"Sulle linee di curvatura della superficie delle onde," Ann. mat. pura ed appl. **2** (1859), 135-136.

On the lines of curvature of the wave surface.

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Translated by D. H. Delphenich

Let l, m, n, φ be functions of two independent parameters u, v. If one considers the enveloping surface of the planes:

(1) $$lx + my + nz = \varphi$$

then one will easily find that the lines u = const., v = const. will be conjugate tangents for that surface if:

(2) $$\begin{vmatrix} l & m & n & \varphi \\ \dfrac{dl}{du} & \dfrac{dm}{du} & \dfrac{dn}{du} & \dfrac{d\varphi}{du} \\ \dfrac{dl}{dv} & \dfrac{dm}{dv} & \dfrac{dn}{dv} & \dfrac{d\varphi}{dv} \\ \dfrac{d^2 l}{du\,dv} & \dfrac{d^2 m}{du\,dv} & \dfrac{d^2 n}{du\,dv} & \dfrac{d^2 \varphi}{du\,dv} \end{vmatrix} = 0,$$

and that the lines of one family will be orthogonal to those of the other if:

(3) $$(l^2 + m^2 + n^2)\left(\frac{dl}{du}\frac{dl}{dv} + \frac{dm}{du}\frac{dm}{dv} + \frac{dn}{du}\frac{dn}{dv}\right)$$
$$= \left(l\frac{dl}{du} + m\frac{dm}{du} + n\frac{dn}{du}\right)\left(l\frac{dl}{dv} + m\frac{dm}{dv} + n\frac{dn}{dv}\right).$$

Thus, the lines u = const., v = const. will be lines of curvature of the surface that is enveloped by the planes (1) when the l, m, n, φ satisfy equations (2), (3).

If one supposes that:

(4) $$f^2 + m^2 + n^2 = 0, \qquad \varphi = u$$

then one will get:

$$l\frac{dl}{du} + m\frac{dm}{du} + n\frac{dn}{du}, \quad l\frac{dl}{dv} + m\frac{dm}{dv} + n\frac{dn}{dv}, \quad \frac{d\varphi}{dv} = 0,$$

so equations (2), (3) will become:

$$\begin{vmatrix} \dfrac{dm}{dv} & \dfrac{dn}{dv} \\ \dfrac{d^2m}{du\,dv} & \dfrac{d^2n}{du\,dv} \end{vmatrix} = 0, \qquad \frac{dl}{du}\frac{dl}{dv} + \frac{dm}{du}\frac{dm}{dv} + \frac{dn}{du}\frac{dn}{dv} = 0;$$

namely, if l, m, n, φ satisfy equations (4) then the necessary and sufficient conditions for the lines u = const., v = const. to be the lines of curvature will be:

$$\frac{1}{dl/dv}\frac{d^2l}{du\,dv} = \frac{1}{dm/dv}\frac{d^2m}{du\,dv} = \frac{1}{dn/dv}\frac{d^2n}{du\,dv}. \tag{5}$$

Now, it is known (Lamé, *Leçons sur la Théorie mathématique*, etc., page 243) that if one supposes that:

$$l^2 = \frac{(a^2-u^2)(a^2-v^2)}{(a^2-b^2)(a^2-c^2)}, \quad m^2 = \frac{(b^2-u^2)(b^2-v^2)}{(b^2-a^2)(b^2-c^2)}, \quad n^2 = \frac{(c^2-u^2)(c^2-v^2)}{(c^2-a^2)(c^2-b^2)}, \quad \varphi = u \tag{6}$$

then the enveloping surface will be the wave surface. However, those values will obviously not satisfy equations (5), so the lines u = const., v = const. will not be lines of curvature for that surface. Rather, the values (6) will satisfy:

$$\frac{dl}{du}\frac{dl}{dv} + \frac{dm}{du}\frac{dm}{dv} + \frac{dn}{du}\frac{dn}{dv} = 0;$$

i.e., the lines u = const., v = const. will be orthogonal.

ZECH – "Die Krummungslinie der Wellenflache zweiaxiger Krystalle," Crelle's Journal **54**.
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