^2 = -c^2.$$

τ is measured from any "cross-section" of the flow onwards.

The ξ, η, ζ shall characterize the individual streamlines, but other than that, we shall leave their meaning undecided. We now set, for the moment:

(16) $$\begin{cases} x(0,0,0,\tau) = \mathfrak{x}(\tau), \\ y(0,0,0,\tau) = \mathfrak{y}(\tau), \\ z(0,0,0,\tau) = \mathfrak{z}(\tau), \\ t(0,0,0,\tau) = \mathfrak{t}(\tau), \end{cases}$$

and consider the filaments of world-lines that surround the world-line (16) $\xi = \eta = \zeta = 0$.

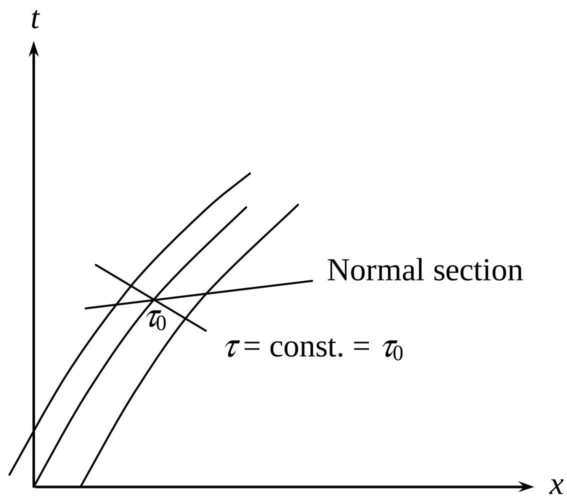

Figure 3.

They can be represented as follows:

(17) $$\begin{cases} x = \mathfrak{x} + x_\xi d\xi + x_\eta d\eta + x_\zeta d\zeta + \cdots, \\ y = \mathfrak{y} + y_\xi d\xi + y_\eta d\eta + y_\zeta d\zeta + \cdots, \\ z = \mathfrak{z} + z_\xi d\xi + z_\eta d\eta + z_\zeta d\zeta + \cdots, \\ t = \mathfrak{t} + t_\xi d\xi + t_\eta d\eta + t_\zeta d\zeta + \cdots, \end{cases}$$

in which we restrict ourselves to the terms that are linear in the increments $d\xi$, $d\eta$, $d\zeta$ (which are initially, small, but ultimately finite). Here, $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, $\mathfrak{t}$ are the functions that are defined by (16), and we have set:

$$x_\xi = \frac{\partial x}{\partial \xi}(0, 0, 0, \tau), \ldots$$

Two space-time vectors with the components x_1, y_1, z_1, t_1 and x_2, y_2, z_2, t_2 are called *normal* when their directions are conjugate relative to the invariant hyperbolic structure:

(18) $$x^2 + y^2 + z^2 - c^2 t^2 = -1,$$

so when one has:

(19) $$x_1 x_2 + y_1 y_2 + z_1 z_2 - c^2 t_1 t_2 = 0.$$

All vectors that are normal to a time-like vector ([1]) x_1, y_1, z_1, t_1 will fill up a three-dimensional linear structure that can be made into the space $t = 0$ by a suitable Lorentz transformation; we call it the *normal section* of the vector. The concept that is thus defined is obviously invariant under Lorentz transformations.

We now consider a well-defined point P on the world-line $\xi = \eta = \zeta = 0$ that belongs to the value τ_0 of proper time. We lay the normal section to the velocity vector $\mathfrak{x}'_0$, $\mathfrak{y}'_0$, $\mathfrak{z}'_0$, $\mathfrak{t}'_0$ at P through that point:

(20) $$\mathfrak{x}'_0(x - \mathfrak{x}_0) + \mathfrak{y}'_0(y - \mathfrak{y}_0) + \mathfrak{z}'_0(z - \mathfrak{z}_0) - c^2\, \mathfrak{t}'_0(t - \mathfrak{t}_0) = 0;$$

hence:

$$\mathfrak{x}' = \frac{d\mathfrak{x}}{d\tau} = \left[\frac{\partial x}{\partial \tau}\right]_{\xi=\eta=\zeta=0}, \ldots,$$

and the index 0 means that $\tau = \tau_0$ has been substituted in the functions.

We replace x, y, z, t in (20) with their expressions (17) as functions of $d\xi$, $d\eta$, $d\zeta$, and τ:

(21) $$\begin{cases} \mathfrak{x}'_0\{\mathfrak{x} - \mathfrak{x}_0 + x_\xi\, d\xi + x_\eta\, d\eta + x_\zeta\, d\zeta + \cdots\} + \cdots \\ \cdots - c^2 \mathfrak{t}'_0\{\mathfrak{t} - \mathfrak{t}_0 + t_\xi\, d\xi + t_\eta\, d\eta + t_\zeta\, d\zeta + \cdots\} = 0. \end{cases}$$

We can regard this as an equation for τ, from which one can calculate the values of proper time t along the neighboring lines $d\xi$, $d\eta$, $d\zeta$ that belong to the normal section τ_0. Since the difference $\tau - \tau_0 = d\tau$ is small, (21) will be a linear equation in $d\tau$. Namely, if one develops:

(22) $$\begin{cases} \mathfrak{x} = \mathfrak{x}_0 + \mathfrak{x}'_0\, d\tau + \cdots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ x_\xi = x_\xi^0 + (x'_\xi)_0\, d\tau + \cdots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

and observes that from (15), one has:

([1]) That is, a vector that meets the structure (18) at a real point.

$$\mathfrak{x}'^2 + \mathfrak{y}'^2 + \mathfrak{z}'^2 - c^2\,\mathfrak{t}'^2 = -\,c^2 \tag{23}$$

identically in τ, it will follow from (21), when one neglects all terms that are quadratic in $d\xi$, $d\eta$, $d\zeta$, $d\tau$, that:

$$c^2\,d\tau = \mathfrak{x}_0'(x_\xi^0\,d\xi + x_\eta^0\,d\eta + x_\zeta^0\,d\zeta) + \cdots - c^2\,\mathfrak{t}_0'(t_\xi^0\,d\xi + t_\eta^0\,d\eta + t_\zeta^0\,d\zeta)\,, \tag{24}$$

or, if one sets:

$$\left\{\begin{aligned} x_\xi^0\,d\xi + x_\eta^0\,d\eta + x_\zeta^0\,d\zeta &= \Xi, \\ y_\xi^0\,d\xi + y_\eta^0\,d\eta + y_\zeta^0\,d\zeta &= \mathrm{H}, \\ z_\xi^0\,d\xi + z_\eta^0\,d\eta + z_\zeta^0\,d\zeta &= \mathrm{Z}, \\ t_\xi^0\,d\xi + t_\eta^0\,d\eta + t_\zeta^0\,d\zeta &= \mathrm{T}, \end{aligned}\right. \tag{25}$$

that:

$$c^2\,d\tau = \mathfrak{x}_0'\,\Xi + \mathfrak{y}_0'\,\mathrm{H} + \mathfrak{z}_0'\,\mathrm{Z} - c^2\mathfrak{t}_0'\,\mathrm{T}\,. \tag{26}$$

We now consider the (one-sheeted) hyperbolic structure:

$$(x - \mathfrak{x}_0)^2 + (y - \mathfrak{y}_0)^2 + (z - \mathfrak{z}_0)^2 + c^2\,(x - \mathfrak{t}_0)^2 = \varepsilon^3 \tag{27}$$

that is drawn through the point $\xi = 0$, $\eta = 0$, $\zeta = 0$, $\tau = \tau_0$ as its center. It cuts the normal section (20) in a figure that one can regard as the "rest form" of the filament through that location.

If we accordingly replace x, y, z, t in (27) with the expressions (17), and then replace the quantities $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, $\mathfrak{t}$, x_ξ, … with the developments (22) then that will give:

$$(\mathfrak{x}_0'\,d\tau + x_\xi^0\,d\xi + x_\eta^0\,d\eta + x_\zeta^0\,d\zeta)^2 + \cdots - c^2(\mathfrak{t}_0'\,d\tau + t_\xi^0\,d\xi + t_\eta^0\,d\eta + t_\zeta^0\,d\zeta)^2 = \varepsilon^2, \tag{28}$$

and in this, the functions of $d\xi$, $d\eta$, $d\zeta$ that are defined by (26) are defined on the normal section $d\tau$; hence, (28) will go to:

$$\left\{\begin{aligned} &\left\{\left(1 + \frac{\mathfrak{x}_0'^2}{c^2}\right)\Xi + \frac{\mathfrak{x}_0'\,\mathfrak{y}_0'}{\mathfrak{c}^2}\,\mathrm{H} + \frac{\mathfrak{x}_0'\,\mathfrak{z}_0'}{\mathfrak{c}^2}\,\mathrm{Z} - \mathfrak{x}_0'\,\mathfrak{t}_0'\,\mathrm{T}\right\}^2 + \cdots \\ &\cdots - c^2\left\{\frac{\mathfrak{t}_0'\,\mathfrak{x}_0'}{c^2}\,\Xi + \frac{\mathfrak{t}_0'\,\mathfrak{y}_0'}{\mathfrak{c}^2}\,\mathrm{H} + \frac{\mathfrak{t}_0'\,\mathfrak{z}_0'}{\mathfrak{c}^2}\,\mathrm{Z} + (1 - \mathfrak{t}_0'^2)\mathrm{T}\right\}^2 = \varepsilon^2. \end{aligned}\right. \tag{29}$$

The rest form is given as a quadratic form in $d\xi$, $d\eta$, $d\zeta$ in this. Since the point $\xi = \eta = \zeta = 0$, $\tau = \tau_0$ was an arbitrary point of the flow, one can drop the indices 0 and replace $\mathfrak{x}'$, … with x_τ, … If we then write (29) in the form:

(30) $$\begin{cases} (c_{11}\,d\xi + c_{12}\,d\eta + c_{13}\,d\zeta)^2 + (c_{21}\,d\xi + c_{22}\,d\eta + c_{23}\,d\zeta)^2 \\ + (c_{31}\,d\xi + c_{32}\,d\eta + c_{33}\,d\zeta)^2 + (c_{41}\,d\xi + c_{42}\,d\eta + c_{43}\,d\zeta)^2 = \varepsilon^2 \end{cases}$$

then the rectangular matrix with four rows and three columns $C = (c_{\alpha\beta})$ will be equal to the product of two matrices S and A that are defined by the derivatives of the functions:

(31) $$C = S\,A;$$

and indeed:

(32) $$S = \begin{pmatrix} 1+\frac{x_\tau^2}{c^2} & \frac{x_\tau y_\tau}{c^2} & \frac{x_\tau z_\tau}{c^2} & -\frac{x_\tau t_\tau}{ic} \\ \frac{y_\tau x_\tau}{c^2} & 1+\frac{y_\tau^2}{c^2} & \frac{y_\tau z_\tau}{c^2} & -\frac{y_\tau t_\tau}{ic} \\ \frac{z_\tau x_\tau}{c^2} & \frac{x_\tau y_\tau}{c^2} & 1+\frac{z_\tau^2}{c^2} & -\frac{z_\tau t_\tau}{ic} \\ -\frac{t_\tau x_\tau}{ic} & -\frac{t_\tau y_\tau}{ic} & -\frac{t_\tau z_\tau}{ic} & 1-t_\tau^2 \end{pmatrix},$$

(33) $$A = \begin{pmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \\ ict_\xi & ict_\eta & ict_\zeta \end{pmatrix}.$$

If we now develop the quadratic form (30) in $d\xi$, $d\eta$, $d\zeta$ then we will get:

(34) $$p_{11}\,d\xi^2 + p_{22}\,d\eta^2 + p_{33}\,d\zeta^2 + 2\,p_{12}\,d\xi\,d\eta + 2\,p_{13}\,d\xi\,d\eta + 2\,p_{23}\,d\xi\,d\eta,$$

in which:

(35) $$P = (p_{\alpha\beta}) = \tilde{C}C = \tilde{A}\tilde{S}\,S\,A.$$

One can further simplify the relation (35) with the help of equation (15), which has the vanishing of the determinant of S as a consequence. Namely, a simple calculation will give:

(36) $$\tilde{S}\,S = S,$$

and with that, (33) will go to:

(37) $$P = \tilde{A}\,S\,A.$$

That is the analogue of the equation (12) that was derived in § 1. The six quantities $p_{\alpha\beta}$ will then be referred to as "deformation quantities" and would be important in a theory of elasticity that is adapted to the principle of relativity.

A filament will be called rigid in the smallest part when its rest form is independent of the proper time τ; i.e., if the following six equations are true:

(38) $$\frac{\partial p_{\alpha\beta}}{\partial \tau} = 0.$$

If those equations are fulfilled in all of space then we will be dealing with the motion of a rigid body.

With that, we have arrived at the general differential conditions for rigidity. Since they are constructed with the help of nothing but concepts that are invariant under Lorentz transformation, they will necessarily have that property, as well.

§ 3. The equation of continuity and the incompressible flow

If ρ is the density that belongs to the flow (1) then it is known that it is linked with the velocity components:

(39) $$w_x = \frac{\partial x}{\partial t}, \qquad w_y = \frac{\partial y}{\partial t}, \qquad w_z = \frac{\partial z}{\partial t}$$

by the equation of continuity.

One can formulate that in two ways: In the **Eulerian** picture, one regards ρ, w_x, w_y, w_z as functions of x, y, z, t; the equation of continuity will then read:

(40) $$\frac{\partial \rho}{\partial t} + \frac{\partial \rho w_x}{\partial x} + \frac{\partial \rho w_y}{\partial y} + \frac{\partial \rho w_z}{\partial z} = 0.$$

In the **Lagrangian** picture, one regards x, y, z, ρ as functions of ξ, η, ζ, t; the condition will then read:

(41) $$\frac{\partial \rho \Theta}{\partial t} = 0,$$

in which Θ is the fundamental determinant:

(42) $$\Theta = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{vmatrix}.$$

The connection between the two formulas will be exhibited by the identity ([1]):

([1]) Cf., say, **Weber-Riemann**, *Die partiellen Differentialgleichungen der mathematischen Physik*, **2**, § 146, 1901.

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho w_x}{\partial x}+\frac{\partial\rho w_y}{\partial y}+\frac{\partial\rho w_z}{\partial z} = \frac{1}{\Theta}\frac{d\rho\Theta}{dt}. \tag{43}$$

Both forms of the equations of continuity can be carried over to the representation of the flow with the help of the proper time by the equations (14). At first, one obviously has:

$$w_x = \frac{x_\tau}{t_\tau}, \qquad w_y = \frac{y_\tau}{t_\tau}, \qquad w_z = \frac{z_\tau}{t_\tau}. \tag{44}$$

If we further replace ρ with the "rest density":

$$\rho^* = \frac{\rho}{t_\tau} \tag{45}$$

then (40) will go to:

$$\frac{\partial\rho^* x_\tau}{\partial x}+\frac{\partial\rho^* y_\tau}{\partial y}+\frac{\partial\rho^* z_\tau}{\partial z}+\frac{\partial\rho^* t_\tau}{\partial t} = 0. \tag{46}$$

We will get the analogue of formula (41) when we verify the validity of the identity that corresponds to (43):

$$\frac{\partial\rho^* x_\tau}{\partial x}+\frac{\partial\rho^* y_\tau}{\partial y}+\frac{\partial\rho^* z_\tau}{\partial z}+\frac{\partial\rho^* t_\tau}{\partial t} = \frac{1}{D}\frac{\partial\rho^* D}{\partial t}, \tag{47}$$

in which D means the functional determinant:

$$D = \begin{vmatrix} x_\xi & x_\eta & x_\zeta & x_\tau \\ y_\xi & y_\eta & y_\zeta & y_\tau \\ z_\xi & z_\eta & z_\zeta & z_\tau \\ t_\xi & t_\eta & t_\zeta & t_\tau \end{vmatrix}. \tag{48}$$

To that end, for the sake of brevity, we replace:

$$\begin{array}{lll} x, y, z, t & \text{with} & x_1, x_2, x_3, x_4, \\ \xi, \eta, \zeta, \tau & \text{with} & \xi_1, \xi_2, \xi_3, \xi_4, \end{array}$$

for the moment. We will then have that the left-hand side of (47) is:

$$\sum_\alpha \frac{\partial\left(\rho^*\frac{\partial x_\alpha}{\partial\xi_4}\right)}{\partial x_\alpha} = \sum_{\alpha,\beta}\frac{\partial\left(\rho^*\frac{\partial x_\alpha}{\partial\xi_4}\right)}{\partial\xi_\beta}\frac{\partial\xi_\beta}{\partial x_\alpha}$$
$$= \sum_{\alpha,\beta}\left(\rho^*\frac{\partial^2 x_\alpha}{\partial\xi_\beta\,\partial\xi_4}+\frac{\partial\rho^*}{\partial\xi_\beta}\frac{\partial x_\alpha}{\partial\xi_4}\right)\frac{\partial\xi_\beta}{\partial x_\alpha}.$$

If we now denote the subdeterminant in the matrix of the determinant D that belongs to $\partial x_\alpha / \partial \xi_\beta$ by S $(\partial x_\alpha / \partial \xi_\beta)$ then successive differentiation of equations (14) with respect to x_α and solving the linear equations that arise will yield:

(49) $$\frac{\partial \xi_\alpha}{\partial x_\alpha} = \frac{S\left(\dfrac{\partial x_\alpha}{\partial \xi_\beta}\right)}{D}.$$

If we substitute that above then we will get:

$$\sum_\alpha \frac{\partial\left(\rho^* \dfrac{\partial x_\alpha}{\partial \xi_4}\right)}{\partial x_\alpha} = \frac{1}{D}\sum_{\alpha,\beta}\left\{\rho^* \frac{\partial^2 x_\alpha}{\partial \xi_\beta\, \partial \xi_4} + S\left(\frac{\partial x_\alpha}{\partial \xi_\beta}\right) + \frac{\partial \rho^*}{\partial \xi_\beta}\frac{\partial x_\alpha}{\partial \xi_4} S\left(\frac{\partial x_\alpha}{\partial \xi_\beta}\right)\right\}$$
$$= \frac{1}{D}\left\{\rho^* \frac{\partial D}{\partial \xi_4} + \frac{\partial \rho^*}{\partial \xi_4} D\right\},$$

from general theorems on determinants. It will then follow that:

$$\sum_\alpha \frac{\partial\left(\rho^* \dfrac{\partial x_\alpha}{\partial \xi_4}\right)}{\partial x_\alpha} = \frac{1}{D}\frac{\partial \rho^* D}{\partial \xi_4},$$

and that is the identity (47) that was to be proved.

Hence, one can write the equation of continuity in the form:

(50) $$\frac{\partial \rho^* D}{\partial \tau} = 0.$$

Formulas (46), (47), and (50) have an invariant character under Lorentz transformations.

The quantity:

(51) $$\rho^* D = \rho_0$$

depends upon only ξ, η, ζ. If D is equal to 1 for $\tau = 0$ (which one can always assume) then ρ_0 will be the "initial value of the rest density."

In the old kinematics, a flow was called incompressible when ρ was constant; i.e., independent of time t. In the new kinematics, we will define that condition as follows:

A flow is incompressible when the rest density ρ^* is constant; i.e., independent of proper time τ.

(46) and (50) then imply two forms for the incompressibility condition:

Namely, one can first write (46) as:

$$\rho^*\left(\frac{\partial x_\tau}{\partial x}+\frac{\partial y_\tau}{\partial y}+\frac{\partial z_\tau}{\partial z}+\frac{\partial t_\tau}{\partial t}\right)+\frac{\partial \rho^*}{\partial x}x_\tau+\frac{\partial \rho^*}{\partial y}y_\tau+\frac{\partial \rho^*}{\partial z}z_\tau+\frac{\partial \rho^*}{\partial t}t_\tau = 0$$

or:

(52) $$\rho^*\left(\frac{\partial x_\tau}{\partial x}+\frac{\partial y_\tau}{\partial y}+\frac{\partial z_\tau}{\partial z}+\frac{\partial t_\tau}{\partial t}\right)+\frac{d\rho^*}{d\tau} = 0.$$

Now, should ρ^* not depend upon τ, the *first form of the incompressibility condition* would then follow:

(53) $$\frac{\partial x_\tau}{\partial x}+\frac{\partial y_\tau}{\partial y}+\frac{\partial z_\tau}{\partial z}+\frac{\partial t_\tau}{\partial t} = 0 .$$

The *second form* then follows directly from (50):

(54) $$\frac{\partial D}{\partial \tau} = 0.$$

Hence, if D is equal to 1 for $\tau = 0$ then D will be equal to 1 identically, and from (51):

$$\rho^* = \rho_0\,(\xi,\ \eta,\ \zeta).$$

§ 4. The rectilinear translation of a rigid body

We would now like to integrate the differential conditions of rigidity (38) for the simplest case of rectilinear translation. If we imagine that rigidity must be identical with incompressibility in this case then we will obtain not just a criterion for our definition of rigidity to make sense, but also, at the same time, a method for integrating it.

We then set:

(55) $$y = \eta,\quad z = \zeta$$

and assume that x and t depend upon only ξ and τ. We then get from (32) and (33) that:

$$S = \begin{pmatrix} 1+\dfrac{x_\tau^2}{c^2} & 0 & 0 & -\dfrac{x_\tau t_\tau}{ic} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\dfrac{t_\tau x_\tau}{ic} & 0 & 0 & 1-t_\tau^2 \end{pmatrix},$$

$$A = \begin{pmatrix} x_\xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ ict_\xi & 0 & 0 \end{pmatrix}.$$

If one forms the matrix:

$$P = \tilde{A} S A$$

from this then one will easily find that:

$$P = \begin{pmatrix} (x_\xi t_\tau - x_\tau t_\xi)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The six rigidity conditions will then reduce to the one equation:

$$\frac{d}{d\tau}(x_\xi t_\tau - x_\tau x_\xi) = 0. \tag{56}$$

On the other hand, the determinant (48) will become:

$$D = \begin{pmatrix} x_\xi & 0 & 0 & x_\tau \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_\xi & 0 & 0 & t_\tau \end{pmatrix} = \begin{vmatrix} x_\xi & x_\tau \\ t_\xi & t_\tau \end{vmatrix}. \tag{57}$$

With that, the incompressibility condition:

$$\frac{dD}{d\tau} = 0$$

will be identical with the rigidity condition (56).

As a result of that, we can also replace the latter with the other form (53) for the incompressibility condition, which assumes the form:

$$\frac{\partial x_\tau}{\partial x} + \frac{\partial t_\tau}{\partial t} = 0 \tag{58}$$

here. The integration will now be easy to perform in this form.

If one sets:

$$x_\tau = p, \quad t_\tau = -q \tag{59}$$

then one will have the following two equations for p, q:

(60) $$\begin{cases} \dfrac{\partial p}{\partial x} - \dfrac{\partial q}{\partial t} = 0, \\ p^2 - c^2 q^2 = -c^2. \end{cases}$$

These are equivalent to one partial differential equation for a function in two independent variables. Namely, if one sets:

(61) $$p = \frac{\partial \varphi}{\partial t}, \qquad q = \frac{\partial \varphi}{\partial x}$$

then the first equation (60) will be fulfilled, and the second one will go to:

(62) $$\varphi_t^2 - c^2 \varphi_z^2 = -c^2.$$

One will get the simplest solution when one sets φ_t and φ_x equal to constants γ and $-\delta$ that must fulfill the condition:

(63) $$\gamma^2 - c^2 \delta^2 = -c^2.$$

One will then have:

$$p = x_\tau = \gamma, \qquad q = -t_\tau = -\delta,$$

from which, it will follow that:

(64) $$\begin{cases} x = W(\xi) + \gamma\tau, \\ t = V(\xi) + \delta\tau, \end{cases}$$

in which W and V mean two arbitrary functions of ξ. As a result of equation (63), the form of equations (64) will indeed remain preserved when one subjects x, t to a Lorentz transformation.

Equations (64), when combined with (55), represent a uniform, rectilinear motion. The functions $W(\xi)$, $V(\xi)$ are determined from the values that x and t should have when $\tau = 0$. Here, it is not convenient to assume that $x = \xi$ when $\tau = 0$, but to determine the functions $W(\xi)$, $V(\xi)$ in such a way that formula (64) represents the same Lorentz transformation that transforms the body to rest; i.e., to set:

(65) $$\begin{cases} x = \alpha\xi + \gamma\tau, \\ t = \beta\xi + \delta\tau, \end{cases}$$

which implies that the conditions:

(66) $$\alpha^2 - c^2 \beta^2 = 1, \qquad \alpha\gamma - c^2 \beta\delta = 0, \qquad \gamma^2 - c^2 \delta^2 = 1$$

must be fulfilled.

As long as one of the two quantities φ_t, φ_z in (62) is independent of t, it must also be independent of the other. In this case, the integration of (62) can be performed easily with the help of a Legendre transformation. Namely, one can then introduce the quantity:

(67) $$\varphi_t = p$$

as an independent variable, along with x, and then think of calculating t as a function of x and p by using (67). If one then introduces the new unknown function:

$$\psi(p, x) = \varphi - p\,t,$$

instead of φ, then:

(68) $$\begin{cases} \psi_p = \varphi_t t_p - p t_p - t = -t, \\ \psi_x = \varphi_x + \varphi_t t_x - p t_x = \varphi_x. \end{cases}$$

With that, (62) will go to the following equation for $\psi(p, x)$:

$$p^2 - c^2\,\psi_x^2 = -\,c^2,$$

and that can be integrated immediately. That will then imply that:

(69) $$\begin{cases} \psi_x = \sqrt{1 + \dfrac{p^2}{c^2}} = q, \\ \psi = qx - w(p), \end{cases}$$

in which w means an arbitrary function. It follows from this by differentiating with respect to p and recalling (68) that:

(70) $$\frac{p}{c^2 q}\,x - w\,(p) = -\,t.$$

If one imagines that p has been calculated as a function of t in this and then substituted in $\varphi = \psi + p\,t$ then one will have the desired general solution of (62):

(71) $$\varphi = qx - w\,(p) + pt.$$

From (59) and (61), one will obviously have:

$$\frac{x_\tau}{t_\tau} = \frac{dx}{dt} = -\,\frac{\varphi_t}{\varphi_x},$$

from which it will follow that every equation φ = const. = $-\,\xi$ represents the world-line of a point of the rigid body. We will then find the following representation of the world-line from (70) and (71):

$$(72) \qquad \begin{cases} \dfrac{p}{c^2}x + qt = qw', \\ qx + pt = w - \xi, \end{cases}$$

or, when it is solved for x and t:

$$(73) \qquad \begin{cases} x = q(w-\xi) - pqw', \\ t = -\dfrac{p}{c^2}(w-\xi) + q^2 w'. \end{cases}$$

In this, the world-lines of rigid bodies are described in such a way that x and t are given as functions of the independent variables ξ, p. We would now like to discuss that representation.

We first remark that uniform translational motion depends upon only *one arbitrary function of one argument w (p)*. One then says that here, as in the old kinematics, *only one degree of freedom* is present. The use of the independent variable $p = x_\tau$ is then essential, which will also have great meaning later on. Furthermore, equations (73) go to the corresponding representation of rectilinear translation in the older kinematics when $c = \infty$. One will then have that $q = \sqrt{1+(p^2/c^2)}$ is equal to 1. It follows from the second equation (73) that p depends upon only t when $c = \infty$, such the first one will assume the form:

$$x = \xi + a\,(t).$$

Finally, we turn to the characterization of the world-lines in the xt-plane. One sees that (72) and (73) have the form of a Lorentz transformation and its inverse that take the variables x, y to the variables $\bar{x} = w - \xi$, $\bar{t} = qw'$ and read:

$$(74) \qquad \begin{cases} \bar{x} = qx + \dfrac{p}{c}ct, \quad & x = q\bar{x} - \dfrac{p}{c}c\bar{t}, \\ c\bar{t} = \dfrac{p}{c}x + qct, \quad & ct = -\dfrac{p}{c}\bar{x} + qc\bar{t}. \end{cases}$$

Equations (66) in the coefficients are obviously fulfilled due to (60) then.

We then have a family of Lorentz transformations that depends upon the parameter p before us. The motion, or rather the associated pencil of world-lines, can now be described thus:

If one gives ξ a well-defined value ξ_1 then x and t will be given as well-defined functions of p by equations (73) that represent the world-line of the point ξ_1. The components of the velocity world-vector along the x and t axes are p, $-q$, resp. All curves of the family are determined by one curve ξ_1. One constructs it as follows: One draws a normal line to the tangent to the curve at a point p in the sense of § 2 (pp. 11). Along with the tangent, it defines a coordinate system that is a transform of the x and t axes. One measures out the line segment $\xi_1 - \xi$ along that x-axis using the unit of the coordinate system ([1]). If one now moves that coordinate system along the curve ξ_1 then the point ξ will describe the world-line that belongs to the parameter value ξ. All of the points of that normal (x-axis) will belong to the same value of p, and will thus have the same velocity.

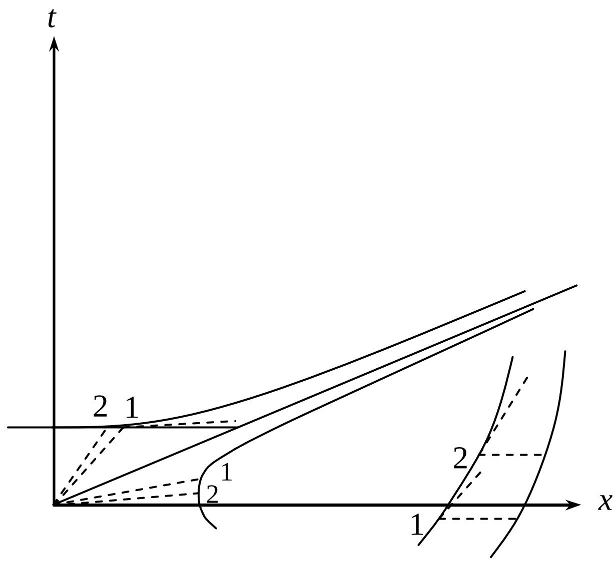

Figure 4.

The rectilinear motion of a rigid body is then arranged such that as long as one transforms *one* point to rest, the same transformation will transform all points to rest; that rest transformation is just (74). In addition to uniform motion, the lines with the same velocity p = const. will always have an envelope; the regularity of the motion is based upon that. For given dimensions of the body, the curvature of the world-lines cannot exceed a certain limit then, and conversely. *It follows from this that a rigid body is necessarily finitely-extended in all directions, and the greater the acceleration it experiences, the smaller it must be.* Here, we have the first proof of the fundamental meaning of atomism in the new dynamics. If the rigid body carries a substance of rest density ρ^* then it will be independent of p and a function of only ξ, η, ζ that we denote by:

$$\rho_0\,(\xi,\,\eta,\,\zeta).$$

§ 5. Hyperbolic motion

We will obtain the simplest motion beyond uniform translation when we set the arbitrary function $w = 0$ in (72) and (73). One will then have:

([1]) Cf., **H. Minkowski**, "Raum und Zeit," *loc. cit.* (rem. pp. 2)

$$\left\{\begin{aligned} x &= -q\xi, \\ t &= \frac{p}{c^2}\xi. \end{aligned}\right. \tag{75}$$

If one eliminates p from this then it will follow that:

$$x^2 - c^2 t^2 = \xi^2. \tag{76}$$

One sees from this that the associated world-lines in the xt-plane and the planes that are parallel to it $y = \eta$, $z = \zeta$ will be hyperbolas that have the lines through the zero point that correspond to the speed of light as asymptotes and cut the x-axis at a distance of ξ from the zero point. A bundle of such hyperbolas represents a motion under which the rigid body comes in from infinity, approaches the zero point, turns around, and once more goes out to infinity, in such a way that its velocity first decreases from c to 0, and after the reversal, it will again increase to c. We would like to call this motion, which is analogous to the uniform, accelerated motion in the older kinematics to some extent, *hyperbolic motion*, for brevity.

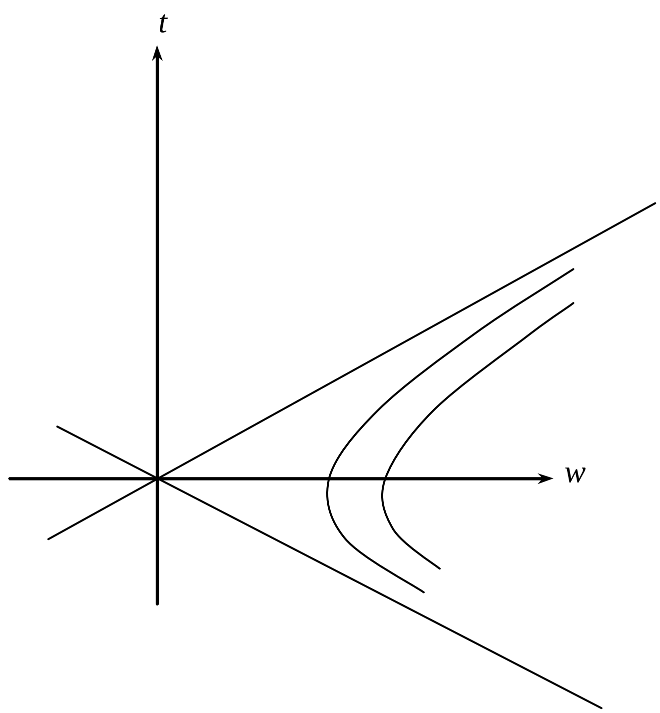

Figure 5.

Since the zero point is an entirely arbitrary point, the hyperbolas:

$$(x - \alpha)^2 - c^2 (t - \beta)^2 = \xi^2 \tag{77}$$

do not represent an essentially different motion, although the velocity will be non-zero for $t = 0$ then. We will then be able to confine ourselves to formulas (75), (76).

This hyperbolic motion proves to be the simplest one not only kinematically, but also dynamically. It is closely connected with the fact that any arbitrary world-line will osculate such a hyperbola at each of its points, namely, the "curvature hyperbola," in which the vector that points from its center to the point P and has magnitude $b = c^2 / \xi$ will represent the acceleration vector of the world-line.

In fact, if we calculate the components of acceleration of the hyperbolic motion then we will next find that:

(78) $$\frac{\partial^2 y}{\partial \tau^2} = 0, \quad \frac{\partial^2 z}{\partial \tau^2} = 0.$$

In order to calculate the x and t-components, we look at the equations:

(79) $$\left\{ \begin{aligned} & \xi_t = -p, \quad p_t = \frac{c^2 q^2}{\xi}, \\ & \xi_t = -q, \quad p_x = \frac{p\,q}{\xi}. \end{aligned} \right.$$

One will then have:

$$\frac{\partial^2 x}{\partial \tau^2} = p_\tau = p_x\, x_\tau + p_t\, t_\tau = \frac{p\,q}{\xi} - \frac{c^2 q^2}{\xi} q\,.$$

We will then get:

(80) $$\frac{\partial^2 x}{\partial \tau^2} = b_x = -\,q\,b,$$

and likewise:

(81) $$\frac{\partial^2 t}{\partial \tau^2} = b_t = \frac{p}{c^2} b,$$

in which:

(82) $$b = \sqrt{b_x^2 - c^2 b_t^2} = \frac{c^2}{\xi}$$

is the magnitude of the acceleration. The assertion above follows from (80), (81), (82) ([1]).

The acceleration is then constant in magnitude for any world-line of the hyperbolic motion. Therein lies its analogy with the uniformly-accelerated motion of the old mechanics, which is represented by parabolic world-lines. It is then the simplest accelerated motion, and every motion can be approximated by hyperbolic motions. Supported by that fact, in what follows, we would like to establish the dynamics of hyperbolic motions more precisely, and above all, to seek to determine the force that an electrically-charged body will exert upon itself in that way. The result will then also give an approximate explanation for all motions for which the magnitude of the acceleration vector varies only slightly.

([1]) **H. Minkowski**, "Raum und Zeit," *loc. cit.* (rem. pp. 2).

CHAPTER TWO

The field of a rigid electron in hyperbolic motion

§ 6. Retarded potentials and field strengths

The forces that are exerted by moving electric charges, which enter into the equations of motion of those charges, are derived from certain auxiliary quantities, namely, the retarded potentials and field strengths. We would like to summarize the expressions for those quantities that will be employed in what follows.

Let an electric current be represented by equations of the form (14); let the initial value of its rest density [cf., § 3, pp. 16, (51)] be:

$$\rho_0\,(\xi,\,\eta,\,\zeta).$$

The retarded potentials will then be given by the following expressions:

$$(83)\quad\begin{cases} 4\pi\,\Phi_x(x,y,z,t)=\iiint\left[\dfrac{\bar\rho_0\,\bar x_\tau}{(x-\bar x)\,\bar x_\tau+(y-\bar y)\,\bar y_\tau+(z-\bar z)\,\bar z_\tau-c^2(t-\bar t\,)\,\bar t_\tau}\right]_{h=0} d\bar\xi\,d\bar\eta\,d\bar\zeta, \\ \qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 4\pi\,\Phi\,(x,y,z,t)=\iiint\left[\dfrac{c\,\bar\rho_0\,\bar t_\tau}{(x-\bar x)\,\bar x_\tau+(y-\bar y)\,\bar y_\tau+(z-\bar z)\,\bar z_\tau-c^2(t-\bar t\,)\,\bar t_\tau}\right]_{h=0} d\bar\xi\,d\bar\eta\,d\bar\zeta. \end{cases}$$

In this, $\bar x$, $\bar y$, $\bar z$, $\bar t$ and $\bar x_\tau$, $\bar y_\tau$, $\bar z_\tau$, $\bar t_\tau$ mean the functions (14) (their derivatives with respect to τ, resp.) when they are taken for the arguments $\bar\xi$, $\bar\eta$, $\bar\zeta$, and in the square brackets, t is replaced with the function of x, y, z, t, $\bar\xi$, $\bar\eta$, $\bar\zeta$ that one obtains by solving the equation:

$$(84)\qquad h = (x-\bar x)^2+(y-\bar y)^2+(z-\bar z)^2-c^2(t-\bar t\,)^2 = 0$$

for τ; indeed, one takes the uniquely-determined solution ([1]) of the equation for which $t > \bar t$. How the expressions (83), which have probably not been used yet for continuous flows in that form, are connected with the usual formulas for the retarded potentials shall be explained briefly in the next paragraph.

The electric field strength $\mathfrak{E}$ and the magnetic one $\mathfrak{M}$ can be derived from the potential by the vector equations:

([1]) That equation has one and only one such solution, since the speed of the flow cannot exceed the speed of light. Cf., **H. Minkowski**, *loc. cit.* (rem. pp. 6).

$$(85) \qquad \begin{cases} \mathfrak{E} = -\dfrac{1}{c}\dfrac{\partial}{\partial t}(\Phi_x, \Phi_y, \Phi_z) - \operatorname{grad}\Phi, \\ \mathfrak{M} = \operatorname{curl}(\Phi_x, \Phi_y, \Phi_z). \end{cases}$$

The potentials (83) are solutions of the equations ([1]):

$$(86) \qquad \begin{cases} \dfrac{\partial}{\partial x}\operatorname{lor}\Phi + \dfrac{1}{c^2}\dfrac{\partial^2 \Phi_x}{\partial t^2} - \Delta\Phi_x = \dfrac{\rho^*}{c} x_\tau, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{\partial}{\partial t}\operatorname{lor}\Phi + \dfrac{1}{c^2}\dfrac{\partial^2 \Phi}{\partial t^2} - \Delta\Phi = \rho^* t_\tau, \end{cases}$$

and indeed especially those solutions for which the quantities:

$$(87) \qquad \operatorname{lor}\Phi \equiv \frac{\partial \Phi_x}{\partial x} + \frac{\partial \Phi_y}{\partial y} + \frac{\partial \Phi_z}{\partial z} + \frac{1}{c}\frac{\partial \Phi}{\partial t}$$

vanish for them.

Equations (86) are the **Lagrangian** equations for the variational problem ([2]) of finding those functions Φ_x , Φ_y , Φ_z , Φ for which the integral:

$$(88) \qquad W = \iiint\!\!\int \left\{ \tfrac{1}{2}(\mathfrak{M}^2 - \mathfrak{E}^2) - \frac{\rho^*}{c}(\Phi_x x_\tau + \Phi_y y_\tau + \Phi_z z_\tau - \Phi t_\tau) \right\} dx\, dy\, dz\, dt$$

is an extremum when it is taken over a domain *G* in the *xyzt*-manifold, and the electrical current and the values of the potentials are given on the boundary of *G*.

§ 7. Comparison of the expressions for the retarded potentials

One can regard the expressions (83) for the potentials as the superposition of the elementary potentials that originate in the individual moving points of the current. Namely, one has the **Liénard-Wiechert** ([3]) expressions for the latter:

([1]) In this representation, I shall follow the procedure of **W. Ritz**, by which I shall regard the potentials as solely the effects of charge on charge in the first approximation and first introduce the partial differential equation in the second one. It is characteristic of this that no use is made at all of the actual **Maxwell** field equations for $\mathfrak{E}$, $\mathfrak{M}$ in my entire theory.

([2]) Cf., **K. Schwarzschild**, *loc. cit.* (rem. pp. 3); **Max Born**, *loc. cit.* (rem., pp. 3).

([3]) **E. Wiechert**, Arch. néerl. (2) **5** (1900), pp. 549; **A. Liénard**, “L’éclairage électrique,” **16** (1898), pp. 5, 53, 106.

(89)
$$\left\{\begin{aligned} 4\pi\,\varphi_x &= \left[\frac{e\,w_x}{c\,r\left(1-\frac{w_r}{c}\right)}\right]_{t-\bar{t}=r/c}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 4\pi\,\varphi &= \left[\frac{e}{r\left(1-\frac{w_r}{c}\right)}\right]_{t-\bar{t}=r/c}. \end{aligned}\right.$$

In this, e means the charge of the active point:

(90) $$x = \bar{x}(t), \qquad y = \bar{y}(t), \qquad z = \bar{z}(t);$$

furthermore:

(91) $$r = \sqrt{(x-\bar{x})^2 + (y-\bar{y})^2 + (z-\bar{z})^2}$$

is the distance from that point to the reference point x, y, z:

(92) $$w_r = \frac{1}{r}\{(x-\bar{x})^2\,\bar{w}_x + (y-\bar{y})^2\,\bar{w}_y + (z-\bar{z})^2\,\bar{w}_z\}$$

is the component of its velocity w_x, w_y, w_z in the direction of r, and in the square bracket, one must set t equal to the value that one gets from the equations:

(93) $$t - \bar{t} = \frac{r}{c}.$$

If one now has a continuous current then the world-line (90) must be replaced with a bundle of world-lines by bringing the functions (90) into the form (1) by the introduction of three parameters ξ, η, ζ, and replacing e with the density $\rho(\xi, \eta, \zeta)$. The functions φ_x, φ_y, φ_z, φ will then be independent of ξ, η, ζ, and one can integrate them over all of space. In this, one must observe that for the spatial integration, one has:

$$dx\,dy\,dz = \Theta\,d\xi\,d\eta\,d\zeta,$$

and that, from § 3, (41), the functional determinant Θ is coupled with the density ρ in the initial density by $\rho_0 = \rho\,\Theta$.

The expressions that arise can be easily brought into the form (83). In order to do that, one needs only to write the equations of motion of the active point homogeneously in the form:

(94) $$x = \bar{x}(\tau), \qquad y = \bar{y}(\tau), \qquad z = \bar{z}(\tau), \qquad t = \bar{t}(\tau),$$

in which τ means the proper time, and ρ is replaced with the rest density ρ^*. Equation (93) then goes to:

(95) $$h = (x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2 - c^2 (t - \bar{t})^2 = 0,$$

from which, τ will be determined uniquely by the auxiliary condition that $t > \bar{t}$ ([1]).

The connection between the expressions (83) and the usual expressions for the potentials is also easy to establish. The latter read ([2]):

(96) $$\begin{cases} 4\pi\,\Phi_x = \iiint \dfrac{d\bar{x}\,d\bar{y}\,d\bar{z}}{r} \left[\dfrac{\rho\, w_x}{c}\right]_{\bar{t}=t-r/c}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 4\pi\,\Phi = \iiint \dfrac{d\bar{x}\,d\bar{y}\,d\bar{z}}{r} [\rho]_{\bar{t}=t-r/c}. \end{cases}$$

In this, the current is thought of as being represented by equations in the form [(1), pp. 5], and one further has:

$$r = \sqrt{(x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2},$$

and all of the arguments in the square brackets must be replaced with $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t} = t - r / c$. The integrations in (95) are extended over all charges, so, since they are moving, over time-like variable limits. The transition from the expressions (95) to the expressions (83) now consists of just performing the integrations over fixed, time-independent limits. That will happen in the following way:

If we replace t with $\bar{t} = t - r / c$ in the current equations (1) then we will get equations of the form:

(97) $$\begin{cases} \bar{x} = \bar{x}\{\xi, \eta, \zeta, \bar{t}\,(\bar{x}, \bar{y}, \bar{z}, t)\}, \\ \bar{y} = \bar{y}\{\xi, \eta, \zeta, \bar{t}\,(\bar{x}, \bar{y}, \bar{z}, t)\}, \\ \bar{z} = \bar{z}\{\xi, \eta, \zeta, \bar{t}\,(\bar{x}, \bar{y}, \bar{z}, t)\}, \end{cases}$$

which couples $\bar{x}$, $\bar{y}$, $\bar{z}$ with ξ, η, ζ, and obviously represent precisely the transformation that will convert the integral to fixed limits when it is applied to (96). That transformation (97) will then represent $\bar{x}$, $\bar{y}$, $\bar{z}$ as functions of their initial values for the time-point that comes under consideration in the square brackets.

In order to compute the functional determinant of the transformation (97):

(98) $$\Delta = \left[\frac{\partial(\bar{x}, \bar{y}, \bar{z})}{\partial(\xi, \eta, \zeta)}\right]_{t=\text{const.}},$$

([1]) Cf., *loc. cit.* (rem. 1, pp. 25).

([2]) Cf., say, **M. Abraham**, *Theorie der Elektrizität*, 2nd ed., v. **2**, formulas (51b), (51c), pp. 57.

we would like to denote the derivative of $\bar{x}$ with respect to ξ for fixed $\bar{t}$ by $(\partial\bar{x}/\partial\xi)_{\bar{t}}$, and by $(\partial\bar{x}/\partial\xi)_t$ for fixed t. If we then differentiate equations (97) with respect to ξ, η, ζ in succession then we will get three systems of equations with coefficients that read the same; e.g., for the differentiation with respect to ξ:

$$\left(\frac{\partial\bar{x}}{\partial\xi}\right)_t = \left(\frac{\partial\bar{x}}{\partial\xi}\right)_{\bar{t}} + \frac{\partial\bar{x}}{\partial\bar{t}}\left\{\frac{\partial\bar{t}}{\partial\bar{x}}\left(\frac{\partial\bar{x}}{\partial\xi}\right)_t + \frac{\partial\bar{t}}{\partial\bar{y}}\left(\frac{\partial\bar{y}}{\partial\xi}\right)_t + \frac{\partial\bar{t}}{\partial\bar{z}}\left(\frac{\partial\bar{z}}{\partial\xi}\right)_t\right\},$$

$$\left(\frac{\partial\bar{y}}{\partial\xi}\right)_t = \left(\frac{\partial\bar{y}}{\partial\xi}\right)_{\bar{t}} + \frac{\partial\bar{y}}{\partial\bar{t}}\left\{\frac{\partial\bar{t}}{\partial\bar{x}}\left(\frac{\partial\bar{x}}{\partial\xi}\right)_t + \frac{\partial\bar{t}}{\partial\bar{y}}\left(\frac{\partial\bar{y}}{\partial\xi}\right)_t + \frac{\partial\bar{t}}{\partial\bar{z}}\left(\frac{\partial\bar{z}}{\partial\xi}\right)_t\right\},$$

$$\left(\frac{\partial\bar{z}}{\partial\xi}\right)_t = \left(\frac{\partial\bar{z}}{\partial\xi}\right)_{\bar{t}} + \frac{\partial\bar{z}}{\partial\bar{t}}\left\{\frac{\partial\bar{t}}{\partial\bar{x}}\left(\frac{\partial\bar{x}}{\partial\xi}\right)_t + \frac{\partial\bar{t}}{\partial\bar{y}}\left(\frac{\partial\bar{y}}{\partial\xi}\right)_t + \frac{\partial\bar{t}}{\partial\bar{z}}\left(\frac{\partial\bar{z}}{\partial\xi}\right)_t\right\},$$

or

$$\left(\frac{\partial\bar{x}}{\partial\xi}\right)_t\left(1+\frac{1}{c}\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}}\right)+\left(\frac{\partial\bar{y}}{\partial\xi}\right)_t\frac{1}{c}\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}}+\left(\frac{\partial\bar{z}}{\partial\xi}\right)_t\frac{1}{c}\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{z}} = \left(\frac{\partial\bar{x}}{\partial\xi}\right)_{\bar{t}},$$

$$\left(\frac{\partial\bar{x}}{\partial\xi}\right)_t\frac{1}{c}\frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}}+\left(\frac{\partial\bar{y}}{\partial\xi}\right)_t\left(1+\frac{1}{c}\frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}}\right)+\left(\frac{\partial\bar{z}}{\partial\xi}\right)_t\frac{1}{c}\frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{z}} = \left(\frac{\partial\bar{y}}{\partial\xi}\right)_{\bar{t}},$$

$$\left(\frac{\partial\bar{x}}{\partial\xi}\right)_t\frac{1}{c}\frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}}+\left(\frac{\partial\bar{y}}{\partial\xi}\right)_t\frac{1}{c}\frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}}+\left(\frac{\partial\bar{z}}{\partial\xi}\right)_t\left(1+\frac{1}{c}\frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{z}}\right) = \left(\frac{\partial\bar{x}}{\partial\xi}\right)_{\bar{t}}.$$

Two times three equations, in which ξ is switched with η (ζ, resp.), must be added to these. We shall now denote the matrices that appear here as follows:

(99)
$$P = \begin{pmatrix} \left(\frac{\partial\bar{x}}{\partial\xi}\right)_t & \left(\frac{\partial\bar{y}}{\partial\xi}\right)_t & \left(\frac{\partial\bar{z}}{\partial\xi}\right)_t \\ \left(\frac{\partial\bar{x}}{\partial\eta}\right)_t & \left(\frac{\partial\bar{y}}{\partial\eta}\right)_t & \left(\frac{\partial\bar{z}}{\partial\eta}\right)_t \\ \left(\frac{\partial\bar{x}}{\partial\zeta}\right)_t & \left(\frac{\partial\bar{y}}{\partial\zeta}\right)_t & \left(\frac{\partial\bar{z}}{\partial\zeta}\right)_t \end{pmatrix},$$

(100)
$$Q = \begin{pmatrix} 1+\frac{1}{c}\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}} & \frac{1}{c}\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}} & \frac{1}{c}\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}} \\ \frac{1}{c}\frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}} & 1+\frac{1}{c}\frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}} & \frac{1}{c}\frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{z}} \\ \frac{1}{c}\frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}} & \frac{1}{c}\frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}} & 1+\frac{1}{c}\frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{z}} \end{pmatrix}$$

(101)
$$R = \begin{pmatrix} \left(\frac{\partial\bar{x}}{\partial\xi}\right)_{\bar{t}} & \left(\frac{\partial\bar{y}}{\partial\xi}\right)_{\bar{t}} & \left(\frac{\partial\bar{z}}{\partial\xi}\right)_{\bar{t}} \\ \left(\frac{\partial\bar{x}}{\partial\eta}\right)_{\bar{t}} & \left(\frac{\partial\bar{y}}{\partial\eta}\right)_{\bar{t}} & \left(\frac{\partial\bar{z}}{\partial\eta}\right)_{\bar{t}} \\ \left(\frac{\partial\bar{x}}{\partial\zeta}\right)_{\bar{t}} & \left(\frac{\partial\bar{y}}{\partial\zeta}\right)_{\bar{t}} & \left(\frac{\partial\bar{z}}{\partial\zeta}\right)_{\bar{t}} \end{pmatrix},$$

so our nine equations can be summarized in the matrix equation:

$$P\,\tilde{Q} = R.$$

The determinant relation:

(102)
$$|\,P\,| \cdot |\,Q\,| = |\,R\,|$$

will follow from that. Now, it is obvious that from (98):

(103)
$$|\,P\,| = \Delta;$$

furthermore, from [§ 3, (42), pp. 14]:

(104)
$$|\,R\,| = [\Theta]_{\bar{t}=t-r/c}.$$

Finally, one easily finds that:

$$|\,Q\,| = 1 + \frac{1}{c}\left(\frac{\partial\bar{x}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{x}} + \frac{\partial\bar{y}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{y}} + \frac{\partial\bar{z}}{\partial\bar{t}}\frac{\partial r}{\partial\bar{z}}\right)$$
$$= 1 + \frac{1}{c}\left(\bar{w}_x\frac{\partial r}{\partial\bar{x}} + \bar{w}_y\frac{\partial r}{\partial\bar{y}} + \bar{w}_z\frac{\partial r}{\partial\bar{z}}\right),$$

and from (92), that can be written:

(105)
$$Q = \left[1 - \frac{w_r}{c}\right]_{\bar{t}=t-r/c}.$$

Hence, we will have:

(106)
$$\Delta = \left[\frac{\Theta}{1 - \frac{w_r}{c}} \right]_{\bar{t} = t - r/c} .$$

If we substitute that in (96) and observe that from [§ 3, (41), pp. 14], one must set:

$$\rho\, \Theta = \rho_0\, (\xi,\ \eta,\ \zeta),$$

then we will get:

(107)
$$\begin{cases} 4\pi\, \Phi_x = \iiint \rho_0(\xi,\eta,\zeta) \left[\dfrac{w_x}{cr\left(1 - \dfrac{w_r}{c}\right)} \right]_{t - \bar{t} = r/c} d\xi\, d\eta\, d\zeta, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ 4\pi\, \Phi = \iiint \rho_0(\xi,\eta,\zeta) \left[\dfrac{w_x}{r\left(1 - \dfrac{w_r}{c}\right)} \right]_{t - \bar{t} = r/c} d\xi\, d\eta\, d\zeta. \end{cases}$$

If we now replace w_x with x_τ / t_τ, etc., as on pp. 15, then formulas (107) will go to the expressions (83) directly. In fact, *only the initial density* ρ_0 will appear in them.

§ 8. Calculating the potentials for hyperbolic motions

We would now like to evaluate the potentials (83) for the hyperbolic motion:

(108)
$$x = -\, q\, \xi, \qquad y = \eta, \quad z = \zeta, \quad t = \frac{p}{c^2}\, \xi.$$

Since $y_\tau = z_\tau = 0$, one also has $\Phi_y = \Phi_y = 0$. Since we have the quantity p as the independent variable, instead of t, in (108), along with ξ, η, ζ we will also regard the equation (84) $h = 0$ as an equation for p. It will then read:

(109)
$$(x + \bar{q}\, \bar{\xi})^2 + (y - \bar{\eta})^2 + (z - \bar{\zeta})^2 - c^2 \left(t - \frac{\bar{p}\, \bar{\xi}}{c^2} \right)^2 = 0;$$

when one introduces the abbreviations:

(110)
$$\begin{cases} s = x^2 - c^2 t^2 = \xi^2, \\ k = -\dfrac{1}{2\xi} [s + \bar{\xi}^2 + (y - \bar{\eta})^2 + (z - \bar{\zeta})^2], \end{cases}$$

one can then write (109) as:

$$\bar{p}\,t + \bar{q}\,x = k;$$

one must add to that:

$$\bar{p}^2 - c^2\,\bar{q}^2 = -\,c^2.$$

One calculates $\bar{p}$ from these equations, and indeed the value of $\bar{p}$ for which $t > \bar{t}$ is chosen. If one substitutes the value:

$$\bar{p} = \frac{k - \bar{q}\,x}{t}$$

that follows from the first equation into the second one then the quadratic equation for $\bar{q}$ will arise:

$$\bar{q}^2 - \bar{q}\,\frac{2kx}{s} = -\,\frac{k^2 + c^2 t^2}{s}.$$

It follows from this that:

$$\bar{q} = \frac{1}{s}\left(kx + ct\sqrt{k^2 - s}\right).$$

If we set the positive root equal to:

(111) $$B = \sqrt{k^2 - s},$$

to abbreviate, and calculate $\bar{p}$ then we will find that:

$$\bar{p} = -\,\frac{c}{s}(k\,c\,t \pm Bx),$$

$$\bar{q} = \frac{1}{s}(k\,x \pm B\,c\,t).$$

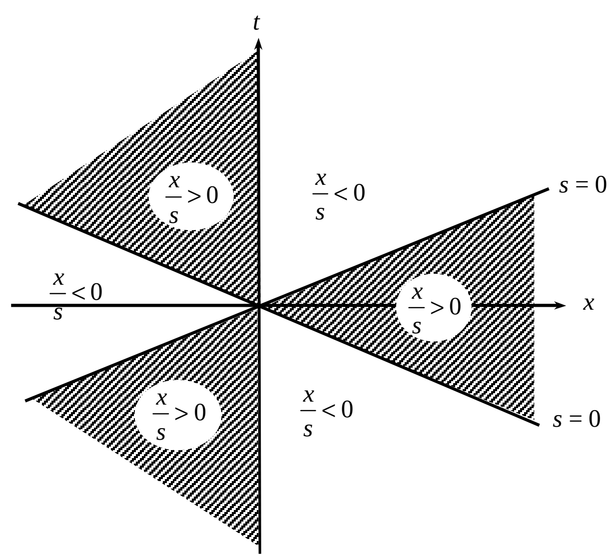

Figure 6.

In this, we must choose the sign that corresponds to the smaller value of $\bar{t}$. Now, since $\bar{t} = \bar{p}\,\bar{\xi}\,/\,c^2$, assuming that the electron moves to the right from the zero point $x = 0$ (i.e., $\bar{\xi} > 0$), that will imply that:

One must take the positive sign for all reference points for which $x / s > 0$.
One must take the negative sign for all reference points for which $x / s < 0$.

The distribution of those references points will then emerge from the Figure.

In what follows, we shall mostly assume that $x / s > 0$; *only those points can be interior points to the electron.* One must then take the positive root B for them. If we occasionally also consider points for which $x / s < 0$ then we will have to replace $+ B$ with $- B$ everywhere.

We then have:

$$
(112) \qquad \begin{cases} \bar{p} = -\dfrac{c}{s}(kct + Bx), \\[2mm] \bar{q} = \dfrac{1}{s}(kx + Bct). \end{cases}
$$

We shall now calculate the denominator in the integral (83) for these values of $\bar{p}$, $\bar{q}$.

Due to the fact that $y_\tau = z_\tau = 0$, $x_\tau = p$, $t_\tau = - q$, that will become:

$$
(x + \bar{q}\,\bar{\xi})\,\bar{p} + c^2\left(t - \frac{\bar{p}}{c^2}\bar{\xi}\right)\bar{q} = x\,\bar{p} + c^2 t\,\bar{q} = - c\,B.
$$

We substitute that in the integral, and that will give:

$$
(113) \qquad \begin{cases} 4\pi\,\Phi_x(x, y, z, t) = \displaystyle\iiint \frac{\bar{\rho}_0}{sB}(kct + Bx)\,d\bar{\xi}\,d\bar{\eta}\,d\bar{\zeta}, \\[2mm] 4\pi\,\Phi\,(x, y, z, t) = \displaystyle\iiint \frac{\bar{\rho}_0}{sB}(kx + Bct)\,d\bar{\xi}\,d\bar{\eta}\,d\bar{\zeta}. \end{cases}
$$

If we set:

$$
(114) \qquad \begin{cases} \psi_1(s) = \dfrac{1}{s}\displaystyle\iiint \bar{\rho}_0\,d\bar{\xi}\,d\bar{\eta}\,d\bar{\zeta} = \frac{e}{s}, \\[2mm] \psi_2(s) = \dfrac{1}{s}\displaystyle\iiint \bar{\rho}_0\,\frac{k}{B}\,d\bar{\xi}\,d\bar{\eta}\,d\bar{\zeta}, \end{cases}
$$

to abbreviate, in which e means the total charge of the electron, then we will easily get:

$$
(115) \qquad \begin{cases} 4\pi\,\Phi_x = \psi_1(s)\cdot x + \psi_2(s)\cdot ct, \\ 4\pi\,\Phi = \psi_2(s)\cdot x + \psi_1(s)\cdot ct. \end{cases}
$$

In this, ψ_1 and ψ_2 are functions of only the coupling of s with x and t.

In particular, those potentials will fulfill equation (87): lor $\Phi = 0$; since $\partial s / \partial x = 2x$, $\partial s / \partial t = -2c^2 t$, one will then have:

$$4\pi \frac{\partial \Phi_x}{\partial x} = \psi_1 + 2\psi_1' x^2 + 2\psi_2' ctx \ ,$$

$$4\pi \frac{\partial \Phi}{\partial t} = \psi_1 c - 2\psi_1' c^2 tx - 2\psi_1' c^3 t^2 \ ;$$

hence:

$$\text{lor}\ \Phi = \frac{\partial \Phi_x}{\partial x} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = \frac{1}{2\pi}(\psi_1 + s\,\psi_1') \ .$$

Now:

$$\psi_1' = -\frac{e}{s^2},$$

so:

(116) $$\text{lor}\ \Phi = 0.$$

We would like to write the potentials (115) in yet another way for later purposes. In order to do that, we imagine that, from (108), we have set:

$$x = -q\,\xi, \qquad t = \frac{p}{c^2}\xi, \qquad \text{hence} \quad s = \xi^2.$$

If we then introduce the following abbreviations:

(117) $$\begin{cases} 4\pi \bar{\Phi}_x = -\xi\,\psi_1 = -\dfrac{e}{\xi}, \\ 4\pi \bar{\Phi} = -\xi\,\psi_2 = -\dfrac{1}{\xi}\iiint \bar{\rho}_0 \dfrac{k}{B}\, d\bar{\xi}\, d\bar{\eta}\, d\bar{\zeta}, \end{cases}$$

in place of the ψ_1, ψ_2, then we can write:

(118) $$\begin{cases} \Phi_x = \quad q\,\bar{\Phi}_x - \dfrac{p}{c}\bar{\Phi}, \\ \Phi = -\dfrac{p}{c}\bar{\Phi}_x + q\bar{\Phi}, \end{cases}$$

instead of (115).

The functions Φ_x, Φ are then connected with the auxiliary functions $\bar{\Phi}_x$, $\bar{\Phi}$ by the same Lorentz transformation that transforms a rigid body to rest [pp. 21, (74)]. We will call $\bar{\Phi}_x$, $\bar{\Phi}$ the *rest potentials*. They are functions of only ξ, η, ζ that no longer depend upon p.

From the relations (118), we see that the electron must move with its field; the rest potential that is perceived by an observer that moves with the electron will depend upon only the rest coordinates ξ, η, ζ.

We would like to give the explicit expression for the scalar rest potential $\bar{\Phi}$:

$$4\pi\bar{\Phi}(\xi,\eta,\zeta) = -\frac{1}{\xi}\iiint\bar{\rho}_0\frac{1}{r}\frac{r^2+2\xi\bar{\xi}}{\sqrt{r^2+4\xi\bar{\xi}}}\,d\bar{\xi}\,d\bar{\eta}\,d\bar{\zeta}\,, \tag{119}$$

in which we have set:

$$r^2 = (\xi-\bar{\xi})^2+(\eta-\bar{\eta})^2+(\zeta-\bar{\zeta})^2. \tag{120}$$

If the reference point lies in the region $x/\,s < 0$ then we must choose the negative sign, instead of the positive one. $\bar{\Phi}_x$, as well as $\bar{\Phi}$, will be infinite for $\xi = 0$. The entire hyperbolic motion will be singular for that value. However, as we shall see, the field strengths will remain finite everywhere and will be defined in all of the *xyzt*-manifold.

§ 9. Field strengths under hyperbolic motion

We would now like to calculate the field strengths from the expressions (115) for the potentials by using formulas (85). Here, they will read:

$$\left\{\begin{aligned} &\mathfrak{E}_x = -\frac{1}{c}\frac{\partial\Phi_x}{\partial t}-\frac{\partial\Phi}{\partial x}, \quad \mathfrak{E}_y = -\frac{\partial\Phi}{\partial y}, \quad \mathfrak{E}_z = -\frac{\partial\Phi}{\partial z},\\ &\mathfrak{M}_x = 0, \qquad\qquad\quad \mathfrak{M}_y = \frac{\partial\Phi_x}{\partial z}, \quad \mathfrak{M}_z = -\frac{\partial\Phi_x}{\partial y}. \end{aligned}\right. \tag{121}$$

If we recall (110) then we will find from (115) that:

$$4\pi\frac{\partial\Phi_x}{\partial t} = c\;\psi_2 - 2\psi_1' c^2 t\,x - 2\psi_2' c^3 t^2,$$

$$4\pi\frac{\partial\Phi}{\partial x} = \quad\psi_2 + 2\psi_2' x^2 + 2\psi_1' c\,t\,x\,.$$

Hence:

$$\mathfrak{E}_x = -\frac{1}{2\pi}(\psi_2 + s\,\psi_2'). \tag{122}$$

It then follows from this that, other than η, ζ, $\mathfrak{E}_x$ depends upon only s (i.e., upon ξ) but not on p. *The z-component of the electric field strength is then constant along any world-line of the electron.*

If one calculates $\mathfrak{E}_x$ then that will give:

(123) $$\mathfrak{E}_x = -\frac{1}{\pi}\iiint \bar{\rho}_0 \frac{\bar{\xi}^2[r^2 - 2\xi(\xi - \bar{\xi})]}{r^3(r^2 + 4\xi\bar{\xi})^{3/2}} d\bar{\xi}\, d\bar{\eta}\, d\bar{\zeta}\,,$$

in which r is defined by (120).

$\mathfrak{E}_x$ will be infinite for $\xi = 0$. However, one can also continue $\mathfrak{E}_x$ along the lines $\xi = 0$; i.e., $x + ct = 0$ and $x - ct = 0$. In that, one must give the opposite sign to the denominator that is identical to B^3 in the region $x / s < 0$. Hence, one must imagine that equations (108) represent the hyperbolas in the xt-plane that are normal to the x-axis for real values of ξ and the hyperbolas that are normal to the t-axis for imaginary ξ. One can write the bracketed expressions in the numerator and denominator of $\mathfrak{E}_x$ in the form:

$$\bar{\xi}^2 - \xi^2 + (\eta - \bar{\eta})^2 + (\zeta - \bar{\zeta})^2,$$
$$[\xi^2 + \bar{\xi}^2 + (\eta - \bar{\eta})^2 + (\zeta - \bar{\zeta})^2]^2 - 4\xi^2\bar{\xi}^2,$$

resp. Only the square of ξ enters into them, such that they will also be real for pure imaginary values of ξ. Furthermore, the expression in the denominator can never be zero in the domain of integration – i.e., for $\bar{\xi}^2 > 0$. If one then sets $\xi = i\alpha$ then it will become:

$$[-\alpha^2 + \bar{\xi}^2 + (\eta - \bar{\eta})^2 + (\zeta - \bar{\zeta})^2]^2 + 4\alpha^2\,\bar{\xi}^2 > 0.$$

Hence, $\mathfrak{E}_x$ is defined in the entire xt-plane.

We would now like to calculate the remaining field components. They will be:

(124) $$\begin{cases} \mathfrak{E}_y = -\dfrac{\partial \Phi}{\partial y} = -\dfrac{x}{4\pi}\dfrac{\partial \psi_2}{\partial \eta} = q\dfrac{\xi}{4\pi}\dfrac{\partial \psi_2}{\partial \eta}, \\[2ex] \mathfrak{E}_z = -\dfrac{\partial \Phi}{\partial z} = -\dfrac{x}{4\pi}\dfrac{\partial \psi_2}{\partial \zeta} = q\dfrac{\xi}{4\pi}\dfrac{\partial \psi_2}{\partial \zeta}, \end{cases}$$

(125) $$\begin{cases} \mathfrak{M}_y = \dfrac{\partial \Phi_x}{\partial z} = \dfrac{ct}{4\pi}\dfrac{\partial \psi_2}{\partial \zeta} = \dfrac{p}{c}\dfrac{\xi}{4\pi}\dfrac{\partial \psi_2}{\partial \zeta}, \\[2ex] \mathfrak{M}_z = -\dfrac{\partial \Phi_x}{\partial y} = -\dfrac{ct}{4\pi}\dfrac{\partial \psi_2}{\partial \eta} = -\dfrac{p}{c}\dfrac{\xi}{4\pi}\dfrac{\partial \psi_2}{\partial \eta}. \end{cases}$$

Now, one easily finds that:

(126) $$\begin{cases} \dfrac{\partial \psi_2}{\partial \eta} = -8\iiint \bar{\rho}_0 \dfrac{\bar{\xi}^2(\eta - \bar{\eta})}{r^3(r^2 + 4\xi\bar{\xi})^{3/2}} d\bar{\xi}\, d\bar{\eta}\, d\bar{\zeta}, \\[2ex] \dfrac{\partial \psi_2}{\partial \zeta} = -8\iiint \bar{\rho}_0 \dfrac{\bar{\xi}^2(\zeta - \bar{\zeta})}{r^3(r^2 + 4\xi\bar{\xi})^{3/2}} d\bar{\xi}\, d\bar{\eta}\, d\bar{\zeta}. \end{cases}$$

Hence, along with η, ζ, the y and z-components of the field strengths will depend upon not merely the combination $\xi^2 = x^2 - c^2 t^2$, but also upon x and t explicitly. One can then regard them as functions of ξ, η, ζ, p.

The sign of the expressions (126) are once more switched in the regions where $x / s < 0$.

One sees that the y and z-components of the field strengths cannot become infinite for $\xi = 0$, either. We shall not go into a more precise discussion of the behavior of the field in the fixed coordinate system. However, one will see directly that $\mathfrak{E}_y$ and $\mathfrak{E}_z$ will vanish for $x = 0$, such that the force lines will be parallel to the x-axis, and that $\mathfrak{M}_y$ and $\mathfrak{M}_z$ will vanish for $t = 0$; i.e., at the moment where the electron turns around, and is therefore instantaneously at rest. One again sees from this that the field of the electron, in fact, moves with it, since the magnetic field will vanish everywhere instantaneously when the electron is at rest for a moment.

§ 10. Transformation of the wave equation, the potentials, and field strengths to a comoving coordinate system

The form (118) that we have given for the retarded potentials then leads to a transformation of the wave equation (86) itself to a coordinate system that moves with the electron; i.e., to the independent variables ξ, η, ζ, p.

In order to simplify the calculations, we transform the variational problem (88), instead of the differential equations (86).

We must then next transform the components of the field strengths.

From (79), we have:

$$\frac{1}{c}\frac{\partial \Phi_x}{\partial t} = \frac{1}{c}\frac{\partial \Phi_x}{\partial \xi}\xi_t + \frac{1}{c}\frac{\partial \Phi_x}{\partial p}p_t = -\frac{\partial \Phi_x}{\partial \xi}\frac{p}{c} + \frac{\partial \Phi_x}{\partial p}\frac{c\,q^2}{\xi},$$

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \Phi}{\partial \xi}\xi_x + \frac{\partial \Phi}{\partial p}p_x = -\frac{\partial \Phi}{\partial \xi}q + \frac{\partial \Phi}{\partial p}\frac{p\,q}{\xi}.$$

We now introduce the rest potentials $\bar{\Phi}_x$, $\bar{\Phi}$ by the same relations (118) as before:

$$(118)\qquad \begin{cases} \Phi_x = q\bar{\Phi}_x - \dfrac{p}{c}\bar{\Phi}, \\[2mm] \Phi = -\dfrac{p}{c}\bar{\Phi}_x + q\bar{\Phi}. \end{cases}$$

One then obviously has:

$$\frac{\partial \Phi_x}{\partial \xi} = q\frac{\partial \bar{\Phi}_x}{\partial \xi} - \frac{p}{c}\frac{\partial \bar{\Phi}}{\partial \xi},$$

$$\frac{\partial \Phi}{\partial \xi} = -\frac{p}{c}\frac{\partial \bar{\Phi}_x}{\partial \xi} + q\frac{\partial \bar{\Phi}}{\partial \xi},$$

and correspondingly for the derivatives with respect to η and ζ. By contrast:

$$\frac{\partial \Phi_x}{\partial p} = \frac{p}{c^2 q}\bar{\Phi}_x - \frac{1}{c}\bar{\Phi} + q\frac{\partial \bar{\Phi}_x}{\partial p} - \frac{p}{c}\frac{\partial \bar{\Phi}}{\partial p},$$
$$\frac{\partial \Phi}{\partial p} = -\frac{1}{c}\bar{\Phi}_x + \frac{p}{c^2 q}\bar{\Phi} - \frac{p}{c}\frac{\partial \bar{\Phi}_x}{\partial p} + q\frac{\partial \bar{\Phi}}{\partial p}.$$

Hence:

$$(127)\qquad \begin{cases} -\mathfrak{E}_x = \dfrac{1}{c}\dfrac{\partial \Phi_x}{\partial t} + \dfrac{\partial \Phi}{\partial x} = -\dfrac{\partial \bar{\Phi}}{\partial \xi} - \dfrac{1}{\xi}\left(\bar{\Phi} - cq\dfrac{\partial \bar{\Phi}_x}{\partial p}\right), \\ -\mathfrak{E}_y = \dfrac{\partial \Phi}{\partial y} = -\dfrac{p}{c}\dfrac{\partial \bar{\Phi}}{\partial \eta} + q\dfrac{\partial \bar{\Phi}}{\partial \eta}, \\ -\mathfrak{E}_z = \dfrac{\partial \Phi}{\partial z} = -\dfrac{p}{c}\dfrac{\partial \bar{\Phi}_x}{\partial \zeta} + q\dfrac{\partial \bar{\Phi}}{\partial \zeta}, \end{cases}$$

$$(128)\qquad \begin{cases} \mathfrak{M}_x = 0, \\ \mathfrak{M}_y = \dfrac{\partial \Phi_x}{\partial z} = q\dfrac{\partial \bar{\Phi}_x}{\partial \zeta} - \dfrac{p}{c}\dfrac{\partial \bar{\Phi}}{\partial \zeta}, \\ -\mathfrak{M}_z = \dfrac{\partial \Phi_x}{\partial y} = q\dfrac{\partial \bar{\Phi}_x}{\partial \eta} - \dfrac{p}{c}\dfrac{\partial \bar{\Phi}}{\partial \eta}. \end{cases}$$

That now gives:

$$(129)\qquad \mathfrak{M}^2 - \mathfrak{E}^2 = -\left[\frac{\partial \bar{\Phi}}{\partial \xi} + \frac{1}{\xi}\left(\bar{\Phi} - c\,q\frac{\partial \bar{\Phi}_x}{\partial p}\right)\right]^2 + \left(\frac{\partial \bar{\Phi}_x}{\partial \eta}\right)^2 - \left(\frac{\partial \bar{\Phi}}{\partial \eta}\right)^2 + \left(\frac{\partial \bar{\Phi}_x}{\partial \zeta}\right)^2 - \left(\frac{\partial \bar{\Phi}}{\partial \zeta}\right)^2.$$

Moreover:

$$(130)\qquad \frac{\rho^*}{c}(\Phi_x x_\tau + \Phi_y y_\tau + \Phi_z z_\tau - \Phi\, t_\tau) = -\rho^*\left(\frac{p}{c}\Phi_x + q\Phi\right) = -\rho^*\bar{\Phi}.$$

Finally, the functional determinant will become:

$$(131)\qquad \frac{\partial(x, y, z, t)}{\partial(\xi, \eta, \zeta, p)} = -\frac{\xi}{c^2 q}.$$

Hence, the variational problem (88) goes to the following one:

(132)
$$W = \iiint\int \left\{ \frac{\xi}{2c^2 q}\left(\left[\frac{\partial \bar{\Phi}}{\partial \xi} + \frac{1}{\xi}\left(\bar{\Phi} - cq \frac{\partial \bar{\Phi}_x}{\partial p} \right) \right]^2 - \left[\left(\frac{\partial \bar{\Phi}_x}{\partial \eta} \right)^2 - \left(\frac{\partial \bar{\Phi}}{\partial \eta} \right)^2 \right] - \left[\left(\frac{\partial \bar{\Phi}_x}{\partial \zeta} \right)^2 - \left(\frac{\partial \bar{\Phi}}{\partial \zeta} \right)^2 \right] \right) + \frac{\xi}{c^2 q}\rho^* \bar{\Phi} \right\} d\xi\, d\eta\, d\zeta\, dp = \text{min.}$$

That will give one the *differential equations of electrodynamics for a hyperbolically-accelerated reference system:*

(133)
$$\left\{ \begin{aligned} & \frac{1}{c}\frac{\partial}{\partial p}\left\{ \frac{\partial \bar{\Phi}}{\partial \xi} + \frac{1}{\xi}\left(\bar{\Phi} - cq\frac{\partial \bar{\Phi}_x}{\partial p} \right) \right\} - \frac{\xi}{q}\left(\frac{\partial^2 \bar{\Phi}_x}{\partial \eta^2} + \frac{\partial^2 \bar{\Phi}_x}{\partial \zeta^2} \right) = 0, \\ & \frac{\partial}{\partial \xi}\left\{ \xi \frac{\partial \bar{\Phi}}{\partial \xi} + \bar{\Phi} - cq\frac{\partial \bar{\Phi}_x}{\partial p} \right\} + \xi\left(\frac{\partial^2 \bar{\Phi}}{\partial \eta^2} + \frac{\partial^2 \bar{\Phi}}{\partial \zeta^2} \right) - \left\{ \frac{\partial \bar{\Phi}}{\partial \xi} + \frac{1}{\xi}\left(\bar{\Phi} - cq\frac{\partial \bar{\Phi}_x}{\partial p} \right) \right\} = \rho^* \xi. \end{aligned} \right.$$

The equations must necessarily be satisfied by the rest potentials (117) that were obtained in § 8, as well. However, they do not depend upon p, like the density $\rho^* = \rho_0(\xi, \eta, \zeta)$; they are then the "static potentials" relative to the accelerated coordinate system. By dropping the derivatives with respect to p, we will get *the differential equations of electrostatics in a hyperbolically-accelerated reference system:*

(134)
$$\left\{ \begin{aligned} & \frac{\partial^2 \bar{\Phi}_x}{\partial \eta^2} + \frac{\partial^2 \bar{\Phi}_x}{\partial \zeta^2} = 0, \\ & \frac{\partial}{\partial \xi}\left(\xi \frac{\partial \bar{\Phi}}{\partial \xi} \right) + \xi\left(\frac{\partial^2 \bar{\Phi}}{\partial \eta^2} + \frac{\partial^2 \bar{\Phi}}{\partial \zeta^2} \right) - \frac{\bar{\Phi}}{\xi} = \rho_0\, \xi. \end{aligned} \right.$$

In addition, however, the potentials must satisfy the equation lor $\Phi = 0$; under the transformation, that will go to:

(135)
$$\frac{\partial \bar{\Phi}_x}{\partial \xi} + \frac{1}{\xi}\left(\bar{\Phi}_x - qc\frac{\partial \bar{\Phi}}{\partial p} \right) = 0.$$

If $\bar{\Phi}_x$, $\bar{\Phi}$ are independent of p then that will become:

(136)
$$\frac{\partial \bar{\Phi}_x}{\partial \xi} + \frac{\bar{\Phi}_x}{\xi} = 0.$$

We would now like to show directly that the expressions (117) [(119), resp.] do, in fact, satisfy equations (134), (136).

For $\bar{\Phi}_x = -\frac{1}{4\pi}\frac{e}{\xi}$, that is clear from the outset; the first equation (134), as well as (136), will be satisfied.

We employ the explicit expression (119) for $\bar{\Phi}$. We will show that it is the precise analogue of the electrostatic potential of the given charges:

$$4\pi u\,(\xi,\,\eta,\,\zeta) = \iiint \frac{\bar{\rho}}{r}\,d\bar{\xi}\,d\bar{\eta}\,d\bar{\zeta}\,,$$

and that it has precisely the same relationship to the differential equation:

$$\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\bar{\Phi}}{\partial\zeta}\right)+\xi\left(\frac{\partial^2\bar{\Phi}}{\partial\eta^2}+\frac{\partial^2\bar{\Phi}}{\partial\zeta^2}\right)-\frac{\bar{\Phi}}{\xi} = f\,(\xi,\,\eta,\,\zeta) \tag{137}$$

that the ordinary potential u has to the equation:

$$\Delta u = f\,(\xi,\,\eta,\,\zeta)\,.$$

Namely, the function:

$$\frac{1}{r}\frac{1}{\xi\bar{\xi}}\frac{r^2+2\xi\bar{\xi}}{\sqrt{r^2+4\xi\bar{\xi}}},$$

which is symmetric in the two series of variables $\xi, \eta, \zeta;\ \bar{\xi}, \bar{\eta}, \bar{\zeta}$, is initially a solution ([1]) to the homogeneous equation (137) ($f = 0$), that corresponds to the solution $1 / r$ of $\Delta u = 0$. One can see that it actually satisfies the equation by a generally lengthy computation. Moreover, it has a singularity of order $1 / r$ for $r = 0$ (i.e., $\xi = \bar{\xi}$, $\eta = \bar{\eta}$, $\zeta = \bar{\zeta}$), and the factor of $1 / r$ will be equal to $1 / \bar{\xi}$ for $r = 0$. Generally, one must exclude the cases in which ξ or $\bar{\xi}$ are equal to zero then; naturally, that value is itself singular for the differential equation (137). Our basic solution now follows from this relationship in precisely the same way as it does in potential theory, in which the expression:

$$4\pi\,\bar{\Phi}(\xi,\eta,\zeta) = \iiint \frac{f(\bar{\xi},\bar{\eta},\bar{\zeta})}{r}\frac{1}{\xi\bar{\xi}}\frac{r^2+2\xi\bar{\xi}}{\sqrt{r^2+4\xi\bar{\xi}}}\,d\bar{\xi}\,d\bar{\eta}\,d\bar{\zeta}$$

satisfies the inhomogeneous equation (137) when f is a function that is defined for $\xi > 0$ or $\xi < 0$ (and naturally, only as long as ξ is non-zero). If one replaces f in this with its value $\rho_0\,\xi$ that corresponds to the comoving charge according to (134) then one will get

---

([1]) Just as $1 / r$ is for $\Delta u = 0$, it is the **Green** function of the differential equation (137) for infinite space, with the boundary condition that the solution must vanish at infinity and the derivative in any direction, when multiplied by r^2, must remain finite.

back to (119). The different signs in the different regions of the reference point then result from the argument that the Φ_x, Φ, which are composed from the $\bar{\Phi}_x$, $\bar{\Phi}$, should be retarded, not advanced potentials in the rest coordinate system. Hence, one can establish the rest potentials $\bar{\Phi}_x$, $\bar{\Phi}$ uniquely in an analogous way by the differential equations (134), (136) and their behavior at infinity, like the usual static potential. Hence, I would not like to go further into that.

If one now observes that $\bar{\Phi}_x$ does not depend upon p, η, ζ then one will get *the following expressions for the field strengths in terms of only the* $\bar{\Phi}$ from (127), (128):

$$\text{(138)} \qquad \left\{ \begin{aligned} \mathfrak{E}_x &= \frac{\partial \bar{\Phi}}{\partial \xi} + \frac{\bar{\Phi}}{\xi}, & \mathfrak{M}_x &= 0, \\ \mathfrak{E}_y &= -q \frac{\partial \bar{\Phi}}{\partial \eta}, & \mathfrak{M}_y &= -\frac{p}{c}\frac{\partial \bar{\Phi}}{\partial \zeta}, \\ \mathfrak{E}_z &= -q \frac{\partial \bar{\Phi}}{\partial \zeta}, & \mathfrak{M}_z &= \frac{p}{c}\frac{\partial \bar{\Phi}}{\partial \eta}. \end{aligned} \right.$$

One easily sees that these expressions are identical with (122), (124), (125).

With **Minkowski** ([1]), we will now introduce the *rest field strengths*, in addition to the rest potentials.

The *rest electric force* is defined by:

$$\text{(139)} \qquad \left\{ \begin{aligned} \bar{\mathfrak{E}}_x &= t_x \mathfrak{E}_x + \frac{1}{c}(y_\tau \mathfrak{M}_z - z_\tau \mathfrak{M}_y), \\ \bar{\mathfrak{E}}_y &= t_x \mathfrak{E}_y + \frac{1}{c}(z_\tau \mathfrak{M}_x - x_\tau \mathfrak{M}_z), \\ \bar{\mathfrak{E}}_z &= t_x \mathfrak{E}_z + \frac{1}{c}(x_\tau \mathfrak{M}_y - y_\tau \mathfrak{M}_x). \end{aligned} \right.$$

One adds the expression for the *rest electric work* to that:

$$\text{(139}^*\text{)} \qquad \bar{A} = x_\tau \mathfrak{E}_x + y_\tau \mathfrak{E}_y + z_\tau \mathfrak{E}_z .$$

Furthermore, the *rest magnetic force* is defined by:

([1]) **H. Minkowski**, *loc. cit.* (rem. 1, pp. 4, cf., pp. 29, *et seq.*).

(140)
$$\begin{cases} \bar{\mathfrak{M}}_x = t_\tau \mathfrak{M}_x - \dfrac{1}{c}(y_\tau \mathfrak{E}_z - z_\tau \mathfrak{E}_y), \\ \bar{\mathfrak{M}}_y = t_\tau \mathfrak{M}_y - \dfrac{1}{c}(z_\tau \mathfrak{E}_x - x_\tau \mathfrak{E}_z), \\ \bar{\mathfrak{M}}_z = t_\tau \mathfrak{M}_z - \dfrac{1}{c}(x_\tau \mathfrak{E}_y - y_\tau \mathfrak{E}_x), \end{cases}$$

and the *rest magnetic work:*

(140*)
$$\bar{B} = x_\tau \mathfrak{M}_x + y_\tau \mathfrak{M}_y + z_\tau \mathfrak{M}_z .$$

If one introduces the expressions (138) into this and imagines that one has set:

$$x_\tau = p, \qquad t_\tau = -q, \qquad y_\tau = z_\tau = 0$$

then one will get:

(141)
$$\begin{cases} \bar{\mathfrak{E}}_x = -q\left(\dfrac{\partial\bar{\Phi}}{\partial\xi} + \dfrac{\bar{\Phi}}{\xi}\right), & \bar{\mathfrak{M}}_x = 0, \\ \bar{\mathfrak{E}}_y = \dfrac{\partial\bar{\Phi}}{\partial\eta}, & \bar{\mathfrak{M}}_y = 0, \\ \bar{\mathfrak{E}}_z = \dfrac{\partial\bar{\Phi}}{\partial\zeta}, & \bar{\mathfrak{M}}_z = 0, \\ \bar{A} = p\left(\dfrac{\partial\bar{\Phi}}{\partial\xi} + \dfrac{\bar{\Phi}}{\xi}\right), & \bar{B} = 0. \end{cases}$$

The rest magnetic force and rest work are then identically zero, as expected. The rest electric force and work, however, can be derived from just the rest potential $\bar{\Phi}$. One finds the following explicit expressions for the rest electric force from the value (119) of $\bar{\Phi}$:

(142)
$$\begin{cases} \bar{\mathfrak{E}}_x = \dfrac{q}{\pi}\iiint \bar{\rho}_0 \dfrac{\bar{\xi}^2[r^2 - 2\xi(\xi - \bar{\xi})]}{r^3(r^2 + 4\xi\bar{\xi})^{3/2}} d\bar{\xi}\, d\bar{\eta}\, d\bar{\zeta}, \\ \bar{\mathfrak{E}}_y = -\dfrac{2}{\pi}\xi\iiint \bar{\rho}_0 \dfrac{\bar{\xi}^2(\eta - \bar{\eta})}{r^3(r^2 + 4\xi\bar{\xi})^{3/2}} d\bar{\xi}\, d\bar{\eta}\, d\bar{\zeta}, \\ \bar{\mathfrak{E}}_z = -\dfrac{1}{\pi}\xi\iiint \bar{\rho}_0 \dfrac{\bar{\xi}^2(\zeta - \bar{\zeta})}{r^3(r^2 + 4\xi\bar{\xi})^{3/2}} d\bar{\xi}\, d\bar{\eta}\, d\bar{\zeta}, \end{cases}$$

and the rest work $\bar{A}$ will emerge from $\bar{\mathfrak{E}}_x$ by switching q with $-p$. These expressions are true inside of the electron itself in any case.

If one compares the expressions (141) with equations (108) then one will see that the quantities:

$$\bar{\mathfrak{E}}_x\,,\ \bar{\mathfrak{E}}_y\,,\ \bar{\mathfrak{E}}_z\,,\ \frac{1}{c^2}\bar{A}$$

emerge from the quantities:

$$\frac{\partial\bar{\Phi}}{\partial\xi}+\frac{\bar{\Phi}}{\xi}\,,\qquad \frac{\partial\bar{\Phi}}{\partial\eta}+\frac{\bar{\Phi}}{\eta}\qquad \text{and } p,\ q,$$

which depend upon only ξ, η, ζ, in precisely the same that x, y, z, t emerge from ξ, η, ζ, and p, q. It follows from this that

$$\bar{\mathfrak{E}}_x\,,\ \bar{\mathfrak{E}}_y\,,\ \bar{\mathfrak{E}}_z\,,\ \frac{1}{c^2}\bar{A}$$

transform in precisely the same way as x, y, z, t ; i.e., as a space-time vector of the first kind.

CHAPTER THREE

The dynamics of rigid electrons under hyperbolic motion

§ 11. The resultant force and the equations of motion

It is known that the product of the rest density with the rest electric force that was defined in the previous paragraphs is referred to as the ponderomotive force of the field and is regarded as equivalent to the usual mechanical forces.

In **Abraham**'s theory of rigid electrons, as in **Lorentz**'s theory of the "deformable" electron, the equations of motion of an electron that is free of ordinary mass are then defined in the following way: By integrating over the space that is filled with an electron at a moment *t*, the resultant of that ponderomotive force with the external one from the force that is generated by the electron itself (the resultant moment, when rotations are considered with it, resp.), and the sum of those resultants will be set equal to zero.

Naturally, that process does not correspond to the path that we have chosen. The resultants thus-defined obviously depend upon the chosen reference system. *We shall seek to exhibit equations of motion that are invariant under Lorentz transformations.*

However, the form of the rest force [(141) and (142), and the remark on pp. 43 about its behavior under Lorentz transformations] that is generated by the electron itself is closely related to the way in which one defines the resultants in order for the equations of motion to be invariant under Lorentz transformations; i.e., in order for the resultants themselves to transform like space-time vectors of the first kind. *We will understand the resultant force of a force field to mean the integral of the product of a rest charge with a rest force over the rest form of the electron; i.e., over ξ, η, ζ for fixed p.*

We would not like to go into the details of how one defines the resultant moment in the case where rotations are allowed.

Furthermore, we will express the *equation of motion of a rigid electron* as follows: *The rigid electron moves in such a way that the resultant of its own field is equal and opposite to the result of the external fields.*

Before we actually calculate the resultant of the internal field for hyperbolic motion, we would like to make a remark on the laws of impulse and energy.

As is known, the following identity is true as a result of the fundamental equations of electromagnetism:

$$(143)\qquad \left\{\begin{aligned} &\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}-\frac{1}{c}\frac{\partial \mathfrak{E}_x}{\partial t}=\rho^*\overline{\mathfrak{E}}_x,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ &\frac{\partial \mathfrak{S}_x}{\partial x}+\frac{\partial \mathfrak{S}_y}{\partial y}+\frac{\partial \mathfrak{S}_z}{\partial z}+\frac{\partial W}{\partial t}=\rho^*\overline{A}. \end{aligned}\right.$$

In this:

$$
(144) \qquad \left\{ \begin{aligned} X_x &= \tfrac{1}{2}(\mathfrak{E}_x^2 - \mathfrak{E}_y^2 - \mathfrak{E}_z^2) + \tfrac{1}{2}(\mathfrak{M}_x^2 - \mathfrak{M}_y^2 - \mathfrak{M}_z^2), \\ X_y &= \mathfrak{E}_x\mathfrak{E}_y + \mathfrak{M}_x\mathfrak{M}_y, \\ X_z &= \mathfrak{E}_x\mathfrak{E}_z + \mathfrak{M}_x\mathfrak{M}_z, \end{aligned} \right.
$$

and two-times three corresponding quantities, are the components of the *Maxwell stresses*, and furthermore:

$$
(145) \qquad \mathfrak{S} = c\,[\mathfrak{E}\,\mathfrak{M}]
$$

is the *radiation vector,* while:

$$
(146) \qquad W = \tfrac{1}{2}(\mathfrak{E}^2 + \mathfrak{M}^2)
$$

is the *energy density.* If one integrates those equations over a space that is bounded by a closed surface then one will get the equations:

$$
(147) \qquad \left\{ \begin{aligned} &\iiint \rho^* \overline{\mathfrak{E}}_x\, dv = \iint T_x\, df - \frac{d}{dt}\iiint \frac{1}{c}\mathfrak{S}_x\, dv, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\iiint \rho^* \overline{A}\, dv = \iint \mathfrak{S}_n\, df - \frac{d}{dt}\iiint W\, dv, \end{aligned} \right.
$$

in which dv means integration over space, df means integration over the boundary, $\mathfrak{S}_n$ means the normal component of $\mathfrak{S}$, and:

$$
T_x = X_x \cos(n, x) + X_y \cos(n, x) + X_z \cos(n, x)
$$

means the normal component of the x-stress on the boundary. Those equations say that the resultant force that is defined in the old sense is equal to the decrease in the electromagnetic quantity of motion $(1 / c)\,\mathfrak{S}$ that one finds in a volume plus the total stress on the boundary of the volume, and that the work that is done by the forces will be equal to the decrease in the total electromagnetic energy plus the radiation that flows through the boundary.

We now first consider **Abraham**'s theory; in it, the ether is assumed to be the absolute rest system and the electron is assumed to be rigid in the older sense. The definition of the resultants by integrating over space for fixed t will be justified for it. If we restrict ourselves to the case of rectilinear translation then all points of the electron will have the same velocity w at a time t. The individual points of the electron will do no work relative to each other, so the forces that act between them cannot enter into consideration, and therefore one can regard the integrals of the force components and of the work over the volume of the electron as the resultant force components and total work, resp. Hence, no relativity principle of any sort is generally fulfilled.

In **Lorentz**'s theory, the electron is regarded as deformable under quasi-stationary motion, and indeed according to the same laws by which rigid, uniformly-moving body (in the sense of the theory that is proposed here) appears to be deformed in a rest system. If one defines the resultant forces, etc., here as the integrals for fixed t then one will get entirely different values according to which reference system one uses as a basis. Equations (143) are generally invariant; equations (143) are generally invariant; i.e., they keep their form when one subjects the coordinates to a Lorentz transformation, but only when one simultaneously transforms the quantities X_x, …, $\mathfrak{S}_x$, …, W in a certain way. That situation causes an energy and quantity of motion of deformation to apparently come about, which **Plank** and **Abraham** have commented upon. Thus, in the kinematics of the principle of relativity, the values of the resultant force, work, stress, and radiation that are defined by integrating at fixed t have no immediate meaning. On explains the contradictions in **Lorentz**'s theory immediately by the fact that this state of affairs is not observed.

We can say that the localization of energy and impulse in the ether in the older sense does not correspond to the principle of relativity. However, for the presentation of the equation of motion, we do not need the laws of energy and impulse in the stated form at all. Rather, the definition that was given on pp. 44 will suffice entirely. The law of energy is then combined with the three equations of motion as the statement that depends upon them that says that the sum of the works that are done by the resultants of the external and internal fields will always be equal to zero.

§ 12. The resultant internal force under hyperbolic motion

We now define the resultants of the internal rest forces from the expressions (141) or (142). If we now denote the spatial element $d\xi\, d\eta\, d\zeta$ by $d\omega$ then we will get:

$$(148)\qquad \left\{\begin{aligned}
K_x^{(i)} &= \int \rho_0\, \overline{\mathfrak{E}}_x\, d\omega = \frac{q}{\pi}\iint \rho_0\, \overline{\rho}_0\, \frac{\overline{\xi}^2[r^2 - 2\xi(\xi-\overline{\xi})]}{r^3(r^2+4\xi\overline{\xi})^{3/2}}\, d\omega\, d\overline{\omega},\\
K_y^{(i)} &= \int \rho_0\, \overline{\mathfrak{E}}_y\, d\omega = -\frac{2}{\pi}\iint \rho_0\, \overline{\rho}_0\, \frac{\overline{\xi}^2[r^2 - 2\xi(\eta-\overline{\eta})]}{r^3(r^2+4\xi\overline{\xi})^{3/2}}\, d\omega\, d\overline{\omega},\\
K_z^{(i)} &= \int \rho_0\, \overline{\mathfrak{E}}_z\, d\omega = -\frac{2}{\pi}\iint \rho_0\, \overline{\rho}_0\, \frac{\overline{\xi}^2[r^2 - 2\xi(\zeta-\overline{\zeta})]}{r^3(r^2+4\xi\overline{\xi})^{3/2}}\, d\omega\, d\overline{\omega}.
\end{aligned}\right.$$

The resultant work $K^{(i)}$ emerges from $K_x^{(i)}$ by switching q with $-p$.

Next, we consider the integral $K_x^{(i)}$. Here, we will introduce the coordinate of any point a of the electron in place of the coordinate ξ, which is computed from the point at which the asymptotes of the hyperbolic motion intersect, and then replace:

$$\begin{array}{lll} \xi & \text{with} & a + \xi, \\ \overline{\xi} & \text{with} & a + \overline{\xi}\,. \end{array}$$

We will likewise prove that the electron must have a center. *We will then choose that center to be the reference point.*

We must then examine the expression under the integral:

$$\frac{(a+\bar{\xi})^2[r^2-2(a+\xi)(\xi-\bar{\xi})]}{r^3[r^2+4(a+\xi)(a+\bar{\xi})]^{3/2}}.$$

Now, from equation (82), the magnitude of the acceleration of the center *a* is equal to:

$$b = \frac{c^2}{a}. \tag{149}$$

If we substitute that in the expression above then it will become the following function of acceleration:

$$f = \frac{(c^2+\bar{\xi}b)[br^2-2(c^2+\xi b)(\xi-\bar{\xi})]}{r^3[r^2b^2+4(c^2+\xi b)(c^2+\bar{\xi}b)]^{3/2}}. \tag{150}$$

If we denote this function of the two points $P(\xi, \eta, \zeta)$, $\bar{P}(\bar{\xi},\bar{\eta},\bar{\zeta})$ by $f(P,\bar{P})$ then f can be decomposed into a symmetric and an skew-symmetric part:

$$f(P,\bar{P}) = f_1(P,\bar{P}) + f_2(P,\bar{P}),$$

in which:

$$f_1 = \tfrac{1}{2}[f_1(P,\bar{P}) + f_2(P,\bar{P})] \qquad \text{is symmetric,}$$

$$f_2 = \tfrac{1}{2}[f_1(P,\bar{P}) - f_2(P,\bar{P})] \qquad \text{skew-symmetric.}$$

Now, it is clear that the integral:

$$\iint \rho_0\, \bar{\rho}_0\, f_2(P,\bar{P})\, d\omega\, d\bar{\omega}$$

vanishes identically.

As a result, we can restrict ourselves to examining f_1. That will imply:

$$f_1 = \frac{b}{2}\frac{r^2[(c^2+\xi b)^2+(c^2+\bar{\xi}b)^2]+2(\xi-\bar{\xi})(c^2+\xi b)(c^2+\bar{\xi}b)^2}{r^3[r^2b^2+4(c^2+\xi b)(c^2+\bar{\xi}b)]^{3/2}}. \tag{151}$$

With that, the six-fold integral of $K_x^{(i)}$ will be proportional to *b*. If we combine *b* with the factor *q* into $-b_x$, according to (80), and form the work done $K^{(i)}$, in which we combine *b* with *p* into $c^2 b_t$ then we can write:

$$\left\{\begin{aligned} K_x^{(i)} &= -\mu b_x, \\ K^{(i)} &= -c^2 \mu b_t. \end{aligned}\right. \tag{152}$$

In this, the rest mass μ is the following quantity, which depends upon only the magnitude b of the acceleration of the center a:

(153) $$\mu = \frac{1}{2\pi}\iint \frac{\rho\bar{\rho}}{r^3[r^2b^2+4(c^2+\xi b)(c^2+\bar{\xi}b)]^{3/2}}$$
$$\left\{r^2[(c^2+\xi b)^2+(c^2+\bar{\xi}b)^2]+2(\xi-\bar{\xi})^2(c^2+\xi b)^2(c^2+\bar{\xi}b)^2\right\}\,d\omega\,d\bar{\omega}.$$

Since b depends upon only the initial coordinate a of the center, it will be constant under any hyperbolic motion. *Hence, μ is a constant that depends upon only the form and charge distribution of the electron for any hyperbolic motion.*

One will get the equations of motion for the x-coordinate from (151) when one sets the sum of the internal force $K_x^{(i)}$ and the external force $K_x^{(e)}$ equal to zero; similarly, $K^{(i)} + K^{(e)} = 0$ is the expression for the energy equation. That will give:

(154) $$\begin{cases} \mu\, b_x = K_x^{(e)}, \\ \mu\, b_t = \dfrac{1}{c^2}K^{(e)}. \end{cases}$$

We must still show that we can give an external force field that is capable of maintaining a hyperbolic motion. One can get that from an electric force E_x that acts in the x-direction and is independent of position and time. Namely, from (139), the rest force is then:

$$\bar{E}_x = t_\tau\, E_x = -\, q\, E_x\,,$$

and likewise the rest work is:

$$\bar{A} = x_\tau\, E_x = p\, E_x\,,$$

and if E_x is constant then integrating $\rho_0\bar{E}_x$ and $\rho_0\bar{A}$ over ξ, η, ζ will yield simply:

$$K_x^{(e)} = -\, q\, e\, E_x\,,$$
$$K^{(e)} = \quad p\, e\, E_x\,.$$

That force can maintain the hyperbolic motion with the acceleration b when one chooses:

(155) $$E_x = \frac{\mu}{e}\,b\,.$$

If the external force field varies only slightly, such that we can regard it as constant inside of the electron, then it will generate a motion that deviates from a hyperbolic motion only slightly. If we were to also regard equations (154) as valid in this case then we would neglect the radiation.

For accelerations that vary only slightly, but are arbitrarily large, we will get the following equations of motion and energy:

$$\text{(156)} \qquad \begin{cases} \mu \dfrac{\partial^2 x}{\partial \tau^2} = t_\tau e E_x, \\ \mu \dfrac{\partial^2 t}{\partial \tau^2} = \dfrac{1}{c^2} x_\tau e E_x, \end{cases}$$

in which x and t refer to the center of the electron. *Those equations are invariant under Lorentz transformations and have the form of the mechanical equations of motion of a mass point.*

If one regards μ as a constant (which will be justified in the next paragraph), introduces the "ordinary" mass by the relation:

$$\text{(157)} \qquad m = \mu\, t_\tau = \frac{\mu}{\sqrt{1 - \dfrac{w^2}{c^2}}},$$

and replaces the derivatives with respect to τ with derivatives with respect to t then one will get:

$$\text{(158)} \qquad \begin{cases} \dfrac{\partial m\, w_x}{\partial t} = e\, E_x, \\ \dfrac{\partial m}{\partial t} \quad = \dfrac{1}{c^2} e\, E_x w_x. \end{cases}$$

The first of these equations is the equation of motion in a form that is analogous to **Newton**'s equations of the older mechanics, and the second one has the form of the energy equation. The quantity $c^2\, m$ then corresponds to the kinetic energy in the old mechanics. *The dependency of the mass m on the velocity will be given by the **Lorentz** formula (157).* What is more essential than that, even for ordinary (i.e., non-electromagnetic) mass, in the relation that is true for the new kinematics is the dependency of the rest mass μ upon the magnitude b of the acceleration in formula (153). We would like to study that dependency more closely in the following paragraphs.

Before that, we must still consider the y and z-components of the internal forces.

If we apply the same argument to $K_y^{(i)}$ that we did to $K_x^{(i)}$ by splitting the integrand into a symmetric and a skew-symmetric part then we will get:

$$\text{(159)} \qquad K_y^{(i)} = -\frac{2}{\pi} b \iint \rho_0\, \bar{\rho}_0 \frac{(\eta - \bar{\eta})(\xi - \bar{\xi})(c^2 + b\xi)(c^2 + b\bar{\xi})}{r^3 \{r^2 b^2 + 4(c^2 + b\xi)(c^2 + b\bar{\xi})\}} d\omega\, d\bar{\omega},$$

and an analogous expression will be true for $K_z^{(i)}$.

If one assumes that the acceleration b is small then:

$$\text{(160)} \qquad [K_y^{(i)}]_0 = -\frac{b}{4\pi c^2} \iint \rho_0\, \bar{\rho}_0 \frac{(\eta - \bar{\eta})(\xi - \bar{\xi})}{r^3} d\omega\, d\bar{\omega}.$$

We will now postulate that for vanishingly-small accelerations the electron will exert no lateral forces upon itself. Namely, if that were not the case then external lateral forces would be required under quasi-stationary translation in order to maintain the motion. However, that contradicts the observations with cathode and Becquerel rays, which move rectilinearly with no external lateral effects on them.

However, in order to have $[K_y^{(i)}]_0 = 0$, the charge distribution must be symmetric to one of the planes $\xi = 0$ or $\eta = 0$. One likewise sees that it must also be symmetric to one of the planes $\xi = 0$ or $\zeta = 0$, in order to have $[K_z^{(i)}]_0 = 0$.

Since the direction of motion is an arbitrary direction in the electron, moreover, the charge must be symmetrically-distributed with respect to each plane that goes through the center. Hence, it must be distributed in concentric layers about the center.

It then follows from the observed fact that no external lateral forces are necessary in order to maintain the quasi-stationary translation that the electron has a center in order for the charge to be distributed in concentric layers.

However, if that were true then we would get, with no further discussion, from (159) that $K_y^{(i)}$ can then vanish for arbitrary values of *b*, and the same thing would be true for $K_z^{(i)}$. Hence, we have the result:

The electron exerts no lateral force upon itself for arbitrary accelerated hyperbolic motions.

The missing part of the law of electrodynamical inertia is also derived from that then. With the same degree of approximation for which equations (156) and (158) are true for motions with weak accelerations, we can also carry over this result to those motions.

The result that one can conclude the existence of a center and the distribution of charge in concentric layers from the behavior of the electron under quasi-stationary translation is one more contribution to the establishment of the atomic picture of electricity. I do not believe that any other theory has given such a close coupling of atomism with the principles of dynamics.

§ 13. The electrodynamical rest mass

We would next like to calculate the value of the rest mass for quasi-stationary motions; i.e., for vanishingly-small values of *b*. If we set $b = 0$ in the expression (153) then it will go to:

$$\mu_0 = \frac{1}{8\pi c^2}\iint \frac{\rho_0\,\bar{\rho}_0}{r}\,d\omega\,d\bar{\omega} + \iint \rho_0\,\bar{\rho}_0\,\frac{(\xi-\bar{\xi})^2}{r^3}\,d\omega\,d\bar{\omega}.$$

The first of these two integrals is the *electrostatic energy of the electron:*

$$8\pi U = \iint \frac{\rho_0\,\bar{\rho}_0}{r}\,d\omega\,d\bar{\omega}. \tag{161}$$

Due to the central symmetry of the electron, the second integral can also be represented in the forms:

$$\iint \rho_0 \, \bar{\rho}_0 \frac{(\eta - \bar{\eta})^2}{r^3} d\omega d\bar{\varpi}$$

and

$$\iint \rho_0 \, \bar{\rho}_0 \frac{(\zeta - \bar{\zeta})^2}{r^3} d\omega d\bar{\varpi}.$$

If we add these three expressions then we will again obtain $4\pi U$. The second integral is then equal to $4\pi / 3\ U$, and we will get:

(162) $$\mu_0 = \frac{4}{3c^2} U$$

for the *rest mass under quasi-stationary motion.*

In particular, if the electron is a homogeneously-charged ball of radius R then that will give:

(163) $$\mu_0 = \frac{1}{5\pi} \frac{e^2}{Rc^2},$$

in which e means the total charge.

That expression agrees with the values that are given by all other theories ([1]).

If the motion is no longer quasi-stationary then one must employ the most general expression (153) for μ. One will then develop μ in powers of b:

(164) $$\mu = \mu_0 + b\,\mu_1 + b_2\,\mu_2 + \ldots$$

We shall now prove that the coefficient μ_1 of b is equal to zero. Namely, one finds the expression for it:

$$\mu_1 = -\frac{1}{16\pi c^2} \iint \rho_0 \, \bar{\rho}_0 \frac{(\xi + \bar{\xi})[r^2 + (\xi - \bar{\xi})^2]}{r^3} d\omega d\bar{\varpi}.$$

Now, since the charge of the electron is distributed in concentric layers, each system of values for $\xi, \eta, \zeta;\ \bar{\xi}, \bar{\eta}, \bar{\zeta}$ will correspond to another one $-\xi, \eta, \zeta;\ -\bar{\xi}, \bar{\eta}, \bar{\zeta}$ for which the integrand will assume the opposite value. It will then follow that $\mu_1 = 0$.

One further finds that:

(165) $$\mu_2 = -\frac{1}{32\pi c^2} \iiint \rho_0 \, \bar{\rho}_0 \left\{ 3r + 2\frac{\xi^2 + \bar{\xi}^2 + 6\xi\bar{\xi}}{r} + \frac{(\xi - \bar{\xi})^2 (3\xi^2 + 3\bar{\xi}^2 + 10\xi\bar{\xi})}{r^3} \right\} d\omega d\bar{\varpi}.$$

([1]) Cf., e.g., **M. Abraham**, *Theorie der Elektrizität*, 2nd ed., v. **2**, pp. 179, formula (117c). A different unit was chosen there; our formula (163) will go to **Abraham**'s:

$$\mu_0 = \frac{4}{5} \frac{e^2}{Rc^2}$$

when we replace e with $\sqrt{4\pi}\, e$.

This value is exceptionally small compared to the value of μ_0 ; hence, whereas this one converges to infinity for decreasing radius R of the electron, μ_2 converges to zero like R. Furthermore, μ_2 has the sixth power of the speed of light in its denominator. We can then say:

In the series for the rest mass:

(166) $$\mu = \mu_0 + b^2 \mu_2 + \dots,$$

the coefficient of μ_2 is so extraordinarily small compared to μ_0 that the term that is quadratic in the acceleration already cannot be noticed by any observation in any case.

Hence, one can regard the rest mass as constant in all practical cases. Its value is given by the expression (162) for μ_0 .

With that, the basic features of the dynamics of rectilinearly-moving electrons are given an electromagnetic basis. Naturally, the domain of validity will be extended directly by the argument that the rectilinear motions can be superimposed with any uniform translation in an arbitrary direction. That was indeed derived only from the transition from one coordinate system to another with the help of a Lorentz transformation, whereby our equations of motion will be invariant. *The theory then encompasses the deflection of electrons by electric fields that have any direction in relation to their velocities and cannot change too fast in space and time. By contrast, they are not immediately true for magnetic deviations.* However, one easily sees that the magnetic deviations will also be reproduced by the theory for quasi-stationary motions.

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