“Tourbillions, corpuscles, ondes avec quelques préliminaires sur le rôle des opérateurs en physique théorique,” Ann. fac. sc. de Toulouse (3) **24** (1932), 1-48.

VORTICES, CORPUSCLES, WAVES

WITH SOME PRELIMINARIES

ON THE ROLE OF OPERATORS IN THEORETICAL PHYSICS

By A. BUHL

Translated by D. H. Delphenich

In what follows, I shall always preserve the fundamental viewpoints that are characteristic of all of my work on theoretical physics. The basic identities have the form:

$$\int_C X\, dY = \iint_A dX\, dY\,, \tag{a}$$

and give some *Stokesian* formulas whose principal one will be formula (4) of the second chapter of this treatise. That formula is appropriate to a space that is divided into *tubes* by two families of surfaces. With a third family, we will have nothing but a way of framing that corresponds to the use of arbitrary curvilinear coordinates. However, I shall address tubes in which the Stokesian analysis makes areas, masses, charges, …, which can be invariant or depend upon time in a certain manner, move in a particularly simple manner.

That is already true for corpuscular propagation.

Corpuscles can range over wave fronts, and all of that is provided analytically by the notion of a vortex, by way of Stokesian formulas. One sees that the title “Vortices, Corpuscles, Waves” is immediately justified. Of course, it will amount to only developing the logical scheme from the identity (a) in a new way. For the moment, I shall not examine how those schemes relate to the numerous experimental facts from latter years in the context of waves and corpuscles, but they are sufficiently plastic – above all, by the *variation of tubes* – in order to show one possible aspect of the typical relationships that unite the propagation of wave and corpuscles.

As for the role of differential operators in theoretical physics, one will perceive it in the context of the preceding considerations in the second chapter and in the first chapter in a more abstract manner.

The principal results of this article have been summarized in five notes that were included in the *Comptes rendus.* I give their titles and dates of publication below:

1. "La Géométrie ondulatoire, Ondes et invariants intégraux propagés," **191** (1930), 545.

2. "La Géométrie ondulatoire. Dévoloppements explicites," **191** (1930), 693.

3. "Considérations dynamiques adjointes à la Géométrie ondulatoire," **191** (1930), 1439.

4. "Propagations conoïdales en Géométrie ondulatoire. Ondes dérivées de l'ellipsoïde," **192** (1931), 323.

5. "La propagation curviligne d'intégrales invariantes. Cas des intégrales doubles. Propagation corpusculaire," **192** (1931), 1006.

In addition, I published three other articles since the manuscript of this memoir was concluded, also in *Comtes rendus:*

6. "Sur une invariance d'intégrales doubles attachée à toute équation différentielle ordinaire du premier ordre," **194** (1932), 822.

7. "Nouvelles invariances intégrales attachées aux équations différentielles contenant plusieurs paramétres," **194** (1932), 1114.

8. "Mouvements multiponctuels attachées à l'équation de Jacobi écrite pour le case d'un seul point," **194** (1932).

Notes 6 and 7 are notes on analysis and pure geometry that will be developed in another report.

Note 8 returns to the considerations of paragraph 10 in Chapter II below with more precision.

___

CHAPTER I

Permutation of integrals of differential systems.

I was led to write the present chapter by at least three different reasons:

First of all, it seemed agreeable to me to return to the main subject of my doctoral thesis: "Sur les Équations différentielles simultanées et la forme aux dérivés partielles adjointes," which was submitted on 14 June 1901. That subject was of interest to numerous geometers, as one can see in the bibliography that I gave in the *Mémorial des Sciences mathématiques* (fasc. XXXIII, pp. 23). Without a doubt, it is still of interest, and meanwhile I am not in a position to offer an example of the commencement of my work to the kind correspondents who have demanded one of me. My thesis was not published in a periodical, and the copies of it that I was given were exhausted long ago. In this chapter, my correspondents will find both a condensation and a perfection of the ideas of my youth.

Secondly, among the geometers that were recently inspired by the topic, let me cite, above all, G. Pfeiffer of Kiev. Pfeiffer has communicated a very important paper to me that I was quite happy to publish last year in the presents *Annales*. I nonetheless believe that the analysis of that Ukrainian mathematician can be simplified and symmetrized; however, be that as it may, it has given me precisely the ambition to return to my own treatise in which not everything was as simple and symmetric as possible.

Thirdly, those permutations of integrals are very short permutations, to begin with, and the permutations are special cases of linear systems. Those systems, with their *determinants* and their *matrices*, are easy to relate to the question, and will lead one to the current apparatus of theoretical physics. The variable operators x_i and the operators of partial differentiation with respect to x_i that are easily associated with them are associated only by way of the theorem of Euler that is another way of resorting to the homogeneity that is usually first put into systems of linear and algebraic equations.

One will see that homogeneity in the second chapter in the context of physical considerations.

Finally, I believe that none of this could have been developed without touching upon some part of the work of the scholar Élie Cartan, which is work that one can imagine will dominate the way that all of the comparisons and citations in this work will be made. I shall confine myself to citing the paper "Sur la réduction à sa forme canonique de la structure d'un groupe fini et continu," Am. J. Math. **18** (1896), which is a citation that was made already in the last year (pp. 140) at the head of the paper by Pfeiffer. That memoir combines algebraic considerations and the use of differential operators in a particularly profound way.

1. Integrals. Jacobi multipler. – In all of what follows, the *definition* of an integral of a differential system:

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n}, \tag{1}$$

is constantly present in spirit.

An *integral* of the system (1) is a function of the variables x_i that remains constant by virtue of that system. If φ is such a function then, since $d\varphi = 0$, one will conclude immediately that:

(2) $$X(\varphi) = X_1 \frac{\partial \varphi}{\partial x_1} + X_2 \frac{\partial \varphi}{\partial x_2} + \cdots + X_n \frac{\partial \varphi}{\partial x_n} = 0.$$

It can have only $n - 1$ *distinct* integrals; say:

(3) $$\varphi_1, \varphi_2, \ldots, \varphi_{n-1}.$$

All of the other ones will be functions of them.

Let φ_u be a function such that:

(4) $$X(\varphi_u) = 1.$$

Set:

(5) $$D = \begin{vmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \cdots & \frac{\partial \varphi_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \varphi_n}{\partial x_1} & \frac{\partial \varphi_n}{\partial x_2} & \cdots & \frac{\partial \varphi_n}{\partial x_n} \end{vmatrix}.$$

D is then the *Jacobi multiplier;* it permits one to write:

(6) $$X(f) = \frac{1}{D} \begin{vmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \varphi_{n-1}}{\partial x_1} & \frac{\partial \varphi_{n-1}}{\partial x_2} & \cdots & \frac{\partial \varphi_{n-1}}{\partial x_n} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{vmatrix}.$$

If one substitutes $\varphi_1, \ldots, \varphi_{n-1}, \varphi_n$ for the variables $x_1, \ldots, x_{n-1}, x_n$ then one will have:

(7) $$X(f) = X(\varphi_i) \frac{\partial f}{\partial \varphi_i} = \frac{\partial f}{\partial \varphi_n},$$

in which i for the index of summation.

For another operator:

(8) $$Y(f) = Y_1 \frac{\partial f}{\partial x_i} + Y_2 \frac{\partial f}{\partial x_2} + \ldots + Y_n \frac{\partial f}{\partial x_n},$$

one will have:

(9) $$Y(f) = Y(\varphi_i) \frac{\partial f}{\partial \varphi_i}.$$

2. Permutability of the operators ***X*** **and** ***Y*****. –** The operators X and Y that were just defined do not permute, in general. With (7) and (9), in the variables φ_i , one will have:

(10) $$XY(f) - YX(f) = \frac{\partial f}{\partial \varphi_i} \frac{\partial}{\partial \varphi_n} Y(\varphi_i).$$

However, the right-hand side will be annulled if:

(11) $$Y(\varphi_i) = F_i(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}),$$

in which the F_i are arbitrary functions of $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$, by not φ_n . The relations (11) can be written more explicitly as:

(12) $$\begin{gathered} Y_1 \frac{\partial \varphi_1}{\partial x_i} + Y_2 \frac{\partial \varphi_1}{\partial x_2} + \ldots + Y_n \frac{\partial \varphi_1}{\partial x_n} = F_1, \\ Y_1 \frac{\partial \varphi_2}{\partial x_i} + Y_2 \frac{\partial \varphi_2}{\partial x_2} + \ldots + Y_n \frac{\partial \varphi_2}{\partial x_n} = F_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ Y_1 \frac{\partial \varphi_n}{\partial x_i} + Y_2 \frac{\partial \varphi_n}{\partial x_2} + \ldots + Y_n \frac{\partial \varphi_n}{\partial x_n} = F_n. \end{gathered}$$

Upon adding (8) to this, viz.:

(8) $$Y(f) = Y_1 \frac{\partial f}{\partial x_i} + Y_2 \frac{\partial f}{\partial x_2} + \ldots + Y_n \frac{\partial f}{\partial x_n},$$

one will conclude from relations (12) and (8) collectively that:

(13) $$\left| \begin{array}{cccc|c} & & & & F_1 \\ & & D & & F_2 \\ & & & & \cdots \\ & & & & F_n \\ \hline \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} & Y(f) \end{array} \right| = 0;$$

hence, one will immediately have:

$$Y(f) = -\frac{1}{D}\left|\begin{array}{ccc|c} & & & F_1 \\ & D & & F_2 \\ & & & \cdots \\ & & & F_n \\ \hline \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots \quad \dfrac{\partial f}{\partial x_n} & Y(f) \end{array}\right|. \tag{14}$$

That is the expression for the most general operator (8) that permutes with the operator X that was originally defined in (2). Independently of any application, that result already has undeniably great esthetic value. In any case, it is remarkably coupled to the Jacobi multiplier.

3. Permutation of integrals. – The present studies had their beginning in the idea of constructing an operator Y of type (8) that would permute the integrals of the system (1) or of equation (2). One sees immediately that when the equality (10) is reduced to:

$$XY(f) - YX(f) = 0, \tag{15}$$

it will suffice that one have $X(f) = 0$ in order for one to also have $X[Y(f)] = 0$. Moreover, the integrals of the equation $Y(f) = 0$ are permuted by the operator X.

In practice, these permutations of the integrals never lead to the discovery of new integrals when one starts with, for example, integrals that are taken from the list (3) and are less than $n - 1$ in number. It is likewise obvious that all of the preceding supposes that all of the integrals of (1) are known. Meanwhile, the study of the operator (14) must be quite interesting. If the integrals tend to form cycles that are impossible to leave then, from that fact, there will be some cyclic properties of the system (1) or equation (2), which will still be very remarkable properties, even if they never aid in the integration, properly speaking. Hence, Galois theory, which relates to algebraic equations, has the main objective of constructing an operator that permutes the roots, but it does not have to give unknown roots upon starting with known roots.

Moreover, we shall see later on that this analogy is precise, so the permutations of integrals can be reduced to the groups of permutations that are used in algebra, in some cases.

Return to (14) and (15). If f is replaced by a true integral of $X(f) = 0$ in (14) then the choice of F_n has no influence on $Y(f)$, because F_n will then have $X(f)$ for a coefficient, and that will be zero, as one sees immediately upon replacing the D that is framed in (14) with its value (5). That will lead one to imagine that Y can be, more generally, of the form:

$$Y = Z + r\,X, \quad \text{with} \quad XZ = ZX,$$

and r arbitrary functions of $\varphi_1, \dots, \varphi_{n-1}, \varphi_n$, or $x_1, \dots, x_n$. One will then have:

(16) $$XY - YX = \lambda X, \qquad \text{if} \qquad \lambda = X(r).$$

One can always consider λ to be an arbitrary coefficient $\lambda(x_1, \ldots, x_n)$. It seems that one has a generalization of (15) in (16). In (16), one again has $X[Y(f)] = 0$ if $X(f) = 0$, but that generalization is easily reduced to the case (15), and one can prefer to always reduce to that case, which at least has the advantage of perfect symmetry. Be that as it may, the simple link between the two cases $\lambda = 0$ and $\lambda \neq 0$ that we just pointed out again is one of the most remarkable things in G. Pfeiffer's paper. It is also one of the very frequent examples in which non-permutability is associated very closely with permutability.

4. Variant. – One can arrive at formula (14) by another path. One starts with:

$$df = \frac{\partial f}{\partial \varphi_1} d\varphi_1 + \frac{\partial f}{\partial \varphi_2} d\varphi_2 + \cdots + \frac{\partial f}{\partial \varphi_n} d\varphi_n,$$

which decomposes into:

(17) $$\begin{aligned} \frac{\partial f}{\partial x_1} &= \frac{\partial f}{\partial \varphi_1}\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial f}{\partial \varphi_2}\frac{\partial \varphi_2}{\partial x_1} + \cdots + \frac{\partial f}{\partial \varphi_n}\frac{\partial \varphi_n}{\partial x_1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \frac{\partial f}{\partial x_n} &= \frac{\partial f}{\partial \varphi_1}\frac{\partial \varphi_1}{\partial x_n} + \frac{\partial f}{\partial \varphi_2}\frac{\partial \varphi_2}{\partial x_n} + \cdots + \frac{\partial f}{\partial \varphi_n}\frac{\partial \varphi_n}{\partial x_n}, \end{aligned}$$

and will give:

(18) $$Y(f) = \frac{\partial f}{\partial \varphi_1} Y(\varphi_1) + \frac{\partial f}{\partial \varphi_2} Y(\varphi_2) + \cdots + \frac{\partial f}{\partial \varphi_n} Y(\varphi_n)$$

upon linear combination.

It is precisely because the latter equation is a consequence of the n preceding ones that one will have:

(19) $$\left|\begin{array}{c|cccc} \frac{\partial f}{\partial x_1} & & & & \\ \frac{\partial f}{\partial x_2} & & D' & & \\ \cdots & & & & \\ \frac{\partial f}{\partial x_n} & & & & \\ \hline Y(f) & Y(\varphi_1) & Y(\varphi_2) & \cdots & Y(\varphi_n) \end{array}\right| = 0.$$

Here, D' is nothing but D with the change of its rows into columns and its columns into rows.

From (18), if f is an integral – i.e., a function of $\varphi_1, \ldots, \varphi_{n-1}$ – then the expression $Y(f)$ will also be an integral if:

$$Y(\varphi_i) = F_i(\varphi_1, \varphi_2, \ldots, \varphi_{n-1}), \qquad i = 1, 2, \ldots, n-1.$$

Since $\partial f / \partial \varphi_n$ has zero for its coefficient, one can repeat the little discussion in the preceding paragraph for $Y(\varphi_n)$ or F_n, and in such a fashion that equation (19) will give (14) again.

When that second method is compared to the first one, one will see that there is a double symmetry in the question. The terms that enter into (12) and (8) *horizontally* are recovered in (17) and (18) *vertically*.

5. One of the aspects of the operator $Y(f)$. – Starting from the operator $Y(f)$ in the form (14), we propose to develop it linearly in the functions F_i.

The result will obviously have the form:

$$Y(f) = U_1(f)\, F_1 + U_2(f)\, F_2 + \ldots + U_n(f)\, F_n ,$$

with

$$\begin{array}{lll} U_1(\varphi_1) = 1, & U_1(\varphi_2) = 0, \ldots, & U_1(\varphi_n) = 0, \\ U_2(\varphi_1) = 1, & U_2(\varphi_2) = 1, \ldots, & U_2(\varphi_n) = 0, \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots, \\ U_n(\varphi_1) = 0, & U_n(\varphi_2) = 0, \ldots, & U_n(\varphi_n) = 1. \end{array}$$

These n^2 equations have an existence that is obviously insured by the symbolic matrix:

$$\frac{1}{D}\left|\begin{array}{cccc|c} & & & & U_1(f) \\ & D & & & U_2(f) \\ & & & & \cdots \\ & & & & U_n(f) \\ \hline \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots & \dfrac{\partial f}{\partial x_n} & \end{array}\right|$$

in which each U_i must be equal to what its *algebraic* minor would be if the column of U_i is the last column of a determinant that is formed, moreover, from the D that is framed and the row of partial derivatives that is placed under that D. Obviously, the preceding n^2 equations can be summarized in:

$$U_i(\varphi_k) = \delta_{ik} = \begin{cases} 1 & \text{if} \quad i = k \\ 0 & \text{if} \quad i \neq k \end{cases}.$$

One sees that what is essential in the construction of $Y(f)$ is the preliminary construction of the operators $U_i(f)$, which are, in short, integrals of the system (1) when f is one, but integrals that reduce to 1 or zero *identically*.

For more developments on the subject of these considerations, one should again refer to G. Pfeiffer's paper.

Here, we add only that if $f(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is an arbitrary integral of (1) then:

$$U_i(f) = U_i(\varphi_k)\,\frac{\partial f}{\partial \varphi_k}$$

will also be one. As a result, no matter what the function Ψ is:

$$\Psi\,[U_1(f), \ldots, U_n(f); \varphi_1, \varphi_2, \ldots, \varphi_{n-1}]$$

will be another one. It is not only *linear* differential operators such as (8) that can permute the integrals of a differential system in an interesting manner. However, the linear operators are the most manageable ones and the ones that are richest in synthetic considerations.

6. New non-permutability. – From now on, in order to abbreviate the writing, we shall set:

$$n - 1 = \nu.$$

On the other hand, for more precision, the symbol (14) will be written $Y_F(f)$ in order to indicate that is it is composed of the functions:

$$F_1(\varphi_1, \varphi_2, \ldots, \varphi_\nu), \quad F_2(\varphi_1, \varphi_2, \ldots, \varphi_\nu), \ldots, \quad F_\nu(\varphi_1, \varphi_2, \ldots, \varphi_\nu). \tag{20}$$

In $Y_F(f)$, thus-constituted, we can put any of the integrals:

$$G_1(\varphi_1, \varphi_2, \ldots, \varphi_\nu), \quad G_2(\varphi_1, \varphi_2, \ldots, \varphi_\nu), \ldots, \quad G_\nu(\varphi_1, \varphi_2, \ldots, \varphi_\nu), \tag{21}$$

in place of f and thus obtain the expressions:

$$Y_F(G_l), \tag{22}$$

which must, of course, be integrals of (1).

Now, imagine that one has permuted the roles of the F in (20) and the G in (21) in the expressions (22). In other words, one forms:

$$Y_G(F_l). \tag{23}$$

The expressions (23) will indeed be integrals of the system (1) again, but in general they will not be the same as in (22). In other words, one will have:

$$Y_F(G_l) - Y_G(F_l) \neq 0. \tag{24}$$

It is that new non-permutability that we propose to study. It is very important, and not just because it includes the *non-commutativity* of the factors in the *matrix* algebraic multiplication as a special case. However, we return to the inequality (24) that must be first established rigorously.

In the φ variables, by virtue of (9) and (11), and with k as the summation index that varies from 1 to v, one will have:

$$Y_F = F_k \frac{\partial}{\partial \varphi_k}, \qquad Y_G = G_k \frac{\partial}{\partial \varphi_k}, \tag{25}$$

so

$$Y_F\, Y_G - Y_G\, Y_F = \left(F_k \frac{\partial G_l}{\partial \varphi_k} - G_k \frac{\partial F_l}{\partial \varphi_k} \right) \frac{\partial}{\partial \varphi_l} = [Y_F\,(G_l) - Y_G\,(F_l)]\, \frac{\partial}{\partial \varphi_l}.$$

Now, the operators (25) are arbitrary operators in v variables; they are certainly not permutable, in general, and the left-hand side of the triple equality that is finally obtained will not be zero. As a result, the bracket in the right-hand side will no longer be zero, in general, and that will prove (24).

7. Permutability of the operators Y_F and Y_G . – Although that cannot be the general case, one can meanwhile propose to study the one in which the operators (25) are permutable. It is clear that one can give them that property by constructing them, one starting with the other, in the way that one constructs $Y(f)$ in (14) by starting with $X(f)$, while the new construction will bring only $v = n - 1$ variables into play, instead of n.

8. Linear transformations. – We now return to the study of the expression (22), upon starting with systems of functions (20) and (21) when those $2v$ functions become linear. One will have, for example:

$$F_k = b_{km}\, \varphi_m, \qquad G_l = a_{lp}\, \varphi_p, \tag{26}$$

in which the indices, such as m and p, that are repeated two times in a monomial term are summation indices. One concludes immediately that:

$$Y_F\,(G_l) = F_k \frac{\partial G_l}{\partial \varphi_k} = a_{lk}\, b_{km}\, \varphi_m. \tag{27}$$

That is the same result as the one that one would get if one had set:

$$\psi_l = a_{lk}\, \theta_k, \qquad \theta_k = b_{km}\, \varphi_m, \quad \text{hence,} \qquad \psi_l = a_{lk}\, b_{km}\, \varphi_m.$$

Hence, to introduce the integrals F_k and G_l in the *linear* form (26) into (14) is to transform a linear form, such as $b_{km}\, \varphi_m$, in such a manner that only the coefficients will be modified. That will give an obvious result. However, it will become quite interesting again when one then returns to the general operator (14). *With the hypotheses* (26), *that operator will contain a theory of linear transformations; it will contain a generalization of those transformations, of a differential physiognomy, when one once more gives F_k and G_l their general significance* (20) *and* (21), *resp*. Before insisting more thoroughly upon that agreement, we shall develop the comparison of (26) and (27) more explicitly.

The relations (26) and (27) will give us the tables of coefficients:

$$\text{(A)}\qquad \begin{matrix} a_{11} & a_{12} & \dots & a_{1v} \\ a_{21} & a_{22} & \dots & a_{2v} \\ \dots & \dots & \dots & \dots \\ a_{v1} & a_{v2} & \dots & a_{vv} \end{matrix} \qquad \begin{matrix} b_{11} & b_{12} & \dots & b_{1v} \\ b_{21} & b_{22} & \dots & b_{2v} \\ \dots & \dots & \dots & \dots \\ b_{v1} & b_{v2} & \dots & b_{vv} \end{matrix},$$

or *matrices*, whose product is another matrix with v rows and v columns, and whose general term is $a_{ik}\,b_{km}$. The factors in that product are not *commutative;* as we stated above, that non-commutativity is a very special case of the non-permutability that is expressed in (24).

9. Matrices and determinants. – The expressions (26) for F_k and G_l are constructed from the *rows* in table (A). Now, imagine that F_k, for example, is constructed from the *rows* in the second table (A), while G_l is constructed from the *columns* of the first one. One will then have:

$$F_k = b_{km}\,\varphi_m\,, \quad G_l = a_{pl}\,\varphi_p\,.$$

Hence:

$$Y_F\,(G_l) = F_k\,\frac{\partial G_l}{\partial \varphi_k} = a_{kl}\,b_{km}\,\varphi_m\,.$$

In general:

$$a_{kl}\,b_{km} \neq a_{km}\,b_{kl}\,,$$

but the determinants that have these unequal sides for their terms will nonetheless be equal, because one passes from one to the other by changing the rows and columns, and *vice versa*. That fact shows that the true multiplication, which has the most generality, is matrix multiplication with non-commutative factors, here. Commutativity appears when the factors are determinants only because the determinant is a matrix with special properties. One can find some other examples of commutativity in the theory of matrices. Hence, if α and β are two matrices that have the identity matrix 1 for their product then one will have both $\alpha\beta = 1$ and $\beta\alpha = 1$.

One sees that there is a theory of the transformation of linear forms of integrals φ of an equation $X = 0$ by differential operators Y, which is a theory that one can utilize with both non-commutative and commutative factors. Nevertheless, everything has its origin in the bordered determinant that in (14), which once more generalizes the two kinds of multiplications here when the forms in φ are no longer linear.

10. Homogeneity and non-commutativity. – The *non-commutativity* of matrix products results immediately from the combination of the *homogeneous* equations:

$$y_l = a_{lk}\, x_k\,, \qquad z_l = b_{lk}\, y_k = b_{lk}\, a_{km}\, x_m\,,$$

so one sees that they are two notions that are quite closely linked, which is a fact that one expects to recover by various paths. It should also be remarked that the variables x_i, which are real or imaginary, have commutative multiplication, and in that regard, they are similar to partial derivation with respect to x_i. However, if one constructs operators that contain both the x_i and the partial derivatives with respect to x_i then the commutativity will disappear, in general. The most elementary example that one can give in that regard is:

$$\frac{d}{dx}(xf) - x\frac{d}{dx}f = f, \tag{28}$$

or symbolically:

$$\frac{d}{dx}x - x\frac{d}{dx} = 1.$$

It is these considerations that the present chapter permits one to generalize in various ways. Recall the differential operator:

$$Y = Z + rX, \qquad \text{with} \qquad XZ = ZX,$$

hence (16):

$$XY - YX = X(r)\, X.$$

Upon taking, for example, $r = \varphi_n$, one will have $X(r) = 1$, but the equality:

$$XY - YX = X$$

will be truly comparable with (28) only if:

$$X(f) = kf, \tag{29}$$

in which k is a constant.

One will then have:

$$XY(f) - YX(f) = kf\,.$$

Euler's theorem on homogeneous functions will realize (29) when one takes:

$$X = x_i\,\frac{\partial}{\partial x_i}\,, \tag{30}$$

and takes f to be a homogeneous function of order k that makes the indeterminacy in f that of an arbitrary function of $n - 1$ variables if the x_i are n in number. Here again, one sees that a construction of non-commutative differential operators is realized thanks to the notions of homogeneity.

In a more immediate manner, one also has:

$$\frac{\partial}{\partial x_j}(x_i f) - x_i \frac{\partial}{\partial x_j} f = \begin{cases} 0 & \text{if} \quad i \neq j, \\ nf & \text{if} \quad i = j. \end{cases}$$

As for the study of the operator Y, when it is finally added to the operator X in (30), it was undertaken by Pfeiffer at the end of his paper in the last year.

We should also remark that, by the preceding, we will also recover the answer that Weyl gave to question of knowing what the more important operators are after partial derivatives with respect to x_i . They are, as that brilliant geometer said ([1]), the factors x_i . Weyl's assertion is justified by some probabilistic considerations. One sees that one can also justify it by some considerations of homogeneity, with Euler's theorem as the main link between the two types of operators. With those two types, one can write down the infinitesimal transformations of numerous groups, notably, the linear groups. With only the operators of differentiation, one can form the linear partial differential equations with constant coefficients that have been written for some time in mathematical physics. With only the x_i, one can form linear, homogeneous algebraic systems, and therefore determinants and matrices. The partial derivatives and all of their combinations, along with determinants and matrices, are then the fundamental instruments of the same theoretical physics in which the considerations of homogeneity are generally respected in one manner or another.

____

([1]) HERMANN WEYL, *Gruppentheorie und Quantenmechanik*, First edition, 1928, pp. 47; second edition, 1931, pp. 49.

CHAPTER II

Tubular spaces.

1. Stokes's formula for tubular spaces. – All of our work on Stokesian formulas rests upon transformations and linear combinations of transformations of the identity:

$$\int_C X\,dY = \iint_A dX\,dY\,, \tag{1}$$

and some analogous identities that relate to spaces with an arbitrary number of dimensions. Without going further in that direction, there is much that can be inferred from (1).

First, let the plane OXY correspond to another plane OPQ by the formulas:

$$X = X(P, Q), \qquad\qquad Y = Y(P, Q).$$

One will have:

$$\int_{C'} X\left(\frac{\partial Y}{\partial P}dP + \frac{\partial Y}{\partial Q}dQ\right) = \iint_{A'} \begin{vmatrix} X_P & X_Q \\ Y_P & Y_Q \end{vmatrix} dP\,dQ\,.$$

One then sees that we employ either the ∂ or simply the index that indicates the differentiated variable for the partial derivatives. Upon setting:

$$U = XY_P\,, \qquad V = XY_Q\,,$$

one will obtain Riemann's formula:

$$\int_{C'} U\,dP + V\,dQ = \iint_{A'} \begin{vmatrix} \dfrac{\partial}{\partial P} & \dfrac{\partial}{\partial V} \\ U & V \end{vmatrix} dP\,dQ\,. \tag{2}$$

Now, perform the new transformation:

$$P = P(x, y, z), \qquad\qquad Q = Q(x, y, z), \tag{3}$$

which will be only a change of variables if z belongs to a surface $z = z(x, y)$. Under those conditions, one introduces the new determinant:

$$\begin{vmatrix} P_x + P_z p & P_y + P_z q \\ Q_x + Q_z p & Q_y + Q_z q \end{vmatrix} = \begin{vmatrix} -p & -q & 1 \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

into the double integral in (2), which will be multiplied by $dx\,dy$. Since:

$$-p\,dx\,dy = \alpha\,d\sigma, \qquad -q\,dx\,dy = \beta\,d\sigma, \qquad dx\,dy = \gamma\,d\sigma,$$

Riemann's formula (2) will become, by definition:

$$\text{(4)} \qquad \int_\Sigma U\,dP + V\,dQ = \iint_\sigma \begin{vmatrix} \dfrac{\partial}{\partial P} & \dfrac{\partial}{\partial Q} \\ U & V \end{vmatrix} \begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} d\sigma .$$

Obviously, in that formula, P and Q have the form (3). The first determinant under the double integral is a function Λ (P, Q).

One can verify directly that the double integral in (4) is Stokesian. For that to be true, it is necessary that one must have:

$$\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \Lambda P_x & \Lambda P_y & \Lambda P_z \\ Q_x & Q_y & Q_z \end{vmatrix} = \begin{vmatrix} \dfrac{\partial \Lambda}{\partial x} & \dfrac{\partial \Lambda}{\partial y} & \dfrac{\partial \Lambda}{\partial z} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} = 0.$$

Now, that is indeed realized, since Λ is a function of only P and Q. One can also verify that (4) is only the ordinary Stokes formula:

$$\int_\Sigma F\,dx + G\,dy + H\,dz = \iint_\sigma \begin{vmatrix} \alpha & \beta & \gamma \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F & G & H \end{vmatrix} d\sigma ,$$

in which one sets:

$$F = U\frac{\partial P}{\partial x} + V\frac{\partial Q}{\partial x},$$
$$G = U\frac{\partial P}{\partial y} + V\frac{\partial Q}{\partial y},$$
$$H = U\frac{\partial P}{\partial z} + V\frac{\partial Q}{\partial z}.$$

Since these three relations can be summarized by:

$$F\,dx + G\,dy + H\,dz = U\,dP + V\,dQ,$$

all of the foregoing can also be attached very simply to the problem of reducing Pfaff forms.

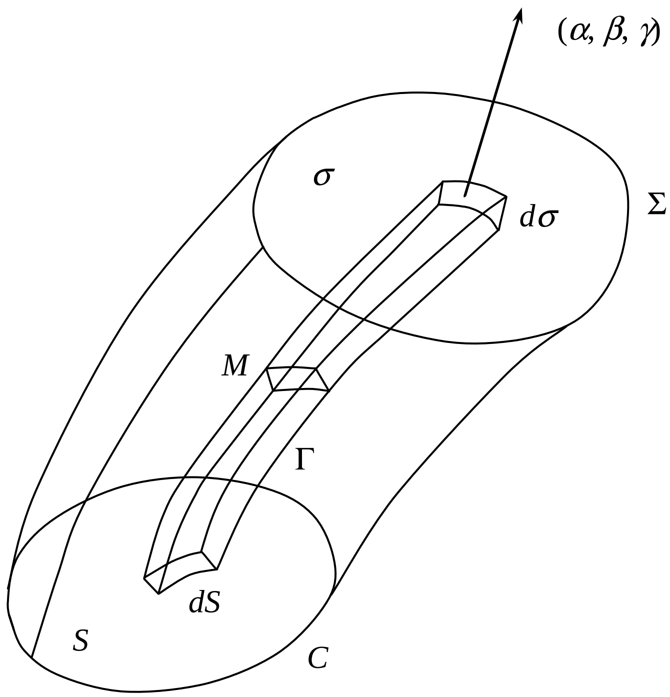

Figure 1.

Now, consider the infinitely-narrow tube Γ with a quadrangular cross-section that is composed of the four surfaces on which:

$$P,\ P + dP,\ Q,\ Q + dQ$$

have temporarily-constant values ([1]). That tube cuts out an element $d\sigma$ with coordinates x, y, z on the end face of the invariable contour Σ. On a surface S whose equation is:

$$\Phi\,(X, Y, Z) = 0, \tag{5}$$

it will cut out the element dS whose coordinates are X, Y, Z. By definition of the tube, one must consider that dP and dQ are constant along the tube Γ in dS, as well as in $d\sigma$. Hence ([2]):

([1]) A. Buhl, "La propagation curviligne d'intégrales invariantes. Cas des intégrales doubles. Propagation corpusculaire," Comptes rendus **192** (1931), pp. 1006. Naturally, the phrase "temporarily-constant values" means that P and Q change when one passes from one tube to another. One can make the definition of a tube more precise. If one has $P(x, y, z) = \lambda$ on one face then one will have:

$$P\,(x + dx, y + dy, z + dz) = \lambda + d\lambda$$

on the opposite face, so $d\lambda = dP$. That is what one expresses by saying that P and $P + dP$ are constant on those faces.

([2]) G. HUMBERT, *Œuvres* published by P. Humbert and G. Julia, t. I, 1929, pp. 442.

$$(6)\quad \Lambda(P,Q)\,dP\,dQ = \Lambda(P,Q)\begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} d\sigma = \begin{vmatrix} \Phi_X & \Phi_Y & \Phi_Z \\ P_X & P_Y & P_Z \\ Q_X & Q_Y & Q_Z \end{vmatrix} \frac{\Lambda(P,Q)\,dS}{\sqrt{\Phi_X^2+\Phi_Y^2+\Phi_Z^2}}.$$

In the middle of this double equality, one must obviously read P (x, y, z) and Q (x, y, z) in Λ (P, Q), which indicates the associated determinant, moreover. In the final expression, one must read P (X, Y, Z) and Q (X, Y, Z) in Λ (P, Q). Now, write:

$$\frac{1}{\sqrt{\Phi_X^2+\Phi_Y^2+\Phi_Z^2}}\begin{vmatrix} \Phi_X & \Phi_Y & \Phi_Z \\ P_X & P_Y & P_Z \\ Q_X & Q_Y & Q_Z \end{vmatrix} = \Delta\,(X, Y, Z).$$

One can express X, Y, Z in terms of Φ, P, Q in this Δ (X, Y, Z), and since the point (X, Y, Z) belongs to the surface S whose equation is (5), one will finally have:

$$\Delta\,(X, Y, Z) = \Delta_1\,(0, P, Q).$$

One then infers from (6) that:

$$(7)\qquad \Delta\,(P, Q)\begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} d\sigma = \Delta_1\,(0, P, Q)\;\Lambda\,(P, Q)\;dS.$$

Since Λ (P, Q) has not been determined up to now, one can profit from that indeterminacy to set:

$$\Lambda\,(P, Q) = \frac{1}{\Delta_1(0,P,Q)};$$

hence:

$$(8)\qquad dS = \frac{1}{\Delta_1(0,P,Q)}\begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} d\sigma.$$

Finally, for the evaluation of the skew area S, one will have the *Stokesian* integral:

$$(9)\qquad S = \iint_\sigma \begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \frac{d\sigma}{\Delta_1(0,P,Q)}.$$

The Stokesian character of that integral is, however, obligatory for purely geometric reasons. The end-face S (Fig. 1) is determined by the *contour* Σ, and the tube with finite section that passes through Σ and is composed of a sheaf of an infinitude of tubes Γ. Just as dS can be called the *tubular* projection of $d\sigma$, the contour C is a tubular projection of Σ. Therefore, C and S are determined by the contour Σ, and not by the end-face σ, which can be replaced by any other end-face *with the same contour* Σ. Therefore, the S in (9) depends upon only the contour Σ.

If one would like to effectively express S in (9) by a line integral that is attached to Σ then it will be sufficient to set:

$$\frac{\partial V}{\partial P} - \frac{\partial U}{\partial Q} = \frac{1}{\Delta_1(0, P, Q)}, \tag{10}$$

and to determine U and V in conformity with equation (10), which is possible in an infinitude of ways. The line integral will then be given immediately by formula (4). One sees that this formula (4) indeed deserves the name of *Stokes formula for tubular spaces.* Moreover, all of the arguments of this paragraph extend easily to arbitrary double integrals that are attached to the area S. Such double integrals can then represent, not only areas, but masses, charges, …; in short, they can take on diverse physical meanings.

2. Invariant, curvilinear, propagating areas. – We now propose to study surfaces on which the finite tube $C\Sigma$ is cut by areas that are equal to S. The existence of such surfaces can be considered to be assured intuitively, but with those preliminaries, we prefer to examine everything analytically and as rigorously as possible.

Now, let there be an element at M (Fig. 1) that must be, by definition, equal to dS while belonging to a surface:

$$\Psi\,(\xi, \eta, \zeta) = 0.$$

The expression:

$$\begin{vmatrix} \Psi_\xi & \Psi_\eta & \Psi_\zeta \\ P_\xi & P_\eta & P_\zeta \\ Q_\xi & Q_\eta & Q_\zeta \end{vmatrix} \frac{\Lambda(P, Q)\, dS}{\sqrt{\Psi_\xi^2 + \Psi_\eta^2 + \Psi_\zeta^2}}$$

must then be equated to any of the expressions in (6). In the factor $\Lambda\,(P, Q)$, one must obviously read $P\,(\xi, \eta, \zeta)$ and $Q\,(\xi, \eta, \zeta)$ for P and Q. Now, write the last expression as:

$$\Delta^*(\xi, \eta, \zeta)\, \Lambda\,(P, Q)\, dS,$$

so, upon reasoning as before:

$$\Delta_1^*\,(0, P, Q)\, \Lambda\,(P, Q)\, dS,$$

and one will have:

$$\Lambda\,(P, Q) \begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} d\sigma = \Delta_1^*\,(0, P, Q)\, \Lambda\,(P, Q)\, dS,$$

instead of (7).

Upon giving $\Lambda\,(P, Q)$ the form:

$$\Lambda\,(P,\,Q) = \frac{1}{\Delta_1^*(0,P,Q)},$$

we will have:

$$\text{(11)} \qquad dS = \frac{1}{\Delta_1^*(0,P,Q)} \begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} d\sigma,$$

instead of (8), which will give:

$$\Delta_1 = \Delta_1^*,$$

when compared with (8).

The equation:

$$\text{(12)} \qquad \begin{vmatrix} \Phi_X & \Phi_Y & \Phi_Z \\ P_X & P_Y & P_Z \\ Q_X & Q_Y & Q_Z \end{vmatrix} \frac{1}{\sqrt{\Phi_X^2 + \Phi_Y^2 + \Phi_Z^2}} = \Delta_1\,(0,\,P,\,Q)$$

then persists for functions Φ other than the ones in equation (5) – viz., $\Phi = 0$ – for the surface S that was considered originally, and the integration of (12), with Φ as an unknown function, will give other surfaces S^* on which the tube Γ cuts out equivalent area elements, where that equivalence obviously extends to the elements that are situated in the same tube.

Furthermore, it is not certain that one will obtain *all* surface S^* in that way. One will further have the expression (9) for S with the surface equation $\Phi = 0$ for which Φ will satisfy the equation:

$$\text{(12}a\text{)} \qquad \begin{vmatrix} \Phi_X & \Phi_Y & \Phi_Z \\ P_X & P_Y & P_Z \\ Q_X & Q_Y & Q_Z \end{vmatrix} \frac{1}{\sqrt{\Phi_X^2 + \Phi_Y^2 + \Phi_Z^2}} = \Gamma_1\,(\Phi,\,P,\,Q),$$

with the single restriction that:

$$\Gamma_1\,(0,\,P,\,Q) = \Delta_1\,(0,\,P,\,Q).$$

However, (12*a*) contains the unknown function Φ *explicitly*, which is not the case in (12). It will then follow that if there is less generality in that then that will be compensated by the fact that (12*a*) is more manageable than (12). Moreover, in what follows, we shall use the latter equation with the aim of obtaining some particularly simple geometric results *effectively*.

Since the surfaces S^* can depend upon an auxiliary parameter t, which we call *time*, in an infinitude of ways, we can say that there is *propagation* of invariant areas in the tube Γ or in the sheaf of tubes Γ.

Let us now fill up the space in Fig. 1 with a sheaf of contiguous tubes Γ. An end-face of invariant area can move in each sheaf, and those end-faces can belong to different surfaces S^* when one passes from one sheaf to a contiguous one, due to the indeterminacy of the surfaces S^* that one infers from the general integral of equation (12). The end-faces that propagate in that way will therefore *not agree* when one passes from a sheaf to a neighboring sheaf, and if those sheaves have small sections then one can imagine a *corpuscular* bombardment, and up to now, the trajectories of the projectiles have depended upon the choice of tube Γ and those projectiles are depicted by small end-faces of invariant *area*.

That geometry can immediately take on a physical aspect: Instead of transporting corpuscles with invariant area, as we have remarked already, one can, by a very simple extension of the preceding theory, just as well transport masses, charges, and in short, integrals of various types that will be invariant or will depend upon time in a certain manner during propagation. For the moment, we shall address areas, due to the very elegant results that are associated with that case.

Finally, if equation (12) is realized with the tubes Γ that have been imposed – i.e., with given functions P and Q – then it will be further clear that these functions P and Q are not the most general ones that satisfy (12). Hence, one can preserve equation (12) by taking Q to be arbitrary and determining P by integrating a first-order *linear* partial differential equation. For the moment, we leave these questions of integration aside, because it is perhaps more remarkable that one can transform the tube Γ into another, more general, one Γ^* without performing any integration, which must be true for at least the examples that shall now treat. We shall ultimately return to the methodical generalities.

3. Archimedean propagation. – The simplest infinitely-thin tubes Γ are obviously rectilinear, and among them, we shall consider, more particularly, the ones that are composed of rays that issue from the Oz axis normally to it. Each tube will then be an infinitely-thin *conoidal* pencil of lines.

It corresponds to:

$$P = \arctan \frac{Y}{X}, \qquad Q = Z. \tag{13}$$

To begin with, we take the surface S to be the circular cylinder that corresponds to:

$$\Phi = X^2 + Y^2 - R^2 = 0. \tag{14}$$

Under those conditions, equation (12) will take the form:

$$\frac{1}{\sqrt{\Phi_X^2 + \Phi_Y^2 + \Phi_Z^2}} \begin{vmatrix} \Phi_X & \Phi_Y \\ P_X & P_Y \end{vmatrix} = \frac{1}{R}. \tag{15}$$

One should remark that the right-hand side of this is a simple constant that is completely independent of P and Q. We now look for surfaces S^* that are more general than the cylinder S of equation (14) and for which one will have:

$$\Phi = Z - f(X, Y) = 0,$$

while keeping equation (15) for that form of Φ. Equation (15), thus-preserved, is:

$$(16)\qquad -\frac{px+qy}{(x^2+y^2)\sqrt{p^2+q^2+1}} = \frac{1}{R},$$

in which the capitalized variables have now been replaced with the lower-case one in such a way that one will get the usual notation for partial derivatives in x, y, z, p, q. Equation (16) is capable of a direct and immediate verification that draws upon the extreme simplicity of the first example. We represent an element dS in the circular cylinder S that corresponds to a conoidal pencil, and in that pencil, the dS^* of an arbitrary surface, where dS^* has the coordinates x, y, z, and a normal with direction coefficients $-p$, $-q$, 1 that makes an angle of λ with the direction of the pencil. One will have:

$$\frac{\cos\lambda\, dS^*}{\sqrt{x^2+y^2}} = \frac{dS}{R}, \qquad \cos\lambda = -\frac{px+qy}{\sqrt{p^2+q^2+1}\sqrt{x^2+y^2}},$$

and since one must have $dS^* = dS$, that will, in fact, bring one back to equation (16).

There are different ways of carrying out such calculations; their plasticity is, moreover, obvious in the generalities of the preceding paragraph. Hence, instead of (13), one can take:

$$P = \frac{Y}{X}, \qquad Q = Z,$$

so, instead of (15), one will have:

$$(17)\qquad \frac{1}{\sqrt{\Phi_X^2+\Phi_Y^2+\Phi_Z^2}}\begin{vmatrix}\Phi_X & \Phi_Y \\ P_X & P_Y\end{vmatrix} = \frac{1}{R}(1+P^*).$$

That equation is closer to the general form of (12), but it provokes one to divide both sides by $1 + P^*$ and to then replace P with arctan P. If one likewise keeps the form (17) then one will be immediately reduced to (16).

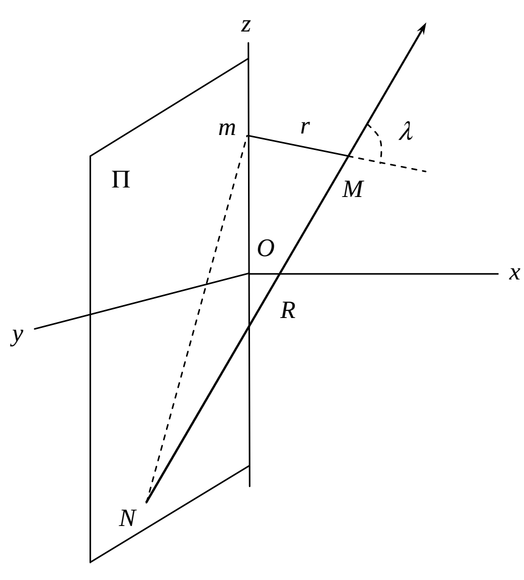

Figure 2.

We now propose to study the most general surfaces S^* that are defined by equation (16). That equation first translates into a very simple geometric property. Let M be a point of a surface S^*.

Project M onto Oz at m, and draw the plane Π that is perpendicular to Mm through Oz. If the normal to S^* at M pierces the plane Π at N then the segment MN of the normal will be constant and equal to R.

Indeed, the figure gives $R \cos \lambda = r$, which is nothing but (16). That property permits one to perceive some important families of surfaces S^* *without calculation.*

First, the property $R \cos \lambda = r$ obviously belongs to the cylinder S of equation (14); the triangle MNm then degenerates into the segment Mm.

It also belongs to the circular cylinders C of *diameter R* for which Oz is a generator. These cylinders are all tangent to the cylinder S internally. Here, MNm is a true triangle, but its plane will always be normal to Oz. Finally, the same property belongs to *spheres* of radius R that are centered on Oz, and thus, inscribed in the cylinder S; the point N is then on Oz. *A right conoid with director Oz will cut out equivalent areas on one of those spheres Σ and on the circumscribed cylinder S.* It is that property that inspired Archimedes ([1]) and gave rise to the figure that is engraved on the tomb of that celebrated geometer that we shall study here once more, but generalized considerably. With those generalizations, it will give various theoretical modes of corpuscular propagation that we shall combine under the name of *Archimedean propagation.*

Even if one sticks to the elementary case – viz., of Archimedean propagation, properly speaking – one can already perceive how a corpuscular propagation by a right conoidal tube Γ that issues from Oz can correspond to a much simpler point-like motion.

It suffices to imagine spheres Σ whose centers describe Oz (rectilinear point-like motion). In an assemblage of sufficiently-thin tubes Γ, the surfaces of those spheres will give a corpuscular propagation of invariant spherical areas.

Another type of propagation can be obtained in those same tubes by rotating the cylinders C around Oz.

One can also associate them with type types. A sphere Σ and a cylinder C intersect along a Viviani curve V. When the curve V turns around Oz, it will generate a sphere Σ; the curve V is then subjected to a translation that is parallel to Oz, which will generate a cylinder C. One can imagine that V is subjected to one of those rotations, then to one of those translations, and then to yet another of those rotations, and so on. The locus of V will then be a *continuous* surface V^*, but one with singular lines. The surface V^* is no less proper for generating a corpuscular propagation of invariant areas in the conoidal tube Γ by motion around and along Oz. However, we shall leave aside these particular geometric images in order to begin the integration of equation (16) or:

$$(x^2 + y^2)^2 (p^2 + q^2 + 1) - R^2 (px + qy)^2 = 0, \tag{18}$$

in as general a manner as possible.

Here, we take the word *integration* in its usual classical sense that it has in the theory of equations in x, y, z, p, q.

([1]) G. LORIA, *Histoire des Sciences mathématiques dans l'Antiquité hellénique*, Gauthier-Villars and Co., 1929, pp. 57.

Whereas the surfaces V^* with singular lines V show that if one abandons the continuity of the tangent plane then the word can take on very different meanings that have the character of G. Bouligand's *contingent* integration. However, for the moment, we shall not stop to compare the two concepts. We simply point out, along the same order of ideas, the beautiful thesis of G. Rabaté that was published again last year (like the Pfeiffer memoir) in these *Annales*, and the no-less-interesting one of G. Durand [J. Math., edited by Henri Villat, **10** (1931), pp. 335]. Durand, in the second page of his expose, addressed the plane curves that were studied by Bouligand, for which $s = f(y)$. Along those lines, the slope, which is continuous according to the classical hypotheses, can nonetheless present as many discontinuities as one pleases. The example of the surfaces V^* is closely coupled with those considerations.

We return to equation (18), which one considers to be the usual one.

If one takes semi-polar coordinates r, θ, z then one will have:

$$p^2 + q^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2, \qquad px + qy = r\,\frac{\partial z}{\partial r}, \qquad p^2 + q^2 = r^2,$$

and (18) will become:

$$(R^2 - r^2)\left(\frac{\partial z}{\partial r}\right)^2 - \left(\frac{\partial z}{\partial \theta}\right)^2 - r^2 = 0.$$

This suggests that we should take $z = a\theta + f(r)$, so:

$$\frac{df}{dr} = \sqrt{\frac{a^2 + r^2}{R^2 - r^2}}, \tag{19}$$

$$z = a\theta + b + \int \sqrt{\frac{a^2 + r^2}{R^2 - r^2}}\, dr. \tag{20}$$

Since b is a constant of integration that is added to the integral in r, we then have a solution to equation (18) with *two* arbitrary constants a and b; it is the *complete* integral. In order to get the *general* integral, one must make b into an arbitrary functions of a, namely, $b(a)$, and then determine the envelope, up to a parameter a, of the family of helicoids (20). That envelope does not seem to be capable of being determined in a completely explicit way; we then confine ourselves to the study of the helicoids (20), for the moment.

Those helicoids simplify considerably for $a = 0$. Equation (20) will then become:

$$r^2 + (z - b)^2 = R^2.$$

This is the equation of the spheres Σ. One sees that these spheres, which were exhibited above by a geometric method, have been revealed by analysis in all manners.

For a general study of the helicoids (20), one can begin with the study of their generating plane curves, which are always situated in a plane that passes through Oz.

That curve will admit the differential equation (19). At an arbitrary point, let r be the radius of curvature, and let v be the segment of the normal that is bounded by Oz. One will have:

$$\rho = \frac{1}{r}\sqrt{(R^2+a^2)(r^2+a^2)}, \qquad v = r\sqrt{\frac{a^2+R^2}{a^2+r^2}}, \qquad \rho v = R^2 + a^2.$$

Hence, the product ρv is *constant*, which would indicate that the plane curve in question experiences a *rotation* around Oz, generates a *surface of revolutions with constant total curvature.* One will arrive at the same result by starting with:

$$\frac{1}{R_1 R_2} = \frac{rt - s^2}{(1+p^2+q^2)^2} = \frac{1}{R^2+a^2},$$

and seeking to verify that Monge-Ampère equation with a solution of the form $z = f(r)$.

We construct those of the curves that pass through the origin. It has a sinusoidal form that oscillates from one side of Oz to the other between two parallels to that axis that have $\pm R$ for their abscissas.

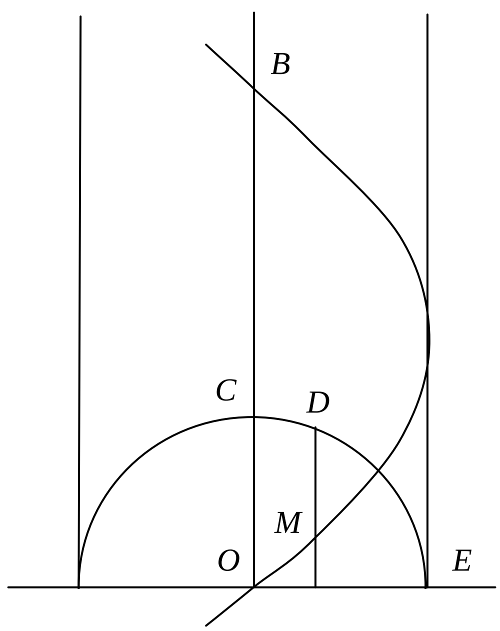

Figure 3.

The arc OM is:

$$s = \sqrt{R^2+a^2}\int_0^r \frac{dr}{\sqrt{R^2-r^2}} = \sqrt{R^2+a^2}\,\arcsin\frac{r}{R}.$$

Now, if DM is parallel to Oz then the circular arc CD will be:

$$s_1 = R \arcsin\frac{r}{R}.$$

Hence:

$$\frac{s}{s_1} = \frac{\sqrt{R^2 + a^2}}{R}.$$

If a is zero then OM will become an arc of the circle that is obviously equal to CD.

The proportionality of the arcs OM and CD in the general case is very remarkable, because that proportionality obviously rises to the same order of ideas as the equivalence of areas that is at the basis for the question.

We further remark that the curve OAB has the intrinsic equation:

$$\frac{\rho^2}{k^2} - \frac{a^2}{R^2 \sin^2 \frac{s}{k}} = 1, \qquad k^2 = R^2 + a^2.$$

In these few developments, one sees the rich harvest of geometric properties that one can expect beyond that of the celebrated Archimedean figure that is composed of the sphere and the circumscribed cylinder. Notably, there are helicoidal surfaces S^* (20) and their envelopes in a when b is replaced with $b(a)$, which have a very simple relationship with the surfaces of revolution with constant total curvature. All of those surfaces S^*, when put into motion around and along Oz, will give propagations of invariant areas in the tube Γ that was defined at the beginning of this paragraph. Among the numerous cases that remain to be studied in detail, we must return to that of the cylinders C for which the plane of the triangle MNm is always normal to Oz. Here, the simplicity and symmetry of the case shows immediately that the propagation of the invariant areas is provided by the propagation of invariant circular *arcs* between two radius vectors that issue from the same point of Oz.

We fix our attention on those two-dimensional propagations that are concerned with invariants *arcs.*

4. Propagation of arcs. Archimedean case. – This case, to which we are led by the force of things, is simpler than the spatial case in Fig. 1. One will have to treat it in the first place, regardless.

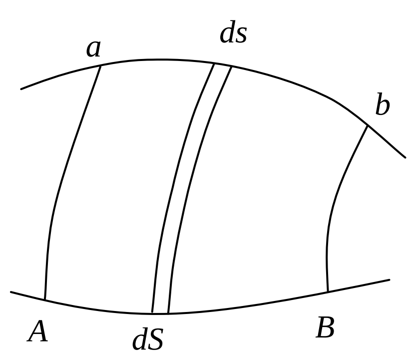

Figure 4.

Fig. 4 is planar. *AB* gives *ab* by *tubular* projection, just as *dS* gives *ds*. The system of tubular projectors is imposed by being given a function in two variables *P* that is constant, as well as *dP*, along the same tube. One has:

$$\Lambda(P)\,dP = \Lambda(P)\left(P_x\frac{dx}{ds}+P_y\frac{dy}{ds}\right)ds = \Lambda(P)\begin{vmatrix}\alpha & \beta\\ P_x & P_y\end{vmatrix}ds,$$

if *a* and *b* are direction cosines of the normal to the arc *ab* at *ds*.

The relations (6) are then replaced with:

$$\Lambda(P)\,dP = \Lambda(P)\begin{vmatrix}\alpha & \beta\\ P_x & P_y\end{vmatrix}ds = \begin{vmatrix}\Phi_X & \Phi_Y\\ P_X & P_Y\end{vmatrix}\frac{\Lambda(P)\,dS}{\sqrt{\Phi_X^2+\Phi_Y^2}}.$$

By an argument that is absolutely analogous to the one that accompanied Fig. 1. One can write:

$$\frac{1}{\sqrt{\Phi_X^2+\Phi_Y^2}}\begin{vmatrix}\Phi_X & \Phi_Y\\ P_X & P_Y\end{vmatrix} = \Delta\,(X, Y) = \Delta_1\,(\Phi, P) = \Delta_1\,(0, P).$$

If one now sets:

$$\Lambda\,(P) = \frac{1}{\Delta_1(0,P)}$$

then one will get *S* – i.e., the arc *AB* – by an integral of the exact differential extended over *ab*, which is an integral that does not depend upon the arc *ab*, but only on its extremities *a* and *b*. The equation:

$$\frac{1}{\sqrt{\Phi_X^2+\Phi_Y^2}}\begin{vmatrix}\Phi_X & \Phi_Y\\ P_X & P_Y\end{vmatrix} = \Delta_1\,(0, P), \tag{21}$$

can serve to determine some other, more general, curves $\Phi = 0$ that can give a propagation of invariant arcs in the tubes that are attached to the function *P*.

We now fix our attention, above all, on the problem of the *variation of the tubes*. Suppose that equation (21) is established with Φ having any degree of generality that one would like. That will not prevent *P* from being imposed at the outset; up to now, the tubes are given. Now, *P* can, in turn, be generalized to P^* in such a way that:

$$\frac{1}{\sqrt{\Phi_X^2+\Phi_Y^2}}\begin{vmatrix}\Phi_X & \Phi_Y\\ P_X & P_Y\end{vmatrix} = \Delta_1\,(0, P^*). \tag{22}$$

The left-hand side of that equation can be expressed by:

$$\Delta^*(X, Y) = \Delta_1^*(\Phi, P^*) = \Delta_1^*(0, P^*),$$

which can be only $\Delta_1(0, P^*)$.

One sees that this generalization of tubes will depend upon an arbitrary function, which is quite simple to explain geometrically. In a family of curves *AB*, an infinitely-thin tube *dS*, *ds*, or one that is not *AaBb* will cut our equivalent arcs (Fig. 4) in the family. However, upon starting with an *arbitrary* curve γ that is transverse to *AB*, one can take equal arcs *along that AB*. The locus of the extremities of those arcs will be a curve γ' that forms a propagating tube of equivalent arcs, along with γ.

Take an example. First, let there be a circle with its center at *O* and a radius of *R*, and when it is considered along with all radii, everything can be represented by:

$$\Phi = \frac{r}{R} - 1 = 0, \qquad P = \theta,$$

with

$$r^2 = x^2 + y^2, \qquad \theta = \arctan\frac{y}{x}.$$

Equation (21) will then give:

$$\Delta_1(0, P) = \frac{1}{R}, \tag{22*}$$

which is independent of *P*. One will get the same result with:

$$\Phi = \frac{x^2 + y^2}{R(x\cos C + y\sin C)} - 1 = \frac{r}{R\cos(\theta - C)} - 1 = 0, \tag{23}$$

and always $P = \theta$. The calculations are simple. One sees that in order to be in better agreement with the usual notations, we replace *X* and *Y* in (21) with *x* and *y*. That is what we did in the spatial case by passing from (15) to (16).

For the function P^* that satisfies (22), it is entirely reasonable to take:

$$P^* = P + f(\Phi),$$

with Φ having the form (23), and *f* being an arbitrary function.

That result can be easily explained in a geometric manner, moreover. In order to evaluate the new circular arcs that are included in the new tube, one must specify the limits of those arcs; i.e., one must seek the points of intersection that are defined by the equations:

$$\Phi = 0, \qquad P + f(\Phi) = k.$$

That system can be replaced with:

$$\Phi = 0, \qquad P + f(0) = k.$$

Now, f (0) is a constant like k; everything happens as it does with rectilinear tubes $P = q = k$, while the constant k simply changes in value. The equation for the new tube:

$$\theta + f\left[\frac{r}{R\cos(\theta - C)} - 1\right] = k$$

can obviously be put into a manageable form, such as:

$$r = R \cos (\theta - C)\ \varphi(\theta - k),$$

for example.

One can infer numerous *algebraic* curves from this that share the property with the lines $\theta = k$ that they give propagating tubes of equal arcs of the circles of the family (23).

5. Conical or central propagation. Buhl-Vincensi surfaces. – After conoidal propagation that issues normally to Oz, the simplest and most elegant one to study is probably the propagation that takes place in rectilinear or conical tubes with summit at O. In order to define those tubes, we begin by setting:

$$P = \frac{z}{\sqrt{x^2 + y^2}}, \qquad Q = \arctan \frac{y}{x}.$$

An infinitely-thin tube is then comprised, on the one hand, between two infinitely-close cones of revolution with summit O and axis Oz, and on the other hand, between two infinitely-close planes that pass through Oz.

The simplest surface on which one can take areas that are preserved by conical propagation is probably the sphere with center at O and constant radius R. We represent it by:

$$\Phi = \frac{1}{R}\sqrt{x^2 + y^2 + z^2} - 1 = 0.$$

Under these conditions, with the forms thus-indicated for Φ, P, Q, one will have:

$$\frac{1}{\sqrt{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}} \begin{vmatrix} \Phi_x & \Phi_y & \Phi_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} = -\frac{1}{R^2}(1 + P^2)^{3/2}. \tag{24}$$

Of course, in the right-hand side, one takes into account the fact that $\Phi = 0$.

Now, while preserving P and Q – i.e., the preceding tubes – determine the surfaces with the equation:

$$\Phi = z - z\,(x, y) = 0$$

for which the relation (24) is still valid. One will then arrive at the partial differential equation:

$$R^2 \frac{z - px - qy}{\sqrt{p^2 + q^2 + 1}} = (x^2 + y^2 + z^2)^{3/2}, \tag{25}$$

and always by very simple calculations.

Of course, that can be formed directly with the aid of:

$$\frac{d\sigma \cos\lambda}{x^2 + y^2 + z^2} = \frac{d\sigma_1}{R^2}, \qquad d\sigma_1 = d\sigma, \qquad \cos\lambda = \frac{z - px - qy}{\sqrt{p^2 + q^2 + 1}\sqrt{x^2 + y^2 + z^2}}.$$

That was recently studied in a methodical manner by Vincensini ([1]).

Let us briefly recall the original geometric considerations that led us from the conical propagation of equivalent areas to spherical areas ([2]).

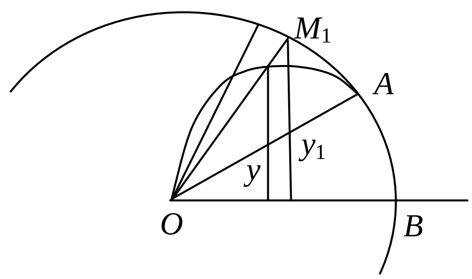

Figure 5.

Take the family of Bernoulli lemniscates:

$$r^2 = R^2 \cos 2(\theta - C).$$

The parameter C is the angle AOB and θ and M_1OB. The radius OM_1 and the infinitely-close radius cut out ds on the lemniscate, and ds_1 on the circumscribed circle. One has, effortlessly:

$$r\, ds = R^2\, d\theta, \quad r \sin\theta\, ds = R^2 \sin\theta\, d\theta, \qquad y\, ds = y_1\, ds_1 .$$

Upon multiplying the two sides of the last equation by 2π, one will see that here there is a propagation of arcs of the lemniscate between two infinitely-close radius vectors that will always give *the same area of revolution* when the revolution takes place around OB. Moreover, that common area of revolution is equal to the are of the spherical zone that is generated by ds_1 .The two circular bands that are generated by ds and ds_1 can be divided into an infinitude of elements by planes that pass through OB, and thus, as Vincensini called it, a *perspective* of area elements that are preserved when one passes from the

([1]) P. VINCENSINI, "Aires courbes en perspective," § 4, Annales de la Fac. des Sc. de Toulouse, 1931.
([2]) A. BUHL, "Sur la formule de Stokes," *ibid.*, (1914), pp. 309.

surface of revolution Σ that is generated by the lemniscate that turns around *OB* to the circumscribed sphere.

All of that extends immediately to arbitrary cones that issue from *O*. As for the surfaces of *revolution* Σ, they are obviously infinite in number, consistent with the arbitrary character of the constant *C*. One can replace then with the Monge surfaces Σ′ that are obtained by rolling the plane of the figure around an arbitrary cone with summit *O*. Those surfaces Σ′ now depend upon an *arbitrary* function (that corresponds to the *arbitrary* cone that is the base for the rolling); they therefore generally integrate equation (25).

That result is extremely remarkable. Indeed, one knows that in regard to equations in *x*, *y*, *z*, *p*, *q*, one most often considers them to be being satisfied in such a way that one can obtain a *complete* integral; i.e., an integral that depends upon two arbitrary constants. As for the general integral, which is a one-parameter envelope of that complete integral, it can rarely be specified. Now, in the present, it is, and the question was in no way prepared to deal with that.

In the case of Archimedean propagations, we are less advanced. Equation (16) has not been integrated *in general* in a purely-geometric manner. In short, one must take into account the complete integral (20).

That comparison obviously carries with it the demand that one must integrate (16), which is already rich in elegant geometric remarks, but meanwhile it cannot be further perfected *from the purely-geometric viewpoint.* Nevertheless, that is a question that we shall pass over for the moment.

Let us return to equation (24). One can obviously write:

$$\frac{1}{\sqrt{\Phi_x^2+\Phi_y^2+\Phi_z^2}}\begin{vmatrix}\Phi_x & \Phi_y & \Phi_z\\ \Pi_x & \Pi_y & \Pi_z\\ Q_x & Q_y & Q_z\end{vmatrix} = -\frac{1}{R^2} \tag{26}$$

by setting:

$$\Pi = \frac{P}{\sqrt{1+P^2}} = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

Meanwhile, equation (26), while still preserving the form (12), has a particularly remarkable aspect to it, now, namely, that the right-hand side Δ_1 (0, *P*, *Q*) depends upon neither *P* nor *Q*; it is a simple constant, as is already true in (15) and (22*). The substitution of Π for *P* changes nothing in the conical tubes that are employed. Along such tubes, Π and *P* will both be constant, since Π is a function of *P*.

However, the reduction of (24) to (26) greatly simplifies the problem of the *variation of tubes* that was encountered before in a simpler case in the preceding paragraph.

One can generalize the functions Π and *Q* of equation (26) into:

$$\Pi^* = \Pi + \varphi(\Phi), \qquad Q^* = Q + \psi(\Phi).$$

One can also keep one of the functions Π or *Q* – say, Π – and generalize *Q* to:

$$Q^* = Q + \chi(\Phi, \Pi).$$

One sees that there is an infinitude of ways of obtaining systems of *curvilinear* tubes in which the Buhl-Vincensini surfaces Σ', which are animated with an arbitrary motion around O, propagate by invariant areas.

We remark that nothing will change in the *rectilinear* tubes that were first envisioned above if one takes:

$$Q^* = Q + \varpi(\Pi),$$

in place of Q, because Q and Π are homogeneous of order zero. The same thing will be true for Q^*, which will once more give a cone when one equates it to a constant – i.e., a rectilinear generator that issues from O.

6. The partial differential equation in Φ, in unitary form. – The replacement of equation (24) with equation (26) and some results of the same nature that were obtained before imply the demand that the equation:

$$\begin{vmatrix} \Phi_x & \Phi_y & \Phi_z \\ \Pi_x & \Pi_y & \Pi_z \\ Q_x & Q_y & Q_z \end{vmatrix} \frac{1}{\sqrt{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}} = \Delta_1(0, P, Q) \tag{12}$$

cannot take the form:

$$\begin{vmatrix} \Phi_x & \Phi_y & \Phi_z \\ \Pi_x & \Pi_y & \Pi_z \\ Q_x & Q_y & Q_z \end{vmatrix} \frac{1}{\sqrt{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}} = 1 \tag{27}$$

in a general manner, with:

$$M = M(P, Q), \qquad N = N(P, Q). \tag{28}$$

It is the form (27) that will be referred to as the *unitary form* of equation (12) when its existence is justified.

The tubes in (12) and (27) are the same, because if P and Q are constants along a tube then the same thing will be true along the same tube at M and N.

That is true because of (28). However, the form (27) lends itself to the problem of the *variation of tubes* much more than (12).

When one says that the tubes in (12) and (27) are similar, that is not to say that they coincide absolutely. Two *similar* tubes can be compared – for example – to two infinitely-thin cylinders that are parallel to Oz, and which both include the same point p of the plane Oxy in their interiors, but which can have different infinitely-small quadrilateral sections around p.

Replacing (12) with (27) amounts to replacing the double equality (6) with:

$$dM\,dN = \begin{vmatrix} \alpha & \beta & \gamma \\ M_x & M_y & M_z \\ N_x & N_y & N_z \end{vmatrix} d\sigma = \begin{vmatrix} \Phi_X & \Phi_Y & \Phi_Z \\ M_X & M_Y & M_Z \\ N_X & N_Y & N_Z \end{vmatrix} \frac{dS}{\sqrt{\Phi_X^2 + \Phi_Y^2 + \Phi_Z^2}},$$

so, from (27), $dS = dM\,dN$, and replacing the area S with a line integral in $M\,dN$.

As for the path that was described above in paragraph 1, it leads one to write (10) and to conclude, from (4), with a line integral in $U\,dP + V\,dQ$.

The two processes can be converted into each other if one can have:

$$U\,dP + V\,dQ = M\,dN. \tag{29}$$

Now, the reduction of the left-hand side of (29) to the right-hand side is always possible.

The simplest case of the Pfaff problem is the question of finding the integrating factor for the first-order differential equation:

$$U\,dP + V\,dQ = 0. \tag{30}$$

In short, in order to put equation (12) into the form (27), conforming to equation (10), one must first set:

$$\frac{\partial V}{\partial P} - \frac{\partial U}{\partial Q} = \frac{1}{\Delta_1(0, P, Q)}. \tag{10}$$

That will yield an infinitude of forms for U and V. One integrates the first-order differential equation (30), in turn, and gets M and N from (29).

Afterwards, one can verify that:

$$\left(\frac{\partial V}{\partial P} - \frac{\partial U}{\partial Q}\right) \begin{Vmatrix} P_X & P_Y & P_Z \\ Q_X & Q_Y & Q_Z \end{Vmatrix} = \begin{Vmatrix} M_X & M_Y & M_Z \\ N_X & N_Y & N_Z \end{Vmatrix},$$

if, from (29):

$$U = M\,\frac{\partial N}{\partial P}, \quad V = M\,\frac{\partial N}{\partial Q},$$

and if:

$$M_X = \frac{\partial M}{\partial P} P_X + \frac{\partial M}{\partial Q} Q_X, \qquad M_Y = \frac{\partial M}{\partial P} P_Y + \frac{\partial M}{\partial Q} Q_Y,$$

$$N_X = \frac{\partial N}{\partial P} P_X + \frac{\partial N}{\partial Q} Q_X, \qquad N_Y = \frac{\partial N}{\partial P} P_Y + \frac{\partial N}{\partial Q} Q_Y.$$

The calculations are not difficult.

In order to return to the practical reduction of (12) to the unitary form (27), one sees that (10) demands at least one quadrature. One must then integrate (30), which is much more difficult, especially when one needs to have explicit results. However, if one can do that then the variation of the tubes can be performed with no new integration by way of (27), precisely as was indicated at the end of the preceding paragraph.

It is obvious that one will not change equation (27) if one replaces M and N with:

$$M^* = M + \varphi(\Phi), \qquad N^* = N + \psi(\Phi),$$

or M with:

$$M^* = M + \mu(\Phi, N),$$

or N with:

$$N^* = N + \nu(\Phi, M).$$

We confine ourselves to these very simple results here. Moreover, it is obvious that the present paragraph can raise many questions and generate advanced research, notably in the case where equations (12) can be reduced to unitary form *explicitly*.

7. Homogeneity and operators x, y, z. – We shall now go on to some considerations of homogeneity that have already entered implicitly into the foregoing, although it is important to exhibit them explicitly. We shall find that this will give a basis for applying the considerations of the preceding chapter.

First of all, if one is given the equation of an arbitrary surface:

$$\Phi(x, y, z) = 0 \tag{31}$$

then one can put it into the form:

$$f(x, y, z) = 1, \tag{32}$$

in which the function f is homogeneous of order *one*.

Indeed, it suffices to write (31):

$$\Phi\left(\frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau}\right) = 0 \tag{33}$$

and to imagine that one has solved it for τ, so:

$$f(x, y, z) = \tau. \tag{34}$$

The latter equation must not change if one replaces x, y, z, τ with $kx, ky, kz, k\tau$, since it is only equation (33) written differently. Therefore:

$$f(kx, ky, kz) = k\tau = kf(x, y, z),$$

and f is homogeneous of order *one*. Now, if one sets $\tau = 1$ then equation (33) will become (31) again and (34) will become (32).

That transformation from (31) to (32), which gives us undeniably interesting results here, also seems to have deep consequences in other domains – notably, in the theory of algebraic surfaces and the integrals that are attached to them.

It hardly needs to be said that the results that are established with three variables will extend to an arbitrary number of variables in the same way.

We return to the Stokes formula (4) for tubular spaces, and to the double equality (6). We have third-order determinants in it that will obviously simplify if we set:

$$\begin{aligned} P_y\, Q_z - P_z\, Q_y &= x\, \Psi\,(x, y, z), \\ P_z\, Q_x - P_x\, Q_z &= y\, \Psi\,(x, y, z), \\ P_x\, Q_y - P_y\, Q_x &= z\, \Psi\,(x, y, z). \end{aligned} \tag{35}$$

The last determinant in (6), with Φ replaced with $f - 1$, will become:

$$\Psi\,(X, Y, Z)\,(X\, f_X + Y\, f_Y + Z\, f_Z) \qquad \text{or} \qquad \Psi\, f.$$

One can imagine that some simplifications must merit attention.

If the relations (35) are true then one will also have:

$$\begin{aligned} x\, P_x + y\, P_y + z\, P_z &= 0, \\ x\, Q_x + y\, Q_y + z\, Q_z &= 0, \\ x\, \Psi_x + y\, \Psi_y + z\, \Psi_z &= -\,3\Psi. \end{aligned}$$

Therefore, P and Q are homogeneous of order zero, and Ψ is homogeneous of order – 3.

Under those conditions, P and Q, when equated to constants, represent cones with summit O that can never intersect except along common generators.

Except for the final variation of the tube, they will first be rectilinear and issue from O.

The double integral in (4) takes the form:

$$\iint_\sigma \left(\frac{\partial V}{\partial P} - \frac{\partial U}{\partial Q}\right) \Psi\,(x, y, z)\,(\alpha x + \beta y + \gamma z)\, d\sigma.$$

Since the first parenthesis is a function of P and Q, it will be homogeneous of order zero, and the entire coefficient of $(\alpha x + \beta y + \gamma z)$ will be homogeneous of order – 3. That double integral is then always identifiable with the one in the formula:

$$\iint_\sigma \left(\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}\right) (\alpha x + \beta y + \gamma z)\, d\sigma = \int_\Sigma \begin{vmatrix} dx & dy & dz \\ x & y & z \\ L & M & N \end{vmatrix}, \tag{36}$$

in which L, M, N are homogeneous functions of order – 2.

In (36), we have the *reduced* Stokes formula, which was given a long time ago ([1]), and of which we have already made numerous geometric applications. It is now interesting to remark that one can interpose an intermediate form between the general

---

([1]) *See*, for example, A. BUHL, *Géométrie et Analyse des Intégrales doubles*, Collect. *Scientia*, 1920, pp. 8.

Stokes formula in a three-dimensional space and the reduced form (36) that is precisely the tubular form (4).

One sees that the considerations of homogeneity associate the operators x, y, z with the partial derivative operators with respect to x, y, z here, where the first association of that type is represented by Euler's theorem on homogeneous functions. Formula (36) is the fundamental *integral* formula that realizes an association of the same nature.

8. Conical propagating areas and the reduced Stokes formula. – With all of the demands of homogeneity in the preceding paragraph, the double equality (6) can be written:

$$\Lambda\,(P, Q)\; dP\; dQ = \Lambda\,(P, Q)\; \Psi(x, y, z)(\alpha x + \beta y + \gamma z)\; ds$$

$$= \frac{\Lambda(P,Q)\Psi(X,Y,Z)\,f^3(X,Y,Z)\,dS}{\sqrt{f_X^2 + f_Y^2 + f_Z^2}}.$$

One can remark that everything in the left-hand side of this refers to the surface S whose equation is:

$$f\,(X, Y, Z) = 1,$$

in which f is homogeneous of order one. Since f is equal to unity on that surface, one can write f^* in place of f, which will make the coefficient of dS homogeneous of order zero. On the other hand, one is not unavoidably dealing with a conic propagation that issues from O; the points (X, Y, Z) and (x, y, z) are on a straight line with O, so:

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z}.$$

One can then replace X, Y, Z in dS with x, y, z, respectively. One will then have:

$$\Lambda\,(P, Q)\; \Psi(x, y, z)(\alpha x + \beta y + \gamma z)\; ds = \frac{\Lambda(P,Q)\Psi(x,y,z)\,f^3(x,y,z)\,dS}{\sqrt{f_x^2 + f_y^2 + f_z^2}}.$$

Under those conditions, the factor $\Lambda\,(P, Q)$ in the right-hand side will become identical to the one on the left-hand side, and one can suppress the common factors of Λ and Ψ. Nevertheless, one must not forget that there are implied double integrals on both sides, and under those conditions, it is more rigorous to isolate dS by arguing as in paragraph 1 – i.e., by setting:

$$\Lambda\,(P,Q) = \frac{\sqrt{f_x^2 + f_y^2 + f_z^2}}{\Psi(x,y,z)\,f^3(x,y,z)},$$

thanks to the indeterminacy in Λ, which is always possible, because the right-hand side is a function of only two variables, since it is homogeneous of order zero.

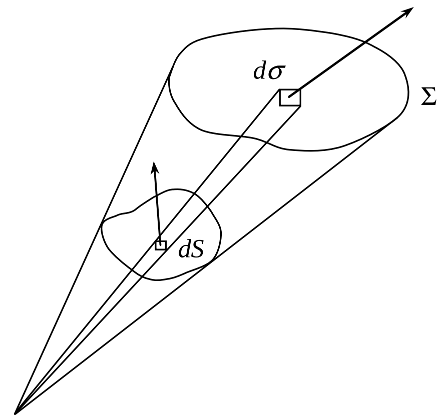

Figure 6.

That formula (37) can be established by direct geometric considerations. With x, y, z on the surface σ, and the X, Y, Z on S, and the usual notations, one will have:

$$\frac{(\alpha x+\beta y+\gamma z)\,d\sigma}{(x^2+y^2+z^2)^{3/2}} = \frac{(Xf_X+Yf_Y+Zf_Z)\,dS}{\sqrt{f_X^2+f_Y^2+f_Z^2}\,(X^2+Y^2+Z^2)^{3/2}}.$$

With:

$$X\,f_X + Y\,f_Y + Z\,f_Z = f$$

and $f(X, Y, Z)$ – which is equal to unity – replaced with f^2, the coefficient of dS can be written with X, Y, Z replaced with x, y, z, respectively; hence, one will get the preceding dS and formula (37).

The double integral in (37) is Stokesian; one can transform it by the reduced formula (36) by setting $L = 0$, $M = 0$ in it:

$$\frac{\partial N}{\partial z} = f^{-3}\sqrt{f_X^2+f_Y^2+f_Z^2},$$

and thus, one will get N by a quadrature. Therefore:

$$S = \int_\Sigma N(y\,dx - x\,dy),$$

and if the contour Σ is traced on a surface (σ) with the equation:

$$N = -\tfrac{1}{2} \qquad \text{or} \qquad 2N + 1 = 0$$

then the area S, which originally belonged to an arbitrary surface, will be made planar by a very remarkable geometric process. It suffices to project from O onto (σ), then to project the end-face that is thus-obtained on (σ) parallel to Oz onto the plane Oxy. The last plane projection will be equal to S.

Meanwhile, for the moment, we shall not insist upon these results, which were already developed at length in some earlier works, but their undeniable importance and precisely those great developments that one can infer from them leads one to demand to know whether that homogeneous geometry is not superior to the one that was first presented with no explicit consideration of homogeneity. However, the superiority does not exist everywhere. A final formula, such as (37), no longer contains any trace of the functions P and Q, nor of Ψ, which is provided by P and Q from (35). The tube O, dS, $d\sigma$ in Fig. 6 is no longer a tube with a well-defined quadrilateral section. It is an arbitrary, infinitely-thin cone that projects dS to $d\sigma$ from O, and conversely, no matter what idea that one might have of those area elements. Under those conditions, the problem of the *variation of tubes* can no longer be posed and solved as it was before.

If we start with a surface S that is well-defined by an equation $f = 1$ and then form, from (37):

$$f^{-3}\sqrt{f_x^2 + f_y^2 + f_z^2} = \lambda\,(x, y, z), \tag{38}$$

in which $\lambda\,(x, y, z)$ is obviously homogeneous of order – 3, then we can afterwards propose to look for the most general solution f^* that is homogeneous of order one and verified (38), while preserving λ. Obviously, the surfaces $f^* = 1$ propagate the area S in the cone $O\Sigma$.

Therefore, suppose one has the sphere:

$$f = \frac{1}{R}\sqrt{x^2 + y^2 + z^2} = 1.$$

Equation (38) is:

$$f^{-3}\sqrt{f_x^2 + f_y^2 + f_z^2} = R^2\,(x^2 + y^2 + z^2)^{-3/2}. \tag{39}$$

That is the equation of the Buhl-Vincensini surfaces – i.e., the equation that gives the most general form for f when one attributes an equation of the form $f = 1$ to the aforementioned surface with f homogeneous of order one. One will easily compare that with (25). However, it is undeniable that the true generality is in (24) or in (26), because, from there one can descend – again, quite easily – to (25) or (39), and with (26), one solves the problem of the variation of tubes immediately.

9. Extensions. Propagation of area integrals. – To conclude this article with all desirable generality, one must envision the case in which there are no longer just areas S that propagate in certain tubes or deformations of them, but integrals of the form:

$$\iint_S \Theta(X, Y, Z)\,dS\,;$$

those integrals will obviously give areas S again when the function Θ reduces to unity identically.

One must then modify the double equality (6) slightly, along with the arguments that follow, and none of that should prove difficult, moreover.

In place of the last two terms in (6), one can set:

$$\Lambda\,(P,\,Q)\begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} d\sigma = \frac{1}{\Theta}\begin{vmatrix} \Phi_X & \Phi_Y & \Phi_Z \\ P_X & P_Y & P_Z \\ Q_X & Q_Y & Q_Z \end{vmatrix} \frac{\Lambda(P,Q)\,\Theta\,dS}{\sqrt{\Phi_X^2+\Phi_Y^2+\Phi_Z^2}},$$

in which, Θ is obviously written for $\Theta\,(X, Y, Z)$. For the last side, we write:

$$\Delta\,(X, Y, Z)\,\Lambda\,(P, Q)\,\Theta\,dS \qquad \text{or} \qquad \Delta_1\,(0, P, Q)\,\Lambda\,(P, Q)\,\Theta\,dS\,.$$

One will profit from the indeterminacy in Λ to set $\Delta_1\,\Lambda = 1$, and one will have:

$$(9^*) \qquad \iint_S \Theta(X,Y,Z)\,dS = \iint_\sigma \begin{vmatrix} \alpha & \beta & \gamma \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \frac{d\sigma}{\Delta_1(0,P,Q)}.$$

That is the extension of formula (9). The Δ_1 in (9^*) coincides with the one in (9) only when $\Theta = 1$. One sees that here, one has set:

$$(12^*) \qquad \frac{1}{\Theta}\frac{1}{\sqrt{\Phi_X^2+\Phi_Y^2+\Phi_Z^2}}\begin{vmatrix} \Phi_X & \Phi_Y & \Phi_Z \\ P_X & P_Y & P_Z \\ Q_X & Q_Y & Q_Z \end{vmatrix} = \Delta_1\,(0,\,P,\,Q),$$

and that is the generalization of equation (12).

Here, one can once more introduce some considerations that are analogous to the ones that are concerned with equation (12*a*) of paragraph 2. However, that would be pointless for what follows.

The problem of the variation of tubes will have exactly the same physiognomy as in paragraph 6.

We now go on to the homogeneous theory. The direct geometric considerations that follow from formula (37) are again modified in an immediate fashion, and show that formula (37) must be replaced with:

$$\iint_S \Theta(X,Y,Z)\,dS = \iint_\sigma \Theta\left(\frac{x}{f},\frac{y}{f},\frac{z}{f}\right) f^{-3}\sqrt{f_x^2+f_y^2+f_z^2}\,(\alpha x+\beta y+\gamma z)\,d\sigma.$$

For an initially-given surface $f = 1$, one will obviously have, as was pointed out in (38):

$$f^{-3}\sqrt{f_x^2+f_y^2+f_z^2} = \lambda\,(x, y, z),$$

in which λ is homogeneous of order – 3. That will entail that the most general forms of f for which the surfaces $f = 1$ will propagate in conical tubes that issue from O as $\Theta\,dS$ or integrals of $\Theta\,dS$ will need to verify equations of the form:

(40) $$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2=\Omega\left(\frac{x}{f},\frac{y}{f},\frac{z}{f}\right).$$

On a surface $f = 1$, one can obviously claim that one has:

(41) $$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2=\Omega\,(x, y, z),$$

in which the function Ω in (41) has the appearance of an arbitrary function of three variables.

10. Vortices. Corpuscles. Waves. – Conversely, any equation (41) that is created with no condition imposed upon Ω can be *homogenized* – i.e., replaced with equation (40), to which one can associate a corpuscular propagation by conical tubes that issue from *O*. One can then construct a more general theory of that propagation with the aid of tubes with well-defined quadrilateral sections (viz., introduce the functions *P* and *Q*), which will permit us to make those tubes *vary* and to finally show that the most general corpuscular propagations that are associated with (41) are curvilinear, as well as rectilinear.

As for the general considerations that give rise to an equation (41) and can approximate a theory of corpuscular propagation, there are at least two types of them:

In the first place, equation (41) can be considered to be a Jacobi equation that relates to the motion of a point, in the sense of classical mechanics. It is already quite important to remark that in classical mechanics a Jacobi equation that is written for the motion of just one point can also govern the entire corpuscular propagation of a number of particles that is as large as one would like.

In the second place, equation (41) governs the propagation of a wave front in such a way that the Stokesian – hence, *vorticial* – considerations of this article will resolve to an ordinary point-like motion, as well as a corpuscular or wave-like propagation.

The search for the basis for the phenomena that are *vortices*, *corpuscles*, or *waves* is an indeterminate question that is probably absurd.

The Stokes formula, which generates so many theories – notably, that of Einstein – is the beginning of a *vorticial* theory. It will become *corpuscular* when it becomes the *formula for tubular spaces*; it will become *wave-like* when it generates equations such as (40) or (41).

In order to make these assertions more precise, we borrow some formulas of Eugène Bloch. Moreover, the summary of that eminent author can be simplified slightly and liberated of all vectorial notation ([1]).

We first recall the ordinary motion of a point with the elementary expressions for kinetic and potential energy:

([1]) E. BLOCH, *L'ancienne et la nouvelle Théorie des Quanta*, Hermann and Co., Paris, 1930, pp. 267-268.

$$T = \frac{m}{2}(x'^2 + y'^2 + z'^2), \qquad U = U(x, y, z).$$

Upon setting:

$$p_x = mx', \qquad p_y = my', \qquad p_z = mz',$$

the Hamiltonian function will have the expression:

$$H = T + U = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + U(x, y, z) = \text{const.} = E.$$

The Jacobi equation is then:

$$\frac{1}{2m}\left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right] + U(x, y, z) = E.$$

One explicitly introduces time by setting:

(42) $$V = S - Et,$$

so one will have the equation for S:

(43) $$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = 2m(E - U),$$

which can be compared with (41).

As for the propagation of a wave-front, it amounts to a function $V(x, y, z, t)$ that must remain constant on that front. One must then have:

$$\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz + \frac{\partial V}{\partial t}dt = 0$$

on it from the outset.

Suppose that one has the point $M(x, y, z)$ on that same front at the time t. During the time dt, the point M will experience a displacement with components dx, dy, dz that is normal to the front and that will have a certain velocity u. Hence:

$$\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = u\,dt\sqrt{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2}.$$

When that equation is compared with the preceding one, one will get:

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = \frac{1}{u^2}\left(\frac{\partial V}{\partial t}\right)^2.$$

Now, if one takes V to be the function (42) then one will have:

$$\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2=\frac{E^2}{u^2}, \tag{44}$$

and a comparison of this with (43) will give:

$$u=\frac{E}{\sqrt{2m(E-U)}}.$$

That is the velocity of propagation of the wave-fronts that correspond to the point-like motion that was envisioned originally. What is interesting here is that equations (43) and (44) are both of type (41). The elementary point-like motion and the associated wave-like propagation then once more correspond to certain corpuscular propagations.

11. Return to conoidal propagation. – The simplifications that we just pointed out for the Stokesian formula (4) and the double equality (6) are certainly not the only ones that one can obtain. We recall the question that was posed in paragraph 7, but posed in a slightly different manner, now.

First, the equation of an arbitrary surface:

$$\Phi(x, y, z)=0 \tag{31}$$

can always be put into the form:

$$f(x, y, z)=1, \tag{45}$$

with f homogeneous of order one *in only x and y*. The argument will be the same, upon just commencing by writing:

$$\Phi\left(\frac{x}{\tau},\frac{y}{\tau},z\right)=0.$$

As for the third-order determinants, from (4) and (6), they will once more simplify greatly, and even more than the first time, if one has:

$$Q=z, \qquad P_x=-y\,\Psi(x, y, z), \qquad P_y=x\,\Psi(x, y, z),$$

instead of (35).

That will give:

$$x\,P_x+y\,P_y=0, \qquad x\,\Psi_x+y\,\Psi_y=-2\,\Psi.$$

Hence, P is homogeneous of order zero, and Ψ is homogeneous of order -2 *in only x and y*. The tubes along which P and Q are constant are conoids that issue from Oz.

The double integral in (4) takes the form:

$$\iint_\sigma \left(\frac{\partial V}{\partial P} - \frac{\partial U}{\partial Q}\right) \Psi(x, y, z)(\alpha x + \beta y)\, d\sigma\,.$$

The entire coefficient of ($\alpha x + \beta y$) is homogeneous of order – 2 *in only x and y*, in such a way that the double integral can always be identified with the one in the formula:

(46) $$\iint_\sigma \frac{\partial N}{\partial z}(\alpha x + \beta y)\, d\sigma = \int_\Sigma N(y\, dx - x\, dy)\,,$$

in which N (x, y, z) is homogeneous of order – 2 *in only x and y*. That is a new type of *reduced* Stokes formula that once more implies numerous geometric applications.

With all of those homogeneity assumptions, the double equality (6) will be written:

$$\Lambda\,(P, Q)\, dP\, dQ = \Lambda\,(P, Q)\, \Psi\,(x, y, z)\,(\alpha x + \beta y)\, d\sigma = \frac{\Lambda(P,Q)\,\Psi(X,Y,Z)\, f^2(X,Y,Z)\, dS}{\sqrt{f_X^2 + f_Y^2 + f^{-2} f_Z^2}}\,.$$

Everything in the right-hand side refers to the surface S whose equation is:

$$f\,(X, Y\, Z) = 1,$$

with f homogeneous or order one *in only X and Y*. We can thus introduce factors f into the right-hand side that are equal to unity and which will render the coefficient of dS homogeneous of order zero *in only X and Y*. Since one has:

$$\frac{X}{x} = \frac{Y}{y}, \qquad Z = z,$$

one can replace X, Y, Z with x, y, z, respectively, in the coefficient of dS.

One will then have:

$$\Lambda\,(P, Q)\, \Psi\,(x, y, z)\,(\alpha x + \beta y)\, d\sigma = \frac{\Lambda(P,Q)\,\Psi(x,y,z)\, f^2(x,y,z)\, dS}{\sqrt{f_x^2 + f_y^2 + f^{-2} f_z^2}}\,.$$

One can isolate dS by suppressing the common factors of Λ and Ψ (or by the more rigorous argument in paragraph 1); thus:

(47) $$S = \iint_\sigma f^{-2} \sqrt{f_x^2 + f_y^2 + f^{-2} f_z^2}\,(\alpha x + \beta y)\, d\sigma\,.$$

Here again, the geometric applications are numerous. Moreover, formula (47) can be established by some direct considerations.

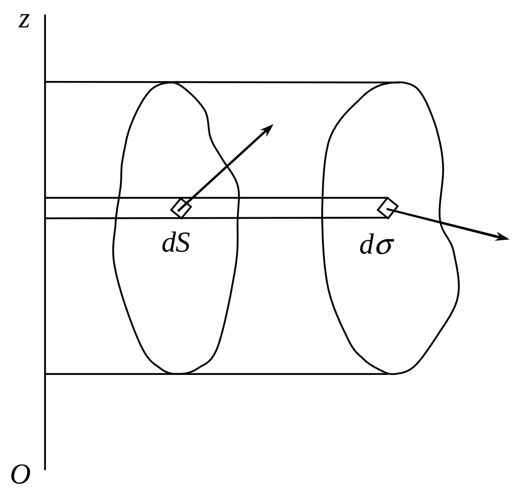

Figure 7.

With x, y, z in $d\sigma$ and then X, Y, Z in dS, and with the usual notations:

$$\frac{\alpha x+\beta y}{x^2+y^2}\,d\sigma = \frac{X\ f_X + Y\ f_Y}{(X^2+Y^2)\sqrt{f_X^2+f_Y^2+f_Z^2}}\ dS.$$

With:

$$X f_X + Y f_Y = f,$$

and such factors of f set equal to unity, one can make the coefficient of dS homogeneous of order zero *in only X and Y*. One can then replace X, Y, Z with x, y, z, resp., so dS and S will agree with the expression (47).

The integral (47) is Stokesian. One can transform it by the reduced Stokesian formula (46) by setting:

$$\frac{\partial N}{\partial z} = f^{-2}\sqrt{f_x^2+f_y^2+f^{-2}f_z^2}\ ,$$

so one will get N by a quadrature, and:

$$S = \int_\Sigma N(y\,dx - x\,dy)\,.$$

If the contour Σ is traced by a surface (σ) whose equation is:

$$2N + 1 = 0$$

then making S planar will again be quire remarkable: If S is projected conoidally onto (σ) then upon starting with Oz, that will give an end-face on (σ) whose projection onto Oxy parallel to Oz will contain S.

One is not dealing with the area S, but an arbitrary integral that is attached to that area, so one easily sees that formula (47) must be replaced with:

$$\iint_S \Theta(X,Y,Z)\,dS = \iint_\sigma \Theta\left(\frac{x}{f},\frac{y}{f},z\right) f^{-2}\sqrt{f_x^2 + f_y^2 + f^{-2} f_z^2}\,(\alpha x + \beta y)\,dS.$$

Once all of that has been arranged, one will see why the coefficient of $(\alpha x + \beta y)$ is homogeneous of order – 2 *in only x and y.* For a given surface S, that coefficient will have a certain expression that also corresponds to a function f^* that is more general than f and is a solution of an equation of the form:

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \frac{1}{f^2}\left(\frac{\partial f}{\partial z}\right)^2 = \Omega\left(\frac{x}{f},\frac{y}{f},z\right). \tag{48}$$

The most general solution f^* that is homogeneous of order one *in only x and y* and satisfies equation (48) must unavoidably be equated to *unity.* Hence, in short, they will be the surfaces $f = 1$ *on which* one has:

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \Omega\,(x, y, z). \tag{41}$$

One then recovers equation (41) with its dynamical and wave-like meanings.

One can now associate a corpuscular propagation of conoidal nature with the motion of a point and the propagation of a wave front.

With the manner of homogenizing (41) that was studied originally, one will get conical propagation. The variation of tubes will permit one to further transform those propagations greatly, which can, perhaps, be made more determinate by a study of the initial conditions for emission. The analysis and geometry that were developed here show simply how wave fronts can collapse to corpuscles and how swarms of corpuscles can range over wave-fronts. An indeterminacy that relates to the Heisenberg uncertainties undoubtedly remains at the basis for these questions.

12. On the Schrödinger equation. – We now have the opportunity to reconcile the end of this chapter with the end of the preceding chapter. It is likewise interesting to borrow a few things from Weyl ([1]). In Newtonian mechanics, we will have:

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) \tag{49}$$

for a free corpuscle.

The correspondence:

([1]) HERMANN WEYL, *Gruppentheorie und Quantenmechanik*, Second edition, Leipzig, 1931. *Cf.*, pp. 45-49. As for the generation of second-order wave equations from sets of operators, such as the ones in table (50), one can also refer to the work that was cited before by EUGÈNE BLOCH, *L'ancienne et la nouvelle Théorie des Quanta.* In particular, see Chapter XIV on the Schrödinger equation.

(50)
$$\begin{cases} H, & p_x, & p_y, & p_z, \\ -\frac{h}{i}\frac{\partial}{\partial t}, & \frac{h}{i}\frac{\partial}{\partial x}, & \frac{h}{i}\frac{\partial}{\partial y}, & \frac{h}{i}\frac{\partial}{\partial z}, \end{cases}$$

when combined with (49), will give the wave equation:

(51)
$$\frac{h}{i}\frac{\partial\psi}{\partial t} - \frac{h^2}{2m}\Delta\psi = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In relativistic mechanics, (49) is replaced by:

$$\frac{H^2}{c^2} - (p_x^2 + p_y^2 + p_z^2) = m^2 c^2.$$

Upon (right) multiplying all of the terms in that equation by ψ and subjecting it to the correspondence (50), this time one will have:

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \Delta\psi = \frac{m^2c^2}{h^2}\psi,$$

instead of (51).

When one is dealing with a corpuscle in a force field, one will have:

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z),$$

in which V denotes the potential energy that was represented by U above. Here, the introduction of the variables x, y, z no longer corresponds to any operator that permutes with the ones in the second row of table (50). Having stated that, Weyl (*loc. cit.*, pp. 49) wrote:

"Despite the non-commutability that was emphasized above, we – with *Schrödinger* – dare to apply the rule for obtaining the wave equation in this case, as well. We then obtain the *Schrödinger differential equation:*

$$\frac{h}{i}\frac{\partial\psi}{\partial t} - \frac{h^2}{2m}\Delta\psi + V(x, y, z)\,\psi = 0."$$

Thus, as the first word of that quotation suggests, it is *despite* certain non-permutabilities that the Schrödinger equation exists.

Under those conditions, it seems interesting to remark that our theory of the *homogenized* Jacobi equation is not constructed *despite* the non-permutabilities and the combinations (such as Euler's theorem) that one will observe in one row or the other of the table:

$$\begin{matrix} \dfrac{\partial}{\partial x}, & \dfrac{\partial}{\partial y}, & \dfrac{\partial}{\partial z}, \\ x, & y, & z, \end{matrix}$$

but rather, *in plain accord with* those non-permutabilities and combinations.

Homogenizing the Jacobi equation or deducing the Schrödinger equation from it are, without fail, distinct processes that are undoubtedly coupled imperfectly in the present article. However, both of them seem to be comparably important and to have physical roles that, while different, must be capable of being associated.

13. Waves derived from an ellipsoid. – The present article touches upon theoretical physics much more than pure geometry, since up to now we have presented only the geometric developments that are necessary for the presentation of physical schemes. From the purely geometric viewpoint, the results can become extremely numerous and can be transformed in a host of ways; there is no reason to elaborate upon those possibilities. Meanwhile, this is the place to touch very briefly upon the ellipsoidal areas and their propagation and some very interesting results that were pointed out in the *Comptes rendus* (9 February 1931), and which can still be claimed by Georges Humbert, since that excellent and dearly-missed geometer dedicated much effort to ellipsoidal areas, which is a question that seems rather difficult, moreover. What follows seems to simplify it considerably and to render the ellipsoidal areas perhaps more manageable than the arcs of the ellipse.

Let the ellipsoid have the equation:

$$AX^2 + BY^2 + CZ^2 = 1. \tag{52}$$

One can associate equation (45) with it:

$$f = \sqrt{\frac{Ax^2 + By^2}{1 - Cz^2}} = 1, \tag{53}$$

in which one sees that f is homogeneous of order *one* in only x and y.

Formula (47) will then give:

$$S = \iint_\sigma \frac{1}{Ax^2 + By^2} \sqrt{(1 - Cz^2)U + C^2 z^2}\,(\alpha x + \beta y)\,d\sigma \tag{54}$$

upon setting:

$$U = \frac{A^2 x^2 + B^2 y^2}{Ax^2 + By^2}. \tag{55}$$

The search for what one can call the *planifying surface* of the ellipsoid leads one to write, always with the considerations of paragraph 11:

$$\frac{\partial N}{\partial z} = \frac{1}{Ax^2 + By^2}\sqrt{U - C(U-C)z^2}\,.$$

One sees that one can get N by a simple circular quadrature. In a more precise manner, upon appealing to:

$$\int\sqrt{a^2 - b^2u^2}\,du = \frac{u}{2}\sqrt{a^2 - b^2u^2} + \frac{a^2}{2b}\arcsin\frac{bu}{a}\,,$$

one will get:

$$N = \frac{z\sqrt{U - C(U-C)\,z^2} + \dfrac{U}{\sqrt{C(U-C)}}\arcsin\sqrt{\dfrac{C(U-C)}{U}}\,z}{2(Ax^2 + By^2)} + \frac{1}{x^2}\varphi\left(\frac{y}{x}\right),$$

with φ an arbitrary function.

The equation of the planifying surface $2N + 1 = 0$ can finally be written:

$$C\,(U - C)\,z^2 = U\sin^2 V, \tag{56}$$

in which U always has the same value that was indicated in (55), and upon setting:

$$\frac{UV}{\sqrt{C(U-C)}} = \varphi\left(\frac{y}{x}\right) - Ax^2 - By^2 - z\sqrt{U - C(U-C)\,z^2}\,,$$

moreover.

Therefore, the planifying surface (σ) in equation (56) approximates the scalene ellipsoid E in equation (52), which leads one to assert:

If a right conoid with director Oz cuts out a certain end-face of area S on the ellipsoid E and cuts out a contour Σ *on the planifying surface* (σ) *then the projection* Σ' *of* Σ *onto the plane Oxy will enclose an area that is equivalent to S.*

Now, compare formulas (47) and (54). One will indeed have:

$$f^{-2}\sqrt{f_x^2 + f_y^2 + f^{-2}f_z^2} = \frac{1}{Ax^2 + By^2}\sqrt{(1 - Cz^2)U + C^2z^2} \tag{57}$$

for the form of f that was indicated in (53), but it is clear that there are other functions f (x, y, z) that are homogeneous of order *one* in x and y that satisfy (57). With those new functions f, the surfaces with the equation $f = 1$ will give end-faces in the conoids of the italicized assertion that are equivalent in area and which can propagate the area S that the same conoid cuts out on the ellipsoid in equation (52).

Obviously, all of the end-faces that are contained in the same conoid can be made planar with the aid of the planifying surface (σ) by the construction that is first concerned with only the ellipsoidal area S.

Meanwhile, the latter results can hardly be made explicit, because equation (57), although of first order, is sufficiently complicated to defy an explicit integration.

Everything will become simple in the case of the sphere with:

$$A = B = C = U = \frac{1}{R^2}.$$

Equation (57) reduces to:

(58)
$$f^{-2}\sqrt{f_x^2 + f_y^2 + f^{-2} f_z^2} = \frac{R}{x^2 + y^2},$$

and one immediately perceives that one can look for a solution $f(x, y)$ to the latter equation that contains only x and y. That is a great source of simplification that does not seem to have an equivalent in the case of equation (57).

For (58), one has:

$$f = \frac{1}{R}\sqrt{x^2 + y^2},$$

and that will bring one back to the Archimedean considerations of the circumscribed cylinder. One see that Archimedes was justly excited about a proposition whose extensions should greatly exceed the case of the figure that was engraved upon his tomb.

____