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FASCICLE LXII

Gravity, Groups, and Mechanics

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TABLE OF CONTENTS

INTRODUCTION

In this fascicle, we once again emphasize the fundamental similarities and dissimilarities that exist between the theories of gravity, groups, and – of course, this latter point is touched upon only lightly – the new mechanics. All of these topics that are of interest to science relate to multiple integral transformations.

We say "theories" of gravity. Indeed, one might have an infinitude of them, since, from Henri Poincaré [**1**], one might have an infinitude of mechanical interpretations of the Universe that lead one to conceive of one.

We briefly review Einstein's theory of gravity in 1929, from the translation of R. Ferrier, a translation that has the advantage of being augmented by preliminary notes by Th. De Donder [**2**].

Since this fascicle was edited, Albert Einstein himself has published – in French – a more developed form of the new theory of gravity [**32**], [**46**].

The very useful work thus presented by Ferrier reveals some difference of interpretation between the conceptions of that scholar and our own, which we do not hesitate to point out. Ferrier writes: "The month of January, 1929 marks an important date in the history of relativity: the one when Einstein abandoned the theory of relativistic mechanics that he had maintained up till then without arriving at a coherent system, which was discussed in a memoir presented to the Berlin Academy of Sciences, and in which he constructed a new one that was based on very different principles."

Now, for us, Einstein abandoned nothing whatsoever, and the new theory of gravity is not based on any principles that differ very much from the old ones.

The first theory of gravity rests on the Bianchi identity [from (23) of Chapter I, when the Λ's are null]; the second one is based on the identity (24). Now, these two identities are, as one says, *conjugate*, and are associated in the most intimate manner: They *both* emerge from the same analytic transformation. Truly, they are not "very different."

If one retains the idea of a difference then one may also remark that the first theory of gravity makes recourse to only Riemann space, therefore, a space without torsion; the second one makes recourse to a space with torsion, in the sense of Élie Cartan. However, T. Levi-Civita has shown [**3**], in March, 1929, that a space with torsion may be represented on a space without torsion by means of the absolute differential calculus and the Ricci rotation coefficients. Once again, the "very different" aspect disappears.

Ferrier also concluded with some pessimistic thoughts. The problem of associating the electromagnetic and gravitational fields can hardly be considered as definitively solved. Of course, we also believe this, but that will not affect anything. We belong to a school of philosophy, which again has Henri Poincaré for its mentor, where one does not believe in the possibility of the existence of synthetic theories of a perfect and definitive character; for us, a theory becomes admissible when it presents a certain extension and, above all, a certain esthetic. Now, from this point of view the Einsteinian theories seem unrivaled.

Where are these theories going? Without a doubt, towards the use of identities that are more and more arduous to extract from the theory of infinite groups, of almost inextricable Pfaff systems, concepts that seem to be outlined in the thoughts of Cartan [**33**], of De Donder, of Weyl,…; do not forget Einstein himself. Next to these identities, those of Bianchi seem quite small indeed. However, as for the truly universal identity that expresses the phenomenological existence of everything that defines its essence, one has the strange contradiction that it seems to be not of this world.

CHAPTER I

LIE GROUPS AND CARTAN SPACES

1. Structural relations and generalizations. – One knows that the fundamental structural relations of finite continuous groups are:

$$c_{ij}^s + c_{ji}^s = 0, \tag{1}$$

$$c_{ij}^\alpha c_{k\alpha}^s + c_{jk}^\alpha c_{i\alpha}^s + c_{ki}^\alpha c_{j\alpha}^s = 0. \tag{2}$$

Here, α is a summation index, like all indices that appear twice in a monomial. The absolute differential calculus, which has made that convention popular, also leads us to remark that the equalities (2) may be subjected to a *contraction* by taking *i*, *j*, or *k* equal to *s*. Let $s = k$; the relation (2) will become:

$$c_{ij}^\alpha c_{s\alpha}^s = 0, \tag{3}$$

with two summation indices, α and *s*, this time.

Obviously, the relations (3) are not distinct from (2), but they may be the point of departure for very important special considerations.

If the constants c_{ij}^s, *in which all three of the indices i, j, s take on all of the integer values from* 1 *to r, satisfy the relations* (1) *and* (2) *then one may always find r infinitesimal transformations* X_s *such that:*

$$(X_i X_j) = X_i X_j - X_j X_i = c_{ij}^s X_s. \tag{4}$$

This is, in summation, Lie's third theorem. Now, recent progress shows that this third and last theorem does not succeed from all points of view; one may cause relations (1), (2) to appear, and likewise functional generalizations of the relations, in various forms, without speaking of groups. One may show that these relations border quite closely to the principles of analysis; for example, they are conditions of simplicity for certain differential systems for certain spaces in which one will then recover the *group spaces*, which were already summarized in fascicle XXXIII of the *Mémorial* on the basis of the significance developments that were published by Élie Cartan [**4**].

As an example of the functional generalization of the relations (1) and (2), we shall first develop Einstein's new theory of gravity in [**2**] in the style that was adopted in fascicle XVI of the *Mémorial* for the first kind of gravitational theory.

2. Fundamental identities. Consequences. – We shall change nothing concerning the choice of these identities. They are:

(5) $$\int_C X\,dY = \iint_S dX\ dY\,, \qquad \iint_S X\,dY\,dZ = \iiint_V dX\,dY\,dZ\,,$$

and all of the analogous relations that are obtained for an arbitrary number of variables X, Y, Z, …; the *two* relations (5) suffice for the construction of a theory of gravity that relates to ordinary spacetime. We know that, by a succession of changes of variables and linear combinations, the identities (5) become the *Stokes formulas* [**5**], which immediately reveal the form of the electromagnetic equations of Maxwell, and what makes Maxwell one of the greatest geniuses to have graced humanity is precisely the fact that he laid the foundations for a theory of electromagnetism that is, at the same time, a theory of geometry.

Such results are codified in a more explicit fashion by the theory of Pfaff forms that are provided with an exterior multiplication. The identities (5) are then replaced by:

(6) $$\int_C P\,dx^i = \iint_S [dP_i dx^i]\,, \qquad \iint_S M_{ij}dx^i dx^j = \iiint_V [dM_{ij}dx^i dx^j]\,.$$

The analogies in notation lead one to think that one has, in (5) and (6), identities that are basically equivalent. Developing the brackets in (6) gives the symbolic determinants:

(7) $$\begin{vmatrix} \dfrac{\partial}{\partial x^i} & \dfrac{\partial}{\partial x^j} \\ P_i & P_j \end{vmatrix},$$

(8) $$\begin{vmatrix} \dfrac{\partial}{\partial x^i} & \dfrac{\partial}{\partial x^j} & \dfrac{\partial}{\partial x^k} \\ M_{i\omega} & M_{j\omega} & M_{k\omega} \\ i & j & k \end{vmatrix},$$

and in the latter, one has:

$$M_{ij} + M_{ji} = 0.$$

As for ω, it is the substitution index, which has an immediately obvious role, and which has been explained many times, moreover. Starting with (7), and using derivatives in D, which are more general than those in ∂, we further set:

(9) $$\begin{vmatrix} \dfrac{D}{Dx^i} & \dfrac{D}{Dx^j} \\ P_i & P_j \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial x^i} & \dfrac{\partial}{\partial x^j} \\ P_i & P_j \end{vmatrix} - \begin{vmatrix} \Gamma^\alpha_{\omega i} & \Gamma^\alpha_{\omega j} \\ {}_iP_\alpha & {}_jP_\alpha \end{vmatrix}.$$

Upon writing:

(10) $$\Lambda^\alpha_{ij} = \Gamma^\alpha_{ij} - \Gamma^\alpha_{ji}\,,$$

the last determinant of (9) is, if one ignores the sign that precedes it, equal to:

(11) $$\Lambda^{\alpha}_{ji} P_{\alpha} .$$

Furthermore, no matter what the expression for (11), one may subdivide (9) into formulas such as:

(12) $$\frac{DP_j}{Dx^i} = \frac{\partial P_j}{\partial x^j} - \Gamma^{\alpha}_{ij} P_{\alpha} .$$

Likewise, with:

(13) $$\begin{vmatrix} \dfrac{D}{Dx^i} & \dfrac{D}{Dx^j} \\ P^i & P^j \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial x^i} & \dfrac{\partial}{\partial x^j} \\ P^i & P^j \end{vmatrix} + \begin{vmatrix} \Gamma^{\omega}_{\alpha i} & \Gamma^{\omega}_{\alpha j} \\ {}^i P^{\alpha} & {}^j P^{\alpha} \end{vmatrix} ,$$

the last determinant is equal to:

$$(\Gamma^{j}_{\alpha i} - \Gamma^{i}_{\alpha j}) P^{\alpha} ,$$

which does not disappear with the expressions (10), but gives the derivatives:

(14) $$\frac{DP^j}{Dx^i} = \frac{\partial P^j}{\partial x^i} + \Gamma^{j}_{\alpha i} P^{\alpha} ,$$

with which:

(15) $$\frac{\partial}{\partial x^i} (P_j P^j) = P_j \frac{DP^j}{Dx^i} + P^j \frac{DP_j}{Dx^i} .$$

One may write an analogous formula with $P_j P^j$ replaced by $P_j Q^j$.

The formulas:

(16) $$\frac{DP_j}{Dx^i} dx^i = dP_j - \Gamma^{\alpha}_{ji} P_{\alpha} dx^i = 0,$$

(17) $$\frac{DP^j}{Dx^i} dx^i = dP^j + \Gamma^{j}_{i\alpha} P^{\alpha} dx^i = 0,$$

define a generalized *parallel displacement* for the vector whose *covariant* components are P_j or whose *contravariant* components are P^j. This affirmation may have a very general sense in which the nature of the functions Γ^{α}_{ji} does not play any role; one recovers ordinary parallel displacement simply when all of these functions vanish identically. Letting two infinitely small vectors issue from the same point, the one d having components dx^j, the other one δ having components δx^j. If one desires that δ, when displaced parallel to itself along d, gives the same point as d, when displaced along δ, then this translates into the equality:

$$dx^j + \delta x^j + d\, \delta x^j = \delta x^j + dx^j + \delta\, dx^j,$$

or, from (17), into:

$$\Gamma^{j}_{\alpha i} \delta x^{\alpha} dx^i = \Gamma^{j}_{i\alpha} dx^{\alpha} \delta x^i .$$

Upon inverting the summation indices i and α on the two sides, one concludes that $\Gamma^j_{\alpha i}$ is equal to $\Gamma^j_{i\alpha}$, which amounts to the nullity of the Λ^α_{ij} written in (10). However, if the quadrilateral contour that we just defined with the aid of d and δ is not closed then the Λ^α_{ij} in (10) are no longer annulled; the space has *torsion*. We now place ourselves in the general case.

The notion of torsion may likewise be revealed in an interesting manner by means of formula (9), without which, we would basically be dealing with something distinct from what we just spoke of.

One may conclude from (9):

$$\iint_S \begin{vmatrix} \dfrac{D}{Dx^i} & \dfrac{D}{Dx^j} \\ P_i & P_j \end{vmatrix} dx^i dx^j = \int_C P_i dx^i + \iint_S P_\alpha \Lambda^\alpha_{ij} dx^i dx^j .$$

If the Λ^α_{ij} are identically null then one has a Stokesian formula with the ∂'s replaced by D's, which rightfully permits us to speak of a Stokesian formula that preserves its ordinary physiognomy in spaces that are devoid of torsion; however, in the spaces with torsion, one must add a complementary term to the formula in D.

One always knows [**5**] that the reasoning made by starting with (7) may be repeated by starting with (8) and then following suit. The rule for derivation that is found in (12) and (14) takes the general form:

$$\frac{D}{Dx^i} A^{****}_{*****} = \frac{\partial}{\partial x^i} A^{****}_{*****} \begin{cases} -\Gamma^\alpha_{\mu i} A^{****}_{**\alpha**} \text{ for each } {}^{****}_{**\mu**} \\ +\Gamma^\mu_{\alpha i} A^{**\alpha*}_{*****} \text{ for each } {}^{**\mu*}_{*****} \end{cases} . \tag{18}$$

One will remark that the index of derivation i is always placed below and in the last place in the Γ coefficients.

With the aid of the derivation rule (18), one will easily recall the calculations that were made in [**5**] (pp. 25), and one finds, notably:

$$\begin{cases} \begin{vmatrix} \dfrac{D}{Dx^i} & \dfrac{D}{Dx^j} \\ \dfrac{DA_k}{Dx^i} & \dfrac{DA_k}{Dx^j} \end{vmatrix} = B^\alpha_{kji} A_\alpha + \Lambda^\alpha_{ij} A_{k\alpha}, \\ \begin{vmatrix} \dfrac{D}{Dx^i} & \dfrac{D}{Dx^j} \\ \dfrac{DA^k}{Dx^i} & \dfrac{DA^k}{Dx^j} \end{vmatrix} = B^k_{\alpha ij} A^\alpha + \Lambda^\alpha_{ij} A^k_\alpha, \end{cases} \tag{19}$$

with the four-index Riemann symbol:

(20) $$B^{\alpha}_{kji} = \frac{\partial}{\partial x^j}\Gamma^{\alpha}_{ki} - \frac{\partial}{\partial x^i}\Gamma^{\alpha}_{kj} + \Gamma^{\beta}_{ki}\Gamma^{\alpha}_{\beta j} - \Gamma^{\beta}_{kj}\Gamma^{\alpha}_{\beta i}.$$

Of course:

$$A_{kj} = \frac{DA_k}{Dx^j}, \qquad A^k_i = \frac{DA^k}{Dx^j}.$$

One likewise has:

(21) $$\begin{vmatrix} \dfrac{D}{Dx^{\tau}} & \dfrac{D}{Dx^{\sigma}} \\ \dfrac{DA_{\mu\nu}}{Dx^{\tau}} & \dfrac{DA_{\mu\nu}}{Dx^{\sigma}} \end{vmatrix} = B^{\rho}_{\mu\sigma\tau}A_{\rho\nu} + B^{\rho}_{\nu\sigma\tau}A_{\mu\rho} + \Lambda^{\alpha}_{\tau\sigma}A_{\mu\nu\alpha}.$$

More simply, one may write [*]:

(22) $$A_{\mu\nu\sigma} - A_{\mu\sigma\nu} = B^{\alpha}_{\mu\sigma\tau}A_{\alpha} + \Lambda^{\alpha}_{\sigma\nu}A_{\mu\alpha},$$

(21) $$A_{\mu\nu\sigma\tau} - A_{\mu\nu\tau\sigma} = B^{\alpha}_{\nu\rho\tau}A_{\alpha\nu} + B^{\alpha}_{\nu\sigma\tau}A_{\mu\alpha} + \Lambda^{\alpha}_{\tau\sigma}A_{\mu\nu\alpha},$$

Now, start with the identity:

$$\left.\begin{matrix} A_{\mu\nu\sigma\tau} - A_{\mu\nu\tau\sigma} \\ +A_{\mu\sigma\tau\nu} - A_{\mu\sigma\nu\tau} \\ +A_{\mu\tau\nu\sigma} - A_{\mu\tau\sigma\nu} \end{matrix}\right| = \left|\begin{matrix} (A_{\mu\nu\sigma} - A_{\mu\sigma\nu})_{\tau} \\ +(A_{\mu\sigma\tau} - A_{\mu\tau\sigma})_{\nu} \\ +(A_{\mu\tau\nu} - A_{\mu\nu\tau})_{\sigma}. \end{matrix}\right.$$

Write the binomials on the right-hand side in the form (21) and the contents of the parentheses in the form (22). Next, differentiate these parentheses with respect to D, as is indicated by the outer indices, upon observing the rule for the derivation of products, of which (15) is a very particular case, but which is always preserved in its ordinary aspect.

After an initial simplification, one obtains:

$$\left.\begin{matrix} B^{\alpha}_{\nu\sigma\tau}A_{\alpha\mu} + \Lambda^{\alpha}_{\tau\sigma}A_{\mu\nu\alpha} \\ +B^{\alpha}_{\sigma\tau\nu}A_{\mu\alpha} + \Lambda^{\alpha}_{\nu\tau}A_{\mu\sigma\alpha} \\ +B^{\alpha}_{\tau\nu\sigma}A_{\mu\alpha} + \Lambda^{\alpha}_{\sigma\nu}A_{\mu\tau\alpha} \end{matrix}\right| = \left|\begin{matrix} (B^{\alpha}_{\mu\nu\sigma})_{\tau}A_{\alpha} + \Lambda^{\alpha}_{\sigma\nu}A_{\mu\alpha\tau} + \Lambda^{\alpha}_{\sigma\nu\tau}A_{\mu\alpha} \\ +(B^{\alpha}_{\mu\sigma\tau})_{\nu}A_{\alpha} + \Lambda^{\alpha}_{\tau\sigma}A_{\mu\alpha\nu} + \Lambda^{\alpha}_{\tau\sigma\nu}A_{\mu\alpha} \\ +(B^{\alpha}_{\mu\tau\nu})_{\sigma}A_{\alpha} + \Lambda^{\alpha}_{\nu\tau}A_{\mu\alpha\sigma} + \Lambda^{\alpha}_{\nu\tau\sigma}A_{\mu\alpha}. \end{matrix}\right.$$

In the second column of the left-hand side, subtract the second column of the right-hand side, taking into account the first equation in (19); i.e.:

$$A_{\mu\nu\alpha} - A_{\mu\alpha\nu} = B^{\beta}_{\mu\nu\alpha}A_{\beta} + \Lambda^{\beta}_{\alpha\nu}A_{\mu\beta}.$$

One obtains:

[*] [DHD]: This was the equation numbering used in the original.

$$\left.\begin{array}{r} B^{\alpha}_{\nu\sigma\tau}A_{\alpha\mu} + \Lambda^{\alpha}_{\tau\sigma}(B^{\beta}_{\mu\nu\alpha}A_{\beta} + \Lambda^{\beta}_{\alpha\nu}A_{\mu\beta}) \\ +B^{\alpha}_{\sigma\tau\nu}A_{\mu\alpha} + \Lambda^{\alpha}_{\nu\tau}(B^{\beta}_{\mu\sigma\alpha}A_{\beta} + \Lambda^{\beta}_{\alpha\sigma}A_{\mu\beta}) \\ +B^{\alpha}_{\tau\nu\sigma}A_{\mu\alpha} + \Lambda^{\alpha}_{\sigma\nu}(B^{\beta}_{\mu\tau\alpha}A_{\beta} + \Lambda^{\beta}_{\alpha\tau}A_{\mu\beta}) \end{array}\right| = \left|\begin{array}{l} (B^{\alpha}_{\mu\nu\sigma})_{\tau}A_{\alpha} + A_{\sigma\nu\tau}A_{\mu\alpha} \\ +(B^{\alpha}_{\mu\sigma\tau})_{\nu}A_{\alpha} + A_{\tau\sigma\nu}A_{\mu\alpha} \\ +(B^{\alpha}_{\mu\tau\nu})_{\sigma}A_{\alpha} + A_{\nu\tau\sigma}A_{\mu\alpha}. \end{array}\right.$$

With some inversions of the summation indices in the left-hand side, one has an equality that must persist independently of the A_α and $A_{\mu\alpha}$; it finally subdivides into two identities. The first one, which generalizes the *Bianchi identity*, is:

$$(B^{\alpha}_{\mu\nu\sigma})_{\tau} + (B^{\alpha}_{\mu\sigma\tau})_{\nu} + (B^{\alpha}_{\mu\tau\nu})_{\sigma} + B^{\alpha}_{\mu\nu\beta}\Lambda^{\beta}_{\sigma\tau} + B^{\alpha}_{\mu\sigma\beta}\Lambda^{\beta}_{\tau\nu} + B^{\alpha}_{\mu\tau\beta}\Lambda^{\beta}_{\nu\sigma} = 0. \tag{23}$$

The second one is:

$$(B^{\alpha}_{\nu\sigma\tau} + B^{\alpha}_{\sigma\tau\nu} + B^{\alpha}_{\tau\nu\sigma}) + (\Lambda^{\alpha}_{\nu\sigma\tau} + \Lambda^{\alpha}_{\sigma\tau\nu} + \Lambda^{\alpha}_{\tau\nu\sigma}) \\ \Lambda^{\beta}_{\nu\sigma}\Lambda^{\alpha}_{\tau\beta} + \Lambda^{\beta}_{\sigma\tau}\Lambda^{\alpha}_{\nu\beta} + \Lambda^{\beta}_{\tau\nu}\Lambda^{\alpha}_{\sigma\beta} = 0. \tag{24}$$

Upon adding them, from (10), one has:

$$\Lambda^{\alpha}_{ij} + \Lambda^{\alpha}_{ji} = 0. \tag{25}$$

One has, in (24) and (25), an obvious extension of the fundamental structural relations (2) and (1), from the theory of finite, continuous groups.

Einstein's theory gravity of the first kind rests on the consideration of a space that is curved but without torsion; the fundamental identity of the theory is the Bianchi identity – i.e., (23) – which is reduced to its first three terms since all of the Λ's with three indices are null. The new theory of gravity [**2**] is that of a space that has torsion, but not curvature; the four-index Riemann *B*'s vanish identically and the identity (23) with them. All that remains is (24) and (25).

The various theories of gravity and the theory of groups thus appear to be closely linked at their foundations; they have exactly the same right to exist.

As for the diversity of the theories of gravity, it is analogous to the diversity of the theories of mechanics. Henri Poincaré has made us sufficiently habituated to admitting the existence of an infinitude of mechanical images if one can conceive of one of them; it seems proper that this conception should triumph today in a general manner. Therefore, in an admirable work that was published in 1929, R.-H. Fowler of Cambridge University [**6**] writes (pp. 4): "It is impossible to argue that the fact that a particular mechanism leads to a state of complete equilibrium in agreement with experimental facts is any evidence for the particular mechanism discussed. It is merely evidence that the laws of this mechanism have been correctly and consistently written down! Any other mechanism would give the same result."

Of the theories of gravity, one may certainly say that they are geometrical theories that are susceptible to being much more varied that the mechanical theories.

We note, moreover, that Fowler writes further in the work cited (pp. 7): "Something more than success and logical rigour appears to be necessary for the acceptance of a model which is to account to our aesthetic satisfaction."

It is certainly in gravity that one encounters most easily that which agrees with our aesthetic satisfaction. It will be difficult to find a mathematical reasoning more elegant than the one that led from the identities (5) to identities (23) and (24).

It seems very interesting to us to recall this reasoning from the Stokesian viewpoint; moreover, it seems that studies such as the one this fascicle must always be carried out only while in very explicit connection with the principles of analysis.

As for the authors that have already given formulas (23) and (24), while deriving them in ways that differed slightly from the preceding one, or likewise more general ways, we cite, in particular: É. Cartan [**7**] (pp. 382), H. Eyraud [**8**] (pp. 21), R. Lagrange [**9**] (pp. 22).

The work of H. Eyraud, which was published in 1926, is actually illuminated very advantageously by the new theory of Einstein. In 1928 [**10**] (pp. 161), we wrote: "A recent thesis of H. Eyraud plainly introduced torsion into electromagnetism, but we lack the necessary hindsight to appreciate the true value of that attempt. Hindsight has been favorable to H. Eyraud in that his theory and that of Einstein are very dissimilar; however, in both cases, one makes recourse to spaces with torsion.

As we have already reproduced in [**11**] (pp. 7), E. Cartan [**7**] (pp. 367) defined the components of torsion and curvature by the formulas:

$$\Omega^i = [\pi^i]' - [\pi^\alpha \pi^i_\alpha] = \Lambda^i_{jk}[\pi^j \pi^k], \tag{26}$$

$$\Omega^i_k = [\pi^i_k]' - [\pi^\alpha_k \pi^i_\alpha] = B^i_{kml}[\pi^m \pi^l]. \tag{27}$$

If one sets:

$$\pi^i = -\,dx^i, \qquad \pi^i_j = \Gamma^i_{j\beta} dx^\beta,$$

upon observing that the accents in (26) and (27) indicate exterior derivations, then one painlessly obtains:

$$\Gamma^i_{\alpha\beta} - \Gamma^i_{\beta\alpha} = \Lambda^i_{\alpha\beta}, \tag{28}$$

which is nothing but (10), and:

$$B^i_{kml} = \frac{\partial}{\partial x^m}\Gamma^i_{kl} - \frac{\partial}{\partial x^l}\Gamma^i_{km} + \Gamma^\beta_{kl}\Gamma^i_{\beta m} - \Gamma^\beta_{km}\Gamma^i_{\beta l}, \tag{29}$$

which coincides with (20). Recall that (29) is skew-symmetric in l, m, as (28) is in α, β. More exactly, if, in the left-hand side of (29), one inverts l and m in the first and third term then one obtains, up to sign, the second and fourth ones. This remark will be used shortly.

3. Bifurcations and *n*-podes. – The principal object of the preceding paragraph was that of obtaining formulas (23) and (24). It is of great use to not separate them in order to judge, from a sufficient altitude, the various aspects of gravity; we repeat, moreover, that they only form a sort of kernel in the set of formulas of the same type, with more and more indices of derivation, sets whose process of formation has been indicated by Cartan [**7**] (pp. 382).

However, having said this, it is convenient to specify that in the present fascicle we shall no longer be occupied with formula (24), to which, of course, (25) will be always be associated. It is likewise very simple to reconstruct (24) in isolation. Start with (29) and form:

(30) $$B^i_{kml} + B^i_{mlk} + B^i_{lkm}\,.$$

One immediately finds that this trinomial may be written:

(31) $$\frac{\partial}{\partial x^m}\Lambda^i_{kl} + \frac{\partial}{\partial x^k}\Lambda^i_{lm} + \frac{\partial}{\partial x^l}\Lambda^i_{mk} + \Gamma^i_{\beta k}\Lambda^\beta_{lm} + \Gamma^i_{\beta l}\Lambda^\beta_{mk} + \Gamma^i_{\beta m}\Lambda^\beta_{kl}\,.$$

If one introduces the derivatives:

$$\Lambda^i_{klm} = \frac{\partial}{\partial x^m}\Lambda^i_{kl} - \Gamma^\alpha_{km}\Lambda^i_{\alpha l} - \Gamma^\beta_{lm}\Lambda^i_{k\alpha} + \Gamma^\beta_{\alpha m}\Lambda^\alpha_{kl}$$

then the expression (31) becomes:

(32) $$\Lambda^i_{klm} + \Lambda^i_{lmk} + \Lambda^i_{mkl} + \Lambda^\alpha_{kl}\Lambda^i_{m\alpha} + \Lambda^\beta_{lm}\Lambda^i_{k\alpha} + \Lambda^\beta_{mk}\Lambda^i_{l\alpha}\,,$$

and the equality of (30) with (32) is nothing but (24).

One may now say that the consideration of the trinomial (30) leads to some very interesting bifurcations. There are at least three great theories that give rise to the vanishing of that trinomial, according to the manner in which it vanishes.

In the first place, the trinomial (30) is null when all of the Λ's are null. This is the case for spaces that are curved, but without torsion, and for theories of gravity of the first type.

In the second place, the trinomial (30) is null when the Γ's, and consequently the Λ's, are certain constant, notably when one has:

(33) $$\Gamma^s_{ij} = c^s_{ij} = -\,c^s_{ji}\,, \qquad\qquad \Lambda^s_{ij} = 2\,c^s_{ij}\,.$$

The expressions (31) are then null if, moreover:

(34) $$c^i_{\beta k}c^\beta_{lm} + c^i_{\beta l}c^\beta_{mk} + c^i_{\beta m}c^\beta_{kl} = 0.$$

In (33) and in (34), one obviously recognizes the relations (1) and (2) − i.e., the fundamental structural relations of the theory of finite, continuous groups.

The *B*'s in (30) are not individually null; they are constants such that:

(34*) $$B^i_{kml} = c^\beta_{kl}c^i_{\beta m} - c^\beta_{km}c^i_{\beta l} = c^i_{\beta k}c^\beta_{ml}\,.$$

In the space thus constituted there exists a displacement by parallelism that is defined by equations (17), which is:

$$dP^j + c^j_{\alpha i} P^\alpha dx^i = 0. \tag{35}$$

If this displacement is valid along the curve with the differential equations:

$$dx^i + \lambda^i \, dt = 0$$

the λ^i being arbitrary functions of t, then equations (35) become:

$$dP^j + c^j_{i\alpha} \lambda^i P^\alpha dt = 0. \tag{36}$$

As we have already shown in fascicle XXXIII of the *Mémorial,* and since we shall recall it shortly with the new developments, this *linear* system may be considered as the fundamental differential system of the theory of groups. It is one of the aspects of the problem of the *linearization* of this theory. We first considered a space in which we have associated a certain notion of parallelism; a theory may emerge and unfold from this just as Euclidian geometry unfolds from the fact that one assumes Euclid's postulate.

In the third place, the trinomial (30) is null when the three B's that comprise it are individually null. It was in order to realize such a nullity that Einstein's new theory introduced the ingenious notion of r-podes, which are tetrapodes (*Vierbeinen*) in four-dimensional space [**2**].

An r-pode is composed of r^2 functions:

$$\begin{array}{cccc} {}^1h_1 & {}^1h_2 & \dots & {}^1h_r\,, \\ {}^2h_1 & {}^2h_2 & \dots & {}^2h_r\,, \\ \dots & \dots & \dots & \dots \\ {}^rh_1 & {}^rh_2 & \dots & {}^rh_r\,, \end{array}$$

which define a determinant h whose minors are *normed*; i.e., when deprived of their sign and divided by h, they form, in due course, the table:

$$\begin{array}{cccc} {}_1h^1 & {}_1h^2 & \dots & {}_1h^r\,, \\ {}_2h^1 & {}_2h^2 & \dots & {}_2h^r, \\ \dots & \dots & \dots & \dots \\ {}_rh^1 & {}_rh^2 & \dots & {}_rh^r. \end{array}$$

One will obviously have:

$$ {}_sh^\lambda \; {}^sh_\nu = \delta^\lambda_\nu\,, \qquad {}^\lambda h_s \; {}_\nu h^s = \delta^\lambda_\nu\,, \tag{37}$$

with δ^λ_ν null, in general, but equal to 1 if $\lambda = \nu$.

One sets, by definition, upon deriving (37):

$$\Gamma^i_{kl} = {}_sh^i \frac{\partial}{\partial x^l}\, {}^sh_k = -\, {}^sh_k \frac{\partial}{\partial x^l}\, {}_sh^i\,. \tag{38}$$

One has, in turn:

$$\frac{\partial}{\partial x^m}\Gamma^i_{kl} = {}_s h^i \frac{\partial}{\partial x^l \partial x^m}{}^s h_k + \frac{\partial}{\partial x^m}{}^s h_k \cdot \frac{\partial}{\partial x^l}{}^s h_k ,$$

$$\Gamma^i_{\sigma m}\Gamma^\sigma_{kl} = -{}^s h_\sigma \frac{\partial}{\partial x^m}{}_s h^i \cdot_t h^\sigma \frac{\partial}{\partial x^l}{}^t h_k ,$$

(39)
$$\frac{\partial}{\partial x^m}\Gamma^i_{kl} + \Gamma^i_{\sigma m}\Gamma^\sigma_{kl} = {}_s h^i \frac{\partial}{\partial x^l \partial x^m}{}^s h_k .$$

The terms that seem to have been omitted in the right-hand side of (39) contain the factor:

(40)
$$\frac{\partial}{\partial x^l}{}^s h_k - {}^s h_k \, {}_t h^\sigma \frac{\partial}{\partial x^l}{}^t h_k ,$$

Therefore, from (29) and (39), *if* Γ^i_{kl} *is defined by* (38) *then one has:*

(41)
$$B^i_{kml} = 0.$$

Before going further, one may make two interesting remarks:

One sees that (39) gives (41) by virtue of the permutability of the ordinary partial derivatives; i.e., the indices *l* and *m* that appear in the right-hand side of (39). Now, H. Weyl, in the case of spaces without torsion, established the nullity of the $\Lambda^i_{\alpha\beta}$ by also appealing to such a permutability [**12**].

One further sees that if the three methods that were employed here to annul the trinomial (30) are ingenious then they are no less particular; the search for more general spaces in which this trinomial is annulled will be, without a doubt, an important subject of study.

Along this order of ideas, we point out a work of G. Mattiloi [**34**]. It contains remarkably symmetric formulas that are comparable to (31).

Let us continue. *The covariant derivatives*, (12) and (14), *of the* ${}^s h_\nu$ *and the* ${}_s h^\nu$ *are identically null.* The verification of this assertion is immediate and amounts to confirming the nullity of expressions like (40). If one sets:

(42)
$$g_{\lambda\mu} = {}^s h_\lambda {}^s h_\mu , \qquad g^{\lambda\mu} = {}_s h^\lambda {}_s h^\mu$$

then the covariant derivatives of these new expressions will again be identically null. The determinant *g* of the $g_{\lambda\mu}$ is the square of the determinant *h*.

The *g* with two indices have the well-known role relating to the raising and lowering of indices; one will observe, moreover, that Einstein often underlined an index before raising or lowering it. Therefore:

$$A_{\underline{\lambda}} = A^\lambda = g^{\lambda\mu} A_\mu , \qquad A^{\underline{\lambda}} = A_\lambda = g_{\lambda\mu} A^\mu .$$

Observe, furthermore, that:

$$\frac{\partial h}{\partial x^\sigma} = \frac{\partial h}{\partial ({}^i h_j)} \frac{\partial ({}^i h_j)}{\partial x^\sigma} = h({}_i h^j) \frac{\partial ({}^i h_j)}{\partial x^\sigma} = h\Gamma^i_{j\sigma} .$$

Being given the derivative, conforming to (18):

$$\frac{DT^{..\sigma}_{..}}{Dx^\tau} = \frac{\partial T^{..\sigma}_{..}}{\partial x^\tau} + \cdots + \Gamma^\sigma_{\alpha\tau} T^{..\alpha}_{..} ,$$

this permits one to contract in σ and τ, multiply by h, and, upon setting $\mathcal{T}$ for hT, to write:

$$h\frac{DT^{..\sigma}_{..}}{Dx^\sigma} = \frac{\partial \mathcal{T}^{..\sigma}_{..}}{\partial x^\sigma} + \cdots + \Lambda^\sigma_{\alpha\tau} \mathcal{T}^{..\alpha}_{..} .$$

This is what Einstein abbreviated to:

(43) $$hT^{..\sigma}_{..;\sigma} = \mathcal{T}^{..\sigma}_{..;\sigma} + \Lambda^\sigma_{\alpha\sigma} \mathcal{T}^{..\alpha}_{..} .$$

One sees that the definition of the symbol:

(44) $$\mathcal{T}^{..\sigma}_{..|\sigma}$$

is quite simple; it is by means of it that the gravitational equations condense in an extremely remarkable manner [**2**].

4. Some gravitational developments. – It does not enter into the plan of this fascicle to go into the physical consequences. We would like to return to the spaces with torsion whose principles, at the present moment, are certainly those of group spaces and the preceding gravitational space. Now that we know how those spaces come about and how we arrive at the fundamental symbol (44) that is attached to them we will be brief in what follows.

The theory of gravity of the first type rests upon a contraction of the Bianchi identity. Now we shall contract the identity (24) while, of course, making an abstraction of the first trinomial or, if you prefer, the equality obtained by annulling (32). This contraction gives, if one uses, as in (43), the semi-colon to indicate the covariant derivative:

(45) $$\Lambda^\alpha_{kl;\alpha} + \varphi_{l;k} - \varphi_{k;l} - \varphi_\alpha \Lambda^\alpha_{kl} = 0, \qquad \varphi_\alpha = \Lambda^\beta_{\alpha\beta} .$$

Set:

(46) $$\mathcal{V}^\alpha_{kl} = h(\Lambda^\alpha_{kl} + \varphi_l \delta^\alpha_k - \varphi_k \delta^\alpha_l) .$$

Consider the equality of the type (43):

$$h(\Lambda^{\alpha}_{kl;\alpha} + \varphi_{k;l} - \varphi_{k;l}) = \mathcal{V}^{\alpha}_{kl|\alpha} + h(\Lambda^{\alpha}_{kl} + \varphi_l \delta^{\alpha}_k - \varphi_k \delta^{\alpha}_l)\varphi_\alpha .$$

The last two terms of the latter parentheses are eliminated under summation over α and the contracted equation (45) is finally written:

(47) $$\mathcal{V}^{\alpha}_{kl|\alpha} = 0.$$

This is the analogue of equation (3) *in the theory of groups.*

The identity:

$$\mathcal{A}^{..ik}_{..|i|k} - \mathcal{A}^{..ik}_{..|k|i} = -(\mathcal{A}^{..ik}_{..}\Lambda^{\sigma}_{ik})_{|\sigma}$$

is easy to prove if one makes recourse to the definition (43) and takes (41) into account. Proofs of this type are, moreover, currently popular in the absolute differential calculus; on this subject, one may further consult the exposé of R. Lagrange [**9**] (pp. 21). Thus:

$$\mathcal{V}^{\alpha}_{k\underline{l}|l|\alpha} - \mathcal{V}^{\alpha}_{k\underline{l}|\alpha|l} = -(\mathcal{V}^{\alpha}_{k\underline{l}}\Lambda^{\sigma}_{l\alpha})_{|\sigma} ,$$

and, from (47), this may be written:

(48) $$(\mathcal{V}^{\alpha}_{k\underline{l}|l} - \mathcal{V}^{\sigma}_{k\underline{\tau}}\Lambda^{\alpha}_{\sigma\tau})_{|\alpha} = 0.$$

In the first approximation, Einstein sets:

$$\mathcal{V}^{\alpha}_{k\underline{l}|l|\alpha} = \mathcal{V}^{\alpha}_{k\underline{l}|\alpha|l} = 0,$$

which, from (47), is indeed null, and then writes:

$$\mathcal{V}^{\alpha}_{k\underline{l}|l} = 0$$

for the law of gravitation *in the first approximation*.

He then considers the expression:

(49) $$\overline{\mathcal{V}}^{\alpha}_{kl} = \mathcal{V}^{\alpha}_{kl} - \varepsilon h(\varphi_l \delta^{\alpha}_k - \varphi_k \delta^{\alpha}_l) ,$$

which differs from $\mathcal{V}^{\alpha}_{kl}$ as slightly as one pleases, and remarks that upon applying the operation $|\,\alpha$ to it and annulling, one recovers the Maxwell equations, which then play the role of electromagnetic equations *in the first approximation.*

We agree to add that one passes to the complete theory [**2**] by replacing the $\mathcal{V}$ in (48) with the overlined $\mathcal{V}$ in (49).

One sees that all of the theory unfolds from the equality (47) and the definition (49).

We repeat that a more developed form of the same theory has recently been given by Einstein himself [**32**], [**46**].

Moreover, we have no need to go further into these aspects of the theory in order to make the most important observations. First, as we already said in the preceding paragraph, it is increasingly obvious that such theories are limitless in number; they may obviously be as varied as the conceptions of space itself. One of the next forms of gravity will undoubtedly involve a space in which curvature and torsion both play a role, as in group spaces.

Consequently, all of these theories with a geometric structure must always recover the Maxwell equations, in one form or another, as the electromagnetic equations *in the first approximation*; it is natural, because these equations were introduced at the beginning of our argument, here, with the symbolic determinants (8) that condense its form. The second formula (6) is the Maxwellian formula *par excellence*.

One may openly affirm that in the present state of Science, a physics of non-Maxwellian electromagnetism, at the ordinary level and in the first approximation, is nevertheless a construction that is as improbable as that of a theory of physics that does not rest upon simple geometry, in the first approximation, such as Euclidian geometry, Cayleyian geometry, or the geometry of Riemann spaces.

In support of this way of looking at things, which, moreover, hardly needs to be defended at the present time, we cite a remarkable work of F.-D. Murnaghan [**13**].

5. The Ricci coefficients. – The recent theory of Einstein, whose premises we just presented, was published in January, 1929. Two months later, in the same *Sitzungsberichte* of the Berlin Academy [**3**], Tullio Levi-Cività published a work of the same nature, which was promptly translated into English and had the same conclusions as those of Einstein, but while making use of only Riemann space. It should come as no surprise that if such a duality is possible then it proves simply that Cartan spaces, which are curved and torsed, may be put into correspondence with spaces without torsion. The correspondence may likewise come about in an infinitude of ways.

We shall stop short of completely analyzing the exposé of Levi-Cività, but we shall content ourselves with showing that the theory of Ricci coefficients, which then come into play, easily assures the preceding relationship, as well as other relationships with the theory of groups. We shall borrow from Levi-Cività not only the results presented in his Berlin note, but also in his *Calcolo differenziale assoluto* [**14**]. This latter work was likewise translated into English and German.

Let there be an *n*-uple of orthogonal congruences. Through each point P of the space V_n there pass n lines that are pairwise orthogonal and denumerated by lower indices in the table:

$$\begin{array}{cccc} \lambda_1^1 & \lambda_1^2 & \cdots & \lambda_1^n, \\ \lambda_2^1 & \lambda_2^2 & \cdots & \lambda_2^n, \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_n^1 & \lambda_n^2 & \cdots & \lambda_n^n. \end{array}$$

In a given row of this table one has the direction parameters for the same row in the *n*-uple. Now, let there be a table of *moments:*

$$\begin{matrix} \lambda_{1|1} & \lambda_{1|2} & \cdots & \lambda_{1|n}, \\ \lambda_{2|1} & \lambda_{2|2} & \cdots & \lambda_{2|n}, \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{n|1} & \lambda_{n|2} & \cdots & \lambda_{n|n}. \end{matrix}$$

The correspondence between the two tables is such that:

(50) $$\lambda_{h|i}\lambda_k^i = \delta_h^k, \qquad \lambda_{h|i}\lambda_h^j = \delta_i^j .$$

One sees that the vertical bar between the lower indices does not have the same significance as it did in (44). If one sets:

$$g_{ik} = \lambda_{h|l}\ \lambda_{h|k}, \qquad g^{ik} = \lambda_h^i \lambda_h^k$$

then one easily obtains:

$$g_k^j = g_{ik}\, g^{ij} = \delta_k^j ,$$
$$g^{ij}\ \lambda_{h|i} = \lambda_h^j, \qquad g_{ij}\lambda_h^j = \lambda_{h|i} .$$

This permits the construction of Riemannian metric for which:

$$ds^2 = g_{ik}\, dx^i\, dx^k.$$

It seems to be entirely unnecessary to show that with the tables above relating to our n-uple of congruences one may derive everything that one derives for Einstein's n-podes. Meanwhile, we pursue the reproduction of certain formulas that ultimately entail some interesting comparisons.

For the n-hedron attached to P, consider, in particular, the directions λ_h and λ_k ; they are orthogonal and give, from (50):

(51) $$\cos \widehat{\lambda_h\lambda_k} = \lambda_{h|i}\lambda_k^i = \delta_h^k .$$

Imagine that λ_h is transported to P', which is infinitely close to P, by the simple device of varying the coordinates; for λ_h , one will then have *local* transport, with the symbol δ'.

On the other hand, λ_k will be transported to P' by *parallelism*, with the symbol δ^*.

What, then, is the variation δ of the expression (51)? One will have:

$$p_{hk}\, ds = \delta \cos \widehat{\lambda_h\lambda_k} = \lambda_k^i \delta' \lambda_{h|i} + \lambda_{h|i}\delta^* \lambda_k^i ,$$

and, from (17):

(52) $$p_{hk}\, ds = \lambda_k^i \left(\frac{\partial \lambda_{h|i}}{\partial x^j} - \Gamma_{ij}^l \lambda_{h|l} \right) \delta x^j = \lambda_k^i \lambda_{h|ij} \delta x^j .$$

Now, take the particularly remarkable case in which the direction δx^j coincides with that of an edge of the n-hedron; for example, take $\delta x^j = \lambda_i^{\;j} ds$. One will have, according to Ricci, the *coefficients of rotation* of the n-hedron:

$$\gamma_{hkl} = \lambda_{h|ij} \lambda_k^{\;i} \lambda_l^{\;j} . \tag{53}$$

Observe that in (52) the Γ's with three indices are the Christoffel bracket symbols, which do not change when one inverts i and j.

In the case $n = 3$, Ricci's theory painlessly gives back that of the moving *trihedron* to which is easily attached, as we have shown in fascicle XXXIII of the *Mémorial* in the theory of parametric groups and the Maurer-Cartan equations. This again leads us to recall that the theory of the trihedron, thanks to the efforts of the brothers François and Eugène and Cosserat, is also becoming a powerful instrument in the synthesis of physical theories.

It is along the same order of ideas that we may unite the theory of gravity in Riemann space with the theory of gravity in Cartan space. Above all, we only have an admirable instrument for synthesis, but one must know how to rise to a sufficient altitude that the view seems logically harmonious; to insist upon criticizing the details seems to be only a testament to one's incomprehension.

Return to the expressions (53). If one has:

$$W = U_i V^i, \qquad V^i = g^{ik} V_k$$

then one likewise has, by covariant derivation:

$$W_l = U_{i|l} V^i + U_i g^{ik} V_{i|l} = U_{i|l} V^i + U^i V_{i|l} . \tag{54}$$

Apply this formula to (50); it becomes:

$$\lambda_{h|ij} \lambda_k^{\;i} + \lambda_h^{\;i} \lambda_{k|ij} = 0.$$

If one multiplies by $\lambda_l^{\;j}$ then one has:

$$\gamma_{hkl} + \gamma_{khl} = 0 \tag{55}$$

Furthermore, note the non-permutability of the derivations that were made at P in the directions of the different edges of the n-hedron. One has:

$$\frac{\partial f}{\partial s_h} = \frac{\partial f}{\partial x^i} \frac{dx^i}{ds_h} = \frac{\partial f}{\partial x^i} \lambda_h^{\;i} = f_i \lambda_h^{\;i} = f_i \, g^{ni} \, \lambda_{h|n} = f^n \, \lambda_{h|n} .$$

By covariant derivation, and from (54):

$$\frac{\partial}{\partial x^j}\frac{\partial f}{\partial s_h} = f^i \lambda_{h|ij} + f_{ij}\lambda_h^i .$$

However:

$$\lambda_h^i \frac{\partial f}{\partial s_h} = f^n \lambda_{h|n}\lambda_h^i = f^n \delta_n^i = f^i .$$

Substituting this f^i in the preceding equations, when multiplied by λ_k^j, one gets:

$$\frac{\partial}{\partial s_k}\frac{\partial f}{\partial s_h} = \gamma_{hlk}\frac{\partial f}{\partial s_l} + \lambda_h^i \lambda_k^j f_{ij} ,$$

and finally:

(56)
$$\frac{\partial}{\partial s_k}\frac{\partial f}{\partial s_h} - \frac{\partial}{\partial s_h}\frac{\partial f}{\partial s_k} = (\gamma_{hlk} - \gamma_{klh})\frac{\partial f}{\partial s_l} .$$

Finally, we point out the formula:

(57)
$$\lambda_{i|\nu\rho} - \lambda_{i|\rho\nu} = \frac{\partial}{\partial x^\rho}\lambda_{i|\nu} - \frac{\partial}{\partial x^\nu}\lambda_{i|\rho} ,$$

which is the analogue of (9) when the expression (10) is null. Since (57) gives:

$$\gamma_{ikl} - \gamma_{ilk} = \lambda_k^\nu \lambda_l^\rho \left(\frac{\partial}{\partial x^\rho}\lambda_{i|\nu} - \frac{\partial}{\partial x^\nu}\lambda_{i|\rho} \right),$$

the difference:

$$\lambda_{i|\nu\rho\sigma} - \lambda_{i|\nu\sigma\rho}$$

transforms like (19) and then exhibits the four-index B's of Riemann. One may then set:

$$\gamma_{ij,hk} = \frac{d\gamma_{ijh}}{ds_k} - \frac{d\gamma_{ijk}}{ds_h} + \gamma_{ijl}\,(\gamma_{lhk} - \gamma_{lkh}) + \gamma_{lik}\,\gamma_{ljh} - \gamma_{lih}\,\gamma_{ljk} ,$$

and confirm that:

$$\gamma_{ij,hk} = -\gamma_{ij,kh} , \qquad \gamma_{ij,hk} = -\gamma_{ji,kh} , \qquad \gamma_{ij,hk} = -\gamma_{hk,ij} ,$$
$$\gamma_{ij,hk} + \gamma_{ih,kj} + \gamma_{ik,jk} = 0,$$

the latter relations being comparable to the ones that were given for the B's of the second type.

Briefly, Ricci's theory, in its fundamental formulas, recalls, at the same time, the theory of groups and that of Riemann spaces. However, the theory of groups, as Cartan [**4**] and Schouten [**15**] have shown, may be a theory of manifolds with torsion; one may thus have, in the theory of Ricci coefficients, everything that one has in that of Riemann spaces, generalized by the appearance of torsion.

6. On certain linear and homogeneous differential systems. – We would now like to focus on the theory of groups more specifically by studying the differential systems (36). We have seen how these systems arise in group spaces with the notion of parallelism. However, we have also shown, in the beginning of our fascicle XXXIII of the *Mémorial,* how they arise from conditions of simplicity.

The system:

$$\frac{d\theta^s}{dt} + C_k^s \theta^k = 0 \qquad (s, k = 1, 2, \ldots, r),$$

where the C_k^s are arbitrary functions of t, is not generally manageable. We attempt to diminish the generality by making the r^2 coefficients C_k^s depend only upon the r functions λ^j. For this, there is no method that is simpler and more intuitive than the one that consists of linearly setting:

$$C_k^s = c_{jk}^s \lambda^j ,$$

the three-index c's being constants. *The difficulty is diminished by linearization*. One thus has the system:

$$\frac{d\theta^s}{dt} + c_{jk}^s \lambda^j \theta^k = 0, \tag{58}$$

which is nothing but (36), up to notation.

We shall now see the essential fact that the search for certain new conditions of simplicity that one might add to the system (58) obligates the three-index c's to satisfy the relations (1) and (2).

Let:

$$f(\theta^1, \theta^2, \ldots, \theta^r, t)$$

be an integral of (58), i.e., an expression that remains constant by virtue of this system. One will have:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \theta^s}\frac{d\theta^s}{dt} = 0,$$

and, from (58):

$$\frac{\partial f}{\partial t} - c_{jk}^s \lambda^j \theta^k \frac{\partial f}{\partial \theta^s} = \frac{\partial f}{\partial t} + \lambda^j E_j(f) = 0.$$

One sees that we introduce the operator:

$$E_j(f) = - c_{jk}^s \theta^k \frac{\partial f}{\partial \theta^s} . \tag{59}$$

We study its properties, notably its permutability properties. One has:

$$E_j\,E_i = c^s_{jk}\theta^k \frac{\partial}{\partial\theta^s}\left(c^t_{il}\theta^l \frac{\partial}{\partial\theta^t}\right) = c^s_{jk}c^t_{il}\theta^k\left(\frac{\partial\theta^l}{\partial\theta^s}\frac{\partial}{\partial\theta^t} + \theta^l\frac{\partial^2}{\partial\theta^s\partial\theta^t}\right),$$

(60) $$(E_j\,E_i) = E_j\,E_i - E_i\,E_j = (c^s_{jk}c^t_{is} - c^s_{ik}c^t_{js})\theta^k \frac{\partial}{\partial\theta^t}.$$

Set:

(61) $$C^t_{ijk} = c^s_{ij}c^t_{ks} + c^s_{jk}c^t_{is} + c^s_{ki}c^t_{js},$$

(62) $$\gamma^s_{ij} = c^s_{ij} + c^s_{ji}.$$

Equation (60) may then be written:

$$(E_j\,E_i) = (C^t_{ijk} + c^s_{ji}c^t_{ks} - \gamma^s_{ij}c^t_{ks} - \gamma^s_{ik}c^t_{js})\theta^k \frac{\partial}{\partial\theta^t}.$$

It is now quite remarkable that this latter expression becomes particularly simple if the expressions (61) and (62) are always null. One has, under these conditions, from (59):

(63) $$(E_j\,E_i) = c^k_{ji}c^t_{sk}\theta^s \frac{\partial}{\partial\theta^t} = c^s_{ji}E_s.$$

It is obviously quite possible for us to study the system (58) when (61) and (62) are non-null, but this nullity seems to be a simple circumstance attached to the system, and which one may exhibit without preliminaries. Nothing will prevent us from comparing the results of this paragraph with those of the preceding ones in an interesting manner. Therefore, upon confronting (60) and (63) with (34*):

$$(E_j\,E_i) = c^s_{ji}E_s = B^t_{kij}\theta^s \frac{\partial}{\partial\theta^t}.$$

The contracted relation:

(3) $$c^i_{is}c^s_{jk} = 0$$

assures us that the system (58) *has the integral:*

$$c^i_{is}\theta^s = \text{const.}$$

The verification is immediate. Along the same order of ideas, with r new constants g_j, and upon setting:

$$g_j\,\lambda^j\,dt = du,$$

the system (58) may be written:

(64) $$\lambda^j A^s_j = 0,$$

with:

(65) $$A_j^s = g_j \frac{d\theta^s}{du} + c_{jk}^s \theta^k .$$

This gives:

$$c_{is}^i A_j^s = 0.$$

The determinant of the A_j^s is null.

We remark that one may make the system (58), with r equations and r unknown functions, correspond to the system (65) of r^2 equations, which, on the other hand, has *constant coefficients.*

If, between these r^2 equations, one may eliminate the r functions θ^s then one will obviously find relations between the A_j^s which are related to the λ^j in (64). One might have in them a means of seeing how the systems (58) are attached to integrable systems by quadratures. However, the elimination involved demands both algebraic and differential or integral considerations; we limit ourselves to pointing out how one might investigate this subject.

7. Inhomogeneous systems. – The filament of simple analogies leads one quite naturally to add to (58) the new system:

(66) $$\frac{d\theta^{s\rho}}{dt} + c_{jk}^s \lambda^j \theta^{k\rho} = \frac{\partial \lambda^s}{\partial \lambda_\rho} .$$

This is only an assemblage of r systems (58) provided with a right-hand side, and as for them, their choice is again truly as simple as possible if one imagines that the functions λ^r contain, besides t, constants λ_ρ that are r in number. Here, as for (58), we commence by studying the system (66) without making any hypothesis on the constants c_{jk}^s .

Moreover, by the same argument as in fascicle XXXIII of the *Mémorial* (Chap. III, § 2) and upon setting:

(67) $$V^{s\rho\tau} = \frac{\partial \theta^{s\rho}}{\partial \lambda_\tau} - \frac{\partial \theta^{s\tau}}{\partial \lambda_\rho} + c_{jk}^s \theta^{k\rho} \theta^{j\tau} ,$$

it easily becomes [**16**]:

(68) $$\frac{\partial^2 \lambda^s}{\partial \lambda_\rho \, \partial \lambda_\tau} - \frac{\partial^2 \lambda^s}{\partial \lambda_\tau \, \partial \lambda_\rho} = \frac{\partial}{\partial t} V^{s\rho\tau} + c_{jk}^s \lambda^j V^{k\rho\tau} - \gamma_{jk}^s \theta^{j\tau} \frac{\partial \theta^{k\rho}}{\partial t}$$
$$- \lambda^m \theta^{l\rho} \theta^{n\tau} (C_{lmn}^s + \gamma_{mj}^s c_{nl}^j + \gamma_{ml}^j c_{jn}^s) .$$

Again, this gives us every reason to believe that one will have a particularly simple theory of differential systems of the type (66), with respect to the constants λ_ρ introduced in the λ^j, when the expressions (61) and (62) are null.

The preceding formula will likewise take a form that is comparable to formula (56) in the theory of Ricci coefficients. Nevertheless, here, the considerations of ordinary analyticity make the left-hand side of (68) identically null in such a way that with the nullity of the expressions (61) and (62) equation (68) reduces to:

$$\frac{\partial}{\partial t}V^{s\rho\tau} + c^s_{jk}\lambda^j V^{k\rho\tau} = 0,$$

which reproduces the form of (58).

We have shown (*loc. cit.*) that the $\theta^{s\rho}$, which integrate the system (66) and are annulled for $t = 0$, likewise render the expressions (67) null. From the integration of the Maurer-Cartan system:

$$\frac{\partial\theta^{s\rho}}{\partial\lambda_\tau} - \frac{\partial\theta^{s\tau}}{\partial\lambda_\rho} = c^s_{kj}\theta^{k\rho}\theta^{j\tau}.$$

This latter system might no longer exist if the expressions (61) and (62) are not always null; the verification is easy [**11**] (pp. 15-16). However, the analysis of the present paragraph explains the fact in a much more profound manner by starting with a differential system (66) that is meaningful no matter what the constant coefficients c^s_{jk}.

In a general manner we arrive at a question of analysis that is as important as it is difficult, that of examining the differential systems constructed with constants and which have properties that are extremely different according to whether or not certain relations of an arithmetic nature exist between these constants. One knows only very little about that subject, moreover.

CHAPTER II

MECHANICS AND NON-COMMUTATIVITY

1. Preliminaries. – We return to the fundamental identities, and notably to the first equality (5) of the preceding chapter. Such an identity takes various forms, by virtue of the fact that:

$$d(XY) = X\,dY + Y\,dX. \tag{1}$$

Now, (1) may be written:

$$\frac{d}{dX}(XY) - X\frac{d}{dX}Y = Y,$$

which proves the existence of symbols q and p such that:

$$qp - pq = \frac{ih}{2\pi} I\,. \tag{2}$$

The Planck factor having simply been introduced, p and q may be treated as constant coefficients. The secret principle of the new mechanics is in (2), which was exhibited by H. Weyl [**17**].

Determinants constitute an essential instrument for the transformation of our fundamental identities. This is why most of the formulas of the absolute differential calculus preserve the symmetry of the determinant. Considerations of the same nature may come about in the context of (2).

Let there be two determinants of the same order:

$$x = |\,a_{hi}\,|, \qquad y = |\,b_{jk}\,|.$$

One has:

$$xy = |\,a_{hm}\,b_{jm}\,|, \qquad xy = |\,b_{jm}\,a_{hm}\,|, \qquad xy = yx.$$

The multiplication of determinants is commutative.

Now let there be *matrices* [**18**]:

$$x = (a_{hi}), \qquad y = (b_{jk}), \tag{3}$$

with which one has:

$$xy = (a_{hm}\,b_{mk}), \qquad yx = (b_{jm}\,a_{mi}) \equiv (b_{hm}\,a_{mk}).$$

Here, there is no general reason for yx to be equal to xy. It is quite imperative that we try to verify (2) with the matrices (3).

2. Non-commutativity and Poisson brackets. – With the action variables J and the angle variables νt of the classical theory, the coordinates are of the form [**19**]:

$$x = \sum x(\alpha_1, \cdots, \alpha_s; J_1, \cdots, J_s)\, e^{2i\pi(\alpha_1\nu_1 + \cdots + \alpha_s\nu_s)t} \,,$$

the summations being taken over all *integers* α_i. This may be written, to abbreviate:

(4) $$x = x(\alpha, J)\, e^{2i\pi(\alpha\nu)t}.$$

In the quantum theory, x is a set of terms:

$$x(n, n-\alpha)\, e^{2i\pi\nu(n,\, n-\alpha)t},$$

such that one has (correspondence principle), for large n:

$$x(n, n-\alpha) \to x(\alpha, J).$$

Now, suppose one has a matrix $xy - yx$, one of whose elements is:

$$\sum_k \begin{vmatrix} x(nk) & y(nk) \\ x(km) & y(km) \end{vmatrix} e^{2i\pi\nu(nm)t}.$$

For large values of m and n, Dirac wrote this in a form equivalent to:

$$\sum_{\alpha+\beta} \sum_{=n-m} \begin{vmatrix} x(n, n-\alpha) & y(n, n-\beta) \\ x(n-\beta, n-\beta-\alpha) & y(n-\alpha, n-\alpha-\beta) \end{vmatrix} e^{2i\pi\nu(n, n-\alpha-\beta)t}.$$

For very large n, with Δ due to $n_r \to n_r + \tau_r$:

$$\frac{\Delta}{h} x(n, n-\alpha) \to \sum \tau_r \frac{\delta}{\delta J_r} x(\alpha, J).$$

Under these conditions, if, in the determinant of the preceding expression, one subtracts the first row from the second one then this expression becomes:

$$-\sum\sum\sum h \begin{vmatrix} \alpha_r x(\alpha, J) & \beta_r y(\beta, J) \\ \dfrac{\delta}{\delta J_r} x(\alpha, J) & \dfrac{\delta}{\delta J_r} y(\beta, J) \end{vmatrix} e^{2i\pi[(\alpha\nu)t + (\beta\nu)t]}.$$

Now let $w_r = \nu_r t + e_r$, with e_r a constant phase. One easily obtains:

$$\frac{\delta}{\delta w_r}[y(\beta, J)\, e^{2i\pi(\beta\nu)t}] = 2i\pi\beta_r\, y(\beta, J)\, e^{2i\pi(\beta\nu)t}.$$

The preceding triple sum then takes a form such that the matrix corresponds to the expression:

(5) $$xy - yx = \frac{ih}{2\pi}\sum\begin{vmatrix}\dfrac{\delta x}{\delta w_r} & \dfrac{\delta y}{\delta w_r}\\ \dfrac{\delta x}{\delta J_r} & \dfrac{\delta y}{\delta J_r}\end{vmatrix} = \frac{ih}{2\pi}[xy].$$

Therefore, *the Poisson brackets, which are classically formed from the coordinates* (4), *correspond, in quantum theory, to a calculus of matrices with non-commutative multiplication.*

We construct the partial derivatives with the δ and write *brackets*, rather than *parentheses*, in order to be in agreement with the notations of Birtwistle [**19**].

The preceding considerations are due to Dirac and Heisenberg; they have been generalized in [**20**] with some extensions of the Poisson brackets.

3. Fundamental double theorem. – The canonical equations of Hamilton and Jacobi play an essential role in the new mechanics. Like all of the essential foundations of physical theories, we attach them to an identity (7), and this follows from a double theorem, both sides of which have been studied over a very long time interval [**21**], [**22**], [**11**].

a. Suppose one has the Green formula:

(6) $$\int \alpha_i \Phi_i d\sigma = \int \frac{\partial \Phi_i}{\partial x_i} d\tau, \qquad \operatorname{div}\Phi = \frac{\partial \Phi_i}{\partial x_i} \qquad (i = 1, 2, \ldots, n)$$

and the identity:

(7) $$\int_{W_{n-1}} X_1 dX_2 \cdots dX_n = \int_{W_n} dX_1 dX_2 \cdots dX_n .$$

One passes from (6) *to* (7) *by the transformation:*

$$X_i = X_i(x_1, x_2, \ldots, x_n)$$

if

$$X(f) = \frac{\Phi_i}{\operatorname{div}\Phi}\frac{\partial f}{\partial x_i} = 0, \text{ with } \quad f = X_2, X_3, \ldots, X_n$$

and if:

$$X(U_1) = 1, \qquad \text{with} \quad U_1 = \log X_1 .$$

b. This permits the construction of the Jacobi multiplier:

$$D = \frac{\partial(U_1, X_2, \cdots, X_n)}{\partial(x_1, x_2, \cdots, x_n)}$$

with which:

$$Y(f) = \frac{1}{D}\left|\begin{array}{cccc|c} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots & \dfrac{\partial f}{\partial x_n} & 0 \\ \hline & & & & F_1 \\ & & & & F_2 \\ & D & & & \cdots \\ & & & & F_n \end{array}\right|,$$

which gives:

(8) $$X[Y(\ )] = Y[X(\ )].$$

The F's are arbitrary functions of $X_2, \ldots, X_n$*, but not* U_1.

On this subject, other than the references that were already pointed out, one may consult two recent memoirs, the one by Pfeiffer [**35**], the other, by the author [**36**].

It is always the evaluation of an extended W, by a very direct transformation, that gives formulas of physical significance. As for the permutability in (8), this may be the germ of quite a lot of non-permutability.

One may attach several current theories to the equation:

(9) $$\operatorname{div} \Phi = \frac{\partial \Phi_i}{\partial x_i} = 0.$$

Bateman [**37**] likewise sees in equation (9) the origin of all of the fundamental equations of physics.

Therefore, with the variables p_i, q_i divided into two subsets, (9) may be written:

(10) $$\frac{\partial P_i}{\partial p_i} + \frac{\partial Q_i}{\partial q_i} = 0,$$

which implies, as naturally as possible:

$$P_i = -\frac{\partial H}{\partial q_i}, \qquad Q_i = \frac{\partial H}{\partial p_i}, \qquad dH = Q_i\, dp_i - P_i\, dq_i = 0,$$

if the motion takes place on the manifold H = const. One is then further led to set, always as simply as possible:

$$P_i = \frac{dp_i}{dt}, \qquad Q_i = \frac{dq_i}{dt},$$

which implies the canonical equations:

(11) $$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}.$$

One may imagine other things by starting with (9). Upon setting, for example:

$$\Phi_i = \frac{\partial V}{\partial x_i},$$

one will have a Laplace equation in n variables, which will give to the mechanics in p, q the aspect of an extension of the theory of the Newtonian potential. Since this Laplace equation itself may take on the aspect of a generalization of the wave equations, the mechanics will take on a wavelike character. (*See*, later on, Chapter III.)

In (10), we also have the famous Liouville theorem:

$$\frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{p}_i}{\partial q_i} = 0,$$

which is fundamental to statistical mechanics.

If x is a function of the p and q then:

$$\frac{dx}{dt} = \frac{\partial x}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial x}{\partial p_i}\frac{\partial H}{\partial q_i} = [x, H].$$

Hence, from (5):

$$\frac{ih}{2\pi}\dot{x} = xH - Hx.$$

These are the Heisenberg equations [**19**]. Compare with [**17**] (page 28).

Now take the Poisson-Jacobi identity:

(12) $$[\varphi_l(\varphi_m , \varphi_n)] + [\varphi_m(\varphi_n , \varphi_l)] + [\varphi_n(\varphi_l , \varphi_m)] = 0,$$

by virtue of which $[\varphi_m , \varphi_n]$ is an integral of (11) if the same is true for φ_m and φ_n .

When one has, *linearly:*

$$[\varphi_m , \varphi_n] = c^s_{mn}\varphi_s ,$$

one concludes, from this and (12) that:

(13) $$\begin{cases} c^s_{mn} + c^s_{nm} = 0, \\ c^s_{lm}c^\alpha_{ns} + c^s_{nm}c^\alpha_{ls} + c^s_{nl}c^\alpha_{ms} = 0. \end{cases}$$

These are the fundamental structural relations of the theory of finite, continuous groups. One sees that these relations (13) are linked to mechanics, just as they are to gravity.

Among the mechanical theories with canonical equations, the most important is truly statistical mechanics [**6**]. In the space of N dimensions, the product $dp_1, \ldots, dq_N$ measures a cellular extent of *weight* $K\, dp_1, \ldots, dq_N$. If there are M systems present then the cell has an extent:

$$(dp_1, \ldots, dq_s)_1 \ldots (dp_1, \ldots, dq_s)_M .$$

One may make each of the parentheses in this product correspond to an *energetic* factor:

$$K_1 e^{-2jE_1}, \ldots, K_M e^{-2jE_M},$$

and imagine that the product of these factors is also a K if:

$$E_1 + \ldots + E_M = \text{const.}$$

Therefore, the *phase* space is complicated; it extends by factorization as long as the energies E_i combine *by addition*. That insight, as rudimentary as it is, demonstrates the fundamental role of the exponential in statistical mechanics. In physics, the exponential appears for a lot of other reasons, for example, in the equation:

$$dA = k\,A\,dt,$$

which governs a host of simple phenomena [**23**]. It is essential for the representation of periodic phenomena, but things might not go to nothing indefinitely as with the *ordinary* exponential. Groups introduce infinitesimal transformations X and corresponding finite transformations e^X; this is a *symbolic* exponential, with non-commutative multiplication.

4. The symbolic exponential. – The first studies of any appreciable profundity on the preceding symbolic differential seem to have been carried out by J.-E. Campbell, who dedicated two interesting memoirs [**24**], [**25**] to them in 1897. One is surprised when one peruses them to find a language and set of preoccupations that strangely resemble the language and preoccupations that one encounters in the work devoted to microphysics in our own time. Furthermore, Campbell played the distinguished role of having inspired Henri Poincaré.

The celebrated memoirs of the latter on groups [**26**] make immediate usage of the preceding symbolic exponential. Henri Poincaré paid the homage to Campbell that is his due; likewise, one must not forget that the symbolic exponential was employed by Lie and his immediate disciples, but only as convenience. Poincaré, whose fundamental ideas we shall soon present, afforded it his inspired spirit of generalization; we can comprehend his work better by first summarizing the much more elementary analysis of the first memoir of Campbell [**24**], a memoir whose notations we shall preserve as much as possible.

J.-E. Campbell first considered two operators, x and y, which are associative and distributive, but *not commutative*. As a consequence, one writes:

$$\begin{aligned} y_1 &= yx - xy, \\ y_2 &= y_1x - xy_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ y_r &= y_{r-1}x - xy_{r-1}\,. \end{aligned}$$

He likewise writes:

$$[yx^r] = yx^r + xy\,x^{r-1} + x^2y\,x^{r-2} + \ldots + x^r\,y$$

and defines constants a_i such that:

$$a_1 = \frac{1}{2}, \qquad a_2 = \frac{1}{12}, \qquad a_3 = 0, \qquad a_1 = -\frac{1}{720}, \quad \ldots,$$
$$(m + 1)\, a_m = a_{m-1} - (a_1\, a_{m-1} + a_2\, a_{m-2} + \ldots + a_{m-1}\, a_1).$$

All of the a's with odd indices are null, except for a_1. One then has:

$$\frac{yx^r}{r!} = \left[\frac{yx^r}{(r+1)!}\right] + a_1\left[\frac{y_1 x^{r-1}}{r!}\right] + \cdots + a_{r-1}\left[\frac{y_{r-1}x}{2!}\right] + a_r y_r .$$

This formula is easily verified for $r = 1, 2, 3, \ldots$; the general proof is made by recurrence upon passing from r to $r + 1$ with no difficulty. Moreover, let:

$$y = y,$$
$$yx = \tfrac{1}{2}[yx] + a_1 y_1 ,$$
$$y\frac{x^2}{2!} = \left[y\frac{x^2}{3!}\right] + a_1\left[y_1\frac{x}{2!}\right] + a_2 y_2 ,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

If one sets:

$$z = y + a_1 y_1 + a_2 y_2 + \ldots$$

then the addition of the preceding formulas gives:

$$y\, e^x = z + \left[z\frac{x}{2!}\right] + \left[z\frac{x^2}{3!}\right] + \ldots$$

If μ is a constant such that one may neglect its square then:

$$(x + \mu\, z)^r = x^r + m[z\, x^{r-1}],$$

(14) $$(1 + m\, y)e^x = 1 + \frac{x+\mu z}{1!} + \frac{(x+\mu z)^2}{2!} + \ldots = e^{x-\mu z}.$$

One may write, with the same approximation:

$$e^{\mu z}\, e^x = e^{x+\mu z}.$$

We shall see later on that Poincaré disdained these formulas as approximations, and that he sought, in a very ingenious manner, to find how one might maintain them when μ ceases to be a very small constant.

Now let:

$$X = \xi_i(x)\frac{\partial}{\partial x_i}, \qquad X' = \xi_i(x')\frac{\partial}{\partial x_i'}.$$

To abbreviate, $\xi_i(x)$ signifies $\xi_i(x_1, x_2, \ldots, x_n)$; analogous remarks apply to the x'.

Likewise:

$$Y = \eta_i(x)\frac{\partial}{\partial x_i}, \qquad Y' = \eta_i(x')\frac{\partial}{\partial x_i'} = Y'(x')\frac{\partial}{\partial x_i'}.$$

We also have:

$$f(x') = f(x) + \frac{t}{1!}X(f) + \frac{t^2}{2!}X^2(f) + \ldots = e^{tX},$$

$$f(x) = f'(x') + \frac{t}{1!}X'(f') + \frac{t^2}{2!}X'^2(f') + \ldots = e^{-tX'}.$$

Hence:

$$x_i = e^{-tX'(x_i')},$$

(15) $$Y'(x_i) = Y'e^{-tX'(x_i')} = e^{tX}Ye^{-tX(x_i)}.$$

Having said this, the principal point of the first memoir of Campbell consists of establishing that one also has:

(16) $$Y'(x_i) = e^{-tY_*},$$

the exponential being developed as in the preceding by nonetheless observing that the n^{th} power of Y_* will be Y_n , and that one will have:

(17) $$\begin{cases} Y_1 = YX - XY, \\ Y_2 = Y_1X - XY_1, \\ \qquad \ldots\ldots\ldots\ldots\ldots\ldots \\ Y_r = Y_{r-1}X - XY_{r-1}. \end{cases}$$

One must therefore establish that under these conditions the right-hand sides of (15) and (16) are equal.

For this, it will suffice that in these expressions the coefficients of t^r are equal. One must then prove that:

$$(-1)^r\frac{Y_r}{r!} = \frac{X^rY}{r!} - \frac{X^{r-1}Y}{(r-1)!}\frac{X}{1!} + \frac{X^{r-2}Y}{(r-2)!}\frac{X^2}{2!} - \ldots$$

This is true for $r = 1$. Rewrite the equality after replacing r with $r - 1$ in it, and then multiplying by X, once on the right, and then on the left. One has:

$$(-1)^{r-1}\frac{Y_{r-1}X}{(r-1)!} = \frac{X^{r-1}YX}{(r-1)!} - \frac{X^{r-2}YX}{(r-2)!}\frac{X^2}{1!} + \frac{X^{r-3}Y}{(r-3)!}\frac{X^3}{2!} - \ldots$$

$$(-1)^{r-1}\frac{XY_{r-1}}{(r-1)!} = \frac{X^rY}{(r-1)!} - \frac{X^{r-1}Y}{(r-2)!}\frac{X}{1!} + \frac{X^{r-2}Y}{(r-3)!}\frac{X^2}{2!} - \dots$$

hence, by subtraction:

$$(-1)^{r-1}\frac{Y_{r-1}X}{(r-1)!} = -\frac{X^rY}{(r-1)!} + r\frac{X^{r-1}YX}{(r-1)!} - r\frac{X^{r-2}Y}{(r-2)!}\frac{X^2}{2!} + \dots;$$

upon dividing by – r one recovers the equality to be established and that this process of recurrence is effectively established for any value of the integer r. Briefly: the fundamental equality (16) is proved.

Now, more explicitly, let there be r infinitesimal transformations X_i such that:

(18) $$X_i X_j - X_j X_i = c_{ij}^s X_s .$$

One must prove that they generate a group; i.e., that if one has:

$$x = e^X, \qquad X = \lambda_i X_i ,$$
$$x' = e^Y, \qquad Y = \mu_i X_i$$

then one also has:

$$x'' = e^Z, \qquad Z = \nu_i X_i .$$

This is what Poincaré called the *Campbell problem* in his first memoir.

One must prove that:

$$e^{\mu_i X_i'} = e^Z.$$

Campbell, likewise in his first memoir, was content to show this equality by assuming that the μ are sufficiently small that one may replace it with:

$$(1+\mu_i X_i')x_i' = e^Z.$$

The expression $\mu_i X_i' = Y'$ is then of the form $\rho_i X_i$, from formula (16), when one develops the right-hand side, while taking into account equalities (17) and (18). Therefore, one must now prove that:

$$(1 + \mu U)\, e^X = e^Z,$$

if U and X are of the form $\rho_i X_i$. Now, one may transform the left-hand side using (14) and write:

$$e^{X+\mu\bar{U}} = e^Z,$$

with:

$$\bar{U} = U + a_1 U_1 + a_2 U_2 + \dots$$

Since the U_i are defined like the Y_i of the table (17), one sees that U is also of the form $\rho_i X_i$, which completes the proof.

This proof is certainly full of interest; it has an undeniable esthetic. However, at the end of it all it seems weak due to the necessity of supposing that μ and the μ_i are sufficiently small for one to be able to neglect their powers and products. Campbell himself remedied this defect in his second memoir.

Henri Poincaré then brought the remedy to perfection, as we shall see in the following paragraphs. His analysis, while often difficult, seems nonetheless more approachable when one is habituated to the reasoning of Campbell, and that is why we have commenced with it. A fundamental notion for Poincaré is that of *regular polynomial*. A symbolic polynomial formed from the X_i is called *regular* when it contains nothing but powers of expressions of the form $\rho_i X_i$.

Any polynomial may be *regularized* by making use of the relations (18). This may be done in only one way. We assume these two assertions, which Poincaré proved in full rigor and not without some length. In (14), we have a regular series; the developments of the Lie exponentials are given from others.

In order to return to the very considerable merit of Campbell's argument, we note that the coefficients a_i are easily expressed in terms of Bernoulli numbers. If:

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + B_1 \frac{t^2}{2!} - B_3 \frac{t^4}{4!} + \ldots$$
$$(2n)!\, a_{2n} = (-1)^{n-1}\, B_{2n-1}.$$

This leads us to the works of Schur that we cited in our fascicle XXXIII. The Bernoulli numbers do not play a fortuitous role in the theory of groups; they quite naturally accompany any exponential analysis, whether ordinary or symbolic.

5. The symbol $\Phi(\theta)$. – Consider the r infinitesimal transformation symbols, or, more briefly, the r operators X_i, and one of their linear combinations:

$$T = t_i\, X_i\, .$$

Then let V be another elementary operator, which might or might not be a linear combination of the operators X. However, V is assumed to be such that:

$$(VX_i) = VX_i - X_iV = b_{ij}X_j\, ,$$

hence:

$$(VT) = VT - TV = b_{ij}\, t_i\, X_j = \theta(T).$$

One may imagine some iterations such as:

$$\theta[\theta(T)] = \theta^2(T), \qquad \ldots, \qquad \theta[\theta^n(T)] = \theta^{n+2}(T), \qquad \ldots$$

Now, let:

$$\Phi(\theta) = \sum g_k\, \theta^k$$

be a polynomial or an ordered series in increasing powers of θ.

Henri Poincaré set:

$$\Phi(\theta)(T) = \sum g_k \, \theta^k(T).$$

Consider the *characteristic* equation:

$$B(\theta) = \begin{vmatrix} b_{11}-\theta & b_{21} & \cdots & b_{r1} \\ b_{21} & b_{22}-\theta & \cdots & b_{r2} \\ \cdots & \cdots & \cdots & \cdots \\ b_{1r} & b_{2r} & \cdots & b_{rr}-\theta \end{vmatrix} = 0. \tag{19}$$

It will convenient in the sequel to denote the elements of the determinant $B(\theta)$ by B_{ij} . Therefore, $B_{ij} = b_{ij}$ when i and j differ, while:

$$B_{ii} = b_{ii} - \theta.$$

The algebraic minor of B_{ij} will be denoted by P^{ij} and the same minor, when *normed*, by B^{ij}. Therefore, $BB^{ij} = P^{ij}$.

If the equation $B(\theta) = 0$ has r distinct roots then there exist r combinations:

$$Y_k = \alpha_{ik} \, X_i \, , \tag{20}$$

such that:

$$VY_k - Y_k V = \theta_k \, Y_k \, . \tag{21}$$

Here, it is obvious that k is not a summation index in the right-hand side, since that index is free in the left-hand side. Therefore:

$$T = t_i \, X_i = t'_k Y_k \, .$$

Multiplying (21) by t'_k, one has:

$$\theta(T) = \theta_k t'_k Y_k \, ,$$

$$\theta^2(T) = \theta_k \, \theta(T) = \theta_k^2 t'_k Y_k \, ,$$

and, in a completely general manner:

$$\Phi(\theta)(T) = \Phi(\theta_k) t'_k Y_k = h_i \, X_i \, .$$

Here, k is indeed a summation index since it will disappear in the left-hand side of the equation. From (20), one deduces, with the habitual notation for normed minors:

$$X_i = \alpha^{jk} \, Y_k \, , \qquad t'_k = \alpha^{jk} \, t_i \, ,$$

$$\Phi(\theta)(T) = \Phi(\theta_k) \, t'_k \, \alpha_{ik} \, X_k \, ,$$

(22) $$h_i = \Phi(\theta_k)\, t_i\, \alpha^{jk}\, \alpha_{ik}\,.$$

This last formula is already quite remarkable; naturally, k is a summation index here. If k is contained only in the sum of products:

$$\alpha^{jk}\,\alpha_{ik} = \alpha_i^j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

then we will obviously have a very simple and elegant result that the absolute differential calculus uses at each step today. However, in the right-hand side of (22), things present themselves in a much more general manner; one must introduce the coefficient $\Theta(\theta_k)$ into the summation over k, where the θ_k are the roots of an algebraic equation, and where Θ is an arbitrary function. Now, the treatment of this case has been the basis for a true stroke of genius on the part of Henri Poincaré. He first proposed to determine the product α^{jk} α_{ik} without summation by provisionally taking:

$$\Theta(\theta) = \frac{1}{\xi - \theta},$$

with ξ denoting an arbitrary constant. Then:

$$\frac{1}{\xi - \theta}(T) = h_i\, X_i = H,$$

(23) $$h_i = \frac{1}{\xi - \theta_k} t_j \alpha^{jk} \alpha_{ik}, \quad (\xi - \theta)(H) = T.$$

This last equation may be written:

(24) $$\xi h_i - b_{ki}\, h_k = t_i\,.$$

In this system, one may solve for the h's as functions of the t's. One finds ([1]):

$$h_i = -\, t_j \frac{P^{ij}}{B(\xi)} = -\, t_i\,,$$

the P^{ij} and B^{ij} containing, of course, ξ in place of θ. Since the h_i thus obtained are rational functions of ξ, they decompose into simple elements. One gets:

$$h_i = -\, t_j \frac{P_k^{ij}}{B'(\theta_k)(\xi - \theta_k)},$$

([1]) The solution of the system (24) leads one naturally to write the determinant in (19) as we did here. It is the determinant in the Mémoire of Poincaré with the rows and columns exchanged.

if P_k^{ij} is what P^{ij} becomes when one replaces ξ with θ_k. Comparing with (23), one has:

$$\alpha^{jk}\,\alpha_{ik} = -\,t_j\,\frac{P_k^{ij}}{B'(\theta_k)},$$

and finally:

(25) $$\Phi(\theta)(T) = -\,t_j\Phi(\theta_k)\,\frac{P_k^{ij}}{B'(\theta_k)}\,X_i\,.$$

One may further write, with a Cauchy integral:

$$\Phi(\theta)(T) = -\frac{1}{2i\pi}\int d\xi\,\Phi(\xi)\,t_i\,\frac{P^{ij}}{B(\xi)}\,X_i\,,$$

or better yet:

(26) $$\Phi(\theta)(T) = -\frac{1}{2i\pi}\int d\xi\,\Phi(\xi)\,t_i B^{ij} X_i\,.$$

The contour of integration must naturally contain all of the roots θ_i in its interior, and the function Φ must be holomorphic on this contour.

The result (26) is certainly more striking than (25); it restores all of the habitual simplicity that the summation indices i and j contribute. The formulas in which the summation index is triple play only a transitory role. Observe, as well, that formulas (26), (25), and the preceding ones are given by Poincaré without the *negative* sign in the right-hand side; this amounts to saying that the illustrious author was not preoccupied with the sign that one attributed to the minors P^{ij} exactly, that sign playing no essential part in the sequel.

It will remain for us to recall the preceding reasoning for the case where the characteristic equation has multiple roots; one will find several suggestions on that subject in the Mémoire of Poincaré.

If V is a linear combination with constant coefficients in the X_i then let:

$$V = v_i\,X_i\,,$$

and if:

$$(X_i\,X_j) = c_{ij}^s X_s\,,$$

one has:

(27) $$\theta(T) = v_j t_i c_{ij}^k X_k = b_{ik}\,t_i\,X_k\,, \qquad b_{ik} = c_{ij}^k v_j\,.$$

6. Fundamental exponential combinations. – Among the most immediate applications of the symbol $\Phi(\theta)(T)$, one must point out the beautiful theorem of Poincaré that is expressed simply by the union of the two formulas:

(28) $$e^{-\alpha V}\,e^{\beta t}\,e^{\alpha V} = e^{\beta U}, \qquad U = e^{-\alpha\theta}(T).$$

The product of three exponentials is a succession of transformations to be read from left to right. The theorem is first established when the constants α, β are assumed to be very small, and such that one may neglect the terms of third order in α and β.

Under these conditions, the left-hand side of the first equation in (28) may be written:

$$\left(1-\alpha V+\tfrac{1}{2}\alpha^2 V^2\right)\left(1+\beta T+\tfrac{1}{2}\beta^2 T^2\right)\left(1+\alpha V+\tfrac{1}{2}\alpha^2 V^2\right)$$

where:

$$1 + \beta T + - \alpha\beta(VT - TV) \equiv e^{\beta T - \alpha\beta\theta(T)},$$

from which:

$$U = T - \alpha\theta(T) \equiv e^{-\alpha\theta(T)}.$$

Now suppose that one pushes the approximation up to terms in β and α^m, inclusively. The left-hand side of (28) must then become a symbolic polynomial in V and T that one may make *regular* in just one way, namely:

$$\varphi(\alpha, \beta) = \sum A\Pi.$$

One will have:

$$\varphi(\alpha + d\alpha, \beta) = e^{-(\alpha+d\alpha)V}\, e^{\beta T}\, e^{(\alpha+d\alpha)V} = e^{-d\alpha V}\, e^{\beta T}\, e^{d\alpha V},$$

$$\varphi(\alpha + d\alpha, \beta) - \varphi(\alpha, \beta) = d\alpha \sum \frac{dA}{d\alpha}\Pi .$$

The left-hand side of this last equality linearly contains the A that therefore satisfy satisfy linear differential equations. Moreover, these A must reduce to the coefficients of $e^{\beta T}$ for $\alpha = 0$; these conditions suffice to determine them. Now, one may satisfy the differential equations in question by taking, conforming to (28):

$$\varphi(\alpha, \beta) = e^{\beta T}, \qquad U = e^{-\alpha\theta}(T).$$

Indeed, this gives:

$$\varphi(\alpha + d\alpha, \beta) = e^{\beta U'}, \qquad U' = e^{-(\alpha+d\alpha)\theta}(T),$$

and one must verify that:

$$e^{-d\alpha V}\, e^{\beta U}\, e^{d\alpha V} = e^{\beta U'}.$$

Now here, since one neglects the square of $d\alpha$, one may write, from (28):

$$e^{\beta U''} = e^{\beta U'}, \qquad U'' = e^{-d\alpha\theta}(U) = e^{-d\alpha\theta}\, e^{-\alpha\theta}(T) = U'.$$

Therefore, the theorem (28) is indeed verified to an approximation of order *one* in β and order m in α.

Now, extend the approximation in β. One will have:

$$\varphi(\alpha + d\alpha, \beta) = e^{-\alpha V}\, e^{(\beta+d\beta)T}\, e^{\alpha V} = \varphi(\alpha, \beta)\varphi(\alpha, d\beta),$$

and, from (28), since one neglects the square of $d\beta$:

$$\varphi(\alpha, d\beta) = e^{d\beta U}, \qquad U = e^{-\alpha\theta}(T).$$

Therefore:

$$\varphi(\alpha + d\alpha, \beta) = \varphi(\alpha, \beta)\, e^{d\beta U}.$$

This is the differential system that is analogous to the one formed in the preceding approximation.

It thus gives rise to the same reasoning. It is in accord with (28), which one sees again by writing:

$$\varphi(\alpha + d\alpha, \beta) = e^{(\beta + d\beta)U} = e^{\beta U}\, e^{d\beta\, U} = \varphi(\alpha, \beta)\, e^{d\beta U}.$$

Briefly, the theorem (28) is now completely established for any α and β.

It is obviously valid if:

$$V = v_i\, X_i\,, \qquad T = t_i\, X_i\,, \qquad (X_i\, X_k) = c_{ik}^{s}\, X_s\,.$$

The same is true for the operators V and X_i, the latter being r in number, if:

$$(VX_i) = b_{ik}\, X_k\,, \qquad (X_i\, X_k) = 0, \tag{29}$$

because this case immediately reverts to the preceding one.

Recall (28). Upon permuting V and T, one has:

$$e^{-\beta T}\, e^{\alpha V}\, e^{\beta T} = e^{\alpha W}, \qquad W = e^{-\beta\eta}(V). \tag{30}$$

The symbol η is formed with T as the symbol θ is with V. Therefore:

$$\eta(Y) = (TY), \qquad \eta(V) = (TV) = -\,\theta(T).$$

If the second of relations (29) enter into play then one has:

$$\eta(X) = 0, \qquad \eta^2(V) = 0, \qquad \eta^m(V) = 0,$$
$$e^{-\beta\eta}(V) = V - \beta\eta(V) = V + \beta\theta(T).$$

Formula (30) thus becomes:

$$e^{-\beta T}\, e^{\alpha V}\, e^{\beta T} = e^{\alpha V \,+\, \alpha\beta\,\theta(T)}. \tag{31}$$

This formula is not true if one abandons the second relation of (29), but the latter may again be considered as being satisfied for very small X, and thus for T that are likewise very small and of first order, one may make the β play the role of infinitely small in (31), and not T, since T appears only with the factor β.

To the same degree of approximation, formula (31) may be written:

$$e^{\alpha V + \alpha\beta\,\theta(T)} \equiv e^{\alpha V} - \beta T\, e^{\alpha V} + e^{\alpha V}\, \beta T \equiv e^{\alpha V} - e^{\beta T}\, e^{\alpha V} + e^{\alpha V}\, e^{\beta T},$$

or, by virtue of (28):

$$e^{\alpha V + \alpha\beta\,\theta(T)} \equiv e^{\alpha V} - e^{\alpha V}\, e^{\beta U} + e^{\alpha V}\, e^{\beta T}, \qquad\qquad U = e^{-\alpha\theta}(T),$$

and, always while neglecting the square of β:

$$e^{\alpha V + \alpha\beta\,\theta(T)} \equiv e^{\alpha V}\,(1 - \beta U + \beta T) \equiv e^{\alpha V}\, e^{\beta(T-U)}\,.$$

If we set:

(32) $$\alpha\theta(T) = W, \qquad T - U = Y$$

then it becomes:

(33) $$e^{\alpha V+\beta W} = e^{\alpha V}\, e^{\beta W}, \qquad Y = \frac{1-e^{-\alpha\theta}}{\alpha\theta}(W)\,.$$

Henri Poincaré posed a question here that was subtle in appearance, but which was meanwhile necessary in order for one to comprehend the necessity of all of the detours that were made in the preceding reasoning, at first sight. In the first of equations (32), where T represents $t_i\, X_i$ and where W represents $w_i\, X_i$, might one determine the t for any w? It is necessary that one have:

$$\alpha\, \beta_{ij}\, t_i = w_j\,,$$

which is obviously possible if only the determinant of the β_{ij} is non-null. However, since one has never reasoned that with the X_i this determinant will always be null, from the second of relations (27), one infers the necessity of commencing with the case where V is forced to be a linear combination of the X_i; the case of V a linear combination of the X may then follow as a limiting case, the characteristic equation (19) having a null root in this limiting case, a fact that is well-known and which does not alter the generalities that are associated with that equation, except to simplify them slightly.

7. Generation of the X_i by starting with the structure. – Therefore, let the r operators X_i be linked by the relations:

$$(X_i\, X_j) = c_{ij}^s X_s\,,$$

in which the c_{ij}^s are the "structure constants," which are linked by fundamental structural relations and are given in advance. Also let, as in the preceding:

$$T = t_i\, X_i\,, \qquad U = u_i\, X_i\,, \qquad v = v_i\, X_i\,, \qquad W = w_i\, X_i\,.$$

Let the series be *regularized:*

(34) $$\varphi(\alpha, \beta) = e^{\alpha V}\, e^{\beta T} = \varphi_0 + \beta\varphi_0 + \beta^2\varphi_2 + \ldots$$

One has:

$$\varphi(\alpha, \beta + d\beta) = e^{\alpha V} e^{\beta T} e^{d\beta T} = \varphi(\alpha, \beta)\, e^{d\beta T} = \varphi(\alpha, \beta)(1 + d\beta\, T);$$

hence:

$$\frac{d\varphi}{d\beta} = \varphi T, \qquad m\varphi_m = \varphi_{m-1} T.$$

These conditions, combined with $\varphi_0 = e^{\alpha V}$, determine φ. Now make:

(35) $$\varphi(\alpha, \beta) = e^W, \qquad \varphi(\alpha, \beta + d\beta) = e^{W+dW};$$

it becomes:

$$e^{W+dW} = e^W e^{d\beta T}.$$

Now, from (33), one may satisfy this upon setting:

(36) $$d\beta\, T = \frac{1-e^{-\eta}}{\eta}(dW),$$

if η is a symbol that is to W what θ is to V. One thus has, in (36), a symbolic representation of a system of differential equations, equations that must be satisfied for the coefficients w_i. From (26), one may write:

$$d\beta\, T = -\frac{1}{2i\pi}\int d\xi \frac{1-e^{-\xi}}{\xi} dw_j B^{ij} X_i,$$

hence:

(37) $$t_i\, d\beta = -\frac{1}{2i\pi}\int d\xi \frac{1-e^{-\xi}}{\xi} dw_j B^{ij}.$$

Here, one has:

$$(W X_i) = c_{ki}^s w_k X_s = b_{is} X_s;$$

the characteristic equation, after a change of rows and columns in the determinant, is (19); i.e., $B(\xi) = 0$. As for B^{ij}, it is the *normed* minor, defined as above in equation (19). Therefore, B^{ij} is a rational fraction in ξ whose denominator $B(\xi)$ is of degree r, while its numerator is of degree $r - 1$. The integrals must be carried out in the plane of the complex variable ξ along a contour that envelops all of the roots of the characteristic equation.

It is now easy to conclude. If the w_j satisfy the differential equations (36), integrated in such a manner that they give $w_i = v_i$ – i.e., $\varphi = \varphi_0$ for $\beta = 0$ – then the two forms of $\varphi(\alpha, \beta)$, as written in (34) and (35), are equal (everything is now arranged so that this is true); finally, one has:

$$e^{\alpha V} e^{\beta T} = e^W.$$

The product of the finite transformations $e^{\alpha V}$ and $e^{\beta T}$ is indeed a transformation of the same form, in which only the parameters are changed; they define a *group*. The Campbell problem is then solved.

Nevertheless, the question, when treated as we just did, does not stop with that. We have not reached the end of the first Mémoire of Poincaré, while the illustrious scholar dedicated three of them to the subject. For the moment, we content ourselves with pointing out two very important transformations of equations (37).

We have:

$$B^{ij}B_{ik} = B^{ij}\, b_{ik} - \xi\, B^{kj} = \begin{cases} 0 & \text{if } k \neq j, \\ 1 & \text{if } k = j, \end{cases}$$

hence:

$$b_{ik}\, t_i\, d\beta = -\frac{1}{2i\pi}\int d\xi(1 - e^{-\xi})\, dw_j B^{kj}\ .$$

The use of the preceding relations for $k = j$ changes nothing in this result, since the Cauchy integral is then augmented with another one that is identically null.

Here is another outcome of (37), which is prodigiously important, this time.

Equation (36) may be written:

$$\frac{dW}{d\beta} = \frac{\eta}{1 - e^{-\eta}}(T) = -\frac{1}{2i\pi}\int \frac{\xi d\xi}{1 - e^{-\xi}} t_j\, B^{kj} X_i\ ,$$

the Cauchy integral providing a new approach to the formula (26). Therefore:

(38) $$\frac{dw_i}{d\beta} = -\frac{1}{2i\pi}\int \frac{\xi d\xi}{1 - e^{-\xi}} t_j\, B^{ij}\ .$$

Now observe that equation (37), when solved for the dw_j, gives a result of the form:

$$\frac{dw_i}{d\beta} = A_{ji}\, t_j\ ,$$

hence:

$$\frac{\partial f}{\partial w_i}\frac{dw_i}{d\beta} = A_{ji}\, t_j\, \frac{\partial f}{\partial w_i} = t_i\, X_j\ , \qquad\qquad X_j = A_{ji}\, \frac{\partial f}{\partial w_i}\ .$$

Comparing this with (38), one has:

$$X_j = -\frac{1}{2i\pi}\int \frac{\xi d\xi}{1 - e^{-\xi}} B^{ij}\, \frac{\partial f}{\partial w_i}\ .$$

This may be considered as a result of the greatest logical and esthetic value. By starting with the given structure, from the characteristic equation and the minors of B^{ij}, one sees that one may very simply construct a Cauchy integral whose contour encloses all of the roots of the characteristic equation, but no the points $2ki\pi$, an integral that represents r infinitesimal transformations that are associated with the structure considered. Nevertheless, one must indeed observe something that Poincaré did not say immediately, and which came to light only at the beginning of his second Mémoire.

We began our reasoning in paragraph 3 with the operators X that one represents, first of all, as each constructed from certain independent variables x, the w used in the last place being only *external* parameters to the X, so to speak. Now, it is not these X that we just constructed, but X that depend upon the w, which meanwhile indeed generate a group having the given structure, but it is a *parametric* group. Briefly, we have here, at last, a fact that, with Élie Cartan, we originally presented our fascicle XXXIII.

The most general construction of groups depend, above all, on the construction of parametric groups; it is notably the latter that generates the *group spaces*, in the sense of Cartan.

Another fundamental remark. – In our fascicle XXXIII and at the end of paragraph 2 of the present chapter, we have recalled that certain authors, notably Schur and Pascal, have attached the theory of finite and continuous groups to that of Bernoulli numbers. Now, the exponential theory that is due to Henri Poincaré, which is essentially constructive, also makes this relationship evident. It suffices, in our last formula giving the operator X_j , to propose to study the integral of the right-hand side upon developing:

$$\frac{\xi}{1-e^{-\xi}},$$

in increasing powers of ξ. The Bernoulli numbers appear immediately as the coefficients of such a development.

8. Terminal comparisons. – The comparisons, with which we are under material obligation to terminate a chapter of limited extent, may be the point of departure of new and great developments. In Chapter I of the present exposé, we recalled that gravity, even with the very recent improvements, is always subordinate to its dependence upon transformations of multiple integrals. We have begun to see, in Chapter II, that the same is true for the new mechanics. The theory of groups, with its exponential character, emerges at the same time as the preceding disciplines and with the same principles. Some of the works of Campbell and Poincaré do not seem to correspond to physical reality, but the first research of Charles Hermite on matrices gives the same impression.

From the bibliographic point of view, we cite the Mémoires of Th. De Donder [**27**], who united gravity and mechanics by many relationships between initial principles that are much than the ones that formed the basis for our theories. The conclusion is also that there is nothing to oppose a link between gravity and wave mechanics [**28**]. The following chapter will confirm that impression.

One of the first developments in the statistical mechanics is comprised of the theory of adiabatic invariants that was presented in the work of R.-H. Fowler (loc. cit.), but reprised by T. Levi-Cività [**29**]. The eminent Italian geometer has made great use of the methods of Lie concerning the transformation of canonical systems.

All of these domains are in a state of active development; one may not encompass or fix anything. If the work of Birtwhistle is translated into French then the fundamental work of Louis de Broglie is translated into German [**30**]. Another exposition arrived in Leningrad that is particularly physical, very clear, very Einsteinian, and is due to J. Frenkel [**31**]. All of this is, without a doubt, only an initial surge.

CHAPTER III

GEOMETRY AND MECHANICS OF CHANNEL SPACES

1. Stokes formula for channel spaces. – The formula thus enunciated is:

(1) $$\int_\Sigma U\,dP + V\,dQ = \iint_\sigma \left(\frac{\partial V}{\partial P} - \frac{\partial U}{\partial Q}\right) \begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} d\sigma\,.$$

An elementary channel has a quadrilateral section. On its lateral faces, P, Q, $P + dP$, $Q + dQ$ are constants.

Let there be a channel containing $d\sigma$ (in x, y, z) and dS (at X, Y, Z). The finite barriers σ and S are intercepted by a sheaf of channels; they are *in channel projection.* Upon calling $\Lambda(P, Q)$, the parenthesis that appears in the double integral in (1), one may write for $\Lambda\,dP\,dQ$:

$$\Lambda(P,Q)\begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} d\sigma = \frac{\Lambda(P,Q)}{\Theta(X,Y,Z)}\begin{vmatrix} \Phi_X & \Phi_Y & \Phi_Z \\ P_X & P_Y & P_Z \\ Q_X & Q_Y & Q_Z \end{vmatrix} \frac{\Theta(X,Y,Z)\,dS}{\sqrt{\Phi_X^2+\Phi_Y^2+\Phi_Z^2}}\,.$$

The barrier S has the equation $\Phi = 0$. Set:

(2) $$\frac{1}{\Theta\sqrt{\Phi_X^2+\Phi_Y^2+\Phi_Z^2}}\begin{vmatrix} \Phi_X & \Phi_Y & \Phi_Z \\ P_X & P_Y & P_Z \\ Q_X & Q_Y & Q_Z \end{vmatrix} = \begin{cases} \Delta(X,Y,Z), \\ \Delta_1(\Phi,P,Q), \\ \Delta_1(0,P,Q) \text{ on } S, \end{cases}$$

since, in order to determine $\Lambda(P, Q)$:

$$\Lambda(P, Q)\,\Delta_1(0, P, Q) = 1.$$

It is the means to have:

$$\iint_S \Theta\,dS = \iint_\sigma \begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \frac{d\sigma}{\Delta_1(0,P,Q)}\,.$$

The double integral in the right-hand side is Stokesian and easily takes the form of the left-hand side of (1).

Briefly, in a channel or a sheaf of channels, the double integrals of $\Theta\,dS$ are invariant under propagation. The general and fundamental equation of the phenomenon is (2) with the right-hand side $\Delta_1(\Phi, P, Q)$. General propagating surfaces S may correspond to that partial differential equation in Θ. They depend upon arbitrary elements; for example, a parameter t that one will call *time*. Therefore, along the channel one may have the propagation of invariant elements (or depending upon time in a certain manner), which are the analogues of *masses* (constants or variables). There are therefore the foundations of a very general mechanics [**38**] in the preceding geometrical considerations. This mechanics will have equation (2) for its fundamental equation, which is analogous to the Jacobi equation written for the motion of a point, but much more plastic.

2. Comparisons with the Jacobi equation. – These comparisons seem to come about in various ways. For the moment, let P and Q be homogeneous of degree zero. Then:

$$XP_X + YP_Y + ZP_Z = 0, \qquad XQ_X + YQ_Y + ZQ_Z = 0.$$

The equation $\Phi = 0$ will take the form $f = 1$, with f homogeneous of order one, which is always possible. On the surfaces $f = 1$, equation (2) may be written, by using Euler's theorem:

$$\frac{1}{\Theta\left(\frac{X}{f}, \frac{Y}{f}, \frac{Z}{f}\right)} \frac{1}{\sqrt{f_X^2 + f_Y^2 + f_Z^2}} \begin{vmatrix} f_X & f_Y & f_Z \\ P_X & P_Y & P_Z \\ Q_X & Q_Y & Q_X \end{vmatrix} = \frac{\Delta(P, Q)}{f^2}, \tag{3}$$

or rather:

$$\left(\frac{\partial f}{\partial X}\right)^2 + \left(\frac{\partial f}{\partial Y}\right)^2 + \left(\frac{\partial f}{\partial Z}\right)^2 = F\left(\frac{X}{f}, \frac{Y}{f}, \frac{Z}{f}\right).$$

This is the *homogenized* Jacobi equation. On any surface $f = 1$, it takes on the ordinary form:

$$\left(\frac{\partial f}{\partial X}\right)^2 + \left(\frac{\partial f}{\partial Y}\right)^2 + \left(\frac{\partial f}{\partial Z}\right)^2 = F(X, Y, Z). \tag{4}$$

However, one will say, with P and Q homogeneous of zero the channel will always be rectilinear and, more exactly, conical with summit O.

Now, one may *vary* them, from a general remark concerning (1). The Pfaffian reduction is:

$$U\,dP + V\,dQ = M\,dN,$$

which translates, in the right-hand side of (1), into:

$$\left(\frac{\partial V}{\partial P} - \frac{\partial U}{\partial Q}\right) \left\| \begin{matrix} P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{matrix} \right\| = \left\| \begin{matrix} M_x & M_y & M_z \\ N_x & N_y & N_z \end{matrix} \right\|.$$

Moreover, (3) may take the form:

(5)
$$\frac{1}{\Theta\sqrt{f_X^2+f_Y^2+f_Z^2}}\begin{Vmatrix} f_X & f_Y & f_Z \\ M_X & M_Y & M_Z \\ N_X & N_Y & N_Z \end{Vmatrix} = \frac{1}{f^2}.$$

The right-hand side obviously reduces to 1 on a surface $f = 1$. However, in (5), M and N are again homogeneous of order zero as functions of P and Q. Observe then that (5) does not change if, for example, one replaces M and N, respectively, by:

$$M^* = M + \rho(f), \qquad N^* = N + \sigma(f).$$

Thus, channels appear anew that are not rectilinear and are very profoundly indeterminate [**39**].

It is essential to remark that in classical mechanics (4) corresponds to the motion *of a single point*, and that here one associates (4) with channel spaces in which the *multi-pointlike* motions are the general case.

3. Jacobi and Schrödinger symbols. – Take these two types of symbols in the respective forms:

(6)
$$J(S) = \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 - S^2\Omega,$$
$$\Sigma(W) = \Delta W + W\Omega.$$

Moreover, set:

$$u = S_1 + S_2, \qquad v = S_1 - S_2.$$

Now, change the notations in the double integral in (1) by setting:

(7)
$$\begin{cases} \Lambda(P,Q)(P_yQ_z - P_zQ_y) = w_x + uv_x + F, \\ \Lambda(P,Q)(P_zQ_x - P_xQ_z) = w_y + uv_y + G, \\ \Lambda(P,Q)(P_xQ_y - P_yQ_x) = w_z + uv_z + H. \end{cases}$$

This system (7), which is collectively elementary, seems fundamental in regard to putting the geometry of mechanics in channel spaces into quantum form, wavelike or corpuscular being the most commonly adopted forms. One is forced to have:

$$F_x + G_y + H_z = 0,$$

which may be written, Δ being the Laplacian in three variables:

$$\Delta w + u\,\Delta v + u_x\,v_x + u_y\,v_y + u_z\,v_z = 0,$$

or indeed:

(8)
$$\Delta w + (S_1 + S_2)\,\Sigma(S_1 - S_2) + J(S_1) - J(S_2) = 0.$$

This relation, which is the only one that is necessary for (7) to be meaningful, couples the Jacobi symbol J with the Schrödinger symbol Σ [**40**] in a *non-symbolic* manner.

This is, without a doubt, the most natural pretext for the introduction of Σ.

4. Quantization. – Recall the Schrödinger equation, with the notations of Weyl [**17**], namely:

$$\frac{h^2}{2m}\Delta\varphi + (E - V)\varphi = 0. \tag{9}$$

The W in (6) is replaced by φ. In (9), V represents the potential energy, and E is the constant total energy.

Set:

$$\varphi = e^{i\nu t}\,\psi(x, y, z).$$

One has:

$$\psi = e^{-i\nu t}\,\varphi, \qquad -\frac{h}{i}\frac{\partial\psi}{\partial t} = h\nu\psi,$$

and equation (9) in ψ becomes:

$$\frac{h^2}{2m}\Delta\psi - \frac{h}{i}\frac{\partial\psi}{\partial t} - V\psi = 0 \quad \text{if} \quad E = h\nu. \tag{10}$$

One sees that merely the fact that one introduces time into the Schrödinger equation in a periodic fashion entails the quantization of energy.

5. Wave equations. – Now recall the Schrödinger equation, with the notation (6):

$$\Delta W + W\Omega = 0.$$

Upon seeking the solutions that are periodic with respect to time, as usual:

$$W = e^{i\nu t}\,w(x, y, z), \qquad \text{hence,} \qquad \frac{\partial^2 W}{\partial t^2} = -\nu^2\,W,$$

one may write:

$$\Delta W - \frac{\Omega}{\nu^2}\frac{\partial^2 W}{\partial t^2} = 0.$$

With the notation employed in (9) for W and $E = h\nu$, it becomes:

$$\Delta W - \frac{1}{C^2}\frac{\partial^2 W}{\partial t^2} = 0, \qquad C = \frac{E}{\sqrt{2m(E-V)}}.$$

This value of C is precisely the result that one obtains by comparing the equation of propagation of a wave front:

$$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2 = -\frac{1}{C^2}\left(\frac{\partial F}{\partial t}\right)^2$$

with the Jacobi equation:

$$\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2 = 2m(E - V),$$

for F when it is replaced by $F(x, y, z) - Et$.

One sees the simplicity with which the mechanics of channel spaces is linked to the fundamental formulas of wave theories, whether corpusclar or quantum.

6. Homogeneity and non-commutativity. – How does the preceding discussion relate to non-commutativity? It is easy to see in paragraph 2 that the latter notion is replaced by that of homogeneity because homogeneity permits one to create non-commutative differential operators in various ways. One first has Euler's theorem:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = mf,$$

which associates the operators:

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$

and:

$$x, y, z.$$

This association was pointed out by Weyl for probabilistic reasons [**17**]. It immediately gives combinations such as:

$$\frac{\partial}{\partial x}(xf) - x\frac{\partial}{\partial x} f = f,$$

a fact that was already pointed out at the beginning of the preceding chapter.

The theory of groups permits us to vary this. One may find, as we saw once more in paragraph 3 of Chapter II, linear differential operators X and Z, such that $XZ = ZX$. Furthermore, let:

$$Y = Z + rX, \qquad \text{hence,} \qquad XY - YX = X(r)X.$$

Euler's theorem gives $X(f) = kf$, and, taking r such that $X(r) = 1$, one might have:

$$XY(f) - YX(f) = kf.$$

With n variables x_i, and f homogeneous of order k, one further has:

$$\frac{\partial}{\partial x_i}(x_i f) - x_i \frac{\partial}{\partial x_i} f = \begin{cases} 0 & \text{if } i \neq j, \\ nf & \text{if } i = j, \end{cases}$$

because the index becomes a summation index for $i = j$.

Briefly, homogeneity is equivalent to non-commutativity, from several points of view.

This is why we find appeals to homogeneity in this chapter, while other authors will take recourse to notion of non-commutativity.

Moreover, the fundamental partial differential equation (2) for Φ is an integrable first order equation if one considers a differential system that is mixed, like all of these systems, with the fundamental constructions of the theory of groups. That theory and mechanics again appear to be inseparable.

7. Probabilistic considerations. – The double integral of (1) or, from (7):

$$\iint_\sigma \left(\frac{dw}{dn} + u\frac{dv}{dn}\right) d\sigma ,$$

obviously expresses a certain probability for a certain phenomenon to take place in the sheaf of channels intercepted by the barrier σ. It is indicated that one let $\varpi(u, v, w)\, d\tau$ express the probability for an analogous phenomenon to be produced in the volume element $d\tau$. In a theory that agrees with (8), which involves only solutions v of the Schrödinger equation, the same probability will be simply $\varpi(v)\, d\tau$. Conditions of simplicity might guide one in one's choice of $\varpi(v)$; we reduce that function to v^2, which is always possible. However, the Schrödinger equation that was invoked here is the time-independent one $\Sigma = 0$, whereas, in general, one must consider the complete equation (10) and likewise imaginary solutions ψ of it. One then has that the probability $v^2 d\tau$ is replaced by:

(11) $$\psi\bar{\psi}\, d\tau ,$$

the ψ surmounted by an overbar denoting the imaginary conjugate of ψ.

The study of the expression (11) and its integrals is a limiting case of the study of the forms with conjugate indeterminates of Charles Hermite:

(12) $$a_{ik}\bar{x}_i x_k , \qquad a_{ki} = \bar{a}_{ik} ,$$

forms that correspond to unitary geometry, whose preliminary study plays a very important role in current microphysics. The works in which such considerations are developed tend to become quite numerous; we confine ourselves to once more citing those of H. Weyl [**17**], as well as the *Leçons* of Élie Cartan [**41**], which lift the purely geometric spirit.

8. Probability fluids and waves. – The considerations of this chapter are essentially in accord with the fundamental remark that Bateman made in regard to equation (9) of the preceding chapter. In order to satisfy:

$$F_x + G_y + H_z = 0, \tag{13}$$

we now set [**42**]:

$$\begin{cases} 2F = uv_x - vu_x = 2\Lambda(P,Q)(P_yQ_z - P_zQ_y), \\ 2G = uv_y - vu_y = 2\Lambda(P,Q)(P_zQ_x - P_xQ_z), \\ 2H = uv_z - vu_z = 2\Lambda(P,Q)(P_xQ_y - P_yQ_x). \end{cases} \tag{14}$$

Equation (13) takes the form:

$$u \sum v - v \sum u = 0, \tag{15}$$

while always denoting the Schrödinger symbol (6) by Σ. Now, (15) may also be written:

$$\frac{\partial}{\partial x}(\rho\lambda) + \frac{\partial}{\partial y}(\rho\mu) + \frac{\partial}{\partial z}(\rho\nu) = 0 \tag{16}$$

on setting:

$$\rho = uv, \qquad \lambda,\ \nu,\ \mu = \frac{\partial}{\partial(x,y,z)}\log\frac{v}{u}. \tag{17}$$

One has in (16) an equation of continuity relative to a permanent fluid motion.
Suppose we have the complete Schrödinger equation (10) and its conjugate:

$$\frac{h^2}{2m}\Delta\psi^* + \frac{h}{i}\frac{\partial\psi^*}{\partial t} - V\psi^* = 0. \tag{18}$$

It gives:

$$\frac{hi}{2m}(\psi\Delta\psi^* - \psi^*\Delta\psi) + \frac{\partial}{\partial t}(\psi\psi^*) = 0,$$

i.e.:

$$\frac{\partial}{\partial x}(\rho\lambda) + \frac{\partial}{\partial y}(\rho\mu) + \frac{\partial}{\partial z}(\rho\nu) + \frac{\partial\rho}{\partial t} = 0, \tag{19}$$

upon setting:

$$\rho = \psi\psi^*, \qquad \lambda,\ \mu,\ \nu = \frac{hi}{2m}\frac{\partial}{\partial(x,y,z)}\log\frac{\psi^*}{\psi}. \tag{20}$$

This time, (19) refers to a general fluid motion and $\rho\, d\tau$ is the probability (11), but reconstructed by a more complete reasoning. In order to be in accord with [**42**] the ψ surmounted by an overbar is replaced by ψ^*.

In (10) and (18) we have equations that relate to *probability waves*, which are imaginary waves in general, and which does not contradict the fact that the results (20) are real.

The search for extensions of (16), (17), (19), and (20) is one of the most current questions of wave mechanics. We cite the results of Crudeli [**43**], [**44**], and Darwin, with comments by Néculcéa [**45**].

Here, we also direct attention to the determination of Λ, P, Q by (14); the functions P and Q are then two distinct integrals of:

$$F\frac{\partial\theta}{\partial x}+G\frac{\partial\theta}{\partial y}+H\frac{\partial\theta}{\partial y}=0.$$

This determines a *channel space* in which the transversal barriers propagate like wave fronts while transporting certain integrals. These fronts, from one channel to a contiguous channel, might not be consistent; they then emit corpuscles.

We again remark that one may imagine that the functions F, G, H in (13) depend upon not only x, y, z, but also a function $f(x, y, z)$ and the partial derivatives of f up to an arbitrary order. One thus has an immense class of partial differential equations of arbitrary order in f, such that they all correspond to channel propagations.

9. Conclusions. – While nonetheless regretting the brevity of this fascicle, we still believe that it has sketched out some essential paths. We first recalled that the principles that form the theory of gravity lead to spatial considerations of extreme generality in which the theory of groups of transformations is included. One does not have to ponder too long on the latter situation to find the origin of symbols that are or are not commutative; this is why we have developed the considerations due to Henri Poincaré, as well as further applying them to physical applications.

Finally, one can go only so far into the channel spaces, which are initially Euclidian, without encountering non-Euclidian considerations that correspond to the forms (12); it remains to pursue an extremely interesting geometry in regard to its invariances, its non-commutative matrix multiplications, and its *representations* by groups.

Luther Pfahler Eisenhart, of Princeton University, just published a very remarkable book on the theory of continuous groups [**47**]. From many points of view, that work develops the present fascicle; it is the quite beautiful analysis that we first pondered that must subsequently occupy theoretical physics.

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