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Proof of Jordan's theorem for an *n*-dimensional space

By

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The theorem to be proved reads as follows:

A Jordan manifold in n-dimensional space R_n – i.e., the one-to-one and continuous image of a closed $(n-1)$*-dimensional manifold – determines two regions and is identical with the boundary of each of these regions.*

We shall denote the Jordan manifold by J and (analogous to what was previously done in the plane *)) divide the theorem into three parts:

1. *The boundary of a region that is determined by J is identical with J*.
2. *J determines at most two regions.*
3. *J determines at least two regions.*

As a result, the first part is included in the results of § 6 of my previous paper, while the third part can be deduced by a method that Lebesgue sketched out **)). The still-remaining second part shall be disposed of in what follows.

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*) Math. Ann. **69**, pp. 169-175.

**) C. R. 27 March 1911. The result that was obtained there, together with the argument of Baire in Bull. des Sc. Math. (2), 31, yields a second proof of the invariance of the *n*-dimensional region.

For those readers whose interest pertains to three-dimensional spaces, I will carry out a very simple proof of the third part of Jordan's theorem that is only valid for $n = 3$.

Let j_1 and j_2 be two closed, continuous curves (in the sense of Schoenflies) in R_3, so we can regard the entire gamut of j_1 (j_2, resp.) as the single-valued and continuous image β_1 (β_2, resp.) of a circle in an infinitude of ways. For a certain choice of β_1 and β_2, the distance between two corresponding points of j_1 and j_2 possesses a maximum M; the lower limit of M that emerges upon varying β_1 and β_2 shall be called the *parametric distance from* j_1 *to* j_2.

When we construct a finite sequence of closed, continuous curves in which j is the first one and an isolated point is the last element, in such a way that the maximum of the parametric distance between two successive elements possesses the value ε, then we will say that j *is contracted with the degree of discontinuity* ε.

We shall briefly call a finite set of closed, continuous curves a *curve system.*

Now, let J' be a Jordan surface in R_3 and let κ be a sphere described around one of its points such that one of part of J' is contained in its exterior and only two-sided sub-regions of J' are included in its interior.

§ 1.

Let E be the infinite region that is determined by J, let I be a finite region that is determined by J, and let P be a point of I. We choose an arbitrary element J' from J, denote the point set that is defined by the remaining elements by J'', the periphery (Umfang) of J' by j, the representative simplex (cf., Math. Ann. **71**, pp. 100) of J' by S and the periphery of S by s, choose a positive sense of the indicatrix in J' and correspondingly (cf., *loc. cit.*, pp 108) in j, connect P and $J'' - j$ inside of I by a path w', and connect $J' - j$ and $J'' - j$ inside of E by a path w_e.

We denote the set of those points of J' that possess a distance $\geq \sqrt{n}/2^{\tau-1}$ from J'' by J'_τ.

We decompose the R_n into homothetic, n-dimensional cubes $\mathfrak{q}_0$ with the edge length 1, each of the $\mathfrak{q}_0$ into 2^n homothetic sub-cubes $\mathfrak{q}_1$ with edge length 1/2, each of the $\mathfrak{q}_1$ into 2^n homothetic sub-cubes $\mathfrak{q}_2$ with edge length 1/4, etc.

We let μ_τ denote the point sets that are defined by the $\mathfrak{q}_\tau$ that contain at least one point of J'_τ in their interior or their periphery, and the set of the points that belong to μ_τ, $\mu_{\tau+1}$, $\mu_{\tau+2}$, … by π_τ; we choose τ to be sufficiently large that w' will *not* meet π_τ. We denote the region that is determined by $J + \pi$ and include P by I_π and the part of the boundary of I_π that is not included in J'' by g.

We let E_τ denote those of the regions that $J'' + g$ determine in which E is included and draw a path w'' in E_τ from E to g. The endpoint R of this path lies in a certain two-sided $(n - 1)$-dimensional pseudo-manifold (cf., my previous paper, pp. 305) γ that belongs to the boundary of E_τ and has planar elements and a boundary that is contained in j *). Thus, two $(n - 2)$-dimensional element sides of γ that coincide in R_n will also be regarded

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By a suitable inversion of R_3, κ goes to a plane k and J' goes to a Jordan surface J. Let $\mathfrak{G}$ be a two-sided region in J that is cut out by k, let γ be the boundary of $\mathfrak{G}$, let $\mathfrak{S}$ be polygonal system that approximates γ to a distance ε_1, and let S be a curve system that lies in k and possesses the parametric distance ε_2 from $\mathfrak{S}$. When we let ε_1, along with ε_2, converge to zero there exists a region G in k that will run through S a non-vanishing number of times c for a sufficiently small ε_1.

In the contrary case, we can, in fact, contract the curve system S in k to a distance $\leq \varepsilon_3$ from γ with the degree of discontinuity ε_4, and on the basis of that, contract the curve system $\mathfrak{S}$ in J to a distance $\leq \varepsilon_5$ from γ with the degree of discontinuity ε_6, where ε_3, ε_4, ε_5, and ε_6, along with ε_1, converge to zero. However, this contradicts the definition of $\mathfrak{S}$.

An altitude l that is erected at a point of G on k will run through S, as well as $\mathfrak{S}$, c times, such that the difference between the numbers of positive and negative crossings of l with an arbitrary sufficiently precise simplicial approximation to $\mathfrak{G}$ is equal to $\pm c$. Thus, we can determine a sub-segment of l that does not enter $J - \mathfrak{G}$ and is bounded by endpoints Q_1 and Q_2 that do not lie in J, for which the difference between the numbers of positive and negative crossings *is not equal to zero* for an arbitrary, sufficiently precise, simplicial approximation to $\mathfrak{G}$. However, an arbitrary segment that connects Q_1 to Q_2 and an arbitrary, sufficiently precise, simplicial approximation to J must necessarily meet, *such that Q_1 and Q_2 will be separated by J.*

*) We understand the boundary of γ to mean the limit points of γ that are not contained in γ. Whether or not such limit points exist will still remain undecided in these paragraphs.

as identical for γ when and only when the two corresponding elements subtend an angle that belongs to E_τ.

We may assume that the point R *does not* belong to an $(n - 2)$-dimensional element of γ.

We choose a positive sense of the indicatrix in g, draw a path w'' in I_τ from P to R that does not meet w', and denote the segment that is defined by w' and w'' by w_{ir}.

The side of γ that belongs to E_τ shall be called its *left* side, while the other one shall be called its *right* side. In turn, the path w_{ir} in I connects the right side of g with $J'' - j$, while the path w''' possesses an end segment w_{il} that connects the left side of g of $J' - j$ in I.

By means of two path segments v' and v'' that lie in an arbitrary vicinity of J' (J'', resp.) and do not meet j, we can extend the paths w_{ir}, w_{il}, and w_e to a γ only at a single crossing point, namely, the points of a polygon w that meets R.

§ 2.

We understand a p-dimensional *net* (*net fragment*, resp.) in R_n to mean the simplicial image of a p-dimensional pseudo-manifold (a p-dimensional fragment, resp.; cf., my previous paper, pp. 306).

We understand the *basic simplexes*, *basic points*, and *basic sides* of a net (net fragment, resp.) to mean the images of the basic simplexes, basic points, and basic sides of the corresponding pseudo-manifold (corresponding fragment, resp.).

If a polygon $\mathfrak{P}$ in R_n is provided with a positive sense of traversal and a closed, two-sided, $(n - 1)$-dimensional net $\mathfrak{N}$ is provided with a positive indicatrix in such a way that no corner point and no sub-segment of $\mathfrak{P}$ lies in $\mathfrak{N}$ and no side of $\mathfrak{P}$ meets an $(n - 2)$-dimensional basic side of $\mathfrak{N}$ then the numbers of positive and negative crossings of $\mathfrak{P}$ with $\mathfrak{N}$ are equal to each other. [*])

For the case in which $\mathfrak{N}$ is one-sided, one can only say that the absolute value of the crossings of $\mathfrak{P}$ and $\mathfrak{N}$ is even.

We now think of a closed, two-sided $(n - 2)$-dimensional net $\mathfrak{R}$ in R_n that is provided with a positive indicatrix and a polygon $\mathfrak{P}$ that is provided with a positive sense of traversal and does not meet $\mathfrak{R}$.

Let $\mathfrak{G}$ be a two-sided, $(n - 1)$-dimensional net fragment that is provided with a positive indicatrix and which possesses $\mathfrak{R}$ as its only boundary and possesses the positive

[*]) Let f be the end point of the crossing polygon sides for a positive sense of traversal of $\mathfrak{P}$, and let i be a positive indicatrix of the crossed basic simplexes of $\mathfrak{N}$. In the event that if represents a positive indicatrix for the R_n, the cross is called positive. In order to realize the property stated in the text, one needs only to let $\mathfrak{P}$ go to infinity in such a way that the paths of the points of $\mathfrak{P}$ meet no $(n - 3)$-dimensional basic side, and especially such that the paths of the corner points of $\mathfrak{P}$ meet no $(n - 2)$-dimensional basic sides of $\mathfrak{N}$.

indicatrix of $\mathfrak{R}$ as the positive boundary indicatrix [**]), while no corner point and no sub-segment of $\mathfrak{P}$ lies in $\mathfrak{C}$ and no side of $\mathfrak{P}$ meets an $(n-2)$-dimensional basic side of $\mathfrak{C}$. If we then let p denote the number of positive crossings of $\mathfrak{P}$ and $\mathfrak{C}$ and let p' denote the number of negative crossings then for a given $\mathfrak{P}$ and $\mathfrak{R}$ the number $c = p - p'$ is independent of the choice of $\mathfrak{C}$.

The number c thus represents a relationship between $\mathfrak{P}$ and $\mathfrak{R}$. We call it the *degree of* $\mathfrak{R}$ *relative to* $\mathfrak{P}$.

§ 3.

Those elements of γ that possess a distance $\geq 1/2^{\nu}$ from j define an $(n-1)$-dimensional fragment that we denote by γ_ν and whose limit η_ν converges uniformly to j for ν increasing without bound.

Along with J, we take a fundamental sequence $z_1, z_2, \ldots$ of simplicial decompositions such that as ν increases without bound the width of the basic simplex associated with z_ν falls below any limit, as long as any $z_{\nu+1}$ is a subdivision of z_ν. Any z_ν determines an $(n-1)$-dimensional net fragment J'_ν (J''_ν, resp.) in R_n as the simplicial image of J' (J'', resp.), and a closed, two-sided $(n-2)$-dimensional net j_ν as the simplicial image of j. We assign a basic point of z_ν to any element corner point of η_ν when it lies in j and possesses the smallest possible distance from that corner point, and correspondingly construct a simplicial map of η_ν out of representative simplexes S of J' that we project from the midpoint of S onto s.

Then, along with η_ν, we take a simplicial decomposition ζ_ν such that under the aforementioned projection any basic simplex of η_ν will be mapped inside of a single basic simplex of s that belongs to z_ν. The corner points of an arbitrary basic simplex of ζ_ν are then assigned to those points of j_ν that are contained in a single basic simplex of j_ν.

By a suitable simplicial decomposition of the limit elements of γ_ν, we extend ζ_ν to a simplicial decomposition of γ_ν, and when we establish all of the basic points of this decomposition that do not belong to η_ν, but replace each basic point that belongs to η_ν with the point of j_ν that corresponds to it, a two-sided, $(n-1)$-dimensional net fragment $\mathcal{F}_\nu$ will be determined as the simplicial image of γ_ν whose limit λ_ν is composed of the simplicial image under η_ν of a finite number of closed, two-sided $(n-2)$-dimensional nets and is contained in j_ν.

§ 4.

For a sufficiently large ν, the polygon w has no point in common with $\mathcal{F}_\nu$, except R, so the total degree of λ_ν relative to w equals ± 1.

[**]) Concerning the relationship between "positive indicatrix" and "positive boundary indicatrix," cf., Math. Ann. **71**, pp. 108.

We denote the degree of the map of λ_ν onto j_ν (cf., Math. Ann. **71**, pp. 105) by c and the degree of j_ν relative to w by c'.

Let $\mathfrak{C}_1$ be a two-sided, $(n - 1)$-dimensional net fragment that is endowed with a positive indicatrix that possesses j_ν as its limit and the negative indicatrix of j_ν as its positive boundary indicatrix, while no corner point and no sub-segment of w lies in $\mathfrak{C}_1$ and no side of w meets an $(n - 2)$-dimensional basic side of $\mathfrak{C}_1$.

Let $\mathfrak{C}_2$, $\mathfrak{C}_3$, …, $\mathfrak{C}_c$ be further net fragments of the same kind, so the numbers of positive and negative crossings of w with $\mathcal{F}_\nu + \mathfrak{C}_1 + \mathfrak{C}_2 + \mathfrak{C}_3 + \ldots + \mathfrak{C}_c$ are equal to each other.

The total degree of λ_ν relative to w will thus be obtained when we multiply the degree of j_ν relative to w by c; i.e., one has the following formula:

$$cc' = \pm 1.$$

However, this formula can only be satisfied when c, as well as c', is equal to ± 1.

§ 5.

We now assume the contrary of the theorem to be proved, that outside of I yet a second finite region I' exists that is determined by J. We then construct γ' and $\mathcal{F}'_\nu$ in I' in analogy to the way that we constructed γ and $\mathcal{F}_\nu$ in I, from which, the limit λ'_ν of $\mathcal{F}'_\nu$, just like the limit λ_ν of $\mathcal{F}_\nu$, covers the net j_ν with the degree ± 1. We may further think of the path segments v' and v'' as being constructed in such a way that they meet γ' as many times as γ, such that γ' has no point in common with w.

Now, on the one hand, any polygon, no corner point and no sub-segment of which lies in $\mathcal{F}_\nu + \mathcal{F}'_\nu$, and no side of which meets an $(n - 2)$-dimensional basic side of $\mathcal{F}_\nu + \mathcal{F}'_\nu$, must be associated with an even number of crossing points, but on the other hand, we can choose ν to be so large that the polygon w has not point in common with $\mathcal{F}'_\nu$, so it crosses $\mathcal{F}_\nu + \mathcal{F}'_\nu$ at only a single point, namely, the point R.

From this contradiction, we infer that *J can determine only a single finite region I.*

Q. E. D.