Schrödinger's wave mechanics: a general method of solution by successive approximations.

Note by **LÉON BRILLOUIN**, presented by Marcel Brillouin

Translated by D. H. Delphenich

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1. Schrödinger, while developing the ideas of L. de Broglie, has recently specified the major features of an atomic wave mechanics ([1]). Suppose one has an atomic system whose potential energy is $V(q^1, \ldots, q^n)$, while its kinetic energy T has the expression:

$$(1) \qquad 2T = \sum_{k,l} m_{kl}\, \dot{q}^k\, \dot{q}^l = \sum_{k,l} m^{kl} p_k\, p_l \, .$$

The m^{kl} are functions of the coordinates $q^k, \ldots,$ and we let m denote the determinant of the m^{kl}. The classical Hamilton equation is written:

$$(2) \qquad \sum_{k,l} m^{kl} \frac{\partial W_0}{\partial q^k} \frac{\partial W_0}{\partial q^l} + 2\,(V - E) = 0,$$

in which E represents the energy constant. Schrödinger arrived at the following general equation:

$$(3) \qquad m^{1/2} \sum_{k,l} \frac{\partial}{\partial q^k} \left\{ m^{-1/2} m^{kl} \frac{\partial \psi}{\partial q^l} \right\} - \frac{8\pi^2}{h^2} + 2\,(V - E)\,\psi = 0,$$

in which h is Planck's constant. The quantized energy levels E are the proper values of that equation; i.e., the ones for which one can find a continuous function ψ that is finite and uniform in the entire extension-in-phase q.

2. I would like to show that equation (3) can be solved by successive approximations, where the first approximation recovers the old quantum mechanics. I set:

$$(4) \qquad \psi = e^{2\pi i W / h} \, ;$$

equation (3) then gives:

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([1]) L. DE BROGLIE, *Thesis*. E. SCHRÖDINGER, Ann. Phys. (Leipzig) **79** (1926), 361-376, 489-527, 734-756. Equation (3) above is found in Schrödinger, pp. 310, equation (18) and pp. 748, equation (31).

(5) $$\sum_{k,l} m^{kl} \frac{\partial W_0}{\partial q^k} \frac{\partial W_0}{\partial q^l} + 2\,(V-E) = -\frac{h}{2\pi i}\, m^{1/2} \sum_{k,l} \frac{\partial}{\partial q^k}\left\{ m^{-1/2} m^{kl} \frac{\partial W}{\partial q^l} \right\}.$$

This equation differs from Hamilton's (2) by the adjunction of the right-hand side, which is very small, due to the value of h. In order for the function ψ to have only one value at each point, while W is a multi-valued function, it is necessary that W must have residues that are integer multiples of h:

(6) $$I_k = n_k\, h.$$

Starting from a point in the extension-in-phase that has a value W and traversing an arbitrary closed circuit, one will determine a value for $W + Nh$ that gives back the same value of ψ. These are the usual quantization conditions, when they are interpreted as resonance conditions, in the sense of L. de Broglie.

3. Equation (5) can be solved by successive approximations by setting:

(7) $$W = W_0 - \frac{h}{2\pi i}\, W_{(1)} + \ldots + \left(\frac{-h}{2\pi i}\right)^n W_{(n)} + \ldots$$

The first approximation is given by the classical Hamilton equation (2). The ones that follow are:

(8) $$2\sum_{k,l} m^{kl} \frac{\partial W_0}{\partial q^k} \frac{\partial W_{(1)}}{\partial q^l} = m^{1/2} \sum_{k,l} \frac{\partial}{\partial q^k}\left\{ m^{-1/2} m^{kl} \frac{\partial W_0}{\partial q^l} \right\}.$$

(9) $$2\sum_{k,l} m^{kl} \frac{\partial W_0}{\partial q^k} \frac{\partial W_{(n)}}{\partial q^l} = F\,(W_0,\, W_{(1)},\, \ldots,\, W_{(n-1)}).$$

The expression F contains the derivatives of functions that are known from the preceding approximations.

We then find that the first approximation is Hamilton's equation, combined with the conditions (6); i.e., the old quantum mechanics. The final approximations involve only linear equations and constitute the novelty in Schrödinger's mechanics.

4. When the variables are separated in Hamilton's function, the first approximation is solved by quadratures that pertain to each variable individually; however, the final approximations will establish a coupling between those variables because one cannot recover such a separation, in general.

There will be complete separation if V and T are presented as sums of terms that each include one variable:

(10) $\qquad V = \sum_k V_{|k|}(q^k)$ and $2T = \sum_k m^{kk}(q^k)\, p_k^2$ with $\qquad m^{kl} = 0\ (k \neq l).$

One will then have:
$$m = |\,m^{kl}\,| = m^{11}\, m^{22} \ldots m^{kk} \ldots m^{nn}.$$

We seek a solution of the form:

(12) $\qquad\qquad\qquad\qquad W = \sum_k u_{|k|}(q^k),$

and we will obtain separated equations of the following type:

(13) $\qquad m^{kk}\left(\dfrac{\partial u_{|k|}}{\partial q^k}\right)^2 + 2V_{|k|} - 2\alpha_{|k|} = -\dfrac{h}{2\pi i}\sqrt{m^{kk}}\,\dfrac{\partial}{\partial q^k}\sqrt{m^{kk}}\,\dfrac{\partial u_{|k|}}{\partial q^k},$

with the auxiliary condition:
$$\sum_k \alpha_{|k|} = E.$$

These equations are solved easily by successive approximations using simple quadratures.

Schrödinger's mechanics thus admits the old quantum mechanics as a first approximation, but generally establishes supplementary couplings between the variables.

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