"Sur une solution du problème relative à l'équation $f(dx_1, dx_2, \ldots, dx_n) = 0$ à coefficients variables," C. R. Acad. Sci. Paris **140** (1905), 1579-1582.

On a solution to the Monge problem that relates to the equation $f(dx_1, dx_2, \ldots, dx_n)$ with variable coefficients

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1. – The problem comes down to finding (without quadrature) the functions $x_1, x_2, \ldots, x_{n+1}$ of one parameter that satisfy the differential equation:

$$(1) \qquad f(x_1, \ldots, x_{n+1}, q_2, \ldots, q_{n+1}) = 0\,, \qquad q_j = \frac{dx_j}{dx_1} \qquad (j = 2, 3, \ldots, n+1)\,.$$

Monge (*Mém. de l'Acad.*, 1784) gave the solution for $n = 2$. J. Serret (*Journ. de Liouville*, 1848) solved the problem when (1) reduced to the expression for the line element of our space. Some new and elegant formulas for that case were given by Darboux (*Journ. de Liouville*, 1887), with method for finding the solution when (1) reduces to a relation with constant coefficients in $q_2, \ldots, q_{n+1}$. For the more general case, quite recently, Zervos (*Comptes rendus*, 19 April 1905) exhibited a system of $n + 1$ equations ($n - 2$ of which are linear differential equations) that are necessary and sufficient conditions that the functions $x_1, \ldots, x_{n+1}$ must satisfy. Zervos started from the complete integral $V = 0$ of the first-order partial differential equation:

$$(2) \qquad F(x_1, \ldots, x_{n+1}, p_1, \ldots, p_{n+1}) = 0\,, \qquad p_j = \frac{\partial x_{n+1}}{\partial x_i} \qquad (i = 1, 2, \ldots, n)$$

that one obtains upon eliminating $q_2, \ldots, q_{n+1}$ from (1) and the relations:

$$-\frac{q_2\,Q_2 + \cdots + q_{n+1}\,Q_{n+1}}{p_1} = \frac{Q_2}{p_2} = \ldots = \frac{Q_n}{p_n} = \frac{Q_{n+1}}{-1}\,, \qquad Q_j = \frac{\partial f}{\partial q_j} \qquad (j = 2, 3, \ldots, n+1)\,.$$

What inspired me to write this note is that I have obtained my own method for solving the problem *in the general case*.

2. – I first showed that the problem (as well as for $n = 2$) is equivalent to the problem of finding (without quadrature) all of the *integral lines* (or envelopes of characteristics) of (2) with the equation $V = 0$.

From a general proposition that I have established in the theory of envelopes in a hyperspace, I then deduced the theorem:

In order for a simply-infinite family Σ *of characteristics of* (2) *to admit an envelope, except for the singular integral, it is* ***necessary and sufficient*** *that one of the following conditions must be fulfilled:*

I. *The equations:*

$$V(x_1, \ldots, x_{n+1}, a_1, \ldots, a_n) = 0 , \qquad \sum_{i=1}^{n} \frac{\partial V}{\partial a_i} a_i' = 0 , \tag{3}$$

$$\frac{\partial V}{\partial a_i} b_h + \frac{\partial V}{\partial a_h} = 0 , \qquad b_h' + \sum_{i=1}^{n} \frac{\partial}{\partial a_i} \left(\frac{\partial V}{\partial a_h} : \frac{\partial V}{\partial a_i} \right) a_i' = 0 \qquad (h = 2, 3, \ldots, n-1) \tag{4}$$

are both verified for the values $a_i(t)$ *and* $b_h(t)$ *that define* Σ, *provided that those values do not identify all of the n equations in the first column identically for any t* ([1]).

II. Σ *belongs to a general integral without belonging to any complete integral of* (2).

III. Σ *belongs to* $n - 1$ *general integrals* Π_s ($s = 0, 1, \ldots, n-2$) *that are each obtained by eliminating the* a_i *from the equations:*

$$V = 0 , \qquad \varphi_s(a_1, \ldots, a_n) = 0 , \qquad \frac{D(V, \varphi_s)}{D(a_1, a_h)} = 0 \qquad (h = 2, 3, \ldots, n) ,$$

in which the $\varphi_s = 0$ *are* $n - 1$ *hypersurfaces in the space of* $a_1, \ldots, a_n$ *that intersect along the line that is the image of the value of* a_i *that defined* Σ *and nowhere else.*

3. – If the $\varphi_s = 0$ verify the latter condition for a simply-infinite family Γ_1 of values of the a_i, and if one supposes, to simplify, that they are both satisfied only for another simple family Γ_2 along which $\varphi_0 = 0$ and $\varphi_1 = 0$, for example, do not intersect. I have shown that one will find the expressions $x_i(a_1)$ for the coordinates of an integral curve of (2) by solving the equations:

$$V = 0 , \qquad \varphi_s = 0 \qquad (s = 0 , 1, \ldots, n-2) , \quad \frac{D(V, \varphi_0)}{D(a_1, a_h)} = 0 \qquad (h = 2, 3, \ldots, n) ,$$

([1]) The set of formulas (3) and (4) from which one deduces the result is basically the same as those of Zervos.

$$\frac{D\left[\frac{D(V,\varphi_0)}{D(a_1,a_2)},\cdots,\frac{D(V,\varphi_0)}{D(a_1,a_n)},\varphi_0+\lambda\,\varphi_1\right]}{D(a_1,\ldots,a_{n-1},a_n)} = 0$$

for the x_i after eliminating the $a_2, \ldots, a_n$ and ultimately setting $\lambda = 0$ in the expressions that are found. When Γ_2 does not exist, one can set $\lambda = 0$ before solving.

4. – From Theorems II and III, the solution to the proposed problem is obtained by seeking the envelope of an arbitrary simply-infinite family Σ of characteristics that belongs to an arbitrary general integral Π_0 of the indicated type. In the case of no. 3, that solution will come down to finding $n - 2$ functions φ_m $(a_1, \ldots, a_n)$ $(m = 1, 2, \ldots, n - 2)$ such that the hypersurfaces $\varphi_s = 0$ $(s = 0, 1, \ldots, n - 2)$ intersect all along an arbitrary simply-infinite family Γ_1 (among the ones that verify the arbitrarily-given equation $\varphi_0 = 0$), and are not all satisfied outside of Γ_1 for another simply-infinite family Γ_2 along which two of those $\varphi_s = 0$ do not intersect.

5. – In order to indicate a general path to follow in the solution of the latter question, I shall suppose that the family Γ_1 is given as the set of all values that verify all of the $n - 1$ independent relations:

$$\varphi_0\,(a_1, \ldots, a_n) = 0\,, \qquad \chi_r\,(a_1, \ldots, a_n) = 0 \qquad (r = 1, 2, \ldots, n - 2)\,.$$

Moreover, to simplify, I shall assume (which is the general case, moreover) that there exists a group $a_1^0, \ldots, a_n^0$ in Γ_1 for which one has:

$$\left(\frac{\partial\varphi_0}{\partial a_j}\right)_0 \neq 0 \qquad\qquad (j = 1, 2, \ldots, n - 1)\,.$$

One can find one of the functions φ_m as the left-hand side of the equation that one obtains by eliminating $\alpha_1, \ldots, \alpha_{n-1}$ from the equations:

$$\varphi_0\,(a_1, \ldots, a_n) = \varphi_0\,(\xi_{k1}, \ldots, \xi_{kn})\,, \qquad \chi_r\,(a_1, \ldots, a_n) = \chi_r\,(\xi_{k1}, \ldots, \xi_{kn}) \qquad (r = 1, 2, \ldots, n - 2)\,,$$

when one sets:

$$\xi_{kn} = a_n^0\,, \qquad \xi_{kn-1} = \sum_{k=1}^{n-1}\left(\frac{\partial\varphi_0}{\partial a_h}\right)_0 \frac{\alpha_r^k - \alpha_{0r}^k}{k\,\alpha_{0r}^{k-1}} + a_{n-1}^0\,, \qquad \xi_{kr} = \left(\frac{\partial\varphi_0}{\partial a_{n-1}}\right)_0 \frac{\alpha_{0r}^k - \alpha_r^k}{k\,\alpha_{0r}^{k-1}} + a_r^0$$

$$(r = 1, 2, \ldots, n - 2)\,,$$

in which the α_{0r} are non-zero constants, k is a positive integer, and the a_i^0 (with the restriction that was posed above) verify the equations:

$$\varphi_0(a_1^0,\ldots,a_n^0) = 0\,, \qquad \chi_r(a_1^0,\ldots,a_n^0) = 0 \qquad (r = 1, 2, \ldots, n-2)\,.$$

In general, it will suffice to take $k = 1, 2, \ldots, n - 2$ in order to obtain the $\varphi_m = 0$ that verify all the conditions, along with $\varphi_0 = 0$.

The $\varphi_0\,(a_1, \ldots, a_n)$, $\chi_r\,(a_1, \ldots, a_n)$ are essentially the $n - 1$ arbitrary functions that enter (if necessary) into the solution that we just indicated for the proposed problem.