# The Grassmann method in projective geometry

Three notes to the Rendiconti del Circolo matematico di Palermo

("Il metodo del Grassmann nella geometria proiettiva,")

by C. Burali-Forti

Translated by D. H. Delphenich

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Note I by C. Burali-Forti, in Turin

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"To treat all of the geometry of position by itself, with no metric concepts that would be extraneous." (*) is the goal that was proposed by Staudt in his book *Geometrie der Lage.*

For some years now, a new direction has been presented in the form of *projective analytic geometry*, and whose goal – at least, as far as metric concepts are concerned – is the opposite of that of Staudt.

The analytic method makes use of coordinates. The projective homogeneous coordinates are difficult to apply to metric questions, just as Cartesian and polar coordinates are difficult to apply to projective questions. A coordinate system, in general, represents a geometric element $P$ by 1, 2, … numbers, which vary with not only $P$ but also with the reference elements, and has only an *indirect* relationship to $P$ that eventually becomes of secondary significance in the calculations that I will consider.

The synthetic method rapidly solves the majority of the projective questions, but the metric method must often be linked to analytic geometry.

If a geometric method succeeds in treating the metric and projective questions indifferently and with equal facility then it will be more perfect and more powerful than the other ones, and more convenient from the didactic aspect, since it would be an instrument that could be applied, not just in many cases, but always (**).

One that can presently deduce the method that satisfies the indicated conditions – and we believe, completely – from Grassmann's book (*Die Ausdehnungslehre*). M. E. Carvallo (***), speaking about Grassmann's work, said: "It synthesizes the known theories of mechanics and geometry…" In particular, it synthesizes the analytic-geometric methods of *quaternions* (Hamilton), *barycenters* (Möbius), and *equipollence* (Bellavitis), and the analytic-geometric method of coordinates in general, without having any need for the use of coordinates, since it can operate upon geometric entities *directly*.

The scope of this paper is to show how Grassmann's work also synthesizes the projective methods (*); i.e., the projective entities *point*, *line*, *plane* immediately take on geometric forms, as well as linear systems and fundamental projective systems of the

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(*) C. Segre: *C. G. C. V. Staudt ed i suoi' lavori* (in the volume "Geometria di Positione di Staudt, traduzione dal tedesco di M. Pieri").

(**) This does not, by any means prejudice the purely scientific question of treating the geometry of position independently of any metric concept. That question seems to have been solved rigorously and completely in the recent papers of Pieri: "Sui principi che reggono la geometria di posizione," Note I, 1895; Note II, 1896. Atti Accademia Torino. – "Un sistema si postulati per la Geometria proiettiva astratta degli iperspazi." Revue de Mathématiques, 1896.

(***) Nouvelles Annales de Mathématiques, 1892. "La méthode de Grassmann."

(*) F. Gaspary. – Bulletin des Sciences Mathématiques. – "Sur une méthode générale de la géométrie qui forme le lien entre la géométrie synthétique et la géométrie analytique."

first, second, and third kind, and *all* of the linear correspondences of ordinary projectivities, and finally, how the general coordinates of the form *coincide* with the homogeneous projective coordinates, so in this way it is also capable of solving the metric problems, provided that a projective entity is substituted for the geometric form that identifies it. That is what defines the objective of this first note.

It will result that, with no need to make an explicit study, the already too rich system of nomenclature that projective geometry currently uses can be simplified and reduced appreciably. However, it is not totally suppressed, because even if the concepts that it expresses do not appear in many of the vast geometric applications of analysis, such concepts are indispensible to *descriptive geometry*, and are of use to *graphical statics*, *optics*, … In regard to descriptive geometry, we will see how all of the methods of representation and fundamental operations (e.g., *projection*, *inversion*) are obtained from a special linear correspondence, and in a completely elementary way. This will define the principal object of Note II.

Since Grassmann's methods are widespread today (**) (not all of which are known), we will dispense with the summary of the parts of the theory that we will use. What remains is a very concise exposition that cannot give an exact impression to those who do not know the theory of forms. Rather than the abstract calculations that are contained in Grassmann's book, which "miraculously apply to geometry" (***), we refer to the *Calcolo geometrico* (****) of Peano, which starts with the elementary concepts of Euclidian geometry, and then obtains the geometric forms and operations that relate to them in a very simple way.

We like to think that we have done something useful – at least, didactically – if we succeed in convincing the reader that Grassmann's method can give an *intimate fusion* of analytic geometry with projective geometry, without the metric and projective questions losing the importance that they have in the analytic and graphic fields.

## § 1. – Projective elements.

A line, as it is considered in projective geometry, is either a line at infinity or consists of all of the points of a Euclidian line plus a point at infinity that is in the direction of the line. A plane is either a plane at infinity or consists of all the points of a Euclidian plane plus the points of a line at infinity that is incident on the plane.

We would like to see how the projective point, line, and plane can be easily obtained from geometric forms.

Let $A$ be a form of the first kind. If $m$ is its *mass* and $m$ is non-zero then one knows (*) that $A$ is reducible to the product of $m$ by a Euclidian point that is called the *barycenter* of $A$.

(**) Rivista di Matematica, 1895. – "Elenco bibliografico sull'Ausdehnunglehre di H. Grassmann."

(***) Carvallo, *loc. cit.*

(****) G. Peano: *Calcolo Geometrico secondo l'Ausdehnungslehre di H. Grassmann*, Bocca, Turin, 1888. We shall cite this book with the symbol C. G.

(*) C. G., pp. 36.

Given the preceding hypotheses, we write posit $A$ (i.e., the position of $A$) (**) for the location of the barycenter of $A$. Therefore, posit $A$ denotes a uniquely-determined Euclidian point.

If $A$, $B$ are forms of the first kind with non-zero mass and equal barycenters then it is obvious that $A$ is equal to the product of $B$ by a real number. Conversely, if $A$ is the product of $B$ by a number then $A$ and $B$ have the same barycenter. The hypotheses that relate to $A$ and $B$ then imply that one can write:

(1) $$\text{posit } A = \text{posit } B \Rightarrow A \in qB,$$

in which $q$ is the locus of the "real numbers" (***).

If $I$ is a vector (which is always intended to be non-zero) – i.e., a form of the first kind with zero mass (****) – then it is not possible to define posit $I$ as in the preceding case.

We let posit $I$ denote an abstract entity that is a function of the vector $I$, and if $J$ is another vector then we will say that posit $I$ = posit $J$ when $I$ is parallel to $J$, or – what is equivalent (†) – when $I$ is the product of $J$ by a number. That is, we set:

(1)′ $$\text{posit } I = \text{posit } J \Rightarrow I \in qJ;$$

i.e., we assume that, by definition, (1) can be proved when $A$, $B$ are not vectors.

It follows from (1)′ that posit $I$ is a function of $I$ that $I$ has in common with all vectors that are parallel to $I$; i.e., ones that have *the same direction* as $I$.

Following the ordinary usage, we say "point at infinity" instead of "position of a vector" and "projective point" in place of "position of a form of the first kind." A projective point is either a Euclidian point or a point at infinity.

Let $a$ be a form of the second kind with zero invariant; i.e., $a\, a = 0$. It is known that it is reducible to either a *line* (i.e., the product of two Euclidian points) or to a *bivector* (i.e., the product of two vectors) (††). A form $A$ of the first kind is said to belong to $a$ when $Aa = 0$.

We write posit $a$ to denote the class of projective points that are positions of the forms of the first kind $A$ such that $Aa = 0$.

If $a$ is reducible to the product of two Euclidian points $P$, $Q$ then posit $a$ will contain all of the Euclidian points of the line that passes through $P$ and $Q$, along with the point at infinity that is the position of the vector $Q - P$.

If $a$ is reducible to the product of two vectors $I$, $J$ then any form $A$ such that $AIJ = 0$ is a vector that is parallel to any Euclidian plane that is parallel to the vectors $I$, $J$. Therefore, posit $a$ is a class of points at infinity.

One easily proves (*), as one does for (1), that if $a$, $b$ are forms of the second kind with zero invariant then:

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(**) The concept of the position of a form, which serves to connect this theory with the theory of projective elements, is discussed in C. G. by Peano (pp. 72, no. 36, 2).

(***) We make use of the logical symbols that were adopted in "Formolario di Matematica," which was published in *Rivista di Matematica*. [Translator's note: These symbols have been updated.]

(****) C. G., pp. 37.

(†) C. G., pp. 41.

(††) C. G., pp. 57.

(*) C. G., pp. 28.

(2) $$\text{posit } a = \text{posit } b \Rightarrow a \in qb.$$

Therefore, if $i$ is a bivector then posit $i$ is a function of $i$ that $i$ has in common with all bivectors that are parallel to $i$.

The preceding justifies our saying, following the common language "line at infinity," instead of "position of the (non-zero) bivector" and "projective line," instead of "position of a (non-zero) form of the second kind with zero invariant."

Analogously, if $\alpha$ is a (non-zero) form of the third kind then posit $a$ will denote the class of projective points that are positions of the forms of the first kind $A$ that lie on $\alpha$; i.e., ones such that $A\alpha = 0$.

If it easily proved (**) that if $\alpha$, $\beta$ are forms of the third kind then:

(3) $$\text{posit } \alpha = \text{posit } \beta \Rightarrow \alpha \in q\beta.$$

If $\alpha$ is reducible to a triangle – i.e., to the product of three distinct, non-collinear, Euclidian points $P$, $Q$, $R$ – then posit $\alpha$ will contain all of the points of the Euclidian plane that are identified by the points $P$, $Q$, $R$ and the points at infinity that are positions of the vectors that are parallel to that plane. As we have seen, these points are on a line at infinity that is *incident* on the plane.

If $\alpha$, $\beta$ are trivectors then $\alpha \in q\beta$, and therefore posit $\alpha$, does not change when one changes the trivector $\alpha$. In order for the form of the first kind $A$ to be such that $A\alpha = 0$, it is necessary and sufficient that $A$ be a vector, and therefore that posit $\alpha$ must contain all points at infinity, and only them.

The preceding justifies our saying, following the common language, "plane at infinity," instead of "position of an arbitrary (non-zero) trivector," and "projective plane," instead of "position of a (non-zero) form of the third kind."

The fact that these projective elements possess the usual properties is easy to comprehend. An example will suffice.

"Two distinct projective planes have a common projective line."

Let $\alpha$, $\beta$ be two planes and let $\alpha_1$, $\beta_1$ be forms of the third kind that have $\alpha$ and $\beta$ for their positions. The regressive product (***) $\alpha_1\beta_1$ is not zero because then $\alpha_1 \in q\beta_1$ would be zero, so $\alpha = \beta$. The regressive product $\alpha_1\beta_1$ is a form of the second kind with zero invariant that contains all of the forms of the first kind that belong to $\alpha_1$ and $\beta_1$. Therefore, posit $\alpha_1\beta_1$ is a projective line that is common to $\alpha$ and $\beta$. If $\alpha_2$, $\beta_2$ are forms of the third kind that have $\alpha$, $\beta$ for their positions then $\alpha_2 \in q\beta_2$, $\beta_2 \in q\beta_1$, $\alpha_2\,\beta_2 \in q\alpha_1\beta_1$, and therefore posit $\alpha_1\beta_1$ = posit $\alpha_2\beta_2$ is the unique projective line that is common to $\alpha$ and $\beta$.

(**) C. G., pp. 28.
(***) C. G., pp. 109, *et seq.*

## § 2. – Linear systems and projective systems.

A system of entities (i.e., classes) is said to be *linear* (*) when the *sum* and *product with a real number* are defined for its elements, and those operations enjoy the properties of the corresponding operations for numbers.

Elements $a_1, a_2, \ldots, a_n$ of a linear system are said to be *linearly-independent*, or simply *independent*, when it is not possible to determine numbers $m_1, m_2, \ldots, m_n$ that are not all zero such that $m_1 a_1 + m_2 a_2 + \ldots + m_n a_n = 0$.

One says that a linear system is "$n$-dimensional" when $n$ independent elements exist, but $n + 1$ of them will always be dependent. This is equivalent to saying that if $a_1, a_2, \ldots, a_n$ are independent elements of the system and $a$ is an arbitrary element of the same system then there are uniquely-defined numbers $x_1, x_2, \ldots, x_n$ such that $a = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n$. The numbers $x$ are called the *coordinates* of $a$ with respect to the reference elements $a_1, a_2, \ldots, a_n$.

For examples, we know (**): The form of the first kind that is incident with a Euclidian line and the vectors that are parallel to a Euclidian plane are linear systems of dimension *two*. The forms of the first and second kind of a Euclidian plane and the vectors and bivectors of space are linear systems of dimension *three*. The forms of first and third kind in space are linear systems of dimension *four*.

If $U$ is a linear system of geometric forms then we let posit $U$ denote the system of projective elements that are positions of the elements of $U$. If, e.g., $U$ is a system of vectors that are parallel to a plane then posit $U$ will be the line at infinity that is incident on that plane.

As one does in ordinary projective geometry, one can, from what was done in § 1, define the "fundamental forms of first, second, and third kind" (***). Because there is no place for misunderstandings here between the two meanings of the term *forms*, we shall say "*projective figures* of the first, second, third kind," in place of the preceding phrase.

Let $n$ be one of the numbers 2, 3, 4, and let $U$ be an $n$-dimensional linear system of geometric forms. posit $U$ is then a projective figure of the $(n - 1)^{\text{th}}$ kind. Conversely, if posit $U$ is a projective figure of the $(n - 1)^{\text{th}}$ kind then $U$ will be an $n$-dimensional linear system.

The reader can prove this general proposition quite easily by making use of the linear systems of two, three, and four dimensions that we just recalled.

In particular:

*a*) The systems of geometric forms whose positions are projective figures of the first kind – viz., *point-like*, *sheaves of rays*, *sheaves of planes* – are linear systems of dimension *two* for the forms of *first*, *second*, and *third* kind, respectively.

*b*) The systems of geometric forms whose positions are projective figures of the second kind – viz., *point-like plane*, *ruled plane*, *stars of lines*, *stars of planes* – are linear systems of dimension *three* for the forms of *first*, *second*, and *third* kind, respectively.

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(*) C. G., pp. 141-144.
(**) C. G., pp. 113-119; 141-144.
(***) Staudt, *Geometria di posizione*, translated by Pieri, Bocca, Torino, 1889; pp. 1-11.

*c*) The systems of geometric forms whose positions are projective figures of the third kind – viz., *spaces of points*, *spaces of planes* – are linear systems of dimension *four* for the forms of *first* and *third* kind, respectively.

Observe that projective systems are not linear systems, but only positions of linear systems.

## § 3. – Double ratios.

Let $U$ be a two-dimensional linear system of geometric forms and let $a$, $b$, $c$, $d$ be elements of $U$.

If posit $U$ is a projective line then $ab$ is a *line* or a *bivector* according to whether posit $U$ is not or is a line at infinity, resp. If $cd$ is not zero then $ab / cd$ (*) is the real number by which one must multiply $cd$ in order to obtain $ab$.

If posit $U$ is a sheaf of lines then the elements of $U$ will be lines that are incident with a Euclidian plane or bivectors, and in one case or the other they are products of a form $W$ of the first kind with a fixed position (i.e., the center of the sheaf) with a form of the first kind. The progressive product of two elements of $U$ is always zero. We always intend that $ab$ shall denote the regressive product (**) of $a$ with $b$. One has that $a = WA$, $b = WB$, $c = WC$, $d = WD$, where $A$, $B$, $C$, $D$ are forms of the first kind whose positions are points of the plane of the sheaf posit $U$. Consequently, if $cd$ is not zero then one has:

$$ab / cd = (W\,AB.\ W) / (W\,CD.\ W) = W\,AB / W\,CD,$$

and $ab / cd$ is a well-defined number that relates to two triangles or two trivectors.

If posit $U$ is a sheaf of planes then the elements of $U$ are forms of third kind and products of a form of the second kind $s$ (i.e., a line or bivector, according to whether the planes of posit $U$ are not or are parallel, resp.) whose position is the axis of the sheaf for forms of the first kind. The product $ab$ is always a regressive product (***).

One has that $a = s\,A$, $b = s\,B$, $c = s\,C$, $d = s\,D$, where $A$, $B$, $C$, $D$ are forms of the first kind. If $cd$ is non-zero then:

$$ab / cd = (s\,AB.\ s) / (a\,CD.\ s) = s\,AB / s\,CD,$$

and $ab / cd$ is a well-defined number that relates to two tetrahedra.

For any posit $U$, if $ab$ is non-zero then we will put $ab / 0 = \pm\infty$, while the choice of sign will remain arbitrary.

Let $u_1$, $u_2$, $u_3$, $u_4$ be elements of $U$ such that no three of them have the same position. Write rat($u_1$, $u_2$, $u_3$, $u_4$), or simply rat $u$, in place of "the double ratio of the sequence $u_1$, $u_2$, $u_3$, $u_4$" and set, by definition:

(1) $$\text{rat}\ u = (u_1\,u_3 / u_2\,u_3)(u_2\,u_4 / u_1\,u_4).$$

(*) C. G., pp. 29.
(**) C. G., pps. 80 and 99.
(***) C. G., pp. 109.

From the hypotheses that were made regarding the position of $u$, it results immediately that none of the ratios that rat $u$ can produce will present themselves in the form 0 / 0, and that rat $u$ cannot present itself in the form $0 \times \infty$. Therefore, rat $u$ is either a real number or it is equal to $\pm\infty$:

(2) $$\text{rat } u \in (q \vee \pm\infty).$$

It results immediately from (1) that:

(3) $$\text{rat } u = 0 \Rightarrow (u_1\, u_3 = 0) \vee (u_2\, u_4 = 0),$$

(4) $$\text{rat } u = \pm\infty \Rightarrow (u_2\, u_3 = 0) \vee (u_1\, u_4 = 0),$$

(5) $$\text{rat}(u_1, u_2, u_3, u_4) = \text{rat } u.$$

Let $u_1$, $u_2$ be independent, while $x_1$, $x_2$, $y_1$, $y_2$ are real numbers, and then consider the double ratio of the sequence $u_1$, $u_2$ , $x_1\, u_1 + y_2\, u_2$, $x_1\, u_1 + y_2\, u_2$ . One then immediately has from (1) that:

(6) $$\text{rat}(u_1, u_2 , x_1\, u_1 + y_2\, u_2, x_1\, u_1 + y_2\, u_2) = x_2\, y_1 / x_1\, y_2 .$$

One easily deduces the following from (5), (6):

(7) $$\text{rat } u = 1 \Rightarrow (u_1\, u_2 = 0) \vee (u_3\, u_4 = 0),$$

(8) $$\text{rat}(u_1, u_3, u_2, u_4) = 1 - \text{rat } u,$$

(9) $$\text{rat}(u_2, u_1, u_3, u_4) = 1 / \text{rat } u.$$

Suppose that the preceding hypotheses are valid for $U$ and $u$, and let $a_1$, $a_2$, $a_3$, $a_4$ be non-zero real numbers. For example, since $(a_1\, u_1)(a_3\, u_3) = (a_1\, a_3)\, u_1\, u_3$, (1) gives:

$$\text{rat}(a_1\, u_1, a_2\, u_2, a_3\, u_3, a_4\, u_4) = \text{rat } u.$$

This says that rat $u$ is a function of *position* of the form $u$, since, e.g., posit $a_1\, u_1$ = posit $u_1$ .

Therefore, if $P_1$, $P_2$, $P_3$, $P_4$ are elements of a projective figure of the first kind such that no three of them coincide then one can define the double ratio of the sequence of $P$ by:

$$\text{rat } P = \text{rat } u,$$

where the $u$ are geometric forms such that:

$$\text{posit } u_r = P_r \qquad (r = 1, 2, 3, 4).$$

This is well-defined, and in general, the double ratio of a sequence of four elements of a projective figure of the first kind. The fact that it coincides with the usual double ratio results, in part from the fact that if $P_1$, $P_2$, $P_3$, $P_4$ are collinear Euclidian points then (1) will give:

$$\text{rat}\, P = (P_1 P_3 / P_2 P_3)\,(P_2 P_4 / P_1 P_4),$$

which is customarily assumed by definition. In § 4, it will be proved that the double ratio of the section of a sheaf is independent of the secant element and equal to the double ratio of the sheaf. The latter property is usually assumed by the definition of the double ratio of four elements of a sheaf. Properties (2)-(9) immediately give the ordinary properties of double ratios.

"If $P_1$, $P_2$, $P_3$, are distinct elements of a projective figure of the first kind and $h$ is either a real number or infinity then there will exist exactly one element $P_4$ of the figure such that rat $P = h$."

Proof. Let $u_1$, $u_2$, $u_3$ be geometric forms that have $P_1$, $P_2$, $P_1$ for their positions, resp. Since $u_1$, $u_2$ are independent and $u_3$ has a position that is distinct from $u_1$, $u_2$, one can determine non-zero numbers $x_1$, $x_2$ such that $u_3 = x_1 u_1 + x_2 u_2$ . Put $u_4 = y_1 u_1 + y_2 u_2$ . One then has rat $u = (x_2 / x_1)\,(y_1 / y_2)$. If rat $u$ has the value $h$ then $y_1 / y_2$ is defined, and consequently, so is *the position* of the form $u_4$ – i.e., the element $P_4$ – which was to be proved (*).

At the end of this section, we will observe that: "The ordinary projective homogeneous coordinates (**) coincide with the general coordinates of the geometric form."

We prove this for the coordinates of the points of a projective plane $\pi$.

Let $A_1$, $A_2$, $A_3$ be forms of the first kind that are linearly independent and have points of the plane $\pi$ for their positions. For a form $P$ that has its position in $\pi$, the coordinates $x_1$, $x_2$, $x_3$ with respect to the reference elements $A_1$, $A_2$, $A_3$ are well-defined, and one has:

$$P = x_1 A_1 + x_2 A_2 + x_2 A_3 \,.$$

If $m$ is a non-zero number then:

$$m\, P = (m\, x_1)\, A_1 + (m\, x_2)\, A_2 + (m\, x_3)\, A_3,$$

and therefore $x_1$, $x_2$, $x_3$ will not be homogeneous coordinates of $P$. However, since posit $(m\, P)$ = posit $P$, $x_1$, $x_2$, $x_3$ will be homogenous coordinates of posit $P$.

If one sets:

$$E = A_1 + A_2 + A_3$$

then one will immediately have that

(*) This is because while the coordinates – e.g., $x_1$, $x_2$ – of $u_3$ with respect to $u_1$, $u_2$ vary when $u_1$, $u_2$, $u_3$ varies, the ratio $x_1 / x_1$ varies only when the positions of $u_1$, $u_2$, $u_3$ vary.

(**) The first systematic treatment of projective, homogeneous coordinates was made by Fiedler (*Darstellende Geometrie*). The concept of a system of homogeneous coordinates is found in Möbius (*Der barycentrische Calcul*, 1827), in Staudt (*Beiträge zur Geometrie der Lage*, 1856), and in Hamilton (*Elements of Quaternions*, 1866).

$$\text{rat}(A_1 A_2, A_1 A_3, A_1E, A_1P) = x_2 / x_3 ,$$
……………………………………..,

and therefore – e.g., $x_2 / x_3$ – will be the double ratio of the line of the sheaf whose center is posit $A_1$ that projects to points that have positions of the forms $A_2$, $A_3$, $E$, $P$. That is, $x_1$, $x_2$, $x_3$ are the projective homogeneous coordinates of posit $P$, since posit $A_1$, posit $A_2$, posit $A_3$ are vertices of the triangle of reference and posit $E$ is the *unit* point. This proves our assertion.

## § 4. Homographies.

Let $\sigma$ be a correspondence between the elements of a linear system $U$ and another linear system $U'$. It is known (*) that $\sigma$ is called a "linear transformation" when for any arbitrary elements $x$, $y$ of $U$ and any arbitrary real number $m$, one has:

$$\sigma(x + y) = \sigma x + \sigma y; \quad \sigma(m\,x) = m(\sigma x).$$

Any real number is a linear transformation of a linear system into itself.

From time to time, we shall recall the properties of such transformations, as required.

Let $n$ be any of the numbers 2, 3, 4, and let $U$, $U'$ be $n$-dimensional linear systems of geometric forms. If $u_1, \ldots, u_n$ are independent elements of $U$ and $u_1', \ldots, u_n'$ are arbitrary elements of $U'$ then exactly one linear transformation of $U$ into $U'$ is defined such that $u_1, \ldots, u_n$ correspond to $u_1', \ldots, u_n'$, with some ordering. If $\sigma$ is that transformation then put:

$$\sigma = \begin{pmatrix} u_1', & \ldots & u_n' \\ u_1, & \ldots & u_n \end{pmatrix}.$$

If the $u'$ are also independent elements of $U'$ then $\sigma$ will admit an inverse. That is, there will exist a linear transformation from $U'$ to $U$ that makes $u_1', \ldots, u_n'$ correspond to $u_1, \ldots, u_n$. This will be denoted by $\sigma^{-1}$; i.e., one will set:

$$\sigma^{-1} = \begin{pmatrix} u_1, & \ldots & u_n \\ u_1', & \ldots & u_n' \end{pmatrix}.$$

If $\sigma$ is an invertible transformation between $U$ and $U'$ then it will be a *single-valued* and *reciprocal* correspondence between $U$ and $U'$.

Linear correspondences between systems of geometric forms will be referred to as *homographies* (**). We shall study their principal properties.

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(*) C. G., pp. 145-151.

(**) In part IV of the paper that was cited above, Carvallo obtained homographies (no. 24) using the coordinates of the geometric forms. Using the method that we presented, one does not make use of coordinates, but must only take into account the number of dimensions of the system, which is not

I. – "If $\sigma$ is an invertible homography between $U$ and $U'$, and $V$ is a linear system that is contained in $U$ then $\sigma V$ will be a linear system that is contained in $U'$ and has the same dimension as $V$."

If $n = 4$ then there will exist an infinitude of linear systems $V$ of dimensions 3 or 2 that are contained in $U$. If $n = 3$ then there will exist an infinitude of two-dimensional linear systems $V$ that are contained in $U$. If $V$ is $n$-dimensional then $V = U$.

To prove that theorem suppose, e.g., that $V$ is a two-dimensional linear system. If $P_1$, $P_2$ are independent elements of $V$ then $\sigma P_1$, $\sigma P_2$ will also be independent elements of $U'$, since otherwise $\sigma$ would not be invertible. If $P$ is an arbitrary element of $V$ then $P = x_1 P_1 + x_2 P_2$ and therefore $\sigma P = x_1 (\sigma P_1) + x_2 (\sigma P_2)$, which proves the theorem.

II. – "If $\sigma$ is an invertible homography between $U$ and $U'$ and $V$ is a two-dimensional linear system that is contained in $U$ then the double ratio of any four elements of $V$ will be equal to the double ratio of the corresponding elements."

Let $P_1$, $P_2$, $P_3$, $P_4$ be elements of $V$. The elements $\sigma P_1$, $\sigma P_2$, $\sigma P_3$, $\sigma P_4$ belong to a two-dimensional system. If one can consider the double ratio of the sequence $P$ then the same thing will be true for the sequence $\sigma P$, since $\sigma$ is an invertible correspondence. If $P_1$, $P_2$, $P_3$, $P_4$ are independent and $x_1$, $x_2$, $x_3$, $x_4$ are the coordinates of $P_1$, $P_4$ with respect to $P_1$, $P_2$ then they will also be the coordinates of $\sigma P_1$, $\sigma P_4$ with respect to $\sigma P_1$, $\sigma P_2$. It follows from this that rat $P$ = rat($\sigma P$) = $(x_2\, y_1) / (x_1\, y_2)$, which was to be proved.

III. – "If the homography $\sigma$ between $U$ and $U$ is such that there exist $n + 1$ united elements of $U$ and any $n$ of them are independent then $\sigma$ will be a number; i.e., every element of $U$ will be united with respect to $\sigma$."

One says that an element $P$ of $U$ is united with respect to $\sigma$ when $\sigma P$ is the product of $P$ with a non-zero number; i.e., when posit($\sigma P$) = posit $P$.

Let $P_r$ ($r = 1, \ldots, n, n + 1$) be $n + 1$ elements of $U$ that are united with respect to $\sigma$. One has:

$$\sigma P_r = h_r\, P_r \qquad (r = 1, \ldots, n, n + 1),$$

$$P_{n+1} = x_1\, P_1 + \ldots + x_n\, P_n\,,$$

where, from the hypotheses that were made, $h_r$ and $x_r$ are non-zero numbers. It follows that:

$$x_1\, (h_1 - h_{n+1})\, P_1 + \ldots + x_n\, (h_n - h_{n+1})\, P_n = 0.$$

However, $P_1, \ldots, P_n$ are independent, and the numbers $x$ are not zeroes, so:

$$h_1 = h_2 = \ldots = h_n = h_{n+1},$$

equivalent to the systematic use of coordinates, which has only an indirect relationship to the geometric entities that they identify.

and consequently $\sigma$ is equal to the numbers $h$, which was to be proved.

IV. – "If $s$ is a geometric form and $s\ U$ is an $n$-dimensional linear system then $s$ will be a homography between $U$ and $s\ U$."

Let $s\ U$ denote the system of forms that is produced (progressively or regressively) from the forms of $U$ by the form $s$. If, e.g., posit $U$ is point-like and $s$ is a form of the first kind that does not belong to $U$ then posit $s\ U$ will be the sheaf of lines that project posit $U$ from posit $s$. If posit $U$ is a star of planes and $s$ is a form of the third kind whose position does not pass through the center of posit $U$ then posit $s\ U$ will be the *ruled plane* that cuts the star posit $U$ with the plane posit $s$.

Proof of the theorem. If $P$, $Q$ are elements of $U$ and $m$ is a number then from the known (*) property of progressive and regressive products, one will have:

$$s(P + Q) = sP + sQ; \quad s(m\ P) = m(s\ P).$$

Therefore, $s$ is the symbol of a linear correspondence between $U$ and $s\ U$. However, by hypothesis, $s\ U$ is an $n$-dimensional linear system like $U$, and by definition, it follows that $s$ is a homography between $U$ and $s\ U$.

Let $P$ be a (non-zero) element of $U$, and let $m$ be a non-zero real number. One has that $\sigma(Pm) = m\ (\sigma P)$, and therefore that posit$[\sigma(m\ P)]$ = posit$(\sigma P)$. That is, posit $\sigma P$ does not change when $P$, which varies in $U$, does not change position.

Therefore, given the homography $\sigma$ between $U$ and $U'$, we can consider the transformation between posit $U$ and posit $U'$ to be such that an element $Q$ of posit $U$ will correspond to the element posit$(\sigma P)$, where $P$ is an arbitrary form in $U$ such that posit $P$ = $Q$.

We denote such a transformation by posit $\sigma$. If $Q$ is an element of posit $U$ then its correspondent in posit $U'$ will be denoted by (posit $\sigma$)$Q$, or more simply by posit $\sigma Q$, but not by posit$(\sigma Q)$, since $\sigma Q$ has no meaning, in general.

We call a correspondence that comes about between two projective figures of the same type a "projective homography," or , when it will not lead to any confusion, simply a "homography."

Projective homographies are not linear correspondences. If $\sigma$ is an invertible homography between $U$ and $U'$ then posit $\sigma$ will also be an invertible, projective homography between posit $U$ and posit $U'$; i.e., posit $\sigma$ will be a single-valued and reciprocal correspondence between posit $U$ and posit $U'$.

Projective homographies, thus defined, comprise all of the correspondences between projective figures that one studies in ordinary projective geometry under the general name of *projectivities*.

(*) C. G., pp. 111-112. Formulas (1)-(9), along with what was said in §§ 1, 2, immediately give the principle of *duality*.

If, indeed, *S*, *S´* are projective figures of the same kind then from propositions I, II, the definition of projective homography, and from what was said in §§ 2, 3, one will have the following proposition:

I′. – "If $\lambda$ is an invertible projective homography between *S* and *S´* and *T* is a projective figure that is contained in *S* then $\lambda T$ will be a projective figure that is contained in *S´* that has the same type as *T*."

II′. – "For any invertible homography between *S* and *S*′, the double ratio of four elements of *S* is equal to the double ratio of the corresponding elements."

If one supposes that *S*, *S´* in II′ are projective figures of the first kind, and one confines oneself to considering double ratios that have the value – 1 (i.e., harmonic double ratios) then one will have the definition of Staudt ([*]). I′ corresponds to the general definition of projectivity for projective figures of the second and third kind that was given in Staudt ([**]).

From prop. III, one deduces (Staudt's theorem):

"If a projective homography transforms the figure *S* of the $(n-1)^{\text{th}}$ kind into itself, and $n + 1$ elements of *S* such that any *n* of them do not belong to a figure of the $(n-2)^{\text{th}}$ kind ([***]) are united then every element of *S* will be united with respect to that homography."

One should notice that in III it is not necessary to suppose that $\sigma$ be an invertible homography, and from the other hypotheses that relate to united elements that imply that $\sigma$ is a number, it also results that $\sigma$ is an invertible homography.

From prop. II, IV, one immediately deduces that the fundamental operations of projective geometry – viz., *projection* and *cutting* (segare) – leave the double ratio invariant. This proves that the double ratio of four elements of a sheaf of lines of planes, as it was defined in § 3, enjoys the property that one ordinarily assumes by definition.

In the following note, we will study the homographies between particular linear systems of geometric forms by the methods that we have given the fundamentals of, and consequently, the projective homographies between particular projective figures. For now, we shall confine ourselves to proving the following two propositions:

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([*]) Staudt, *loc. cit.*, pp. 42.

([**]) Staudt, *loc. cit.*, pp. 51. There exists a slight difference that relates to the way of considering the correpondence that can be examined later on. For now, we confine ourselves to this example. According to Staudt's method, if *S*, *S´* are stars of *lines and planes* then one can consider the projectivity that makes any line (plane, resp.) of *S* correspond to a plane (line, resp.) of *S´*. We must split it into two that have, as we shall see, an intimate relationship between them, since a star of lines and planes is not the position of a linear system of forms, but rather the (logical) sum of a star of lines with a star of planes.

([***]) We intend that a point should be a projective figure of type *zero*. Otherwise, for $n = 2$, one says "the three given elements of *S* and *S´* are distinct."

"An invertible projective homography between the projective figures $S$, $S'$ of type $n$ – 1 is defined by the condition that $n + 1$ given elements of $S$ will correspond to $n + 1$ given elements of $S'$, as long as any $n$ of the given elements in $S$ and $S'$ do not belong to a projective figure of type $n - 2$."

Let $U$, $U'$ be linear systems of geometric forms, such that posit $U = S$, posit $U' = S'$. The systems $U$, $U'$ are $n$-dimensional.

Let $P_r$, $P'_r$ ($r = 1, \ldots, n, n + 1$) be forms of $U$ and $U'$ that have the given elements of $S$ and $S'$ for their positions. From the hypotheses that were made and from § 2, one deduces that any $n$ of the $P$ and $P'$ are independent.

Let $h$ be a non-zero number, and set:

$$\sigma = \begin{pmatrix} P'_1, & P'_2, & \ldots & hP'_n \\ P_1, & P_2, & \ldots & P_n \end{pmatrix}.$$

$\sigma$ is then an invertible homography between $U$ and $U'$, and $\sigma P_{n+1}$ is a well-defined form in $U'$. If one desires that posit($\sigma P_{n+1}$) = posit $P'_{n+1}$ then the number $h$ will be determined uniquely, and posit $\sigma$ will be a homography that satisfies the conditions that were posed in the theorem.

Let $\lambda$ be a projective homography that satisfies the theorem. If $Q$ is an arbitrary element of $S$ then one can consider the correspondence between $S'$ and $S'$ that makes posit $\sigma Q$ correspond to $\lambda Q$. This is a projective homography, but it has $n + 1$ united elements, so no $n$ of them will belong to a projective figure of type $n - 2$, and therefore posit $\sigma = \lambda$. The theorem is thus proved.

"If $S$, $S'$ are projective figures of the first kind and a single-valued and reciprocal correspondence between $S$ and $S'$ preserves the double ratios then that correspondence will be a projective homography."

This is an immediate consequence of the preceding proposition and the fact that when one is given the value of a double ratio and three of its projective elements, the fourth one will be determined uniquely (§ 2).

Turin, January 1896.

C. BURALI-FORTI

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Note II by C. Burali-Forti, in Turin (*)

Communicated on 14 February 1897

In this note, we propose to study the property of homographies that any form $P$ of the first kind will correspond to a form of the first kind that is a linear function of $P$ and a fixed form $W$ that is also of the first kind (**). The projective homography that this corresponds to (Note I, pp. 192) contains the ordinary *homologies* and *perspectivities*, and rapidly gives a method of representation that descriptive geometry makes use of, along with the fundamental theorems that it continually uses.

For brevity of notation, and to symbolically express some propositions, we write:

$F_1$, $F_2$, $F_3$, instead of forms of the *first*, *second*, and *third* kind, respectively.

$v$, $v^2$, $v^3$, instead of *vectors*, *bivectors*, *trivectors*, respectively; $\omega$ will denote the unit trivector (***).

For the homographies, recall the following definitions and properties: Let $U$, $U'$, $U''$ be $n$-dimensional linear systems of geometric forms.

If $\sigma$, $\lambda$ are homographies between $U$ and $U'$ then we will say that $\sigma = \lambda$ when for any form $P$ of $U$, one has that $\sigma P = \lambda P$. With the same hypotheses, set $(\sigma + \lambda)\, P = \sigma P + \lambda P$, so $\sigma + \lambda$ is a homography, since $(\sigma + \lambda)(P + Q) = (\sigma + \lambda)\, P + (\sigma + \lambda)\, Q$ and $(\sigma + \lambda)(m\, P) = m[(\sigma + \lambda)\, P]$. If $\sigma$ is a homography between $U$ and $U'$, and $\lambda$ is a homography between $U'$ and $U''$ then set $\lambda\sigma P = \lambda(\sigma P)$, so $\lambda\sigma$ will be a homography between $U$ and $U''$. If $\sigma$ is a homography between $U$ and $U$ then set $\sigma^1 = \sigma$, and if $n$ is a positive integer then set $\sigma^{n+1} = \sigma^n\sigma$. If $\sigma$ is invertible then set $\sigma^{-n} = (\sigma^n)^{-1}$, and one easily proves that the powers of $\sigma$ will enjoy the same properties that powers of numbers do.

(*) See Note I in volume 10, pp. 177-195, in these Rendiconti.

(**) By the principle of duality, one obtains, in an analogous way, the homography that makes any form $\pi$ of the third kind correspond to a form of the third kind that is a linear function of $\pi$ and a fixed form $\theta$ that is also of the third kind.

(***) According to what we did in Note I, we have that:

$$\text{projective point} = \text{posit}(F_1 \neq 0), \quad \text{projective line} = \text{posit}(F_1\, F_1 \neq 0), \quad \text{projective plane} = \text{posit}(F_3 \neq 0),$$
$$\text{point at infinity} = \text{posit}(v \neq 0), \quad \text{line at infinity} = \text{posit}(v^2 \neq 0),$$
$$\text{plane at infinity} = \text{posit}(v^3 \neq 0) = \text{posit}\ \omega.$$

## § 5. Collinear homographies.

We say that a linear correspondence $\sigma$ between forms of the first kind and forms of the first kind is a *collinear homography* when there exists a fixed form $W$ of the first kind such that for any form $P$ of the first kind one has that $\sigma P$ is a linear function of $P$ and $W$ (i.e., $\sigma P \in qP + qW$).

The form $W$ is called the *central form* of the homography $\sigma$.

**Theorem I. –** If $\sigma$ is a collinear homography whose central form is $W$ then a number $s$ will be determined, along with at least one form $\alpha$ of the third kind such that for any form $P$ of the first kind one will have:

(1) $$\sigma P = sP + (P\alpha)\, W.$$

Proof. If $P$, $P'$ are $F_1$ then, by hypothesis, numbers $s$, $s'$, $s''$, $l$, $l'$, $l''$ will be determined such that:

$$\sigma P = sP + \lambda W, \qquad \sigma P' = s'P' + l'W, \qquad \sigma(P + P') = s''\,(P + P') + l''W.$$

However, by hypothesis, $\sigma$ is a homography, and therefore $\sigma(P + P') = \sigma P + \sigma P'$. Consequently:

$$sP + s'P' + (l + l')W = s''\,(P + P') + l''W.$$

If one multiplies the two sides of the equation by a form $\beta$ of the third kind that contains $P'$ and $W$ (i.e., $P'\beta = W\beta = 0$) then one has that $sP\beta = s''P\beta$; i.e., $s = s''$. One proves that $s' = s''$in an analogous way.

From that, one deduces that: "There exists exactly one number $s$ such that $\sigma P = sP + lW$."

Now, let $P_r$ ($r = 1, 2, 3, 4$) be four independent $F_1$ . If numbers $l_r$ are determined such that:

$$\sigma P_r = sP + l_r\, W,$$

and a form $\alpha$ of the third kind is determined such that (*):

$$P_r \alpha = l_r$$

then $P_1P_2\,P_3\,P_4$ will be a unit tetrahedron. If $P$ is an arbitrary $F_1$ then numbers $x_r$ will be determined such that:

$$P = x_1\,P_1 + x_2\,P_2 + x_3\,P_3 + x_4\,P_4\,.$$

If $P\alpha = l_1\,x_1 + l_2\,x_2 + l_3\,x_3 + l_4\,x_4$ then one will have that:

$$\sigma P = sP + (P\alpha)\, W,$$

(*) $\alpha$ is the form that has the numbers $l_r$ for its coordinates with respect to the $P_r$ ; i.e., one has:

$$\alpha = l_1\,P_2\,P_3\,P_4 + l_2\,P_3\,P_4\,P_1 + l_3\,P_4\,P_1\,P_2 + l_4\,P_1\,P_2\,P_3\,.$$

which proves the theorem.

We intend the notation:

(2) $$\sigma = [s, W, \alpha]$$

to mean that $\sigma$ is the collinear homography such that formula (1) is valid for any form $P$ of the first kind. Theorem I shows that any collinear homography can assume the form (2).

We call the form $\alpha$ of the third kind that appears in (2) the "base form" of the homography $\sigma$.

**Theorem II.** – The homography $[s, W, \alpha]$ is the identity (i.e., $[s, W, \alpha] = 1$) only when $s = 1$, and either the central form ($W$) or the base form ($\alpha$) is zero. In symbols:

$$[s, W, \alpha] = 1 \Rightarrow (s = 1) \wedge (W = 0 \vee \alpha = 0).$$

Proof. If $s = 1$ and $W = 0$ or $\alpha = 0$ then from (1) one has that for any $P$, $\sigma P = P$; i.e., $\sigma = I$.

Converse. If $s = 1$ then from (1) if one has that $(1 - s) P = (P\alpha) W$, from which, upon multiplying by $P$, one deduces that $(P\alpha)(PW) = 0$ for any $P$, then this will be true only when either $W = 0$ or $\alpha = 0$. However, if $W = 0$ or $\alpha = 0$ then $(1 - s) P = 0$; i.e., $s = 1$.

**Theorem 2.** – If the collinear homographies $\sigma = [s, W, \alpha]$, $\sigma_1 = [s_1, W_1, \alpha_1]$ are not identities then $\sigma = \sigma_1$ only when $s = s_1$ and there exists a real, non-zero number $m$ such that $W_1 = mW$ and $\alpha = m\alpha_1$ .

$$(\sigma \neq 1) \wedge (\sigma_1 \neq 1) \Rightarrow (\sigma = \sigma_1) \Rightarrow (s = s_1) \wedge (\exists\, m \in q \neq 0) \wedge (W_1 = mW) \wedge (\alpha = m\alpha_1 \neq m\Lambda)$$

Proof. – If $s = s_1$ and there exists a number $m$ such that $W_1 = mW$, $\alpha = m\alpha_1$ then from (1) one will have that $\sigma P = \sigma_1 P_1$ for any $P$; i.e., $\sigma = \sigma_1$ .

Converse. If $\sigma = \sigma_1$ then from (1) one will have:

($a$) $$sP + (P\alpha)\, W = s_1 P + (P\alpha_1)\, W_1,$$

or, after multiplying by $P$:

($b$) $$(P\alpha)(PW) = (P\alpha_1)(PW_1).$$

If $W$ and $W_1$ are non-zero forms then this will show (Theor. II) that the line $PW$ will always pass through the point $W_1$; i.e., it will prove that: "There exists a number $m$ such that $W_1 = mW$." If one takes $mW$ in ($b$) instead of $W_1$ then one will have that $P\alpha = m(P\alpha_1)$; i.e., $\alpha = m\alpha_1$ .

If one substitutes that in ($a$) then one will have that $s = s_1$ .

We will see some applications of this theorem in the following section. For now, we confine ourselves to pointing out some properties of collinear homographies that are easily deduced from the notation (2).

Set:

$$[W,\ \alpha] = [1,\ W,\ \alpha], \tag{3}$$

and from (1) one will easily obtain the formulas:

$$[s,\ W,\ \alpha] = [W,\ \alpha] + s - 1, \qquad [s,\ W,\ \alpha] = s\left[W, \frac{\alpha}{s}\right] = s\left[\frac{W}{s}, \alpha\right]. \tag{4}$$

The first of (4) is true for any $s$ and reduces any general collinear homography to the sum of a homography $[1, W, \alpha]$ and a number. The second of (4) is true only for non-zero $s$ and reduces the general collinear homography to the product of a homography $[1, W, \alpha]$ with a number. Except for the homography $[0, W, \alpha]$, which is, moreover, devoid of any interest, all of the other collinear homographies can be reduced to the form (3).

If $m$, $n$ are numbers such that $m + n$ is non-zero then from (1) one easily finds that:

$$\frac{m[W,\alpha]+n[W,\alpha_1]}{m+n} = \left[W, \frac{m\alpha+n\alpha_1}{m+n}\right], \tag{5}$$

$$\frac{m[W,\alpha]+n[W_1,\alpha]}{m+n} = \left[\frac{mW+nW_1}{m+n}, \alpha\right], \tag{5$'$}$$

which gives the barycentric property of the homography $[W, \alpha]$ and therefore that of the general homography $[s, W, \alpha]$ for non-zero $s$, as well.

## § 6. – Collineations.

We call any collinear homography of the form $[W, \alpha]$ [See § 5, (3)] a *collineation*.

In the sequel, it will always be implicit that:

$$\sigma = [W,\ \alpha],$$

where $W$ is a form of the first kind and $\alpha$ is a form of the third kind.

I. $P \in F_1 \Rightarrow \sigma P = P + (P\alpha)W$.

II. $P \in F_1 \Rightarrow \sigma P = (W\alpha + 1)\,P + (PW)\,\alpha$.

I is an immediate consequence of (1) in § 5. II is deduced from I by observing that $(P\alpha)W = (PW)\alpha + (W\alpha)\,P$. One obtains the position of $\sigma P$ from I by barycentric construction of the point $P$ and the point $W$. One obtains the position of $\sigma P$ from II by barycentric construction of the point $P$ and the point of intersection of the line $WP$ with the position of the base form of $\sigma$.

One easily proves the following propositions by using propositions I, II:

III. $P \in F_1 \Rightarrow [P(\sigma P) = 0 \Rightarrow (PW = 0) \vee (P\alpha = 0)]$.

IV. $(P \in F_1) \wedge (\sigma \neq 1) \wedge (P(\sigma P) \neq 0) \Rightarrow \text{rat}[W, (PW)\alpha, P, \sigma P] = W\alpha + 1$.

V. $(P_1, P_2, P_3, P_4 \in F_1) \wedge (P_1 P_2 P_3 P_4 \neq 0) \Rightarrow \dfrac{(\sigma P_1)(\sigma P_2)(\sigma P_3)(\sigma P_4)}{P_1 P_2 P_3 P_4} - W\alpha + 1$.

III shows that $P$ is united with respect to $\sigma$ only when it lies in the central form or the base form of $\sigma$. IV says that the double ratio that is defined by the point $W$, the point of intersection of the line $PW$ with the base form, the point $P$, and the correspondent to $P$ is constant (and equal to $W\alpha + 1$).

We let $\omega$ denote a unit trivector – i.e., a trivector such that if $O$ is an arbitrary point then the tetrahedron $O\omega$ is right-handed, and its volume is 1.

VI. $F_1 \wedge \bar{X} \in \{\sigma X \in v\} \Rightarrow F_1 \wedge \bar{X} \in \{X\,[\omega + (W\alpha)\,\alpha] = 0\}$.
VII. $P \in F_1 \Rightarrow [\sigma P \in v \Rightarrow \omega + (W\omega)\,\alpha = 0]$.
VIII. $[\omega + (W\omega)\,\alpha] \in v^3 \Rightarrow W \in v \vee \alpha \in v^3$.

VI expresses the idea that: "A form $X$ of the first kind has a vector for its correspondent under $\sigma$ only when $X$ lies in the form of the third kind $\omega + (W\omega)\,\alpha$." This is deduced immediately from I upon observing that the conditions $(\sigma P)\omega = P\omega + (P\alpha)(W\omega) = P\omega + P[(W\omega)\alpha] = P[\omega + (W\omega)\alpha]$ and $(\sigma P)\omega = 0$ are equivalent to the condition "$\sigma P$ is a vector." The form $\omega + (W\omega)\,\alpha$ is called the "limit form" of the collineation $\sigma$.

VII expresses the idea that any $F_1$ has a vector for its correspondent only when the limit form of $\sigma$ is zero. VIII expresses the idea that the limit form of $\sigma$ is a trivector only when the central form is a vector or the base form is a trivector (in fact, $[\omega + (W\omega)\alpha]\,\omega = (W\alpha(\alpha\omega))$.

When $\sigma$ is not the identity, we set:

(1) center $\sigma$ = posit $W$, base $\sigma$ = posit $\alpha$,

and Theorem III of § 5 proves that center $\sigma$ is a point function of $\sigma$ and that the base $\sigma$ is a plane function of $\sigma$.

The number $W\alpha + 1$ that appears in propositions II, IV, V is called the "double ratio of $\sigma$" and is denoted by the notation rat $\sigma$; i.e., we set:

(2) $$\text{rat } \sigma = W\alpha + 1.$$

Theorem III of § 5 proves that rat $\sigma$ is a function of $\sigma$.

We call the locus of points that are positions of $F_1$ and have vectors for their correspondents the "limit of $\sigma$" and denote it by lim $\sigma$. If the limit form of $\sigma$ is not zero then from prop. IV, one has that:

$$\lim \sigma = \text{posit}[\omega + (W\alpha)\ \alpha].$$

If the limit form of $\sigma$ is zero then (prop. VII) lim $\sigma$ is the totality of all points. Theorem III of § 5 proves that lim $\sigma$ is a function of $\sigma$.

If the limit form of $\sigma$ is not a trivector then the base form of $\sigma$ will not be a trivector, either, and one will have that lim $\sigma$ is a projective plane that is parallel to the plane base $\sigma$, because the triangle $\omega + (\Omega\omega)\ \alpha$ is obtained from the triangle $(\Omega\,\omega)\ \alpha$ (which has the mean position of $\alpha$) by means of a translation.

Observe that the elements $W$, $\alpha$ can vary for the collineation $\sigma = [W, \alpha]$, without $s$ varying, which explains the importance of the projective elements center $\sigma$, base $\sigma$, rat $\sigma$, lim $\sigma$, since they are, in a way, invariants of the homography $\sigma$.

If, in the above, one considers the (three-dimensional) system of $F_1$ that have their positions in a projective plane $\pi$ instead of the general system of $F_1$ then one will obtain collineations of the plane $\pi$. In the sequel, we will suppose that one knows the properties of plane collineations that one obtains from the foregoing by substituting the $F_2$ of the fixed plane for the $F_3$ .

## § 7. – Homologies.

We shall say *homology*, instead of "collineation with non-zero double ratio." If we write Collin and Homol, instead of "collineation" and "homology" then we will have:

$$\text{Homol} = \text{Collin} \wedge \overline{\sigma \in}\ \{\text{rat } \sigma \neq 0\}.$$

I. $(\sigma \in \text{Homol}) \wedge (\sigma \neq 1) \wedge (m \in \{n \neq 0\}) \Rightarrow \sigma^m \in \{\text{Homol} \neq 1\}.$

$$(\text{center } \sigma^m = \text{center } \sigma) \wedge (\text{base } \sigma^m = \text{base } \sigma) \wedge (\text{rat } \sigma^m = (\text{rat } \sigma)^m).$$

This proposition expresses the idea that: "If $\sigma$ is a homology that is not the identity and $m$ is a non-zero (positive or negative) integer then $\sigma^m$ will also be a homology that is not the identity that hasthe same center and base as $\sigma$, and whose double ratio is the $m^{\text{th}}$ power of the double ratio of $\sigma$."

Proof. – If $\sigma = [W, \alpha]$ and $W, A, B, C$ are independent $F_1$ then, from I of § 6, one will have that $(\sigma W)(\sigma A)(\sigma B)(\sigma C) = (W\alpha + 1)WABC$, and therefore $\sigma$ will be an invertible homography, or – in other words – an arbitrary power of $\sigma$ will be a homography.

For any $m$, one has from I of § 6 that:

(1) $$\sigma^m W = (\text{rat }\sigma)^m\, W.$$

From the middle proposition, one also has, after multiplying by $\sigma^{m-1}$ and taking (1) into account, that:

$$\sigma^m P = \sigma^{m-1}\, P + (\text{rat }\sigma)^m\, (P\alpha)\, W.$$

From this formula for positive $m$, one deduces that:

(2) $$\sigma^m P = P + [1 + \text{rat }\sigma + (\text{rat }\sigma)^2 + \ldots + (\text{rat }\sigma)^{m-1}]\, (P\alpha)W,$$

(2)′ $$\sigma^{-m} P = P - [(\text{rat }\sigma)^{-1} + (\text{rat }\sigma)^{-2} + \ldots + (\text{rat }\sigma)^{-m}]\, (P\alpha)W,$$

and these formulas prove the theorem (*).

Let $a$, $b$ be two projective geometric elements for which one can give meaning to the phrase "the distance from $a$ to $b$." We shall write dist($a$, $b$) instead of that phrase. One has the theorem:

II. $(s \in \{\text{Homol} \neq 1\}) \wedge (\text{center }\sigma \notin \text{posit }\omega) \wedge (\text{base }\sigma \notin \text{posit }\omega) \wedge (m \in N)$
$\Rightarrow \text{dist}(\text{center }\sigma, \lim \sigma^m) = \text{dist}(\text{base }\sigma, \lim \sigma^{-m})$.

"If $\sigma$ is a homology that is not the identity whose center and base are proper elements, and if $m$ is a positive integer then the distance from the center of $\sigma$ to the limit of $\sigma^m$ will be equal (in absolute value) to the distance from the base of $\sigma$ to the limit of $\sigma^{-m}$."

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(*) If we set $h$ = rat $\sigma$ in (2), (2)′ then we will have, for positive or negative $m$:

$$\sigma^m\, P = P + \frac{h^m - 1}{h - 1}\, (P\alpha)\, W \qquad \text{or} \qquad \sigma^m\, P = P + m\, (P\alpha)\, W,$$

according to whether $h$ is not equal to 1 or is equal to 1, respectively.

If we set $\sigma = [W, \alpha]$ then we will have that:

$$\sigma^m = \left[W, \frac{h^m - 1}{h - 1}\alpha\right] \qquad \text{or} \qquad \sigma^m = [W, m\alpha],$$

respectively.

It is known (G. Peano, *Calcolo Geometrico*) that $e^\sigma = 1 + \sigma + \frac{\sigma^2}{2!} + \frac{\sigma^3}{3!} + \ldots$ is a convergent series.

From I, one deduces by a simple calculation that:

$$e^{\sigma - 1} = \left[W, \frac{e^{h-1} - 1}{h - 1}\alpha\right] \qquad \text{or} \qquad e^{\sigma - 1} = \sigma.$$

It then results that $e^{\sigma-1}$ is a homology that has the same center and base as $\sigma$ and whose double ratio is $e^{h-1}$.

Proof. – If center $\sigma$ and base $\sigma$ are proper elements then, from Theorem VIII of § 6 and the preceding theorem, one can deduce that lim $\sigma^m$ and lim $\sigma^{-m}$ are proper planes that are parallel to base $\sigma$.

Set $\sigma = [W, \alpha]$ and $h$ = rat $\sigma$. Fix the positive sense of rotation on the plane $\alpha$ and let mag $\alpha$ be the signed area of the triangle $\alpha$. Let $\Delta$ be the signed distance from center $\sigma$ to base $\sigma$ and let $\Delta_m$ be the signed distance from center $\sigma$ to lim $\sigma^m$.

Observe that for $h \neq 1$ the limit form of $\sigma^m$ is:

$$\beta = \omega + \frac{h^m - 1}{h - 1} (W\alpha)\ \alpha,$$

so one has that

$$\frac{1}{3}\Delta = \frac{W\alpha}{\text{mag}\ \alpha}, \qquad \frac{1}{3}\Delta_m = \frac{W\beta}{\text{mag}\ \beta}.$$

However, the triangle $\beta$ is obtained from the product of $\alpha$ with the number $(W\omega)(h^m - 1) / (h - 1)$ by means of a translation, and therefore:

$$\text{mag}\ \beta = W\omega\, \frac{h^m - 1}{h - 1}\, \text{mag}\ \alpha.$$

As a consequence, one has:

$$\Delta_m = \frac{h^m}{h^m - 1}\Delta \qquad \text{and} \qquad \Delta_{-m} = \frac{-1}{h^m - 1}\ \Delta.$$

One thus has that:

$$\Delta_m + \Delta_{-m} = \Delta,$$

and this formula, which is also valid for $h = 1$, proves the theorem.

III. $(\sigma \in \text{Homol}) \Rightarrow [\sigma^2 = 1 \Rightarrow \text{rat}\ \sigma = -1]$.

"A homology is involutory only if its double ratio is equal to –1."

Proof. – The condition $\sigma^2 = 1$ is equivalent to $\sigma = \sigma^{-1}$. For any $P$, this is equivalent to $P\alpha = -\frac{1}{\text{rat}\ \sigma} P\alpha$, which then proves the theorem.

IV. $(\sigma \in \text{Homol}) \wedge (m \in N) \wedge (\sigma^m = 1 \neq \Lambda) \Rightarrow (\sigma^2 = 1)$.

"If a homology is cyclic then it will be involutory," and this is proved as before.

V. $(\sigma \in \text{Homol}) \Rightarrow [(\text{lim}\ \sigma = \text{lim}\ \sigma^{-1}) \Rightarrow (\sigma^2 = 1) \vee (\text{lim}\ \sigma = \text{posit}\ \omega)]$.

"The limit plane (*) of a homology coincides with the limit plane of its inverse only when either the homology is involutory or its limit plane is at infinity."

Proof. – The condition lim $\sigma$ = lim $\sigma^{-1}$ is equivalent to:

$$[\omega + (W\omega)\alpha]\left[\omega - \frac{1}{\text{rat}\,\sigma}(W\omega)\alpha\right] = 0,$$

which is developed from:

$$\left(1 + \frac{1}{\text{rat}\,\sigma}\right)(W\omega)(\alpha\omega) = 0.$$

We call any homology that is not the identity and has an improper center and a proper base an *affinity*. Any proper point will correspond to a proper point, because $W\omega = 0$ and $(\sigma P)\omega = P\omega$. The ratio between a tetrahedron and its correspondent is equal to the inverse of the double ratio of the affinity (§ 6, prop. V).

We call any homology that is not the identity and has a proper center and an improper base a *homothety*. If $W$ is a point ($W\omega = 1$) then any proper point $P$ will correspond to a form of the first kind whose mass is the double ratio of a homothety, because (§ 6, II) $(\sigma P)\omega = (W\alpha + 1)P\omega = \text{rat}\ \sigma$. It follows from this and prop. V of § 6 that the ratio of a tetrahedron to its correspondent (with respect to posit $\sigma$) is equal to the cube of the inverse of the double ratio of a homothety. If $W$ is a proper point and $P$ is a proper point then one will have that:

$$\text{posit}\ \sigma P = \frac{\sigma P}{\text{rat}\,\sigma} = W + \frac{1}{\text{rat}\,\sigma}(P - W),$$

so from the usual construction of the homothety of a point it is proved that any figure will correspond to a figure that is similar to itself.

We call any homology whose center and base are improper elements a *congruence*. If $\sigma = [W, \alpha]$ then $W\alpha = 0$ and therefore rat $\sigma = 1$. One also has that $\alpha = k\omega$, where $k$ is a number, and therefore $\sigma P = P + kW$; i.e., any congruence is a *translation*.

The reader will easily recognize that the projective homographies that are the positions of the homologies that are now studied are the ordinary projective homologies. We have replaced the usual term "plane of the homology" with the generic term "base," so the theorems that were stated are also applicable to plane homologies with some minor changes. We have replaced the usual term "characteristic of the homology" with the term "double ratio of the homology," because rat $\sigma$, by itself, *does not characterize* the homology $\sigma$ (*). Moreover, we have not followed the common practice of considering

(*) One will have that if $[W, \alpha]$ is a homology of the limit form then it will be non-zero, because $W[\omega + (W\omega)\alpha] = (W\omega)(W\alpha + 1)$.

(*) The reader can easily state and prove the customary theorems in regard to the various ways of defining a homology with the methods that methods that have been used so far.

the homology (like the rest of the projective transformations) to be a *double* transformation between a *first* and *second* figure and a *second* and *first* figure (**), which leads, e.g., to the consideration of *two* planes that are limited by just one homology.

## § 8. – Perspectivities.

We shall say *perspectivity,* instead of "collineation with double ratio zero." Writing Persp instead of perspectivity, we will have:

$$\text{Persp} = \text{Collin} \wedge \overline{\sigma \in} \{\text{rat } s = 0\},$$

or, more simply:

$$\text{Persp} = \text{Collin} \wedge \overline{\text{rat}}\ 0.$$

If $\sigma = [W, \alpha]$ and $W, A, B, C$ are independent $F_1$ then $(\sigma W)(\sigma A)(\sigma B)(\sigma C) = 0$ when rat $\sigma = W\alpha + 1 = 0$, and therefore: "perspectivities are not invertible," namely, if $\sigma$ is a pespectivity then $\sigma^{-1}$ will not be a homography.

I. $\sigma \in$ Persp $\Rightarrow$ center $\sigma \in$ lim $\sigma$.

II. $(\sigma \in \text{Persp}) \wedge (P \in F_1 \neq 0) \Rightarrow$ posit $\sigma P \in$ base $\sigma$.

Proposition I expresses the idea that the center of any perspectivity belongs to the limit figure. This is obvious if lim $\sigma$ is all of space. If lim $\sigma$ is a plane then $\omega + (W\omega)\ \alpha \neq 0$ and $W[\omega + (W\omega)\ \sigma]$ = rat $\sigma(W\omega) = 0$, which proves the theorem.

Proposition II expresses the idea that any form of the first kind has a form with its position in base $\sigma$ for its correspodent.

If $W$ is a proper point and base $\sigma$ is not the plane at infinity then posit $\sigma$ will be the usual CENTRAL PROJECTION (***), for which $W$ will be the *center* and base $\sigma$ will be

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(**) The concept of correspondence is presented by the ordinary method in a form that is incomplete and not at all precise. Regarding the notations center $\sigma$, base $\sigma$, lim $\sigma$, observe that they permit one to *represent* two or more homologous planes in a drawing and to *interpret* the figure with no need for other information.

(***) We point out a system of notation for central projections that seems to be useful to us and would substitute for the ordinary notation, although it seems very incomplete. If $F$ is a figure (i.e., a class of points) then one lets (as one ordinarily does) $F'$ denote the locus of images (or projections) of points of $F$. This notation is incomplete because it should contain some hint about the center and the image (quadro); it is, however, sufficient in practice, because it is unnecessary, in any case, to change the reference elements. We write, e.g., $J$, instead of the point at infinity. Then, if $a$ is a line then $Ja$ (instead of $J \wedge a$) means "point at infinity of $a$." Therefore $(Ja)'$ means "vanishing point of $a$." In place of $(Ja)'$, one can write $J'a$, but not $Ja'$, which would mean "point at infinity for the image of $a$." Analogously, if $\alpha$ is a plane then $J\alpha$ will mean the "line at infinity of $\alpha$" and $(J\alpha)'$ or $J'\alpha$ will mean the` "vanishing line of $\alpha$." We indicate the image by a fixed letter – e.g., $\pi$ – so $a\pi$, $\alpha\pi$ (instead of $a \wedge \pi$, $\alpha \wedge \pi$) will denote the *trace* of $a$ and $\alpha$. In a drawing, e.g., a point with the notation $a\pi = J'a$ will represent the line $a$ that passes through the center of the *image*. If $W$ is a vector – i.e., center $\sigma$ is a point at infinity – then posit $\sigma$ will be an ordinary PARALLEL PROJECTION.

For any non-zero form $P$ of the first kind, set:

$$\Delta_P = \frac{P\alpha}{\operatorname{mag}\alpha} \quad \text{and} \quad \Delta = \Delta_W .$$

If $P$ is a proper point then $\Delta_P$ will be the signed distance to base $\sigma$. If $P$ is a vector and $O$ is an arbitrary proper point of base $\sigma$ then one will have that $\Delta_P = \Delta_{O+P}$ ; i.e., $\Delta_P$ will be the distance from base $\sigma$ to the extremity of the vector $P$ when the origin of $P$ is a point of base $\sigma$. From prop. I of § 6, we observe that $W\alpha = -1$, so one has that:

$$\sigma P = P - \frac{\Delta_P}{\Delta} W. \tag{1}$$

If one takes $\Delta$ for the unit of measure then one will have, from (1), that:

$$P = \sigma P + \Delta_P W. \tag{2}$$

If $W$ is a fixed element then (2) will prove that the form $P$ is the *representative* that gives the form $\sigma P$ on the proper plane base $\sigma$ and the number $\Delta_P$ . If $P$ is a proper point or a vector then under central projection $\sigma P$ will be a form of mass $1 - \Delta_P$ or $-\Delta_P$ , respectively. Under parallel projection, $\sigma P$ will be a proper point or a vector; in the latter case, if $W$ is normal vector to base $\sigma$ then one will have that posit $\sigma$ is the ordinary VALUED PROJECTION (proiezione quotata).

If $Q$, $R$ are arbitrary $F_1$ then, from (2), one will have:

$$PQ = (\sigma P)(\sigma Q) + [\Delta_Q(\sigma P) - \Delta_P(\sigma Q)]W, \tag{2$'$}$$

$$PQR = (\sigma P)(\sigma Q)(\sigma R) + [\Delta_P(\sigma Q)(\sigma R) + \Delta_Q(\sigma R)(\sigma P) + \Delta_R(\sigma P)(\sigma Q)]W, \tag{2$''$}$$

and one thus obtains the representation of the form $PQ$ of the second kind and the form $PQR$ of the third kind in terms of the elements that represent $P$, $Q$, $R$. One can obtain the usual properties of central and parallel projection from formulas (2), (2)′, (2)″. However, it is more interesting to notice that one can deduce the *descriptive geometry of geometric forms* from formulas (2), (2)′, (2)″, the importance of which will be obvious if one observes that $F_2$ can represent systems of forces, and therefore a simple method of representing $F_2$ will lead to a rapid solution of the problem of the composition of forces in spaces, a problem that is very complicated when one uses the usual methods (Monge).

Let $\sigma = [W, \alpha]$ be a perspectivity. If $\beta$ is a non-zero $F_3$ then we can define, in an infinitude of ways, three forms $A$, $B$, $C$ of the first kind such that $A$ and $B$ will belong to $\alpha$ and $\beta$, respectively, and the tetrahedron $WABC$ will be not zero. One has that (§ 6, I):

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projection. Two parallel lines with the notations $\alpha\pi$, $J'\alpha$ will represent the plane $\alpha$, and this notation is much clearer than the usual one ($s$, $q'$).

$$\sigma A = A, \qquad \sigma B = B, \qquad \sigma C = C + rW,$$

where $r = C\alpha$ is a number. One can therefore give the following formula for $\sigma$:

(3)
$$\sigma = \begin{pmatrix} 0, & A, & B, & C+rW \\ W, & A, & B, & C \end{pmatrix},$$

and $\sigma$ will be a correspondence between the $F_1$ in space and the $F_1$ in the plane $\alpha$, as well as a correspondence between the $F_1$ in the plane $\beta$ and the $F_1$ in the plane $\alpha$. If we let $\lambda$ denote this linear transformation between two three-dimensional systems of $F_1$ then we will have:

(4)
$$\lambda = \begin{pmatrix} A, & B, & C+rW \\ A, & B, & C \end{pmatrix}.$$

The homography $\lambda$ is called a perspectivity between the $F_1$ in $\beta$ and the $F_1$ in $\alpha$. It is invertible, and $\lambda^{-1}$ is a perspectivity between the $F_1$ in $\alpha$ and the $F_1$ in $\beta$. We say center $\lambda$ for center $\sigma$, base $\lambda$ for the line $\alpha\beta$ = line $AB$, and lim $\lambda$ for the geometric element that is common to the plane $\beta$ and lim $\sigma$. If lim $\lambda$ is a proper line then lim $\lambda^{-1}$ will also be a proper line, and lim $\lambda$ and lim $\lambda^{-1}$ will be parallel to base $\lambda$.

It is easy to recognize that posit $\lambda$ is the usual perspectivity whose center is the point $W$ between the plane $\beta$ and the plane $\alpha$.

If $I$, $J$, $K$ are unit vectors (or of equal modulus), and if $J$ and $K$ are normal to $I$, and if $O$ is a proper point then if one sets:

$$\lambda = \begin{pmatrix} 0, & I, & K \\ 0, & I, & J \end{pmatrix}$$

then one will have that $\lambda$ is a *rotation* of the plane $OIK$ around the line $OI$ through the angle ($K$, $J$), or an *inversion* of the plane $OIK$ onto the plane $OIJ$. The other inversion is given by:

$$\lambda_1 = \begin{pmatrix} 0, & I, & -K \\ 0, & I, & J \end{pmatrix},$$

and therefore the two inversions of one plane onto another preserve the sense when one is given the sense of positive rotation on the plane that is normal to the given planes.

## § 9. – Theorems.

In the sequel, $\sigma_1$, $\sigma_2$, $\sigma_3$, … will be collineations, and we will set:

$$\sigma_r = [W_r, \alpha_r], \qquad h_r = \text{rat } \sigma_r,$$

for $r = 1, 2, 3, \ldots$

**Theorem I.** – If $R$, $S$ are linear systems of $F_1$, $\sigma_2$ is an invertible collineation between $R$ and $S$, $\sigma_2 W_1 \neq 0$, and $\lambda = \sigma_2\sigma_1\sigma_2^{-1}$ then:

1. $\lambda$ will be a collineation between the $F_1$ in $S$ and the $F_1$ in $\sigma_2\sigma_1 R$.
2. The center of $\lambda$ will be the position of the correspondent with respect to $\sigma_2$ of the form that is central to $\sigma_1$ [i.e., center $\lambda$ = posit $\sigma_2$ (center $\sigma_1$)].
3. The base of $\lambda$ will be the locus of positions of the correspondents with respect to $\sigma_2$ of the $F_1$ that lie in $R$ and in base $\sigma_1$ .

Proof. – Let $Q$ be an $F$ in $R$. From $I$ of § 6, one has:

$$\sigma_2 (\sigma_1 Q) = \sigma_1 Q + [(\sigma_1 Q)\, \alpha_2]\, W_2 .$$

If one replaces $\sigma_1 Q$ in the right-hand side with the value that was given by I in § 6 then one will have, after some simple calculations, and once more from I of § 6:

$$\sigma_2\, \sigma_1\, Q = \sigma_2\, Q + (Q\, \alpha_1)\, \sigma_2\, W_1 . \tag{1}$$

If we set $\sigma_2\, Q = P$ then we will have that $Q = \sigma_2^{-1} P$, because $\sigma_2$ is invertible between $R$ and $S$, and therefore (1) will become:

$$\sigma_2\sigma_1\sigma_2^{-1}P = P + [(\sigma_2^{1}P)\alpha_1]\sigma_2 W_1 . \tag{1$'$}$$

This formula proves the theorem, because $\lambda P$ is a linear function of $P$ and $\sigma_2\, W_1$, and since $\sigma_2\, W_1 \neq 0$, the $P$ that belongs to the base of $\lambda$ are such that $(\sigma_2^{1}P)\alpha_2 = 0$.

**Corollary.** – If, under the hypotheses of Theorem I, one has that $\sigma_1$, $\sigma_2$ are homologies then $\lambda$ will be a homology, center $\lambda$ = posit $\sigma_2$ (center $\sigma_1$), base $\lambda$ = posit $\sigma_2$ (base $\sigma_1$), and rat $\lambda$ = rat $\sigma_1$ . Indeed, in this case, $R = S = F_1$, and therefore $\lambda\sigma_2\, W_1 = \sigma_2\, \sigma_1 W_1 = \sigma_2\,(h_1 W_1) = h_1(\sigma_2 W_1)$.

The reader can easily verify that if one is given in Theorem I that posit $R$ = base $\sigma_1$, posit $S$ = base $\sigma_2$ and $\sigma_1$, $\sigma_2$ are perspectivities then $\lambda$ will be a homology onto the plane base $\sigma_2$, and for posit $\lambda$ one will obtain the known property of projections of a plane figure into a plane from two different centers. Analogously, if one sets posit $R$ = base $\sigma_2$ and $\sigma_1$, $\sigma_2$ are perspectivities then $\lambda$ will be a homology onto the plane base $\sigma_2$, and for posit $\lambda$ one will obtain the known property of projections of two systems of perspective planes into a plane. It is known that the two properties that we just recalled are fundamental for the solution of perspective problems in descriptive geometry.

**Theorem II.** – If $R$, $S$, $T$ are linear systems of $F_1$, if $\sigma_1$ is a collineation between $R$ and $S$, and $\sigma_2$ is a collineation between $S$ and $T$, and $\left\{\begin{array}{c}\text{base}\,\sigma_1 = \text{base}\,\sigma_2 \\ \text{center}\,\sigma_1 = \text{center}\,\sigma_2\end{array}\right\}$ then:

1. $\sigma_2\sigma_1$ will be a collineation between $R$ and $T$.

2. $\left\{\begin{array}{c}\text{base } \sigma_2\sigma_1 = \text{base } \sigma_1 \\ \text{center } \sigma_2\sigma_1 = \text{center } \sigma_1\end{array}\right\}$.

3. $\left\{\begin{array}{l}\text{The center of the collineations } \sigma_1, \sigma_2, \sigma_2\sigma_1 \text{ will be in a line.} \\ \text{The bases of the collineations } \sigma_1, \sigma_2, \sigma_2\sigma_1 \text{ will pass through a line.}\end{array}\right\}$.

4. rat($\sigma_2$ $\sigma_1$) = (rat $\sigma_1$)(rat $\sigma_2$).

Proof. – If $P$ is an $F_1$ in $R$ then, from I of § 6, one will easily find that:

$$\sigma_2\,\sigma_1\,P = P + (P\alpha_1)\,W_1 + [P\alpha_2 + (P\alpha_1)(W_1\alpha_2)]\,W_2\,. \tag{2}$$

If one supposes that base $\sigma_1$ = base $\sigma_2$ then:

$$\alpha_2 = k\,\alpha_1\,,$$

where $k$ is a number, and (2) will become:

$$\sigma_2\,\sigma_1\,P = P + (P\alpha_1)\,[W_1 + kh_1W_2], \tag{3}$$

and this formula will immediately prove the first three parts of the thesis. From (3), it results that $\sigma_2\,\sigma_1 = [W, \alpha_1]$, where $W = W_1 + kh_1\,W_2$ . One has that:

$$\text{rat}\,(\sigma_2\,\sigma_1) = W\alpha_1 + 1 = W_1\,\alpha_1 + 1 + kh_1\,W_2\,\alpha_1 = h_1 + h_1\,W_2\,\alpha_2 = h_1\,h_2\,,$$

and this proves the fourth part of the thesis.

One proves the theorem in an analogous way when center $\sigma_1$ = center $\sigma_2$ ; i.e., $W_2 = k\,W_1$ .

**Corollary I.** – Under the hypotheses of Theorem II, if one has that $\sigma_1$, $\sigma_2$ are homologies then $\sigma_2\sigma_1$ will be a homology. Indeed, $h_1$, $h_2$ will be non-zero, and therefore rat($\sigma_2\sigma_1$) will be non-zero.

**Corollary II.** – Under the hypotheses of Theorem II, if, in addition, one has that $\sigma_1$, $\sigma_2$ are perspectivities, and that center $\sigma_2$ is a point at infinity then lim $\sigma_2\sigma_1$ = lim $\sigma_1$ . Indeed, the limit form of $\sigma_2\sigma_1$ is, for the preceding situation, $\omega + [(W_1 + kh_1\,W_2)\,\omega]\,\alpha_1 = \omega + (W_1\,\omega)\,\alpha_1$ , which is the limit form of $\sigma_1$ .

The reader can easily verify that if $\sigma_1$ is a perspectivity with a proper base and center, and $T = R$, and $\sigma_2$ is either a rotation of $S$ around base $\sigma_1$ or an inversion of $S$ in $R$ then the property of the projective homography posit ($\sigma_2\sigma_1$) will give the known property of the center and limit line of a perspectivity between two plane systems when one of the planes turns around the base of the perspectivity (*). It is known that the property that we just recalled is fundamental for the solution of metric problems in descriptive geometry.

Turin, January 1897.

C. BURALI-FORTI.

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(*) Preserve the notation that was pointed out in the note on pp. 23 for central projection. Let $\alpha$ be a plane that is not parallel to the image and does not issue from the center of projection, and let $s$ be a line in front of $\alpha$. Let $\Omega_{\alpha,s}$ be the homology (from the plane onto the image) that has the image of $s$ for its base, the vanishing line $J'\alpha$ of the plane $\alpha$ for its limit line, and its center at the inversion onto the image of the center of projection, where that inversion is done about $J'\alpha$ in a given sense. The homology $\Omega_{\alpha,s}$ , when applied to the image of a figure in $\alpha$, will give the image of the inversion of that figure onto the front plane that passes through the line $s$. If $\alpha_1$, $\alpha_2$ are parallel planes that satisfy the conditions on $\alpha$, and $s_1$, $s_2$ are front lines of $\alpha_1$ and $\alpha_2$ then the homologies $\Omega_{\alpha_1,s_1}$ , $\Omega_{\alpha_2,s_2}$ will have their limit lines and centers in common if the inversion is performed in the same sense. Consequently, $\Omega_{\alpha_1,s_1} = \Omega_{\alpha_2,s_2}$ only when the lines $s_1$, $s_2$ lie in a plane that issues from the center of projection. As we have already pointed out (Rivista di matematica, vol. II), this property is useful in descriptive geometry. One gets, e.g., the image of the apparent contour of a surface of revolution by applying the inverse of a homology $\Omega_{\alpha,s}$ to the envelope of a system of circumferences that is the image of the inversion of the parallel in a system of front planes, a system that remains defined under the homology $\Omega_{\alpha,s}$ , etc.

Note III by C. Burali-Forti, in Turin (*)

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## § 10. – Polarity in the plane.

Let $A$, $B$, $C$ be $F_1$ and let $a$, $b$, $c$ be $F_2$ in a projective plane. If $ABC \neq 0$ then the homography:

(1)
$$\sigma = \begin{pmatrix} a & b & c \\ A & B & C \end{pmatrix}$$

will transform the points of the plane into lines in the plane. If one also has that $abc \neq 0$ then $\sigma^{-1}$ will transform the lines into points.

$\sigma$ is called a *polarity* when:

1. There exists at least one $F_1$ that is not united with respect to $\sigma$.
2. Let the forms $P$, $Q$ of the first kind be arbitrary. If $P$ is in $\sigma Q$ then $Q$ will also be in $\sigma P$. [i.e., if $P(\sigma Q) = 0$ then $Q(\sigma P) = 0$.]

The line $\sigma P$ is said to be the *polar* of $P$, and if $\sigma$ is invertible then $P$ will be the *pole* of $\sigma P$; i.e., if $p$ is an $F_2$ then $\sigma^{-1}p$ will be a point whose polar is $p$. It immediately results that the polars of the points of $\sigma P$ form a sheaf with center $P$ that is projective to the points of $\sigma P$, and if $\sigma$ is invertible then the poles of the lines that pass through $\sigma^{-1}p$ are on the line $p$ and form a point-set (*puntegiatta*) that is projective to the sheaf whose center is $\sigma^{-1}p$ (Note. I).

**Theorem I.** – If $\sigma$ is a polarity, and $A$ is an $F_1$ that is not united [i.e., $A(\sigma A) \neq 0$], and any point $P$ of $\sigma A$ is united [i.e., $P(\sigma P) = 0$] then $\sigma P = 0$.

Let $B$, $C$, with $BC \neq 0$, be two $F_1$ in $\sigma A$. If $ABC \neq 0$ then, by the hypotheses made, one must have:

$$\sigma = \begin{pmatrix} BC & mBA & nCA \\ A & B & C \end{pmatrix}.$$

If $P = xB + yC$ is an arbitrary point of $\sigma A$ then one will have, by hypothesis, $P(\sigma P) = xy\,(n - m)\,ABC = 0$ and therefore $n = m$.

Now, if $Q = xA + yB + zC$, $R = x'A + y'B + z'C$ are arbitrary points of the plane then one will have, recalling that $n = m$:

(*) See Notes I and II in **10** (1896), 177-195 and **11** (1897), 64-82, resp., in these Rendiconti.

$$Q(\sigma R) = (xx' + myz' - mzy')\,ABC,$$
$$R(\sigma Q) = (xx' + mzy' - myz')\,ABC,$$

but $\sigma$ is a polarity, and therefore if $Q(\sigma R) = 0$ then one will also have $(R\sigma Q) = 0$; i.e., one must have $m = 0$, which proves precisely that $\sigma P = 0$.

**Theorem II.** – If $\sigma$ is a polarity then one can determine, and in an infinitude of ways, three independent forms of the first kind $P$, $Q$, $R$ and three numbers $h$, $k$, $l$ that are not all zero such that:

$$\sigma = \begin{pmatrix} hQR & kRP & lPQ \\ P & Q & R \end{pmatrix}. \tag{2}$$

If $\sigma$ is a polarity then there will exist a $P$ that is not united. If a point $Q$ that is not united exists in $\sigma P$ then $\sigma Q$ will pass through $P$ and will cut $\sigma P$ at a point $R$ such that $PQR \neq 0$, and therefore $\sigma$ will have the form (2). If any point of $\sigma P$ is united then there will be fixed points $Q$, $R$ in $\sigma P$ such that $QR \neq 0$, and one will have, from Theorem I, $\sigma Q = \sigma R = 0$, and therefore $\sigma$ will again have the form (2).

The triangle $PQR$ of (2) is said to be *auto-conjugate* with respect to the polarity $\sigma$.

**Theorem III.** – If $\sigma$ is a polarity and $A$, $B$ are any $F_1$ in the plane then one will always have $A(\sigma B) = B(\sigma A)$.

Give $\sigma$ the form (2) and set:

$$A = xP + yQ + zR, \qquad B = x'P + y'Q + z'R,$$

so one has:

$$A(\sigma B) = (xx' + yy' + zz')\,PQR, \qquad B(\sigma A) = (x'x + y'y + z'z)\,PQR.$$

**Theorem IV.** – If the homography (1) admits at least one $F_1$ that is not united then it will be a polarity only when it satisfies any of the following conditions:

1. The determinant:

$$\begin{vmatrix} Aa & Ab & Ac \\ Ba & Bb & Bc \\ Ca & Cb & Cc \end{vmatrix} = ABC \,.\, abc$$

is symmetric.

2. If the $F_1$, $P$, $Q$ are arbitrary then one will always have $P(\sigma Q) = Q(\sigma P)$.
3. $BC.\, a + CA.\, b + AB.\, c = 0$.

The first of the two conditions are immediate consequences of Theorem III.

As for the third one, one observes that $BC.\, a = BA.\, c - Ca.\, B$, and analogously for the other two products. Therefore, when one sums them:

$$BC.\, a + CA.\, b + AB.\, c = (Bb - Bc)\,A + (Ac - Ca)\,B + (Ba - Ab)\,C.$$

Now, if $\sigma$ is a polarity then the right-hand side will be zero, because the determinant is symmetric. If the right-hand side is zero then the coefficients of $A$, $B$, $C$ must be zero, because $ABC \neq 0$, and therefore the determinant will be symmetric; i.e., $\sigma$ will be a polarity.

Condition 3 says that the points $BC.\,a$, $CA.\,b$, $AB.\,c$ are collinear; i.e.: *Any triangle is homologous to the polar trilateral.*

**Theorem V.** – If the polarity $\sigma$ is not invertible then the locus of united $F_1$ is either a point or a line or two distinct lines. If $\sigma$ is invertible then there will either exist no united $F_1$ or they will define a curve that is described by a point function of a numerical variable that is continuous, along with its derivative.

If we give $\sigma$ the form (2) and set $A = xP + yQ + zR$ then $A$ will be united – i.e., $A(\sigma A) = 0$ – only when:

(2) $$hx^2 + ky^2 + lz^2 = 0.$$

Suppose that $\sigma$ is not invertible – i.e., $hkl = 0$. If $l = 0$ and $hk \neq 0$ then $\sigma R = 0$, so the polar of any point will pass through $R$, and the equation of the united $F_1$ will be $hx^2 + ky^2 = 0$, and when $h$ and $k$ have the same sign that will give only $R$, but when $h$ and $k$ have different signs then two distinct lines will issue from $R$. If $l = k = 0$ and $h \neq 0$ then any point will have $QR$ for its polar, and only the points of $QR$ will be united (Theorem I).

On the other hand, if $\sigma$ were invertible then (3) would prove the theorem immediately.

**Theorem VI.** – If the polarity $\sigma$ is invertible then the polars of the points $P$ of a curve will envelop a curve whose points have the tangents to the given curve for their polars.

Let the point $P$ be a function of the numerical variable $t$ and denote its derivative with respect to $t$ by a prime.

Since $\sigma$ is a distributive operation, one has:

$$(\sigma P)' = \sigma P'.$$

The tangent to $P$ is the line $PP'$; the pole of this line is the point $(\sigma P)(\sigma P')$, or $(\sigma P)(\sigma P)'$, which is the precisely the point at which the curve that is enveloped by $\sigma P$ touches the line $\sigma P$; i.e., the polar to $P$.

**Theorem VII.** – If the polarity $\sigma$ is invertible and admits a curve of united points then the polar to an arbitrary point $P$ of it will be the tangent at $P$ to the curve.

If $P$ is united then $P(\sigma P) = 0$. Differentiating gives $P'(\sigma P) + P(\sigma P') = 0$. However, by Theorem III, $P'(\sigma P) = P(\sigma P')$, and therefore $P'(\sigma P) = 0$; i.e., the line $PP'$, which is tangent to $P$, will coincide with the line $\sigma P$, which is the polar to $P$.

## § 11. – Conics.

In this section, suppose that $\sigma$ is an invertible polarity on a proper plane and that it admits a curve of united points that one calls a *conic*.

Let $P$ be a point and let $p$ be a line in the plane of the polarity. The notations $p\sigma$, $P\sigma^{-1}$ denote two homographies that are applied to the $F_1$ and $F_2$, respectively, in the plane that goes through $p$ or passes through $P$. If $p\sigma$ is applied to the points of the line $p$ then it will be an involution for that line; i.e., its square will be a number and will give the *conjugate points* with respect to $\sigma$ that lie in $p$. Analogously, $P\sigma^{-1}$ will be an involution for the line that passes through $P$ and will give the conjugate *line* that issues from $P$ (*).

If the pole of the line at infinity, which is the center of the conic, is a proper point $O$ then $O\sigma^{-1}$ will be the involution of the conjugate diameters, and its orthogonal rays will be the two axes. If $O$ is at infinity then all of the diameters will be parallel, etc., from the usual projective properties of conics.

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Let the proper point $O$ be the center of the conic and let the unit vectors $I$, $J$ be two conjugate directions. The polarity $\sigma$ will then have the form:

$$s = \begin{pmatrix} IJ & hJO & kOI \\ O & I & J \end{pmatrix}.$$

The point $P = O + xI + yJ$ will belong to the conic when:

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(*) An involution $\lambda$ of the two-dimensional system (viz., a projectivity of the first kind) has the form $\lambda = \begin{pmatrix} hb & ka \\ a & b \end{pmatrix}$ and $\lambda^2 = hk$. The element $p = xa + yb$ will be united if:

$$p(\lambda p) = (hx^2 - ky^2)\, ab = 0.$$

Therefore the involution $\lambda$ will be hyperbolic, elliptic, or parabolic, according to whether:

$$\lambda^2 > 0, \quad \lambda^2 < 0, \quad \lambda^2 = 0, \text{ resp.}$$

If $\lambda^2 > 0$ then the united elements are $\sqrt{k}x \pm \sqrt{h}y$, and therefore two arbitrary conjugate elements are harmonically separated.

If $a$, $b$ are vectors then in order for there to exist two orthogonal conjugate elements, one must have $p \mid \lambda p = hx^2\, a \mid b + kxy\, a^2 + hxy\, b^2 + ky^2\, a \mid b = 0$, or, taking $a^2 = b^2 = 1$, $a \mid b = \cos \varphi$:

$$hx^2 \cos \varphi + (h + k)\, xy + ky^2 \cos \varphi = 0.$$

This will have real roots for $x$, $y$ when:

$$(h + k)^2 - 4hk \cos^2 \varphi = (h - k)^2 + 4\, hk \sin^2 \varphi \geq 0,$$

and since this condition is always satisfied, there will exist a pair (or an infinitude for $\varphi = \pi/2$ and $h + k = 0$) of orthogonal conjugate directions.

$$P(\sigma P) = (1 + hx^2 + ky^2)\ OIJ = 0,$$

and therefore the equation of the conic will be:

$$hx^2 + ky^2 + 1 = 0,$$

and since $\sigma$ admits united forms, one must have:

$$h = -\frac{1}{a^2}, \qquad k = \mp\frac{1}{b^2}.$$

One therefore has for $\sigma$ that:

$$\sigma = \begin{pmatrix} IJ & -\frac{1}{a^2}JO & \mp\frac{1}{b^2}OI \\ O & I & J \end{pmatrix} = \frac{1}{ab}\begin{pmatrix} abIJ & -bJO & \mp aOI \\ O & aI & bJ \end{pmatrix}.$$

If one sets:

$$U = aI, \qquad V = bJ,$$

and one neglects the numerical factor $1/ab$ in $\sigma$ then one will have:

(1) $$\sigma = \begin{pmatrix} UV & -VO & \mp OU \\ O & U & V \end{pmatrix},$$

and if $P = O + xU + yV$ then the equation of the conic will be:

$$x^2 \pm y^2 = 1.$$

The curve that is given by the upper sign (+) is called an *ellipse*, and the one that is given by the lower sign (–) is called a *hyperbola*.

The vector $mU + nV$ has a point of the curve for its position when $(mU + nV)\ \sigma(mU + nV) = -(m^2 \pm n^2)\ OUV = 0$, and therefore: *The ellipse has no points at infinity, while the hyperbola has two points at infinity*.

From (1), one has, for any number $m \neq 0$:

$$s\,(O + mU) = -\,mV\,(O + \frac{1}{m}U), \qquad \sigma(O + mV) = \pm\, mV\,(O \pm \frac{1}{m}V),$$

from which, it results that (*): *The extremes of the conjugate diameters are on the ellipse, and only two are on the hyperbola. The polars of the extremes of two conjugate*

(*) The same result can be arrived at by observing that the involutions $(OU)\sigma^{-1}$, $(OV\sigma^{-1}$ of the lines $OU$, $OV$ are $\begin{pmatrix} U & O \\ O & U \end{pmatrix}$, $\begin{pmatrix} V & \mp O \\ O & V \end{pmatrix}$, respectively. The point $O$ is the center of the two involutions, and the preceding *diameters define a parallelogram that has the extremes of the diameters for middle points of the sides.*

If results from (1) that the involution:

$$\lambda = \begin{pmatrix} \mp V & U \\ U & V \end{pmatrix},$$

when applied to the vector $P - O$, where $P$ is a point of the conic, gives the semi-diameter that is conjugate to $P - O$ in magnitude and direction, and since:

$$(P-O)^2 \pm [\lambda(P-O)]^2 = (x^2 \pm y^2)\, U^2 \pm (x^2 \pm y^2)\, V^2 = U^2 - V^2,$$

$$(P-O)[\lambda(P-O)] = \mp\, (x^2 \pm y^2)\, UV = \mp\, UV,$$

$$(P-O) \mid \lambda\,(P-O) = xy\,(U^2 - V^2) + (y^2 \mp x^2)\, U \mid V,$$

it immediately results that: *For the ellipse or the hyperbola, the sum or difference of the squares of two conjugate diameters is constant. The parallelogram that is constructed from two conjugate diameters has constant area. Only the circle has an infinitude of axes, and all pairs of orthogonal conjugate diameters.*

If one supposes, as one can do, that $U \mid V = 0$ then one will have that two vectors $P - O$, $\lambda(P - O)$ have equal moduli only when $(x^2 - y^2)(U^2 - V^2) = 0$, and therefore: *For the hyperbola, two conjugate diameters either always have different lengths or they always have equal lengths* (viz., an equilateral hyperbola). *For the ellipse, there exist two conjugate diameters* $\left[\frac{\sqrt{2}}{2}(U \pm V)\right]$ *of equal length* ($\sqrt{2a^2 + 2b^2}$) *that have the axes for bisectors.*

The asymptotes have the united elements of $\lambda$ for their directions, and since:

$$(mU + nV)\, \lambda\, (mU + nV) = \mp\, (m^2 \pm n^2)\, UV,$$

one will have immediately that: *The ellipse has no asymptotes. The hyperbola has two asymptotes, whose directions are $U \pm V$, and are therefore the diagonals of any parallelogram that has two conjugate diameters for its medians. Only the equilateral hyperbola has orthogonal asymptotes.*

Since:

$$P = O + xU + yV = O + (x+y)\frac{U+V}{2} + (x-y)\frac{U-V}{2},$$

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formulas show that: *The product of the distances from the center of an involution to two conjugate points is constant, and for the hyperbolic involutions it equals the square of the distance from the center to a double point.*

and $x^2 - y^2 = (x + y)(x - y) = 1$ for the hyperbola, one has that: *The equations of the hyperbola that refers to the asymptotes is $XY = m$. The triangle that is defined by the asymptotes and a tangent to a proper point has constant area.*

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Let $A$ be a point of the conic, $I$, a unit vector that is parallel to the diameter that issues from $A$, $J$, a unit vector that is parallel to the tangent to $A$, $mA + I$, a form of the first kind whose positions is the center of the conic, and finally, let $p$ be a positive, non-zero number.

By virtue of Theorem IV of section 10, one can set:

$$\sigma = \begin{pmatrix} AJ & J(mA+I) & \frac{1}{p}AI \\ A & I & J \end{pmatrix}, \tag{2}$$

and if $P = A + xI + yJ$ is a point of the conic then one will have that $P(\sigma P) = 0$; i.e.:

$$y^2 = 2px - mpx^2 \tag{3}$$

will be the equation of the conic.

For $m = 0$, the center is at infinity, and the conic is a hyperbola. For $m \neq 0$, the center is $O = A + \frac{1}{m}I$, and if one takes this point to be the origin of the axes then (3) will become:

$$\frac{X^2}{\frac{1}{m^2}} + \frac{Y^2}{\frac{p}{m^2}} = 1, \tag{4}$$

which gives an ellipse for $m > 0$ and a hyperbola for $m < 0$, and for $mp$, it gives the circle.

One says that the point $F = A + kI$ is a *focus* when the involution $F\sigma^{-1}$ is circular. As one usually does, one proves immediately that $F$ can be a focus only when $AI$ is an axis. Therefore, in all of what follows, we will assume that $I \mid J = 0$.

If $r$ is a line that issues from $F$ then one can set:

$$r = F(uI + vJ) = hvIJ - vJA + uAI,$$

and one will therefore have:

$$F(\sigma^{-1}r)\mathbf{.}\ w = v(h^2m - 2h)\ I + upJ,$$

which is normal to $uI + vJ$ for any $u$, $v$ only when:

$$mh^2 - 2h + p = 0. \tag{5}$$

This gives a real $h$ when $1 - mp \geq 0$, and therefore: *For the parabola* ($m = 0$), *there is just one focus, which is at a distance from the vertex of* 1/4 *the parameter* ($2p$). *For the hyperbola* ($m < 0$), *there are two foci on the transverse axis, and for the ellipse* ($m > 0$), *there are two foci on the major axis* [cf, (4)].

(5) says that (3) is true for $x = h$ and $y = p$, so: *The chord that is normal to the axis that passes through the focus is the parameter.*

If $F_1 = A + h_1 I$, $F_2 = A + h_2 I$ are the two foci for $m \neq 0$ then $h_1$, $h_2$ will be the roots of (5) and therefore:

$$\frac{F_1 + F_2}{2} = A + \frac{h_1 + h_2}{2} I = A + \frac{1}{m} I = 0.$$

That is: *The middle point of the foci on a conic with a center is the center of the conic.*

The director that relates to the focus $F$ is the polar of $F$ – i.e., the line:

$$\sigma F = [(1 - mh)\, A - hI]\, J,$$

and therefore it cuts the focal axis at the point (*):

$$H = A - \frac{h}{1 - mh} I.$$

If one now observes that (5) gives:

$$(mh - 1)^2 = 1 - mp, \qquad h - p = h(mh - 1)$$

then one will have, taking (3) into account:

$$(P - F)^2 = (1 - mp) \left[ x + \frac{h}{1 - mh} \right]^2,$$

and if one observes that $x + \dfrac{h}{1 - mh}$ is the distance from $P$ to the director of $F$, one will have that: *The ratio of the distances from a point of the conic to the focus and to the director is constant and equal to* $\sqrt{1 - mp} = e$ (viz., the *eccentricity*). *For the parabola, ellipse, hyperbola, and circle,* the eccentricities will be 1, < 1, > 1, and 0, respectively.

If $m \neq 0$ and $r_1$, $r_2$ are the signed distances from $P$ to the two foci then:

$$r_1 = e\left( x + \frac{h_1}{1 - mh_1} \right), \qquad r_2 = e\left( x + \frac{h_2}{1 - mh_2} \right),$$

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(*) The points $F$, $H$ are conjugate for the involution $(AI)\sigma^{-1}$, and therefore on a conic with a center ($m \neq 0$), the distances from the center to a focus and to the corresponding directors have the semi-focal axis for their geometric mean.

and therefore $r_1 - r_2 = 2/m$; i.e., *For the ellipse and hyperbola, the sum or difference of the focal rays is the focal axis.*

The point $P$ of the conic is a function of one numerical variable $t$, so if:

$$\text{mod}(P - F_1) \pm \text{mod}(P - F_2) = \text{const.}$$

then one will have (*), upon differentiating with respect to $t$:

$$\left[\frac{P-F_1}{\text{mod}(P-F_1)} \pm \frac{P-F_2}{\text{mod}(P-F_2)}\right] \Big| \frac{dP}{dt} = 0,$$

and therefore: *The bisector of the focal rays that issue from a point P of the conic are the tangent and normal to the conic at P.*

If $O$ is the base of the perpendicular that goes from $P$ to the director that relates to $F$ then one will have:

$$\text{mod}(P - F) = e \,\text{mod}(P - Q),$$

and if one observes that $(P - Q) \mid dQ / dt = 0$ then one will have:

$$\left[\frac{P-F}{\text{mod}(P-F)} - e\frac{P-Q}{\text{mod}(P-Q)}\right] \Big| \frac{dP}{dt} = 0,$$

which gives a simple construction for the normal at $P$ in terms of the focus, the director, and the eccentricity.

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The theorems of PASCAL and BRIANCHON are easily obtained with the usual methods. Thus, we shall examine the theorems of DESARGUES and STURM, instead.

Let $\sigma$, $\sigma_1$ be two polarities of the same plane that admit united elements. The two conics that are defined by the two polarities will pass through either four points, or three points that have a common tangent at one of them, or two points that have common tangents. Under these hypotheses, the conics that satisfy these conditions are united for the polarity $m\sigma + n\sigma_1$ and a line that cuts them at conjugate points for an involution. One states the theorem that is dual to that of STURM in an analogous way.

One can set:

$$\sigma = \begin{pmatrix} a & b & c \\ A & B & C \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} a_1 & b_1 & c_1 \\ A & B & C \end{pmatrix}.$$

$m\sigma + n\sigma_1$ is then (Theorem IV of § 10) a polarity. Moreover, if $P(\sigma P) = 0$ and $P(\sigma_1 P) = 0$ then one will also have $P[(m\sigma + n\sigma_1)P] = 0$. If $(\sigma P)(\sigma_1 P) = 0$ then one will also have

(*) Cf., C. BURALI-FORTI, *Introduction à la Geométrie différentielle suivant la méthode de* H. GRASSMANN, Gauthier-Villars, Paris, 1897.

$\sigma P.\ (m\sigma + n\sigma_1)\ P = 0$. Conversely, a polarity $\sigma_2$ that satisfies the same conditions can be of the form $m\sigma + n\sigma_1$, because a point will determine one of the conics considered in just one way, and therefore the first part of the theorem is proved.

Let $e$ be the line that is considered. Just one of the conics $m\sigma + n\sigma_1$ will pass through a point $P$ of it, and if it cuts $r$ at $P'$ then $P'$ will be a linear function of $P$; i.e., the operation $\lambda$ is such that $\lambda P = P'$ is a homography, and since $\lambda^2$ is a number, it will be an involution.

## § 12. – Second-degree equations.

The coordinates of a united form for a polarity will annul a function of degree two in its coordinates. Conversely, if:

$$f(x, y) = h_{11}\, x^2 + 2h_{12}\, xy + h_{22}\, y^2 + 2h_{13}\, x + 2h_{12}\, y + h_{33} \tag{1}$$

is a function of degree two then a polarity $\sigma$ will always be determined, up to a numerical factor, and its curve of united points will have the equation $f(x, y) = 0$.

Indeed, if $O$ is a fixed point, and $I$, $J$ are non-parallel unit vectors, and one sets:

$$a_r = \frac{1}{OIJ}(h_{r1}\, JO + h_{r2}\, OI + h_{r3}\, IJ)$$

for $r = 1, 2, 3$, and one supposes that $h_{rs} = h_{sr}$ then the homography:

$$\sigma = \begin{pmatrix} a_1 & b_1 & c_1 \\ I & J & 0 \end{pmatrix}$$

will be a polarity, and $P = O + xI + yJ$ will be a united point when $f(x, y) = 0$.

In the analytical theory of conics, one considers only the following elements:

$$H = \begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix} = \begin{vmatrix} Ia_1 & Ja_1 & Oa_1 \\ Ia_2 & Ja_2 & Oa_2 \\ Ia_3 & Ja_3 & Oa_3 \end{vmatrix} = OIJ.\ a_1\, a_2\, a_3\,,$$

$$H_{33} = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} = \begin{vmatrix} Ia_1 & Ja_1 \\ Ia_2 & Ja_2 \end{vmatrix} = IJ.\ a_1\, a_2 = IJ.\ (\sigma I)(\sigma J),$$

and if one sets:

$$IJ = \sin\varphi, \qquad \text{from which,} \quad I\ i\ J = \cos\varphi,$$

then one can also consider the number:

$$K = h_{11} + h_{22} - 2h_{12}\cos\varphi\,.$$

One can give the number $K$ the form:

$$K = Ia_1 + Ja_2 - (Ia_2 + Ja_1)\, I\, i\, J,$$

and since $I\, i\, J.\, J = IJ.\, iJ + I$, one will have:

$$K = IJ\,[i\, I.\, \sigma J - iJ.\, \sigma I].$$

One will then have:

$$\frac{H}{\sin^2\varphi} = \frac{\sigma O.\sigma I.\sigma J}{4OIJ},$$

$$\frac{H_{33}}{\sin^2\varphi} = \frac{IJ.(\sigma I)(\sigma J)}{(IJ)^2},$$

$$\frac{K}{\sin^2\varphi} = \frac{iI.\sigma J - iJ.\sigma I}{IJ},$$

for the usual *invariants*, and since any vector can be expressed *linearly* in terms of $I$, $J$, the right-hand sides will give the invariant character of the left-hand sides *immediately*, and with no need for the usual lengthy proof. The superiority of GRASSMANN's method over the usual analytical method is then given a new proof, should the ease by which we obtained the properties of conics in the preceding sections not be enough.

Turin, April 1901.

C. BURALI-FORTI.

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