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GEOMETRICAL OPTICS

BY

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WITH 11 FIGURES

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Foreword.

This booklet contains that part of geometrical optics that can be regarded as an immediate consequence of the principles of FERMAT and HUYGENS. The description of the ray map in the first approximation may be informally integrated with the general theory, and for that reason, it will likewise be considered. By comparison, I have left aside the theory of optical errors of third order, upon which the calculation of optical instruments rests, because I was forced to choose a much too scant presentation for the foundations of ray optics. This sacrifice was, however, alleviated for me by the fact that precisely those things had been treated in a classical way a long time ago by K. SCHWARZSCHILD (see footnote 59, pp. 39). In addition, one finds them in all of the books that are dedicated to geometrical optics, and thus, above all, in the following two works: CZAPSKI-EPPENSTEIN: *Grundzüge der Theorie optischer Instrumente*, 3rd ed., issued by H. ERFLE and H. BOERGEHOLD, and M. HERZBERGER: *Strahlenoptik*. Since these books include very thorough and almost flawless bibliographies, I can restrict myself to just the most necessary ones in my own references.

I am indebted to G. PRANGE, who corrected this booklet, for numerous essential improvements that I cannot detail individually. My thanks also go out for the editors of the "Ergebnisse" and the publishers, for accommodating all of my wishes.

October 1937

C. CARATHÉODORY.

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Introduction

With the onset of the Nineteenth century, a view of geometrical optics began to prevail that had already been initiated by CHR. HUYGENS (1629-1695) (cf., *infra*, footnote 37), but was then completely forgotten. Until then, one had to content oneself with treating the validity of the ray refraction in the first approximation on the axis of a rotationally-symmetric system ([1]), but now one turns to the posing of more general questions. In the year 1808, E. L. MALUS (1775-1812) stated the theorem that a stigmatic light bundle, after reflection or refraction from a curved surface, would be converted into a normal congruence ([2]). MALUS was of the opinion that this theorem was true only for stigmatic light bundles, and therefore when light rays passed through an instrument it would only apply to the first reflection or refraction. However, the theorem generally extends to arbitrary normal congruences; this was established for the case of reflection by CH. DUPIN (1784-1873) in the year 1816 and for the case of refraction by L. A. J. QUETELET (1796-1874) in the year 1825, and almost simultaneously by J. D. GERGONNE (1771-1859) ([3]).

From the extended theorem of MALUS, the law of ray mapping can be achieved for an arbitrary optical instrument, when one ignores a similarity transformation (cf., § 27). This path is, however, very tedious and really only a detour. Nonetheless, it was still used occasionally and in remarkable ways for a long time after the discovery of a direct path (cf., BRUNS, footnote 18).

The natural approach to the theory of geometrical optics in its full generality was first found by Sir WILLIAM ROWAN HAMILTON (1805-1865) ([4]). HAMILTON himself already had been interested in optical problems for thirteen years. However, should this folklore itself seem to be only a legend, it is therefore sufficiently amazing that before his entrance into Trinity College of Dublin (July, 1823) he had already done his first work on caustics ([5]) and before the conclusion of his studies he had presented his great work "Theory of Systems of Rays" to the Irish Academy (April, 1827). This rare gift was, moreover, immediately recognized by everyone. In the same year, before he had time to get over the final exams, he was made professor of astronomy, a post that his teacher Dr. BRINKLEY, who had meanwhile been named the Bishop of Cloyne, had occupied ([6]).

([1]) Cf., M. HERZBERGER: "Geschichtlicher Abriss der Strahlenoptik,: Z. Instrumentenkde., Bd. 52 (1932), pp. 429-435. 485-493. and 534-542.

([2]) MALUS, "Optique, Dioptrique," J. École polytechn., **7** (1808), pp. 1-44, 84-129 – MALUS, E. L. "Traité d'optique," Mém. prés. à l'Institut par divers savans **2** (1811), pp. 244-302.

([3]) One finds a detailed history of MALUS's theorem , with all of the necessary references, on pp. 463 of the *Collected Papers of HAMILTON* (see footnote 16).

([4]) One finds the best introduction to HAMILTON's ideas in G. PRANGE: "W. R. HAMILTONs Arbeiten zur Strahlenoptik und analytischen Mechanik" NOVA Acta. Abh. Leop. Carol. Deutsche Akad. d. Naturforscher, Bd. 107, no. 1, pp. 1-35. Also very useful is J. L. SYNGE: "Hamilton's method on Geometrical Optics," J. Opt. Soc. Amer., **27** (1937), 75-82.

([5]) This paper was announced for the first time in 1931 under the title of "On Caustics, Part First, 1824;" in his *Mathematical Papers*, v. 1, pp. 345-363.

([6]) ROBERT PERCIVAL GRAVES: *Life of Sir W. R. HAMILTON including selections from his poems, correspondence and miscellaneous writings.* 3 vols. (Dublin, Trinity College 1882-1889, Dublin Univ. Press, Ser.) F. KLEIN: *Vorlesungen über die Enwicklung der Mathematik im 19. Jahrhundert* (Berlin, Springer 1926), Bd. 1, esp., pp. 182 et seq.

The first youthful labor of HAMILTON on caustics already included many of the ideas that would later make him famous. In the "Theory of Systems of Rays," the first paper of what would be followed by three significant "Supplements," we find, above all, the concept of *characteristic function* ([7]). HAMILTON had the inspiration of regarding the optical length of a light ray that linked a point in the object space with a point in the image space as a function of the positions of these two points. HAMILTON had first seen the true basis for this situation, in general, in 1832 when he derived the properties of characteristic function on the basis of the formulas for the variation of a curve integral with variable endpoints ([8]). This formula had been discovered by J. L. LAGRANGE (1746-1813) ([9]) and L. EULER (1707-1783) had, in turn, written it down for curve integrals in three-dimensional space whose integrands were completely general expressions ([10]), but neither of the two had treated the subject with the simplicity and clarity that HAMILTON first invested it with.

A second, extraordinary accomplishment of HAMILTON consisted in the fact that, along with the first characteristic function that he had worked with, he also found three other functions of the same type, for which the roles of position coordinates and direction coordinates were switched with the aid of a so-called LEGENDRE transformation ([11]).

The affinity between the conception of geometrical optics, as HAMILTON had founded it, and the treatment of mechanics by the methods that LAGRANGE had developed in his *Mécanique Analytique* is so great that HAMILTON could carry over all of the methods that he had devised for the theory of optical instruments effortlessly to the most general problems of mechanics ([12]).

These latter papers attracted the interest of C. G. J. Jacobi (1804-1851), who immediately realized the advance over LAGRANGE and who presented the HAMILTONian theory to greater mathematical public ([13]) in a newly-minted form ([14]). By contrast, the optical works that had defined HAMILTON's starting point would not be noticed even once by the specialists outside of England up until the end of the Nineteenth century ([15]). This was surely connected with the fact that, on the one hand, the *Irish Transactions*, that included these treatises was hard to obtain outside of England, but above all, also the fact that HAMILTON, for whom new ideas were always erupting, had

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([7]) *Mathematical Papers*, v. 1, pp. 17.

([8]) *Ibid.*, pp. 168.

([9]) See R. WOODHOUSE: *A Treatise on Isoperimetrical Problems*, pp. 90, Cambridge, 1810.

([10]) EULER, L: *Instit. Calculi Integralis*, pp. 555, Petersburg, 1770.

([11]) "Third Supplement to an Essay on the Theory of Systems of Rays," presented to the Irish Academy in 1832. *Mathematical Papers*, pp. 164-293, esp. pp. 175 and 268.

([12]) HAMILTON, W. R.: "On a General Method in Dynamics," Philos. Trans. Roy. Soc. London 1834, Pt. 2, pp. 247-308. – "Second Essay on a General Method in Dynamics," *ibid*., 1835, Pt. 1, pp. 95-144.

([13]) One finds a complete and very precise presentation of this entire historical development in G. PRANGE: "Die allgemeinen Integrationsmethoden der analytischen Mechanik," Enzklop. d. math. Wiss. mit Einschl. ihrer Anwend, IV, 12 and 13, released Dec. 1933, Bd. 4/2, pp. 505-804, esp., pp. 593-615.

([14]) On the contradiction to the conceptions of JACOBI and HAMILTON, cf., A. W. CONWAY and A. J. McCONNELL: "On the Determination of Hamilton's Principal Function," Proc. Roy. Irish Acad., 41, Sect. A (1932), pp. 18-25.

([15]) At the naturalist convention in Halle in 1891, F. KLEIN gave a talk with the title "Über neuere englische Arbeiten zur Mechanik," in which he notably emphasized the significance of the papers of HAMILTON on ray optics. Cf., Jber. Deutsch Math.-Vereinig., Bd. 1 (1891/92) or FELIX KLEIN *Gesammelte mathematische Abhandlungen*, Bd. II, pp 601-602. Berlin, Julius Springer 1922. Despite the authority of KLEIN, this talk still did not have the desired effect.

squeezed so many different things into these papers that they were, to some extent, quite tedious to read.

It is only recently that anyone with an interest in ray optics has been in a position to study HAMILTON's treatises comfortably. Above all, he can do this in the monumental publication of those works, which contain many new ideas that were published for the first time in the manuscript, and which is provided with an excellent apparatus of comments ([16]), or also in German translation by PRANGE, which is distinguished by his detailed commentary ([17]).

From the general ignorance of HAMILTON's papers, it is not surprising that his results were discovered many more times. Above all, one must cite the paper of H. BRUNS (1848-1919), who had the greatest influence on the further development of ray optics ([18]). In addition, he provided the inducement for F. KLEIN to steer the attention of the general scientific world to HAMILTON's work in optics ([19]). The starting point of BRUNS is clumsier than HAMILTON's original method, because, starting from MALUS's theorem, he dragged along the entire apparatus of SOPHUS LIE's (1842-1899) contact transformations with him. On the other hand, BRUNS had an obvious trick that the esteemed HAMILTON had not thought of, and which simplified the theory of the ray map in two regards. He first simplified it by the fact that he had the ray system fall on a screen and then characterized the individual rays by their determining data when they went through the screen. By that means, he could employ the eikonals, which depend upon only four variables, in place of the characteristic functions of HAMILTON, while only one of the four characteristic functions of HAMILTON could be regarded as a function of four variables ([20]). One observes that this number of variables cannot be reduced further, since the ray space is also indeed four-dimensional.

The second simplification that BRUNS had, however, involuntarily arrived at rests upon the fact that any individual system of formulas that arises from an arbitrarily given eikonal, and which serves to describe the ray map, can be used for all possible optical spaces and any choice of coordinates. By contrast, the mapping formulas that were calculated from a HAMILTON characteristic function always belong to a single specific problem (cf., § 32 and 64).

One must also consider this result of BRUNS when one would like to give a modern presentation to the HAMILTONian theory after more than a hundred years has passed. One must also consider many other things, such as, e.g., the study of canonical transformations whose beginnings one admittedly finds in HAMILTON himself, but first experienced its systematic definition at the hands of JACOBI and S. LIE. It is further

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([16]) *The Mathematical Papers of Sir William Rowan Hamilton*, Cunningham Memoir No. XIII, v. 1, *Geometrical Optics*, ed. for the Royal Irish Academy by A. W. CONWAY and J. L. SYNGE, Cambridge University Press, 1931 4°, XXVIII and pp. 534.

([17]) "W. R. Hamiltons Abhandlungen zur Strahlenoptik," Transl. and comments by G. PRANGE, Leipzig, Akad. Verlagsges., 1933. pp. 429 and comment on pp. 116.

([18]) BRUNS, H. *Das Eikonal*, Abh. math. phys. Cl. sächs. Akad. Wiss., Bd. 21 (1895), 323-436.

([19]) KLEIN, F.: "Über das BRUNsche Eikonal," Z. Math. u. Phys., Bd. 46 (1901) or Ges. math. Abh., Bd. II, pp. 603-606.

([20]) On this, one confers the polemic between M. HERZBERGER: "On the Characteristic Function of HAMILTON, the Eiconal of BRUNS and Their Use in Optics," J. Opt. Soc. Amer., **26** (1936), 177-180, and J. L. SYNGE: "HAMILTON's Characteristic Function and BRUNS Eiconal," *ibid.*, **27** (1937), 138-144.

convenient to derive the main result of HAMILTON, namely, the equivalence of FERMAT's and HUYGENS's principles, by a path that is the opposite of the one that was followed by HAMILTON.

Namely, in place of FERMAT's principle, we will take HUYGENS's principle as the starting point, and show the equivalence of the two theories with the help of CAUCHY's theory of characteristics (1819). This has the advantage that the theorem of the conservation of the integral invariants of POINCARÉ and CARTAN, which, as we will see, not only replaces the celebrated theorem of MALUS, but virtually completes it, will follow almost by itself. I have, moreover, summarized some of these things two years ago in my book *Variationsrechnung und partielle Differentialgleichungen erster Ordnung* (Leipzig, Teubner, 1935), and I will cite that book without an author name in the sequel.

I have treated the ray map with the help of the eikonal and the coupling of the individual line elements of optical spaces – two problems that are all too easily and all too often confused – in different chapters, in order to clearly separate them from each other, and I hope that this will contribute to the clarity of the presentation. In addition, I have taken great pains to clarify some points that, although they do not seem to have a fundamental significance, still do not seem trivial. For instance, at many times we will assume that every possible optical ray map is realizable in terms of at least one of the three remaining eikonals. This supposition is, however, false, although I have compiled *all* ray maps for which this does not pertain, such that one now knows all of the cases for which a representation of the map by this eikonal is not possible. In this way, one can verify that this hypothesis is always true, at least, for rotationally-symmetric systems. I have also considered a term in the eikonals of these latter systems that was, inconceivably, always forgotten.

Chapter I

The principles of FERMAT and HUYGENS

1. The discovery of FERMAT's principle ([21]). Once GALILEO GALILEI (1564-1642) had invented the telescope † in the year 1609, the problem of the true explanation for the refraction of light was popular topic of the era that attracted the best minds ([22]). The first one to correctly describe the laws of refraction by a geometric construction on the basis of many measurements was WILLEBROD SNELL (1581-1626); however, the manuscript of SNELL that HUYGENS could refer to was lost without a trace, and the fact that SNELL had discovered the law of refraction was first recognized a century after his death ([23]). The discovery of SNELL had no further influence on the development of optics. In the meantime, RENÉ DESCARTES (1596-1850) had rediscovered the same law, and described by a simple mathematical law that he announced in the year 1637 ([24]). DESCARTES found this formula by an ingenious device, namely, with the aid of the hypothesis (later proved to be false) that under the change of velocity that the light suffers when it passes through a medium, the component of the velocity that is parallel to the separation surface must remain constant, while the absolute velocities on both sides of this surface must have a fixed ratio.

Immediately after the appearance of the book of DESCARTES – i.e., in the same year 1637 – PIERRE FERMAT (1601-1665) attacked the physical foundations of the CARTESIAN theory fiercely ([25]). It created a controversy that lasted a century and can still provoke a limited degree of interest to this day. We need only to point out that FERMAT, et al., rejected the theory of DESCARTES, and indeed, with justification, on the grounds that in it the velocity of light in a dense medium would have to be greater than that in air.

In the course of time, FERMAT came to the idea of founding Dioptrics on a *minimum principle*, similar to the one that HERON of ALEXANDRIA (c. 200 A.D.) had employed

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([21]) For the historical details of this chapter, see also C. CARATHÉODORY: *The Beginning of Research on the Calculus of Variations*, Osiris, v. 2 (1937).

† [D.H.D. Not true; this is a popular misconception about Galileo.]

([22]) Including JOHANNES KEPLER (1571-1630), who had already written his *Dioptrik* in 1610. (Cf., M. CASPAR: *Bibliographia Kepleriana*, no. 40, pp. 61. Munich, Beck 1936.)

([23]) Cf., HUYGENS: *Opuscula posthuma*, v. 1, Amstelodami 1728. *Dioptika*, pp. 2.

([24]) In the anonymous work: *Discours | de la Methode | pour bien conduire sa raison, and chercher | la verité dans les sciences | Plus | la Dioptrique | les Meteores | et | la Geometrie | qui sont des essais de cette Methode.* – A Leyde | De l'Imprimerie de Ian Maire | CIC.IC.CXXXVII. Avec Privelege.

It was long asserted that DESCARTES knew the result of SNELL and had employed it in his own investigations without mentioning his predecessor. It was first in the source reference of the Dutch historian D. J. KORTEWEG that it was proved that this opinion is obviously false [see D. J. KORTEWEG: "Descartes et les manuscrits de Snellius d'aprés quelques documents nouveaux," Rev. Métaphys. et Morale 4th Année (1896), pp. 489-501]. KORTEWEG had attributed the discovery of the law of refraction to DESCARTES rather precisely, and showed that at the time of the manuscript of SNELL, who had died already, not one of his best friends knew of its existence, and it was first rediscovered many years later. Cf., also, E. GERLAND: *Geschichte der Physik*, pp. 481, Munich, Oldenbourg, 1913.

([25]) Cf., the letter to MERSENNE on September 1637 (*Oeuvres de Fermat*, t., pp. 106. Paris, Gauthier-Villars 1891-1922).

for the treatment of Catoptics (refraction) ([26]). During reflection, a light ray remains in the same medium and it suffices to postulate that the ordinary duration of the light path must be as short as possible (cf., on this, § 4). On the contrary, under refraction, the light traverses two different media, and FERMAT now prescribed that the duration of the light ray, when one evaluated it with different velocities on both pieces, should generally be a minimum.

FERMAT had already expressed these thoughts in the year 1657 ([27]). At the time it had not been established that light propagated with a finite velocity, and FERMAT then left this question open. However, he chose his constants in such a way that if one interpreted the expression that was used for the minimum as a light duration then the velocity in dense media would be smaller than in sparse ones.

In the meantime, the DESCARTES law of refraction had been confirmed quite precisely by experiment. Since FERMAT was of the opinion that his Ansatz, which was indeed diametrically opposite to that of DESCARTES, must then lead to a law of refraction that was incompatible with observations, he was therefore too discouraged by this situation to follow through with the analytical consequences of his minimum principle ([28]). It was first at the end of the year 1661 that he finally gathered, from the repeated urging of his friends, that it was true, and was extremely surprised to find that his principle led to precisely the same law of refraction as the hypothesis of DESCARTES ([29]).

2. Generalization and formulation of FERMAT's principle. FERMAT had assumed that the velocity of light was the same for all points of a transparent medium and in all directions. However, the investigations of Chr. HUYGENS and I. NEWTON (1642-1727) ([30]) showed that this velocity is indeed independent of the prevailing intensity of the light, but it does depend upon the color of the light, and in crystalline media, on the direction of the light ray. In addition, it is interesting to consider those media in which the density varies from point to point, as is the case for, e.g., the Earth atmosphere. In such media, the light velocity v is also a function of position. If one lets c

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([26]) HERONIS ALEXANDRINI, *opera quae supersunt omnia*, 5 vols., Teubner, Leipzig (1899-1914), with German translation. "De Speculis," v. II 1, pp. 301-365. This paper, which presently exists only in the form of a Latin translation from the 13th Century, had long been attributed to Cl. PTOLEMAEUS. It was first in the 18th Century that critics showed that it went back to HERON. The evidence of DAMIANOS (4th Century A. D.) is important in connection with this, and is found in his book (Greek characters) (*Principal Facts of Optics*), ed. R. SCHÖNE, Greek and German, Berlin 1897. In chap. 14, pp. 20 of this book (also cited in Heronis Al. opera II, pp. 303) the minimum principle of HERON was discussed and set down in words: (Greek quotation); i.e., *At the conclusion of his proof, he said: If the nature of light leads us to believe that our eyes are not unnecessary then it must reflect with equal angles* (meas. from the normal to the mirror). [In the opinion of Greek physics, light did not originate from the things that one saw, but from the eye of the beholder. One then had the inversion (Greek characters).]

([27]) Letter to CUREAU DE LA CHAMBRE of August 1657 (*Ouevres*, t. 2, pp. 354).

([28]) Cf., the passages in his letter in the *Oeuvres*, t. 2, pp. 460 and 486.

([29]) Letter on Sunday, 1 January 1662 to CUREAU DE LA CHAMBRE, Oeuvres, t. 2, pp. 457. The proof that was attributed to this letter found a broader synthetic proof in t. 1, pp. 170, in which the property of the minimum was confirmed. *ibid.*, pp. 173.

([30]) I. NEWTON: *Opticks, or a treatise on the reflexions, refractions, inflections, and colours of light.* London, 1704.

denote the constant light velocity *in vacuo* and lets v denote the velocity in the medium under consideration then one introduces the quantity:

$$n = \frac{c}{n}, \tag{2.1}$$

which one calls the *index of refraction* ([31]). We will, however, assume throughout that the light whose spreading we will examine is *monochromatic*, so the index of refraction n should depend upon only the geometric data (position and direction).

A medium for which n does not depend upon position is called *homogeneous*. If the index of refraction does not depend upon the direction then the medium will be called *isotropic.*

3. In the theory of optical instruments, one almost exclusively considers the passage of light rays through isotropic, piecewise homogeneous media. This situation had moved some authors to treat geometrical optics in the vector notation, a choice of notation that is advisable only when invariance under rigid rotations in space is expressly specified. However, this invariance plays an entirely subordinate role in ray optics, since most optical instruments – if one ignores prisms, etc. – possess a symmetry axis whose position must be specified by a suitable choice of coordinates. In the following, we will therefore distinguish an axis and denote it with the symbol t – by which, a parallelism between our formulas with the ones in analytical mechanics becomes especially obvious – and establish the points of space with the aid of this variable t, along with two other variables x_1 and x_2 .

We will often have to consider the case in which the three axes t, x_1, and x_2 define a rectangular coordinate cross, and in which the medium that we examine is isotropic, but not homogeneous. The index of refraction:

$$n = n(t, x_1, x_2) \tag{3.1}$$

will then appear to be a function of three variables (t, x_i), and the time T that light requires to describe a piece of the curve:

$$x_i = x_i(t), \qquad (i = 1, 2;\ t' < t < t'') \tag{3.2}$$

will be represented by the integral:

$$T = \int_{t'}^{t''} \frac{ds}{v} = \frac{1}{c}\int_{t'}^{t''} n(t, x_1, x_2)\sqrt{1 + \dot{x}_1^2 + \dot{x}_2^2}\, dt\,. \tag{3.3}$$

([31]) The definition above of the index of refraction is true for the undulatory theory of light. For the emission theory, one must set n proportional to the velocity itself. Cf., P. STÄCKEL: "Elementare Dynamik der Punktsysteme und starren Körper," Encykl. d. mathem. Wiss. IV.7, Bd. 4/1, pp. 490.

In this, ds means the differential of the arc length of our curve, and we let $\dot{x}_i$ denote the derivative of the functions (3.2). In order to confirm the validity of (3.3), one must imagine that one has set:

$$\frac{1}{v} = \frac{n}{c}, \qquad ds^2 = dt^2 + dx_1^2 + dx_2^2 . \tag{3.4}$$

The function in the integral (3.3) will, by our argument, play an entirely similar role to that of the LAGRANGIAN function for holonomic problems of classical mechanics. In order to express this analogy rigorously, we would like to introduce the notation:

$$L(t, x_i, \dot{x}_i) = n(t, x_1, x_2) \sqrt{1 + \dot{x}_1^2 + \dot{x}_2^2} . \tag{3.5}$$

The formulas that we will derive, will be, moreover, independent of this special form (3.5) of the function $L(t, x_i, \dot{x}_i)$; they will remain valid for an arbitrary form of the function L, and will correspondingly be just as well-suited to the case in which the medium in question is crystalline, and thus, anisotropic, as for the case in which one uses a curvilinear coordinate system in an isotropic medium – and, in turn, also in an anisotropic one – and converts the function (3.5) into such coordinates.

4. FERMAT's principle shall now be formulated for the general problem of that sort. A precise translation of the demand that FERMAT proposed for the special case, and which he alone had considered, will read as follows: *Let A and B be two given points of space. If one considers the totality of curve arcs γ that connect these points and computes the integral:*

$$\int_\gamma L(t, x_i, \dot{x}_i)\, dt \tag{4.1}$$

for these curves then the light ray that connects A with B will be the curve for which the expression (4.1) possesses the smallest possible value.

The consideration of special optical instruments has show that the principle that is formulated in this way is consistently useful. Namely, in some situations, one can construct light rays that go through the instrument and choose the two points A and B such that no other curve that γ that connects these two points and that runs through the given instrument and the integral (4.1) will attain a minimum value. Similar phenomena were already familiar for the problems of catoptics in the time of FERMAT ([32]). FERMAT himself wished to overcome this difficulty by replacing the curved mirror with a plane mirror that contacted it at the place where the ray was reflected ([33]). If one overlooks the fact that this represents a stopgap that is hard to justify then similar constructions in the case of refraction would not lead to the desired objective.

([32]) One needs only to consider a light ray that starts from the midpoint A of a spherical hollow mirror and after reflection is led from the point A to an arbitrary endpoint B. Any other light ray from A to B consists of two rectilinear line segments and has a smaller total duration along the path that is bent at the sphere.

([33]) *Oeuvres*, t. 2, pp. 355.

One must then modify FERMAT's principle. Since the formulas of the calculus of variations will be developed later on, one must propose that the requirement of a minimum of the integral (4.1) be replaced with the requirement of the vanishing of the first variation of this integral, a proposal that has generally been agreed to up to the present time. From a purely mathematical standpoint, there is absolutely nothing to prevent this procedure. In this way, one obtains all curves that come under consideration as light rays. The method, however, possesses two drawbacks: First, the generally understood and elementary concept of a minimum is replaced with a complicated and artificial concept, since the first variation of an integral can be made more intuitive only by means of much work and many words. The second drawback consists in the fact that the way that one obtains the differential equations for the light rays from the condition for the vanishing of the first variation must likewise seem extraordinarily artificial, when one follows it through with the necessary diligence.

Fortunately, the difficulty that had engendered so many paradoxes can be eliminated by means of a minor modification of the problem statement. FERMAT, and also all of his followers, had considered a fixed piece of the light ray and had drawn all of the comparison curves through the two endpoints of the curve segment. Meanwhile, if these endpoints are noticeably far apart – e.g., when they lie on both sides of the instrument – it can happen that the postulated minimum property is not present. The choice of endpoints is, however, completely arbitrary and thoroughly artificial. One avoids all of the difficulties when one assumes that the light ray is *unbounded* (which it is, in reality) and the choice of endpoints *is left open*. One then postulates something less than what FERMAT had done, but something more than what would suffice for the vanishing of the first variation: One demands that FERMAT's principle should retain its original content when one considers arc segments along a given light ray whose positions are arbitrary, but whose lengths are sufficiently short. One then arrives at the following formulation of FERMAT's principle:

FERMAT's principle: *A curve e can coincide with the path of a light ray if and only if each point P of e is an interior point to at least one arc segment of these same curve e that possesses the following property: The integral* (4.1), *when taken along this arc segment between its endpoints P' and P'', will have a smaller value than the same integral when one computes it along a curve γ that differs from e, but possesses the same endpoints, namely, P' and P'', and lies in a certain narrower neighborhood of e.*

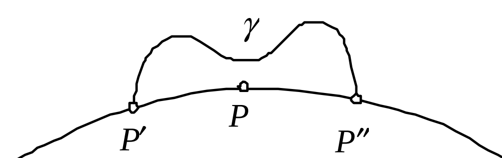

Fig. 1.

The last condition means that one may restrict the choice of comparison curve very strongly without corrupting the fact that the curve *e* that it adheres to must be a possible light ray: Namely, one may choose two arbitrary positive numbers ε and η, and arrive at the fact that only such curves γ can be drawn for comparison for which the distance between two points Q and Q^* that lie on *e* (γ, resp.) and possess the same abscissa *t*, is less than ε, and for which the angle that the tangents to both curves at these points subtend is, at the same time, less than η. The necessity of such a restriction, for which the values of the numbers ε and η is not prescribed once and for all, is rooted in the nature of the problems that we would like to treat. If one would always like to leave the curve γ be completely arbitrary, or choose permanently fixed numbers $\varepsilon = \varepsilon_0$ and $\eta = \eta_0$, then it can happen that for certain positions of curves, which one would like to establish, if they are to represent a light ray according to FERMAT's principle then one must consider comparison curves that do not lie in the field of the instrument; e.g., they might fall on a screen.

5. The discovery of HUYGENS's principle. FERMAT's principle represents a geometric theorem that is, in fact, suitable for characterizing the form of the light rays that pass through an optical instrument. For the further development of optics, however, it is worthwhile to point out that, right from the beginning, physicists were satisfied with the ideas that had guided FERMAT.

FERMAT had stated the theorem: *"La nature agit toujours par les voies les plus courtes"* ([34]) ["Nature always acts along the shortest path."]. One could immediately respond to it ([35]) by saying that this is moral principle, not a physical one, and that many times by such a choice of "shortest path," Nature can certainly find itself in a quandary. The fact that HUYGENS, who lived in Paris at the time, and constantly socialized with the academia there, had made the same sort of argument was shown in a letter that was written at the time ([36]). One can, in this sense, practically say that reasoning that was completely similar to the kind that moved physicists to replace action-at-a-distance in electricity with the FARADAY-MAXWELL theory 200 years later had also led HUYGENS to the theory of light by thinking through it first. The result was the book "vom Licht" that first appeared in 1690, but had been already written completely twelve years before ([37]). The 124 little quarter-pages that he had devoted to this work on optics included, in principle, everything that would be accomplished during the next five quarter-centuries of progress in the theory of the propagation of light. The most celebrated part of this paper was the first chapter, in which the phenomenon of light was described as a process of oscillation, and from which HUYGENS's principle would be derived. For our special purposes, it is the chapter that followed it that is more important: It showed, in particular, that HUYGENS's principle could also be applied when one declined to pursue the details of the oscillation process and thus agreed to work in the first approximation by establishing that the velocity of light was a function of position and direction ([38]). This approximation subsumes the content of both the principles of

([34]) In the *Catoptics* of OLYMPIODOR (6th Century A. D), which HERON had revised, one finds the statement, loc. cit., footnote 26, v. II, 1, pp. 368: (Greek characters), i.e., *Nature does nothing superfluous and does no unnecessary work.*

([35]) Letters from CLERSELIER to FERMAT on 6 and 13 May 1662 (*Oeuvres*, t. 2, pp. 464, et seq.).

([36]) On 22 June 1662, *Oeuvres complètes de Christiaan Huygens*, publiées par la Sociétés Hollandaise des Sciences, t. 4, pp. 157, Lettre 1025. La Haye, Martinus Nijhoff 1894.

([37]) *Traité | De la Lumiere. | Ou sont expliquées | Les causes de ce qui luy arrive | Dans la Reflexion, & dans la | Refraction | Et particularement | Dans l'etrange Refraction | Du Cristale d'Islande.* | Par C. H. D. Z. (Chr. Huygens de Zuilyck) | Avec un Discours de la Cause | De la Pesanteur. | A Leide | Chez Pierre vander Aa, Marchand Libraire | MDCXC.

([38]) Precisely the same fact had led E. SCHRÖDINGER, in our own time, to discover the relationship between classical mechanics and wave mechanics.

FERMAT and HUYGENS, and one can, as is generally customary in the procedure of W. R. HAMILTON, derive HUYGENS's principle from only FERMAT's. However, it is not merely a convenient crutch for the intuition when one works with both principles from the outset. Moreover, one can, in this way, liberate the calculations from all slag and bothersome repetitions, and present a philosophical scheme that is not easy to surpass in terms of simplicity and clarity.

For that reason, we would first like to briefly sketch HUYGENS's train of thought in the simplest case of the spherical waves and express certain consequences that arise from our argument by analytical formulas.

6. HUYGENS's principle. If a light signal were given at a point O in a homogeneous, isotropic medium with an index of refraction n at the time T_0 then at the time $T > T_0$ the light excitation would be noticeable on the outer surface of a sphere $\chi_0(T)$, that has O for its center and possesses the radius (Fig. 2):

$$R = \frac{c}{n}(T - T_0). \tag{6.1}$$

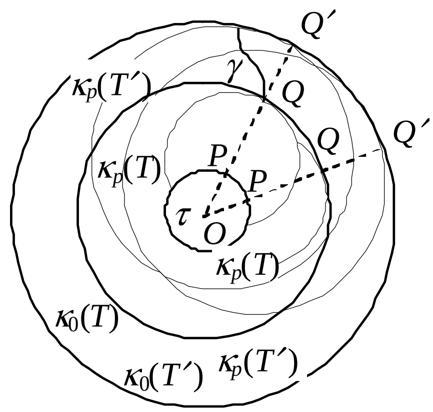

Figure 2.

We consider a convex surface t that contains O in its interior and lies completely within $\chi_0(T)$, and denote the distance between an arbitrary point P of τ and the point O by $\rho(P)$. A light signal that is given at the point P and the time:

$$T_P = T_0 + \frac{n}{c}\,\rho(P) \tag{6.2}$$

generates a light excitation that is found at the time T on the outer surface of a sphere $\chi_P(T)$. This sphere $\chi_P(T)$ contacts the sphere $\chi_0(T)$ at the point Q, at which the light ray from O to P meets the sphere $\chi_0(T)$. All of these spheres were called *light waves* by HUYGENS, and he deduced two different consequences from the construction above.

First, when one fixes T and lets the point P describe the surface t, the light wave $\chi_0(T)$ will appear to be the envelope of light waves $\chi_P(T)$ that were generated by the light excitations at the various points of the surface τ.

Second, if one fixes the point P and lets T vary then the contact points Q, Q', …, of the light waves $\chi_P(T)$, $\chi_P(T')$, … with their current envelopes $\chi_0(T)$, $\chi_0(T')$, …will describe the light ray that go from O to P.

Finally, we remark that the length of an arbitrary curve γ that links concentric spheres $\chi_0(T)$ and $\chi_0(T')$ can never be less that the length of a light ray QQ' that links the same two spheres together.

7. The family of spheres $\chi_0(T)$ can be represented by an equation of the form:

$$S(t, x_1, x_2) = T. \tag{7.1}$$

The two-parameter family of light rays through O are solutions of a system of differential equations:

$$\dot{x}_i = y_i(t, x_j) \qquad (\mathrm{i, j} = 1, 2), \tag{7.2}$$

through which, the direction of the light ray is expressed as a function of position and direction. Then, from (7.1), the time that the light needs in order to traverse an arbitrary piece of this light ray will be equal to the difference between the two values of S at its endpoints; it can be represented by the curve integral:

$$\int_{t'}^{t''} dS = \int_{t'}^{t''} (S_i + \psi_i S_{x_i})\, dt\,. \tag{7.3}$$

From (3.3) and (3.5), however, this time can also be expressed by the integral:

$$\int_{t'}^{t''} L(t, x_j, \psi_j)\, dt\,, \tag{7.4}$$

and the two integrals (7.3) and (7.4) are equal to each other for all possible pairs of values (t', t'') and all light rays if and only if the identity exists:

$$L(t, x_i, y_j) - S_i - \psi_i S_{x_i} = 0. \tag{7.5}$$

From the remark at the end of § 6, one sees with the help of an entirely similar argument that one must consistently have for an arbitrary line element t, x_i, $\dot{x}_i$:

$$L(t, x_j, \dot{x}_j) - S_i - \dot{x}_i S_{x_i} \geq 0. \tag{7.6}$$

8. Generalizations. The last result can be generalized in various ways. First, we can replace the spherical waves $\chi_0(T)$ with any other light waves. One now obtains the general light waves that occur in an isotropic and homogeneous medium by the following construction: We assume that we are given an arbitrary surface t and set:

$$T_P = s(P), \tag{8.1}$$

in place of (6.2), in which $s(P)$ means an arbitrary continuous function. From this, we determine the wave surfaces $\chi(T)$, which are no longer spheres, from the envelope of the spheres $\chi_P(T)$ whose centers lie on τ and which possess the radius:

$$\frac{c}{n}(T - s(P))\,.$$

Second, we can liberate ourselves from the assumption that the medium is homogeneous and isotropic. HUYGENS himself had considered inhomogeneous media in his book, as well, when he treated the air refraction of the Earth atmosphere, aand also an anisotropic medium, namely, calcite (Iceland spar), and the birefringence that is produced by it.

In order to obtain these generalizations, we will, however, set out on a new path: Namely, we will seek to present the most general system of functions S, ψ_1, ψ_2 for which the relations (7.5) and (7.6) are true for an arbitrarily given function $L(t, x_i, \dot{x}_i)$. The solutions of the differential equations (7.2) are then light rays, by means of FERMAT's principle, and this shows that one will obtain all possible light rays and possible families of wave surfaces in this way.

Chapter II.

The foundations of geometrical optics

9. The fundamental equations. The question that was just posed should now be treated for the case in which the function $L(t, x_i, \dot{x}_i)$ in § 3 is differentiable arbitrarily many times. The treatment of discontinuity surfaces, on which refraction or reflection of light occurs, will thus not be a problem for us (§ 23). We must then determine the functions S, ψ_1, ψ_2 in such a way that the relations (7.5) and (7.6) are simultaneously true. The expression on the left-hand side of (7.6) must then possess a minimum when one takes $\dot{x}_j = \psi_j$. As a result, the first derivatives of this expression with respect to the $\dot{x}_j$ must vanish for $\dot{x}_j = \psi_j$, and one will obtain the equations:

$$S_{x_i} = L_{\dot{x}_i}(t, x_j, \psi_j) \qquad (i, j = 1, 2). \tag{9.1}$$

With these values for S_{x_i}, in place of (7.5), one can write down the equation:

$$S_t = L(t, x_j, \psi_j) - \psi_i\, L_{\dot{x}_i}(t, x_j, \psi_j)\,. \tag{9.2}$$

If one substitutes these values in the left-hand side of (7.6) then one will obtain a function:

$$E(t, x_i, \psi_i, \dot{x}_i) = L(t, x_j, \dot{x}_j) - L(t, x_j, \psi_j) - (\dot{x}_i - \psi_i) L_{\dot{x}_i}(t, x_j, \psi_j)\,, \tag{9.3}$$

and one can easily show that for all of the special functions L that occur in optics it is *never negative* and vanishes only when the equations $\dot{x}_i = \psi_i$ are true [39].

With the help of equations (9.1) and (9.2), one can write:

$$L(t, x_j, \dot{x}_j) = S_t + S_{x_i} \dot{x}_i + E(t, x_j, \psi_j, \dot{x}_j), \tag{9.4}$$

in place of (9.3), and obtain this identity by integration along an arbitrary curve γ from t' to t'':

$$\int_\gamma L(t, x_i, \dot{x}_i)\, dt = S'' - S' + \int_\gamma E(t, x_i, \psi_j, \dot{x}_i)\, dt\,. \tag{9.5}$$

When one observes that $E \geq 0$ and vanishes only for line elements $(t, x_i, \dot{x}_i)$ that lie on a curve of the family that arises by integration of the differential equations $\dot{x}_i = \psi_i$, one will see that the curves of this family must represent light rays, by FERMAT's principle.

([39]) This is connected with the fact that the so-called "ray surfaces of optics" are convex. For each problem in optics, however, the ray surface is nothing but the indicatrix (or the metric) of the corresponding problem in the calculus of variations. Cf., *Variationsrechnung*, § 225.

The problem of geometric optics accordingly comes down to that of determining functions S, ψ_1, and ψ_2, for which the "fundamental equations" (9.1) *and* (9.2) *are true.*

10. Calculation of the HAMILTONIAN function. When one calculates ψ_j as functions of t, x_i, from equations (9.1) and substitutes these values in (9.2), one will obtain a first-order partial differential equation for the function S. This elimination is particularly simple when one has determined the HAMILTONIAN function H from the LAGRANGIAN function L that it is associated with ([40]).

To that end, we introduce new variables y_i, which we will call *canonical direction coordinates*, and which will play the same role in optics as the impulse coordinates do in mechanics. These quantities will be defined by the defined by the two equations:

$$y_i = L_{\dot{x}_i}(t, x_j, \dot{x}_j) \qquad (i, j = 1, 2), \tag{10.1}$$

which, when solved for $\dot{x}_j$:

$$\dot{x}_j = \varphi_j(t, x_i, y_i) \qquad (i, j = 1, 2). \tag{10.2}$$

With these functions, one defines:

$$H(t, x_i, y_i) = - L(t, x_j, \varphi_j) + y_1\varphi_1 + y_2\varphi_2 , \tag{10.3}$$

and obtains the HAMILTONIAN function H, which is then, in other words, the LEGENDRE transform of L. By partial differentiation of (10.3) with respect to t, x_i, y_i one obtains, in that sequence, the identities:

$$H_t = - L_t(t, x_j, \varphi_j), \qquad H_{x_i} = - L_{x_i}(t, x_j, \varphi_j), \qquad (i, j = 1, 2) \tag{10.4}$$

$$H_{y_i} = \varphi_i(t, x_j, y_j) = \dot{x}_i . \tag{10.5}$$

11. For **isotropic media**, one has:

$$L = n(t, x_j)\sqrt{1 + \dot{x}_1^2 + \dot{x}_2^2} , \tag{11.1}$$

and one has:

$$y_i = \frac{n\,\dot{x}_j}{\sqrt{1 + \dot{x}_1^2 + \dot{x}_2^2}}, \qquad n^2 - y_1^2 - y_2^2 = \frac{n^2}{1 + \dot{x}_1^2 + \dot{x}_2^2}, \tag{11.2}$$

$$\varphi_j = \frac{y_i}{\sqrt{n^2 - y_1^2 - y_2^2}}, \qquad H = -\sqrt{n^2 - y_1^2 - y_2^2} . \tag{11.3}$$

It is very easy to verify equations (10.4) and (10.5) in this special case directly.

12. Derivation of the differential equations for light rays. A comparison of equations (9.1) and (9.2) with (10.1) and (10.3) yields the equations:

[40] *Variationsrechnung*, § 235.

$$y_i = S_{x_i}, \qquad S_t + H(t, x_j, y_j) = 0, \tag{12.1}$$

from which, it next follows that the function S, by which the wave surfaces were determined, must always satisfy the partial differential equation:

$$S_t + H(t, x_1, x_2, S_{x_1}, S_{x_2}) = 0. \tag{12.2}$$

From (9.1), one obtains, in addition, when one observes the formulas of § 10, $\psi_i = H_{y_i}(t, x_j, S_{x_j})$, and the light rays that run through the system of wave surfaces, are then solutions of the ordinary differential equations:

$$\dot{x}_i = H_{y_i}(t, x_j, S_{x_j}) \qquad (i = 1, 2). \tag{12.3}$$

13. We now assume that we have ascertained a solution $S(t, x_1, x_2)$ to the partial differential equation (12.2) in *any* way, computed its derivatives S_{x_j}, and substituted them in (12.3). The general integral of the system of differential equations (12.3) that we obtained in this way will then be represented by equations of the form:

$$x_i = \xi_i(t, u_k) \qquad (i, k = 1, 2), \tag{13.1}$$

in which the u_k mean integration constants that may choose arbitrarily. We now introduce the new functions:

$$\sigma(t, u_k) = S(t, \xi_j(t, u_k)), \qquad \eta_i(t, u_k) = S_{x_i}(t, \xi_j(t, u_k)). \tag{13.2}$$

Identities exist between the functions (13.1) and (13.2), which we would now like to present.

First, one can compute the total derivative of σ and obtain:

$$d\sigma = S_t\, dt + S_{x_j}\, d\xi_j ,$$

or, when one considers (12.2), (13.1), and (13.2):

$$d\sigma = -H(t, \xi_j, \eta_j) + \eta_j\, d\xi_i . \tag{13.3}$$

Second, one expresses the fact that the ξ_i are solutions of (12.1) with the equations:

$$\frac{\partial \xi_i}{\partial t} = H_{y_i}(t, \xi_j, \eta_j) \qquad (i = 1, 2). \tag{13.4}$$

Third, when one partially differentiates the second equation (13.2) with respect to t, one will obtain:

$$\frac{\partial \eta_i}{\partial t} = S_{t x_i} + S_{x_i x_j} \frac{\partial \xi_j}{\partial t} = S_{t x_i} + S_{x_i x_j} H_{y_j} . \tag{13.5}$$

On the other hand, it follows from (12.2) by partial differentiation with respect to x_i that:

$$S_{t x_i} + S_{x_i x_j} H_{y_j}(t, x_k, S_{x_k}) = - H_{x_i}(t, x_k, S_{x_k}) ;$$

when one substitutes the functions ξ_k in this for the x_k the left-hand side will become identical with the right-hand side of (13.5), and one will therefore have the equation:

$$\frac{\partial \eta_i}{\partial t} = -H_{x_i}(t, \xi_j, \eta_j) . \tag{13.6}$$

We obtain a final relationship between the ξ_i and the η_i when we partially differentiate the second equation in (13.2) with respect to u_2 and write:

$$\frac{\partial \eta_i}{\partial u_2} = S_{x_i x_j} \frac{\partial \xi_j}{\partial u_2} .$$

We multiply both sides of this equations by $\partial \xi_i / \partial u_1$, sum over i, and obtain:

$$\frac{\partial \xi_i}{\partial u_1}\frac{\partial \eta_i}{\partial u_2} = S_{x_i x_j} \frac{\partial \xi_i}{\partial u_1}\frac{\partial \xi_j}{\partial u_2} . \tag{13.7}$$

One now remarks that the right-hand side of this equation will remain unchanged when one switches i with j and simultaneously u_1 with u_2 .

If one then introduces the symbol:

$$[u_1, u_2] = \frac{\partial \xi_i}{\partial u_1}\frac{\partial \eta_i}{\partial u_2} - \frac{\partial \xi_i}{\partial u_2}\frac{\partial \eta_i}{\partial u_1} = \frac{\partial(\xi_1, \eta_1)}{\partial(u_1, u_2)} + \frac{\partial(\xi_2, \eta_2)}{\partial(u_1, u_2)} \tag{13.8}$$

then the relation follows from the latter remark:

$$[u_1, u_2] = 0. \tag{13.9}$$

LAGRANGE (1736-1813) introduced the expression (13.8) when he developed the method of variation of constants in celestial mechanics ([41]), and the symbol also originated with him. For that reason, one calls $[u_1, u_2]$ a *LAGRANGE bracket.*

14. Equations (13.4) and (13.6) state that the functions ξ_i, η_i must be solutions of the system of ordinary differential equations:

([41]) LAGRANGE, J. L.: "Mémoire sur la théorie générale de la variation des constants arbitraries dans tous les problèmes de la mécanique," (1808), *Oeuvres*, t. 6, pp. 771-805.

$$\dot{x}_i = H_{y_i}(t, x_j, y_j), \qquad \dot{y}_i = -H_{x_i}(t, x_j, y_j), \tag{14.1}$$

which one calls the *canonical equations*. From § 10, the first of these equations is equivalent to (10.1); with the help of (10.4), one then sees that the second equation can be described as:

$$\frac{d}{dt} L_{\dot{x}_i} = L_{x_i} \qquad (i = 1, 2). \tag{14.2}$$

These are the *EULER equations* of the variational problem with the basis function L. We see that the light rays that we have introduced as solution of the differential equations (12.3) must necessarily also be solutions of this system of second-order differential equations. However, for our later purposes, it will be much more convenient to start with the canonical differential equations (14.1), which are indeed equivalent to the EULER differential equations.

The functions $\xi_i(t, u_j)$, $\eta_i(t, u_j)$ that belong to the light rays, which arise for a particular light propagation as a result of HUYGENS's principle, must still satisfy the condition (13.9). The two-parameter ray manifold or, as one also says, the *ray congruence:*

$$x_i = \xi_i(t, u_1, u_2)$$

is not arbitrary then. However, before we deduce consequences from the condition (13.9), we must examine certain properties of general ray manifolds.

15. Peculiarities of the solutions of the canonical equations. We let:

$$x_i = \xi_i(t, u_\alpha), \quad y_i = \eta_i(t, u_\alpha), \qquad (i = 1, 2;\ \alpha = 1, 2, \dots m;\ 2 \le m \le 4) \tag{15.1}$$

denote a solution of the canonical differential equations that depends upon *arbitrarily many* integration constants u_α. It is now no longer generally possible to find a function $\sigma(t, u_\alpha)$ for which equation (13.3) is verified, when one substitutes the function (15.1) in its right-hand side.

However, if one restricts the relation that follows from (13.3):

$$\frac{\partial \sigma}{\partial t} = -H(t, \xi_j, \eta_j) + \eta_i \frac{\partial \xi_i}{\partial t} \tag{15.2}$$

then it will always be possible to determine functions $\omega(t, u_\alpha)$ by a quadrature that satisfy the condition:

$$\frac{\partial \omega}{\partial t} = -H(t, \xi_j, \eta_j) + \eta_i \frac{\partial \xi_i}{\partial t}. \tag{15.3}$$

The association of these functions $\omega(t, u_\alpha)$, which are defined only up to an arbitrary additive function, moreover, with the solution (15.1) is fundamental to the entire theory ([42]).

In order to obtain the relation enters in place of (13.3), we calculate the total differential $d\omega$ and convert it. One first obtains from (15.3):

$$d\omega = \frac{\partial\omega}{\partial t}dt + \frac{\partial\omega}{\partial u_\alpha}du_\alpha = -H\,dt + \eta_i\frac{\partial\xi_i}{\partial t}dt + \frac{\partial\omega}{\partial u_\alpha}du_\alpha . \tag{15.4}$$

If one multiplies both sides of the equation:

$$\frac{\partial\xi_i}{\partial t}dt + \frac{\partial\xi_i}{\partial u_\alpha}du_\alpha = d\xi_i$$

by ξ_i , sums over i, and adds the result to (15.4) term-by-term then that will yield the relation:

$$d\omega = -H\,dt + \eta_i\,d\xi_i - \lambda_\alpha\,du_\alpha , \tag{15.5}$$

in which we have set:

$$\lambda_\alpha = -\frac{\partial\omega}{\partial t}dt + \eta_i\frac{\partial\xi_i}{\partial u_\alpha}du_\alpha . \tag{15.6}$$

16. The most important fact in our theory now consists of the knowledge that the functions λ_α no longer depend upon t; i.e., that the quantities $\partial\lambda_\alpha / \partial t$ vanish identically. In fact, one has:

$$\frac{\partial\lambda_\alpha}{\partial t} = -\frac{\partial^2\omega}{\partial t\,\partial u_\alpha} + \frac{\partial\eta_i}{\partial t}\frac{\partial\xi_i}{\partial u_\alpha} + \eta_i\frac{\partial^2\xi_i}{\partial t\,\partial u_\alpha} . \tag{16.1}$$

On the other hand, it follows by differentiating (15.3) with respect to u_α that:

$$\frac{\partial^2\omega}{\partial t\,\partial u_\alpha} = H_{x_i}\frac{\partial\xi_i}{\partial u_\alpha} - H_{y_i}\frac{\partial\eta_i}{\partial u_\alpha} + \frac{\partial\eta_i}{\partial u_\alpha}\frac{\partial\xi_i}{\partial t} + \eta_i\frac{\partial^2\xi_i}{\partial t\,\partial u_\alpha} . \tag{16.2}$$

Now, since ξ_i, η_i are solutions of the canonical differential equations (14.1), one sets:

([42]) The presentation in this text leans very essentially on CAUCHY (cf., § 16, footnote 43). One can, however, ponder the fact that our function $\omega(t, u_\alpha)$ has the greatest affinity with with HAMILTON's characteristic functions (see, Introduction). If one introduces, e.g., the quantities $x_i' = \xi_i(t', u_\alpha)$ and $x_i = \xi_i(t, u_\alpha)$ in the characteristic function $V(t', x_i', t, x_j)$, and then replaces the quantity t' with $t' = \varphi(t, u_\alpha)$ in the result of the substitution, where φ means an arbitrary function, then one will obtain a solution of equation (15.3).

$$- H_{x_i} = \frac{\partial \eta_i}{\partial t}, \qquad - H_{y_i} = - \frac{\partial \xi_i}{\partial t} \tag{16.3}$$

in (16.2) and obtains:

$$\frac{\partial^2 \omega}{\partial t\,\partial u_\alpha} = \frac{\partial \eta_i}{\partial t}\frac{\partial \xi_i}{\partial u_\alpha} + \eta_i \frac{\partial^2 \xi_i}{\partial t\,\partial u_\alpha}, \tag{16.4}$$

from which, it will follow that the right-hand side of (16.1) vanishes.

A remarkable relationship exists between the quantities λ_α and the LAGRANGE brackets of § 13. Namely, if we differentiate equation (15.6) with respect to u_β then we will obtain:

$$\frac{\partial \lambda_\alpha}{\partial u_\beta} = \left[- \frac{\partial^2 \omega}{\partial u_\alpha\,\partial u_\beta} + \eta_i \frac{\partial^2 \xi_i}{\partial u_\alpha\,\partial u_\beta} \right] + \frac{\partial \xi_i}{\partial u_\alpha}\frac{\partial \eta_i}{\partial u_\beta}.$$

The bracketed part is symmetric in α and β and comparison with (13.3) yields:

$$\frac{\partial \lambda_\alpha}{\partial u_\beta} - \frac{\partial \lambda_\beta}{\partial u_\alpha} = [u_\alpha, u_\beta]. \tag{16.5}$$

The LAGRANGE brackets $[u_\alpha, u_\beta]$ are likewise independent of t. This result was already obtained by LAGRANGE in the year 1808; the quantities λ_α, or at least equivalent functions, were first employed by CAUCHY (1789-1857) for his theory of characteristics ([43]).

17. Refining the formulas with the help of the initial values. If the functions (15.1) are known then the function ω will be defined by (15.3) only up to an additive constant that depends upon the parameters u_α arbitrarily. As a result, the λ_α will also not be defined uniquely, and one can normalize the right-hand side of (15.5) in various ways.

One obtains a very important normalization of that sort when one is given the initial values by which the solutions (15.1) are established uniquely.

We assume that for $t = \tau(u_\alpha)$ the following equations are true:

$$\xi_i(\tau(u_\alpha), u_\alpha) = A_i(u_\alpha), \qquad \eta_i(\tau(u_\alpha), u_\alpha) = B_i(u_\alpha). \tag{17.1}$$

If we then substitute $t = \tau(u_\alpha)$ in (15.5) then it will follow from the relation $\omega(\tau(u_\alpha), u_\alpha) = \omega_0(u_\alpha)$ that:

$$- H(\tau, A_j, B_j)\, d\tau + B_i\, dA_i = d\omega_0 + \lambda_\alpha\, du_\alpha . \tag{17.2}$$

Here, we have used the fact that λ_α does not depend upon t. We now introduce the relation:

$$\Omega(t, u_\alpha) = \omega(t, u_\alpha) - \omega_0(u_\alpha), \tag{17.3}$$

([43]) Bulletin des Sciences par la Société Philomatique de Paris (1819), 10-21. This important treatise is still not included in the volumes of the *Oeuvres Complètes* of CAUCHY that have appeared up to now.

and obtain, when we subtract (17.2) from (15.5) term-by-term:

$$d\Omega = -H(\tau, \xi_j, \eta_j)\, d\tau + \eta_i\, d\xi_i - (-H(\tau, A_j, B_j)\, d\tau + B_i\, dA_i). \tag{17.4}$$

Clearly, this relation is only a special form of equation (15.5). The function Ω is the solution of the differential equation (15.3) for which one has:

$$\Omega(\tau(u_\beta), u_\alpha) = 0; \tag{17.5}$$

here, we must write:

$$\lambda_\alpha = -H(\tau, A_j, B_j)\frac{\partial\tau}{\partial u_\alpha} + B_i\frac{\partial A_i}{\partial u_\alpha} \tag{17.6}$$

for the λ_α. If one calculates the LAGRANGE brackets $[u_\alpha, u_\beta]$ from equation (16.5) then one will find, when one then employs the notations:

$$H_{x_i}(\tau, A_j, B_j) = H_{x_i}^0, \qquad H_{y_i}(\tau, A_j, B_j) = H_{y_i}^0, \tag{17.7}$$

that:

$$[u_\alpha, u_\beta] = \sum_{i=1}^{2} -H_{x_i}^0\frac{\partial(\tau, A_i)}{\partial(u_\alpha, u_\beta)} + H_{y_i}^0\frac{\partial(B_i, \tau)}{\partial(u_\alpha, u_\beta)} + \frac{\partial(A_i, B_i)}{\partial(u_\alpha, u_\beta)}. \tag{17.8}$$

18. Determination of the wave surfaces for given initial values. It is now very simple to respond in a completely general way to the question that we treated in § 8 for homogeneous, isotropic media by a geometric construction. One deals with the problem of finding a solution to the partial differential equations (12.2) that, on the surface:

$$t = \tau(u_1, u_2), \qquad x_i = A_i(u_1, u_2) \qquad (i = 1, 2), \tag{18.1}$$

assumes the initial values:

$$S(\tau(u_j), A_i(u_j)) = s(u_1, u_2). \tag{18.2}$$

We must next determine the initial values $B_i(u_j)$ of the functions η_i in § 13. For that, we remark that, from (13.3) and (18.2), one must have, in any case:

$$\frac{\partial s}{\partial u_i} = -H(\tau, A_j, B_j)\frac{\partial\tau}{\partial u_\alpha} + B_i\frac{\partial A_i}{\partial u_\alpha}. \tag{18.3}$$

These are two equations, from which one can calculate (§ 19) the $B_i(u_j)$. We then integrate the canonical equations (14.1) with these initial values and determine the function $\Omega(\tau, u_1, u_2)$ by quadrature; Ω is uniquely determined due to the condition (17.5). From (18.3), it then follows that the bracket on the right-hand side of (17.4) must equal ds. If one then sets:

$$\sigma(t, u_1, u_2) = \Omega(\tau, u_1, u_2) + s(u_1, u_2) \tag{18.4}$$

then one will get the equation:

$$-H(t, \xi_j, \eta_j)\, dt + \eta_j\, d\xi_j = d\sigma, \tag{18.5}$$

which is identical with (13.3).

We calculate the u_j from the equations:

$$x_i = \xi_i(t, u_j) \qquad (i, j = 1, 2) \tag{18.6}$$

and get:

$$u_j = \chi_j(t, x_i). \tag{18.7}$$

We further set:

$$S(t, x_i) = \sigma(t, \chi_j(t, x_i)), \qquad Y_i = \eta_i(t, \chi_j(t, x_k)), \tag{18.8}$$

and obtain from (18.5):

$$-H(t, x_j, Y_j)\,dt + Y_j\,dx_j = dS. \tag{18.9}$$

This relation shows that S satisfies the partial differential equation (12.2), so one has:

$$S_{x_i} = Y_i, \qquad S_i = -H(t, x_j, Y_j). \tag{18.10}$$

Furthermore, S possesses the desired initial value. Namely, the identity $\chi_j(t, \xi_j(t, u_\kappa)) \equiv u_j$ follows from (18.6) and (18.7), so one can write:

$$\sigma(t, \xi_j(t, u_j)) = \sigma(t, u_i),$$

in place of the first equation (18.8). However, if one sets $t = \tau(u_j)$ in this equation then equation (18.2) will follows from (17.5), (17.1), along with (18.4), which will serve to verify it.

19. One obtains the condition for one to be able to calculate the B_j as single-valued functions of the parameters u_i from equations (18.3) when one writes down that the functional determinant of second order satisfies:

$$\left| -H_{y_i}^0 \frac{\partial \tau}{\partial u_j} + \frac{\partial A_i}{\partial u_j} \right| \neq 0. \tag{19.1}$$

However, from (17.1):

$$\frac{\partial A_i}{\partial u_j} = \frac{\partial \xi_i}{\partial t}\frac{\partial \tau}{\partial u_j} + \left.\frac{\partial \xi_i}{\partial u_j}\right|_{t=\tau},$$

such that the condition (19.1) is equivalent to the relation:

$$\left| \frac{\partial \xi_i}{\partial u_j} \right|_{t=\tau} \neq 0, \tag{19.2}$$

from which, the solubility of equations (18.6) for the u_j will follow. The two coincident relations (19.1) and (19.2) can also be written with the help of a three-rowed determinant in the form:

$$\left|\begin{array}{cc} 1 & \dfrac{\partial \xi_i}{\partial t} \\ \dfrac{\partial \tau}{\partial u_j} & \dfrac{\partial A_i}{\partial u_j} \end{array}\right|_{t=\tau} \neq 0. \tag{19.3}$$

This latter relation is very easy to interpret geometrically; it says that the light rays that run through the figure should not contact the surface (18.1).

20. If a single (in general, arbitrary) solution to the canonical differential equations (14.1) is given that runs through the surface (18.1), but does not contact it, then one can give functions $s(u_1, u_2)$ in infinitely many ways so that in the calculations of §18, this given solution will be included in the figure that we constructed there. *One concludes from this that any solution of this sort is a possible light ray for which FERMAT's principle is valid* ([44]).

21. Optical equidistance. Field-like structures. How can one geometrically interpret all of these formulas? In § 13, we had a family of wave surfaces $S(t, x_i) = \text{const.}$ and a two-parameter family of light rays $x_i = \xi_i(t, x_i)$ that ran through these wave surfaces. The normal to the wave surfaces had the direction of a vector with the components:

$$S_t,\ S_{x_1},\ S_{x_2}, \tag{21.1}$$

and since S must be a solution of the partial differential equation (12.2), the direction of the normal vectors was uniquely determined when one knew the S_{x_i}. The tangent to the light ray that ran through the wave surface had the direction of the vector:

$$1,\ \frac{\partial \xi_1}{\partial t},\ \frac{\partial \xi_2}{\partial t}, \tag{21.2}$$

and equations (13.4) were true for $\eta_i = S_{x_i}$. In any event, when this is the case for a surface and a ray, one will say that the wave surfaces intersect the light rays *transversely*.

If the medium is isotropic then it will follow from the formulas of § 11 that a surface intersects a light ray transversally when the vectors (21.1) and (21.2) have the same direction; i.e., when the surface is run through *orthogonally* by the ray.

From equation (9.5), the optical length along any light ray that is intersected transversally by wave surfaces at any of its points is equal to the difference of the values of S at its endpoints; E is then consistently equal to zero along such a ray. This optical length also remains constant when the endpoints slide along two fixed wave surfaces. For that reason, the surfaces of the family $S(t, x_i) = \text{const.}$ are called *optically equidistant.*

([44]) The proof of the converse conclusion, that any possible light ray – i.e., that any curve for which FERMAT's principle is valid – is a solution to the canonical equations is somewhat more complicated (Cf., *Variationsrechnung*, § 245).

If the medium is not only isotropic, but also homogeneous, then these surfaces will also be equidistant in the ordinary sense ([45]).

The LAGRANGE bracket (13.9) is identically zero for the two-parameter ray manifold (13.1). Any ray manifold for which this is the case shall be called a *field-like* manifold. In the neighborhood of a point at which the relation (19.2) is true, a given field-like manifold will simply cover the space (t, x_1, x_2) with light rays. In addition, it will follow from $[u_1, u_2] = 0$ that the expression $\lambda_1\, du_1 + \lambda_2\, du_2$ in (17.2) is a complete differential. As a result, one can determine the solutions of the $S(t, x_i)$ of the partial differential equation (12.2) that intersect the rays of our manifold transversely by using the methods of § 19. In this case, one says that the ray manifold defines a *field.*

If one examines the values of the functional determinant:

$$\frac{\partial(\xi_1, \xi_2)}{\partial(u_1, u_2)}$$

along a particular ray of a field-like manifold then the points at which this determinant vanishes will define the only exceptional places, in whose neighborhood the *field-like structure* cannot also be regarded as a *field.*

One can prove that these exceptional places are *isolated* along any individual ray. Therefore, I would not like to go into this question here, since I have recently treated this topic quite thoroughly ([46]).

Among the field-like structures, we must point out the ones that consist of all light rays that go through a fixed point t^0, x_i^0. These ray manifolds will be called *stigmatic* ([47]), or also *distinguished field-like manifolds*. The fact that these manifolds are field-like follows immediately from the fact that the derivatives $\partial\xi_i / \partial u_j$ must vanish for $t = t_0$, and as a result, one must have $[u_1, u_2] = 0$. Here, the light waves $S(t, x_j)$ = const. are precisely the "optical" spheres that HUYGENS had employed (§ 6).

One obtains a further important class of field-like structures when one takes the functions τ, B_1, B_2 to be constant in the formulas of § 17. Namely, in homogeneous media the wave surfaces will be planar and the light will consist of parallel light rays.

The main result to which we will be led consists in an inversion of the result in § 13: *From HUYGENS's principle, any field-like congruence of light rays represents possible paths of propagation for light.*

22. Introduction of arbitrary curvilinear coordinates. For many problems, it is practical to employ curvilinear coordinates that are defined by the equations:

$$t = t(t', x'_j), \qquad x_i = x_i(t', x'_j) \qquad\qquad (i, j = 1, 2). \tag{22.1}$$

([45]) The optical equidistance of the wave surfaces corresponds to the "geodetic equidistance" that one encounters in the calculus of variation (cf., *Variationsrechnung*, § 298).

([46]) *Variationsrechnung*, §§ 313-327.

([47]) In place of the word *stigmatic*, one often, especially in older papers on optics, finds the term *anastigmatic* used, which includes a superfluous double negation.

It is not difficult to calculate the new LAGRANGIAN function $L'(t', x'_i, dx'_i / dt')$ directly. However, the computations becomes simpler when one first exhibits the transformed HAMILTONIAN function $H'(t', x'_i, y'_i)$. Namely, one needs only to employ the fact that by substituting the functions (22.1) in a solution $S(t, x_j)$ of the HAMILTON-JACOBI partial differential equation (12.2) must necessarily be a solution $S'(t', x'_j)$ to the transformed partial differential equation. By means of equations (22.1), one then has $dS = dS'$ and therefore also:

$$- H' \, dt + y'_j \, dx'_j = - H \, dt + y_i \, dx_i \,. \tag{22.2}$$

This latter equation is equivalent to the following system:

$$y'_j = - H(t, x_k, y_k) \frac{\partial t}{\partial x'_j} + y_i \frac{\partial x_i}{\partial x'_j} \qquad (j = 1, 2), \tag{22.3}$$

$$H' = H(t, x_k, y_k) \frac{\partial t}{\partial t'} - y_i \frac{\partial x_i}{\partial t'} \,. \tag{22.4}$$

One then obtains H' when one first computes the y_i as functions of (t', x'_j, y'_j) using (22.3) and substitutes these values in (22.4).

From the equation (22.2), one can, in addition, read off an important property of the canonical direction coordinates. Namely, if one considers that the differentials:

$$dt, \; dx_1, \; dx_2$$

can be interpreted as the components of a *contravariant vector* then it will follow from (22.2) that the three quantities:

$$- H(t, x_j, y_j), \quad y_1, \quad y_2$$

transform like the components of a *covariant vector* ([48]).

23. Derivation of the general law of refraction. We now assume that two different media are separated by a discontinuity surface:

$$\mathfrak{D}: t = \tau(u_j), \quad x_i = A_i(u_j) \qquad (i, j = 1, 2). \tag{23.1}$$

For the one side of the surface $\mathfrak{D}$ – e.g., for the points:

$$t < \tau(u_j), \qquad x_i = A_i(u_j), \tag{23.3}$$

the LAGRANGIAN and HAMILTONIAN functions will be denoted by $L(t, x_j, \dot{x}_j)$ [$H(t, x_j, y_j)$, resp.], as before. In the second medium, the same functions will be denoted by $L'(t, x_j, \dot{x}_j)$ [$H'(t, x_j, y'_j)$, resp.].

([48]) *Variationsrechnung*, § 83.

We now consider an arbitrary propagation of light through this combined system that is generated by a family of wave surfaces. At the various points of the discontinuity surface $\mathfrak{D}$ this light excitation will be noticeable at a time that one can establish with the aid of a function $s(u_1, u_2)$.

However, from HUYGENS's principle the propagation of light in the two media will be uniquely determined by the function $s(u_1, u_2)$. The associated light rays will be refracted at the discontinuity surface $\mathfrak{D}$ during this propagation. From (18.3), the equations:

$$\frac{\partial s}{\partial u_i} = - H(\tau, A_j, B_j)\,\frac{\partial \tau}{\partial u_i} + B_k\,\frac{\partial A_k}{\partial u_i} \qquad (i = 1, 2) \tag{23.3}$$

between the derivatives of $s(u_j)$ and the functions τ, A_i, B_i must be fulfilled. One finds, in exactly the same way, that one also must have:

$$\frac{\partial s}{\partial u_i} = - H'(\tau, A_j, B'_j)\,\frac{\partial \tau}{\partial u_i} + B'_k\,\frac{\partial A_k}{\partial u_i}. \tag{23.4}$$

However, from these two equations, there follows the relation:

$$- [H'(\tau, A_j, B'_j) - H(\tau, A_j, B_j)]\,\frac{\partial \tau}{\partial u_i} + [B'_k - B_k]\,\frac{\partial A_k}{\partial u_i} = 0, \qquad (i = 1, 2) \tag{23.5}$$

in which the derivatives of the function $s(u_j)$ no longer appear.

The system of equations (23.5) represents the law of refraction for light rays at the discontinuity surface $\mathfrak{D}$. Namely, if two rays are drawn through a point τ, A_i of the surface $\mathfrak{D}$ that have the canonical direction coefficients B_i (B'_i, resp.) and fulfill equations (23.5) then one can give functions $s(u_j)$ in infinitely many ways for which the ray field that is defined by equations (23.3) and (23.4) contains these two prescribed rays.

Remark. It is very easy to establish that one also will be led to the same law of refraction (23.5) by FERMAT's principle. Namely, if e and e' are two light rays, each one of which belongs to the two fields of rays that were just considered, and one denotes the time it takes light to go from A to B along these rays by T, then one will get:

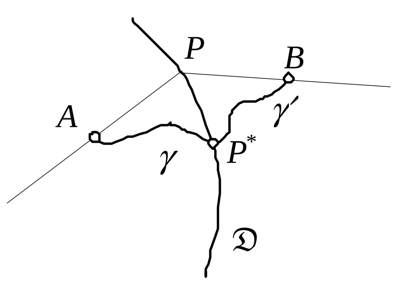

Figure 3.

$$T = [s(P) - S(A)] + [S'(B) - s(P)] = S'(B) - S(A).$$

For the time duration T^* along another path AP^*B, however, one will get, from (9.5):

$$\begin{aligned} T^* &= (s(P^*) - S(A)) + \int_\gamma E\,dt + (S'(B) - s(P^*)) + \int_{\gamma'} E'\,dt \\ &= T + \int_\gamma E\,dt + \int_{\gamma'} E'\,dt\,. \end{aligned}$$

Now, since the functions E and E' are always ≥ 0, it will follow from this that $T^* \geq T$, such that the ray APB must be a light ray, from the definition of § 4.

One can complete this result by saying that one then shows that FERMAT's principle is no longer true when one extends the ray AP in another direction from the one that e' possesses at the point P. We would therefore like to go into this detail.

24. Consequences of the law of refraction. Clearly, the law of refraction must be independent of the choice of coordinates. This property can be verified effortlessly with the aid of our formulas: namely, from the ones in § 22, the numbers:

$$-[H'(t, A_j, B'_j) - H(t, A_j, B_j)], \qquad [B'_j - B_j] \tag{24.1}$$

are components of a covariant vector, and equations (23.5) state simply that this covariant vector should be orthogonal to each of the two contravariant vectors:

$$\frac{\partial \tau}{\partial u_j}, \quad \frac{\partial A_i}{\partial u_j} \quad (j = 1, 2). \tag{24.2}$$

This is, however, a condition that remains invariant under any change of coordinate system.

For the special case that is based upon rectangular Cartesian coordinates, the covariant vectors cannot be distinguished from the contravariant ones. The condition above simply states that the vector (24.1) should always be perpendicular to the discontinuity surface, and one obtains the ordinary law of refraction in the case of isotropic media ([49]).

25. We now consider an arbitrary ray manifold in the first medium that depends upon two, three, or four parameters u_α, and whose rays run through the discontinuity surface (23.1). One can characterize each individual ray of this manifold by the line element of the ray that one finds at the point at which the ray pierces the discontinuity surface. The first two parameters u_1, u_2 of the u_α, which we will denote by Latin indices when they are to be considered separately from the other ones, can then be employed as position coordinates on the discontinuity surface. As a consequence, the ray manifold itself can always be represented by solutions (15.1) of the canonical differential equations that are established by the initial conditions:

$$\xi_i(\tau(u_j), u_\alpha) = A_i(u_j), \qquad \eta_j(\tau(u_j), u_\alpha) = B_i(u_\alpha), \tag{25.1}$$

in which the $\tau(u_j)$, $A_i(u_j)$ have the same meaning as in (23.1). From (17.2), one can then write:

$$-H(\tau, A_j, B_j)\, d\tau + B_i\, dA_i = d\omega_0 + \lambda_\alpha\, du_\alpha\,, \tag{25.2}$$

([49]) One observes that this result leads to a construction that agrees precisely with the rule of DESCARTES in § 1 when one meanwhile sets the index of refraction proportional to the velocity of light, as the emission theory of light would suggest (cf., § 2, footnote 31).

just after one calculates with a function $\omega(t, u_\alpha)$ and has determined the functions λ_α and ω_0 with its help. If one associates each ray of the manifold in question with the broken ray that arises from its advance into the second medium then one will obtain (with entirely similar notations) the equation:

$$-H'(\tau, A_j, B'_j)\,d\tau + B'_i\,dA_i = d\omega'_0 + \lambda'_\alpha\,du_\alpha . \tag{25.3}$$

Now, it follows from the law of refraction (23.5) that the left-hand sides of the last two equations must always be equal to each other. It follows from this, when one introduces the notation:

$$\Psi(u_\alpha) = \omega_0(u_\alpha) - \omega'_0(u_\alpha), \tag{25.4}$$

that:

$$\lambda'_\alpha\,du_\alpha = d\Psi + \lambda_\alpha\,du_\alpha . \tag{25.5}$$

For the derivation of this latter relation, we have chosen the parameters u_α very specially. However, this equation will remain correct for any arbitrary choice of the parameters u_α. Namely, if one introduces new parameters v_β through the equations:

$$u_\alpha = u_\alpha(v_\beta), \tag{25.6}$$

for which new functions $\mu_\alpha(v_\beta)$, $\mu'_\alpha(v_\beta)$ appear in place of the functions $\lambda_\alpha(u_\beta)$, $\lambda'_\alpha(u_\beta)$, then one will always have:

$$\lambda_\alpha\,du_\alpha = \mu_\alpha\,dv_\alpha + dM, \qquad \lambda'_\alpha\,du_\alpha = \mu'_\alpha\,dv_\alpha + dM', \tag{25.7}$$

such that the relation (25.5) will always preserve the same form under any arbitrary choice of parameter.

Equation (25.5) is equivalent to the system:

$$\lambda'_\alpha = \lambda_\alpha + \frac{\partial\Psi}{\partial u_\alpha} \qquad (\alpha = 1, 2, \ldots). \tag{25.8}$$

It then follows from (16.5) *that the* LAGRANGE *brackets* $[u_\alpha, u_\beta]$ *remain unchanged under arbitrary refraction of the light rays. They represent differential invariants whose values do not change under the passage of light through any instrument along the entire light ray.*

Incidentally, one remarks that, from § 16, one can always normalize the functions λ'_α such that equations (25.8) will be replaced with $\lambda'_\alpha = \lambda_\alpha$. One can then always choose the notations such that the λ_α themselves remain constant along each ray.

26. Integral invariants. MALUS's theorem. For the applications that we will make of the invariance of the brackets $[u_\alpha, u_\beta]$, it would be very advantageous that this theorem remain true for any choice of the parameters u_α, since one could then make the most convenient choice of parameter in each special case. On the other hand, this theorem has no immediate geometric significance, since one can describe each ray manifold in infinitely many different ways with the help of parameters u_α and then obtain different values of $[u_\alpha, u_\beta]$ each time.

However, we arrive at any theorem that is geometrically meaningful in the following way: For a two-dimensional manifold that is represented with the help of the parameter u_1, u_2 one has, from (13.8):

$$[u_1, u_2] = \frac{\partial(\xi_1, \eta_1)}{\partial(u_1, u_2)} + \frac{\partial(\xi_2, \eta_2)}{\partial(u_1, u_2)}. \tag{26.1}$$

If we now introduce new parameters (v_1, v_2) by equations of the form (25.6) then, as must follow from this, for each ray of the manifold in question, the relation:

$$[v_1, v_2] = [u_1, u_2]\,\frac{\partial(u_1, u_1)}{\partial(v_1, v_2)} \tag{26.2}$$

must exist between the old LAGRANGE brackets and the transformed ones. If G_u and G_v are two regions in the $u_1\,u_2$-plane (v_1v_2-plane, resp.) that go to each other by means of the transformation (25.6) then one will have:

$$\iint_{G_u} [u_1, u_2]\, du_1 du_2 = \iint_{G_v} [v_1, v_2]\, dv_1 dv_2\,. \tag{26.3}$$

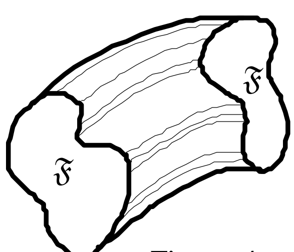

Figure 4.

The value of the double integral (26.3) is thus independent of the choice of parameter.

If one interprets the double integral as an integral over a surface patch 𝔉 that a pencil of rays runs through (Fig. 4) then the value of the integral depends upon only the pencil, but not on the location or form of the surface patch over which one integrates. For that reason, the integral will be called an *integral invariant.*

27. If the ray manifold is field-like then $[u_1, u_2] = 0$, and the integral invariant (26.3) will vanish identically. Conversely, if the integral invariant vanishes for all possible regions G_u then one must have $[u_1, u_2] = 0$ and the ray manifold will be field-like. This result includes the theorem that MALUS expressed in the year 1808, with the generalizations that DUPIN and QUETELET made of it later on ([50]), which states that when one cuts through an arbitrary instrument with a two-dimensional ray manifold then the ray manifold in the object space will be field-like if and only if the ray manifold in the image space has the same property.

In the past, MALUS's theorem was very strictly observed. It seemed that it was even believed that the optical ray map could be characterized by this theorem alone. Naturally,

([50]) Cf., the Introduction. It is noteworthy that the treatise of LAGRANGE appeared in the same year 1808 (cf., § 13, footnote 41), which already essentially included the invariance of the brackets $[u_\alpha, u_\beta]$.

this is not the case, because the ray maps for which MALUS's theorem is true without restrictions are *less* general than the ones for which all of the LAGRANGE brackets $[u_\alpha, u_\beta]$ remain invariant. Namely, one has the following theorem:

If the rays of two homogeneous and isotropic optical spaces are associated with each other in a one-to-one way such that every field-like ray manifold in the first space goes over to the same kind of manifold in the image space then one can, by a similarity transformation, and possibly also a reflection through one of the coordinate planes that one of these spaces is subject to, always arrive at the fact that after performing these operations the LAGRANGE *bracket itself will remain invariant.*

From (25.2) and (25.3), ray manifolds are considered to be field-like if and only if the expressions $\lambda_\alpha \, du_\alpha$ ($\lambda'_\alpha \, du_\alpha$, resp.) are complete differentials. We must then now demand that any time when one selects a two-parameter family of light rays in the object space in such a way that $\lambda_\alpha \, du_\alpha$ becomes a complete differential the corresponding expression $\lambda'_\alpha \, du_\alpha$ will possess the same property. From a theorem on PFAFF forms ([51]), there must then be a *constant* number ρ such that:

$$\lambda'_\alpha \, du_\alpha = \rho(\lambda_\alpha \, du_\alpha) + d\Psi \tag{27.1}$$

exists identically. By assumption, however, the last relation goes to (25.5) by a similarity transformation (and possibly a reflection, in case $\rho < 0$), from which the assertion is proved.

Starting from the requirement that MALUS's theorem must be valid, one can then – at least, in isotropic, homogeneous space – study the form of all possible ray maps, and this would clarify the role that this theorem has played in the history of ray optics.

28. The integral invariants. (26.1) can be brought into a form that admits a very intuitive geometric interpretation.

Namely, due to the relation (16.5), one can write, when one denotes the boundary of the region G_u by γ:

$$I = \iint_{G_u} [u_1, u_2] \, du_1 du_2 = \int_\gamma (\lambda_1 du_1 + \lambda_2 du_2) \, . \tag{28.1}$$

The closed curve γ will be represented in the $u_1 \, u_2$-plane by the equations:

$$u_i = u_i(s) \qquad (0 \leq s \leq 2\pi), \tag{28.2}$$

where the functions $u_i(s)$ refer to periodic functions. To these functions, we add a third one:

$$t = t(s), \tag{28.3}$$

([51]) *Variationsrechnung*, § 145.

which does not need to be periodic, and consider the curve c that is defined by equations (28.3) and:

$$x_i = x_i(s) = \xi_i(t(s), u_j(s)). \tag{28.4}$$

A comparison of (28.1) with our previous equation (15.4) then yields:

$$J = \int_c (-H\,dt + \eta_i d\xi_i) - \int_c d\omega. \tag{28.5}$$

Therefore, one can, from our construction of the curve c, choose any curve that encircles the bundle of light rays in question once and whose endpoints lie on the *same* light ray.

One can make two applications of this formula: First, if the function $t(s)$ in (28.3) is periodic with period 2π then the curve will be closed and the integral in (28.5) will vanish. One will then have:

$$J = \int_Q (-H\,dt + \eta_i d\xi_i)\,. \tag{28.6}$$

In the terminology of POINCARÉ, the right-hand side of (28.3) is called a *relative* integral invariant, because it is true only for closed curves. Moreover, POINCARÉ considered only such closed curves that laid in the planes t = const. The integral invariant (28.6) was considered, in particular, by ÉLIE CARTAN ([52]).

Second, one can, however, choose the curve c to be on the boundary of the ray pencil in such a way that its tangent at each point intersects the light ray, which includes that point, *transversally* (§ 21). The condition for this is:

$$-H\,dt + \eta_i\,d\xi_i = 0 \tag{28.7}$$

and one has, as a consequence, in place of (27.5):

$$J = -\int_c d\omega = \omega_1 - \omega_2, \tag{28.8}$$

in which ω_1 and ω_2 mean the values of ω at the endpoints of c. The quantity J is then equal to the optical distance h between the two endpoints of a curve that entwines the ray pencil and intersects each ray on the boundary of this bundle (Fig. 5). The invariance of J will be expressed by saying that the optical distance between the endpoints of c is independent of the arbitrary choice of the initial points of this curve.

This extraordinarily intuitive interpretation for J goes back to G. PRANGE ([53]).

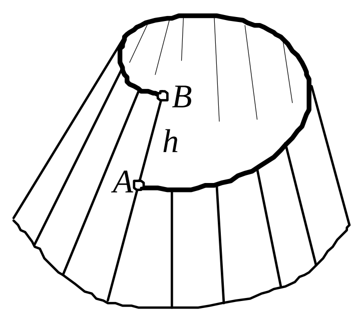

Figure 5.

([52]) É. CARTAN: *Leçons sur les invariants intégraux*, Paris, Hermann, 1922.

([53]) PRANGE, G.: "Die allgemeinen Integrationsmethoden der analytischen Mechanik," Enzykl. d. Mathem. Wiss., Bd. 4, II, Art. 12 and 13, pp. 622.

29. DESCARTES surfaces. A certain converse of MALUS's theorem was treated right at the beginning of Dioptics by DESCARTES for a special case. He addressed the following problem: Consider two arbitrary field-like ray congruences that lie in optically different media $\mathfrak{M}$ and $\mathfrak{M}'$, and assume that they permeate the media. Let a surface D go through a point (t^0, x_i^0), such that when one lets one of the media exist on one side of the surface, while the other one is on the other side, the former ray congruence will go to the latter one by refraction.

Since the two ray congruences are field-like, one can construct families of light rays that are represented by the equations:

$$S(t, x_i) = \text{const.}, \qquad S'(t, x_i) = \text{const.},$$

and intersect these ray congruences transversally. From § 23, any surface $t = \tau(x_1, x_2)$ that belongs to the family:

$$S(t, x_i) = S'(t, x_i) + C \tag{29.1}$$

is a possible discontinuity surface. The surface will go through the point (t^0, x_i^0) when one determines C from the equation:

$$C = S(t^0, x_i^0) - S'(t^0, x_i^0). \tag{29.2}$$

By this construction, a part of the light rays will be cut out from the media $\mathfrak{M}$ and $\mathfrak{M}'$. One calls these pieces that were cut out *virtual* light rays; the remaining part of the light rays will be called *real* light rays.

DESCARTES treated this problem for the special case in which the two media were isotropic and homogeneous and the two field-like ray congruences were stigmatic. One can always, by a suitable choice of the axes, give equation (29.1) the form:

$$n\sqrt{t^2 + x_1^2 + x_2^2} = \pm\, n'\sqrt{(t-a)^2 + x_1^2 + x_2^2} + C. \tag{29.3}$$

In this case, the DESCARTES surface is a surface of rotation whose meridian curve is an algebraic curve of fourth order. However, only a piece of this surface is useful as a discontinuity surface, namely, the piece that is represented by the equation (29.3) itself (while preserving the signs of the roots).

30. The aplanatic points of the sphere. A noteworthy special case of the DESCARTES surface was discovered by HUYGENS. Namely, if the constant C in (29.3) is equal to zero then one will obtain the equation of a sphere when one removes the square roots by squaring. Moreover, this connection may be conveniently established in an elementary geometric manner.

Namely, if F and F' are two inverse points of a sphere of radius r that one finds on a line MF through the center then, by definition, there will exist the equation:

$$\frac{\alpha}{\rho} = \frac{\rho}{a+\alpha}, \tag{30.1}$$

from which, one will assume that the two triangles *PMF* and *F′MP* are similar. It follows from this that the angle *MFP* will be equal to the angle i', and that one can therefore write:

$$\frac{\sin i'}{\sin i} = \frac{MP}{FM} = \frac{\rho}{a+\alpha} = \frac{\alpha}{\rho}. \tag{30.2}$$

In particular, the ratio $\sin i' : \sin i$ will be independent of the position of the point *P*. Should this ratio be equal to $n : n'$ then one would have:

$$n'\rho - n\,\alpha = n\,a, \qquad n\,\rho - n'\alpha = 0, \tag{30.3}$$

from which, one would arrive at the equations:

$$\rho = \frac{a\,n\,n'}{n'^2 - n^2}, \qquad \alpha = \frac{a\,n^2}{n'^2 - n^2}, \qquad \alpha + a = \frac{a\,n'^2}{n'^2 - n^2}, \tag{30.4}$$

which one can also verify directly from (29.3).

We now imagine a solid of rotation that consists of glass in air, with the index of refraction $n = 1$, whose meridian *ABPCD′F′A′* is composed of two concentric circles of radius ρ and $\rho : n'$ and two rectilinear line segments (Fig. 6). In this, n' is the index of refraction of the glass, which will be taken to equal 1.5 in the picture. All light rays *F′P* that start from a point *F′* of the smaller spherical surface that bounds the solid will be refracted at *P* on the large sphere in such a way that their starting point seems to be the point *F*. The spherical surface with the great circle *D′F′E′* will, as a consequence, be mapped stigmatically, and the virtual image, which lies on the spherical surface *DFE*, will be linearly magnified with a ratio $n'^2 : 1$ and undistorted.

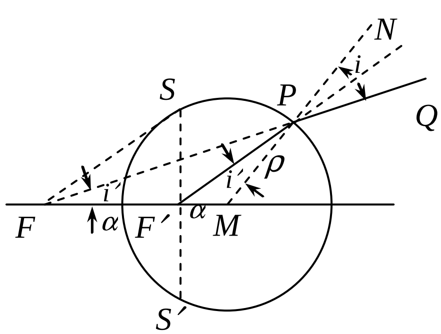

Figure 6.

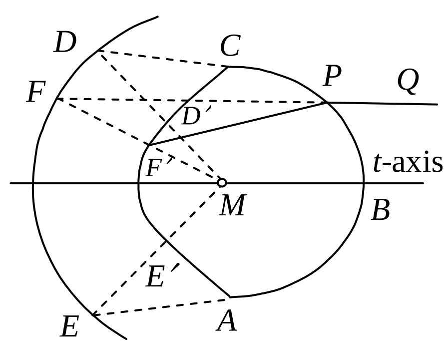

Figure 7.

Chapter III

The ray map

31. Definition and representation of the ray map. We consider an optical instrument that consists of an arbitrarily complicated system of lenses (or mirrors). The light excitation originates in one space – the *object space* – whose points will be represented by arbitrary (Cartesian or also curvilinear) coordinates (t, x_1, x_2) and leads into a second space – the *image space* – that is described by the same sort of coordinates (t', x'_1, x'_2). We call the two Hamiltonian functions that determine the form of the light rays in the interior of these two spaces $H(t, x_i, y_i)$ and $H'(t', x'_i, y'_i)$.

First of all, under the passage of light through this instrument, any ray in the object space that goes through the instrument will be associated with an image ray.

Secondly, from § 25, *under this association, the* LAGRANGE *bracket of an arbitrary ray congruence in the object space and the* LAGRANGE *bracket of the corresponding ray congruence in the image space possess the same values on any two associated rays when the congruences are represented in terms of the same parameters.*

The idea that any ray map that satisfies the two conditions above can be realized, at least approximately, by some suitable system of lenses is generally broadened. We will give examples in which this does not always need to be the case (§§ 57 and 61). The separation between the maps of this type, which can be realized optically, and which is, moreover, a mathematical problem, is never taken into account and might be exceptionally difficult.

This remark should not, however, lead one to think that the study of the most general maps of rays, in which the LAGRANGE brackets remain invariant, are of merely theoretical interest. On the contrary: Almost all practical applications that one can make of the general theory were inconceivable when one did not have these general maps to work with as a foundation.

In order to represent such a ray map, we consider the most general solution in object space:

$$x_i = \xi_i(t, a_1, a_2, b_1, b_2), \qquad y_i = \eta_i(t, a_1, a_2, b_1, b_2), \qquad (i = 1, 2) \qquad (31.1)$$

of the canonical equations (14.1) that satisfies the initial conditions:

$$\xi_i(t^0, a_j, b_j) = a_i\,, \qquad \eta_i(t^0, a_j, b_j) = b_i\,, \qquad (i = 1, 2) \qquad (31.2)$$

for $t = t^0$. Correspondingly, we consider the analogous solution in image space:

$$x'_i = \xi'_i(t', a'_1, a'_2, b'_1, b'_2), \qquad y'_i = \eta'_i(t', a'_1, a'_2, b'_1, b'_2) \qquad (i = 1, 2) \qquad (31.3)$$

to the associated canonical equations that is established by the initial conditions:

$$\xi_i'(t'^0, a_j', b_j') = a_i', \qquad \eta_i'(t'^0, a_j', b_j') = b_i'. \tag{31.4}$$

The one-to-one association of the rays of these two spaces will then be expressed by four equations:

$$a_i' = A_i(a_j, b_j), \qquad b_i' = B_i(a_j, b_j) \qquad (i, j = 1, 2), \tag{31.5}$$

in which it is self-explanatory that the functional determinant must satisfy:

$$\frac{\partial(A_1, A_2, B_1, B_2)}{\partial(a_1, a_2, b_1, b_2)} \neq 0. \tag{31.6}$$

32. We represent a ray congruence in the object space by considering the quantities a_j, b_j to be functions of two parameters u_1 and u_2 and substituting these values in (31.1). With the help of equations (31.5) and (31.3), one then computes the associated ray congruence in image space. From the reasoning in § 17, one will then have, when one further notices that $dt = dt^0 = 0$ here:

$$d\omega_0 + \lambda_1\, du_1 + \lambda_2\, du_2 = b_1\, da_1 + b_2\, da_2, \tag{32.1}$$

such that one can write, from (16.5):

$$\left.\begin{aligned} [u_1, u_2] &= \frac{\partial}{\partial u_2}\left(b_1 \frac{\partial a_1}{\partial u_1} + b_2 \frac{\partial a_2}{\partial u_1}\right) - \frac{\partial}{\partial u_1}\left(b_1 \frac{\partial a_1}{\partial u_2} + b_2 \frac{\partial a_2}{\partial u_2}\right) \\ &= \frac{\partial(a_1, b_1)}{\partial(u_1, u_2)} + \frac{\partial(a_2, b_2)}{\partial(u_1, u_2)}. \end{aligned}\right\} \tag{32.2}$$

The expression for the LAGRANGE bracket $[u_1, u_2]'$ in image space is entirely similar, and we must exhibit the most general transformation (31.5) for which one always has:

$$[u_1, u_2]' = [u_1, u_2] \tag{32.3}$$

for any choice of functions $a_1(u_1, u_2), \ldots, b_2(u_1, u_2)$.

It is worth mentioning that the form of the condition that we have obtained in this way is completely independent of the form of the HAMILTONIAN *functions H and H'. Our theory is therefore also valid for arbitrary curvilinear coordinates and can therefore also be applied to the cases in which arbitrary curved surfaces will be represented by the conditions $t = t^0$ and $t' = t'^0$.*

33. Connection with canonical transformations. In the investigation that now follows, the notations that we used up to now shall be replaced with ones that are better adapted to what one finds in the literature. Namely, we would like to replace a_1, a_2 with x, y and b_1, b_2 with ξ, η, and likewise denote the coordinates of the line elements in the image space by x', y', ξ', η'. The most general transformation for which the LAGRANGE brackets remain invariant shall then be given in the form:

$$\left.\begin{array}{ll} x' = X(x, y, \xi, \eta) & y' = Y(x, y, \xi, \eta) \\ \xi' = \Xi(x, y, \xi, \eta) & \eta' = \mathrm{H}(x, y, \xi, \eta). \end{array}\right\} \tag{33.1}$$

If one denotes the parameters of a ray congruence by u, v then one will have, from (32.2):

$$\left.\begin{array}{l} [u,v] = \dfrac{\partial(x,\xi)}{\partial(u,v)} + \dfrac{\partial(y,\eta)}{\partial(u,v)}, \\ [u,v]' = \dfrac{\partial(X,\Xi)}{\partial(u,v)} + \dfrac{\partial(Y,\mathrm{H})}{\partial(u,v)}. \end{array}\right\} \tag{33.2}$$

We now choose u and v to be any two of the four variables x, y, ξ, and η, and keep the remaining two variables constant. The six relations:

$$[x, y] = 0, \quad [x, \eta] = 0, \quad [y, \xi] = 0, \quad [x, \eta] = 0, \quad [x, \xi] = 1, \quad [y, \eta] = 1 \tag{33.3}$$

then arise from the first equation in (33.2). Due to the requirement that $[u, v]' = [u, v]$, one must then have, from the second equation in (33.2):

$$[x, y]' = \frac{\partial(X,\Xi)}{\partial(x,y)} + \frac{\partial(Y,\mathrm{H})}{\partial(x,y)} = 0, \tag{33.4}$$

and one further obtains five similar first-order partial differential equations from (33.3) that are easy to write down.

We would now like to show that conversely when these six equations are fulfilled the equation $[u, v]' = [u, v]$ will be valid for not only the six special ray congruences that we have considered up to now, but in complete generality. To that end, we calculate the coefficients λ, μ, ρ, σ in the differential form:

$$\Xi\, dX + \mathrm{H}\, dY - \xi\, dx - \eta\, dy = \lambda\, dx + \mu\, dy + \rho\, d\xi + \sigma\, d\eta, \tag{33.5}$$

and obtain:

$$\begin{array}{ll} \lambda = \Xi\, X_x + \mathrm{H}\, Y_x - \xi, & \mu = \Xi\, X_y + \mathrm{H}\, Y_y - \eta, \\ \rho = \Xi\, X_\xi + \mathrm{H}\, Y_\xi\,, & \sigma = \Xi\, X_\eta + \mathrm{H}\, Y_\eta\,. \end{array}$$

From the last equations, it follows by differentiation that:

$$\left.\begin{array}{lll} \lambda_x - \mu_x = [x, y]', & \lambda_\xi - \mu_\xi = [x, \xi]' - 1, & \lambda_\eta - \sigma_\eta = [x, \eta]', \\ \mu_\xi - \rho_x = [y, \xi]', & \mu_\eta - \sigma_y = [y, \eta]' - 1, & \rho_\eta - \sigma_\xi = [\xi, \eta]'. \end{array}\right\} \tag{33.6}$$

Therefore, if the LAGRANGE brackets $[x, y]'$, …, have the same values as $[x, y]$, …, then it will follows from (33.3) that the left-hand side of all equations (33.6) must vanish, and that means the same thing as the demand that the right-hand side of (33.5) must be a complete differential. One can then write:

$$\Xi\, dx + \mathrm{H}\, dy - \xi\, dx - \eta\, dy = d\Psi. \tag{33.7}$$

Conversely, if equation (33.7) is fulfilled then one will immediately calculate that $[u, v]' = [u, v]$ for all possible ray congruences, and this is precisely the result that we wanted to prove ([54]).

34. When the four functions $X, \ldots, \mathrm{H}$ satisfy equation (33.7), one calls the transformation (33.1) a *canonical transformation*. The *fundamental result* that we have obtained can then be expressed as follows:

The requirement that the LAGRANGE *brackets should remain invariant for the ray maps of optics is equivalent to the requirement that the association of the line elements be represented by a canonical transformation for $t = t^0$ and $t' = t^{0\prime}$.*

There is very extensive literature ([55]) on canonical transformations, which also play an important role in mechanics. Some of the most important results for optics will be summarized here. For the further details, one can also look them up in my *Variationsrechnung*.

The first important property of canonical transformations consists in the fact that each arbitrary canonical transformation is always, in fact, a transformation for which the functional determinant:

$$D = \frac{\partial(X, Y, \Xi, \mathrm{H})}{\partial(x, y, \xi, \eta)} \tag{34.1}$$

can never vanish. Namely, one proves that $D = +1$ ([56]). The proof of this fact is not entirely simple when one would like to be careful about the sign of D; however, it suffices for most purposes to show that $D = \pm 1$, and one achieves this by an entirely elementary calculation. Namely, one remarks that one can also obtain D from the following equation:

$$D = \frac{\partial(\Xi, \mathrm{H}, -X, -Y)}{\partial(\xi, \eta, -x, -y)}. \tag{34.2}$$

If one now multiplies the two determinants (34.1) and (34.2) *by columns* then one will get:

([54]) In the derivation above, we employed the fact that the functions $X, \ldots, \mathrm{H}$ are at least twice continuously differentiable. In chap. 6 of my *Variationsrechnung*, I showed that the result above, as well as the entire theory of canonical transformations can be derived without assuming that these functions possess two derivatives.

([55]) See PRANGE, *loc. cit*. 14, esp., pp. 748 *et seq.*

([56]) *Variationsrechnung*, § 102.

$$D^2 = \begin{vmatrix} [x,\xi]' & [y,\xi]' & 0 & [\eta,\xi]' \\ [x,\eta]' & [y,\eta]' & [\xi,\eta]' & 0 \\ 0 & [x,y]' & [x,\xi]' & [x,\eta]' \\ [y,x]' & 0 & [y,\xi]' & [y,\eta]' \end{vmatrix}. \tag{34.3}$$

Therefore, if the transformation is canonical then one will have, as we have announced:

$$D^2 = 1. \tag{34.4}$$

It then follows from this that the inverse of a canonical transformation always exists and is obviously also canonical, and since, from (33.7), one obtains a canonical transformation from the composition of two canonical transformations, the totality of all canonical transformations defines a group ([57]).

35. POISSON brackets. The second main property of canonical transformations now consists of the fact that one can also characterize these transformations by the construction of POISSON *brackets*. These POISSON brackets are, when one compares them with the LAGRANGE brackets that have considered exclusively up to now, dual in a certain sense. Namely, in order to define the LAGRANGE brackets (33.2), we had to consider *four* variables x, y, ξ, η of *two* parameters u and v. Now, we take *two* functions F and G of *four* variables x, y, ξ, η and define the POISSON bracket (G, F) by the formula:

$$(G, F) = \frac{\partial(F,G)}{\partial(x,\xi)} + \frac{\partial(F,G)}{\partial(y,\eta)}. \tag{35.1}$$

The calculations with POISSON brackets are made simpler when one notes the following two properties of these brackets, which follow immediately from (35.1). First, we have:

$$(G, F) = -(F, G), \tag{35.2}$$

and second, in the event that $\Phi(F_1, \ldots, F_n)$ is a function of arbitrarily many functions $F_k(x, y, \xi, \eta)$:

$$(G, \Phi) = \sum_{k=1}^{n} \frac{\partial \Phi}{\partial F_k}(G, F_k). \tag{35.6}$$

36. We arrive at a relation between POISSON and LAGRANGE brackets in the following way: For an arbitrary function $F(x, y, \xi, \eta)$, we define the following expression:

$$(\Xi, F)\, dx' + (\mathrm{H}, F)\, dy' = (X, F)\, dx' - (Y, F)\, d\eta', \tag{36.1}$$

in which we have set:

([57]) *Variationsrechnung*, § 94.

$$dx' = \frac{\partial X}{\partial x}dx + \cdots + \frac{\partial X}{\partial \eta}d\eta, \qquad \text{etc.,} \tag{36.2}$$

develop the POISSON brackets that enter into (36.1), and collect the coefficients of the first derivatives of F. In this way, we find that the expression (36.1) *is always identical with* [58]:

$$\left.\begin{aligned} &([x,\xi]'dx + [y,\xi]'dy + [\eta,\xi]'d\eta)F_x \\ &+([x,\eta]'dx + [y,\eta]'dy + [\xi,\eta]'d\xi)F_y \\ &+([x,y]'dx + [x,\xi]'d\xi + [x,\eta]'d\eta)F_\xi \\ &+([y,x]'dx + [y,\xi]'d\xi + [y,\eta]'d\eta)F_\eta. \end{aligned}\right\} \tag{36.3}$$

If the transformation is now canonical then this latter expression has the value:

$$F_x\,dx + F_y\,dy + F_\xi\,d\xi + F_\eta\,d\eta = dF; \tag{36.4}$$

conversely, if this is the case for all possible functions F then $[x, \xi]'$, $[y, \eta]'$ must equal unity, and the remaining LAGRANGE brackets must vanish. However, the theorem follows from this:

A necessary and sufficient condition for the transformation (33.1) *to be canonical is the existence of the identity:*

$$(\Xi, F)\,dx' + (\mathrm{H}, F)\,dy' - (X, F)\,d\xi' - (Y, F)\,d\eta' = dF \tag{36.5}$$

for all possible functions $F(x, y, \xi, \eta)$.

37. We would now like to employ the fact that the functional determinant (34.1) is necessarily non-zero, and that one can then compute the quantities x, y, ξ, η as functions of the x', y', ξ', η' from equations (33.1). From this, it follows that one can associate each function $F(x, y, \xi, \eta)$ with a function $F'(x', y', \xi', \eta')$ for which the identity exists:

$$F(x, y, \xi, \eta) = F'(x', y', \xi', \eta'). \tag{37.1}$$

One can also write the total differential dF as:

$$dF = dF' = F'_{x'}\,dx' + F'_{y'}\,dy' + F'_{\xi'}\,d\xi' + F'_{\eta'}\,d\eta',$$

and one sees that equation (36.5) means the same thing as the four equations:

$$(\Xi, F) = F'_{x'}, \quad (H, F) = F'_{y'}, \quad (F, X) = F'_{\xi'}, \quad (F, Y) = F'_{\eta'}. \tag{37.2}$$

([58]) When one reverts to the index notation, this completely elementary, if somewhat lengthy, computation becomes so self-evident that it can be done in one's head. *Variationsrechnung*, § 91.

If we set, in sequence, x', y', ξ', η' for F' and X, Y, Ξ, H for F then we will obtain a number of equations that reduce to the following six:

$$(X, Y) = 0, \qquad (\Xi, \mathrm{H}) = 0, \tag{37.3}$$
$$(X, H) = 0, \qquad (\Xi, Y) = 0, \tag{37.4}$$
$$(\Xi, X) = 1, \qquad (\mathrm{H}, Y) = 1. \tag{37.5}$$

38. The last conditions are then necessary for the transformation (33.1) to be canonical. We must now show that these conditions are also sufficient; i.e., that any transformation (33.1) is canonical as long as equations (37.3) to (37.5) exist identically. For that, we first remark that when one multiplies the determinants (34.1) and (34.2) with each other *by rows* then one will get:

$$D^2 = \begin{vmatrix} (\Xi, X) & (\Xi, Y) & 0 & (\Xi, \mathrm{H}) \\ (\mathrm{H}, X) & (\mathrm{H}, Y) & (\mathrm{H}, \Xi) & 0 \\ 0 & (Y, X) & (\Xi, X) & (\mathrm{H}, X) \\ (X, Y) & 0 & (\Xi, Y) & (\mathrm{H}, Y) \end{vmatrix}.$$

Thus, if the conditions (37.3) to (37.5) are fulfilled then one must have $D^2 = 1$, so $D \neq 0$. One can then associate each function $F(x, y, \xi, \eta)$ with a function $F(x', y', \xi', \eta')$, for which (37.1) is true. When one employs (35.3), one will then have:

$$(\Xi, F) = F'_{x'}(\Xi, X) + F'_{y'}(\Xi, Y) + F'_{\eta'}(\Xi, \mathrm{H}).$$

If equations (37.3) to (37.5) are fulfilled then the first equation (37.2) must be, as well. One verifies the remaining equations (37.2) in exactly the same way, from which, (36.5) then follows. With the help of the results of § 36, one then obtains the theorem:

The existence of equations (37.3) *to* (37.5) *is necessary and sufficient for the transformation* (33.1) *to be canonical.*

39. An invariant property is true for the POISSON bracket that is also similar to the one that we started with for the LAGRANGE bracket. Namely, if we set:

$$G(x, y, \xi, \eta) = G'(X, Y, \Xi, \mathrm{H}),$$

similar to what we did in (37.1), then we will have:

$$(G, F) = G'_{x'}(X, F) + G'_{y'}(Y, F) + G'_{\xi'}(\Xi, F) + G'_{\eta'}(\mathrm{H}, F),$$

and it will follow with the help of formulas (37.2) that:

$$(G, F) = (G', F')',$$

which is a relation that implies equations (37.3) to (37.5), moreover, from the result of the previous paragraphs, and to which, it is equivalent.

40. Construction of the canonical transformations. The relations (37.3) to (37.5) also show that if only *one* of the four functions X, Y, Ξ, H is prescribed then none of the remaining ones can be prescribed arbitrarily when the transformation is canonical.

Namely, a canonical transformation can – e.g., with the help of two functions $X(x, y, \xi, \eta)$ and $Y(x, y, \xi, \eta)$ – be constructed only when the POISSON bracket first satisfies:

$$(X, Y) = \frac{\partial(X,Y)}{\partial(x,\xi)} + \frac{\partial(X,Y)}{\partial(y,\eta)} \equiv 0, \tag{40.1}$$

and secondly when the first two rows of the functional determinant (34.1) are not proportional to each other, since otherwise that determinant would vanish. The latter says that at each point at least one of the six second-order functional determinants:

$$\frac{\partial(X,Y)}{\partial(x,y)}, \quad \frac{\partial(X,Y)}{\partial(x,\eta)}, \quad \frac{\partial(X,Y)}{\partial(y,\xi)}, \quad \frac{\partial(X,Y)}{\partial(\xi,\eta)}, \tag{40.2}$$

$$\frac{\partial(X,Y)}{\partial(x,\xi)}, \quad \frac{\partial(X,Y)}{\partial(y,\eta)} \tag{40.3}$$

may not vanish. On the other hand, these two conditions are also sufficient for one to calculate the functions Ξ, H, which, together with X, Y, define a canonical transformation. However, before we prove this, we shall establish a lemma that is also useful for later purposes.

41. From the four functional determinants (40.2), one can, in fact, select pairs of these expressions in four different ways, such that one of the variables x, y, ξ, η appears twice in the denominator. We would now like to show that when both functional determinants of one such pair vanish at a point, the first derivatives of X and Y with respect to the selected variables must also vanish at this point when the conditions of the previous paragraphs are fulfilled.

In particular, we must then prove, for instance, that from:

$$\frac{\partial(X,Y)}{\partial(x,y)} = 0, \quad \frac{\partial(X,Y)}{\partial(x,\eta)} = 0, \tag{41.1}$$

it must necessarily follow that:

$$\frac{\partial X}{\partial x} = 0, \qquad \frac{\partial Y}{\partial x} = 0. \tag{41.2}$$

However, one sees this immediately. Namely, if either of these two quantities were non-zero then, from (41.1), the existence of two finite numbers l, m would follow for which the four equations:

$$\left.\begin{array}{ll} X_y = \lambda X_x & Y_y = \lambda Y_x \\ X_\eta = \mu X_x & Y_\eta = \mu Y_x \end{array}\right\} \tag{41.3}$$

are simultaneously valid. If one substitutes these values in the second functional determinant (40.3) then it must vanish. Due to (40.1), the first determinant in (40.3) must also vanish. The comparison of (40.2) with (41.3) then further yields:

$$\frac{\partial(X,Y)}{\partial(y,\xi)} = \lambda\,\frac{\partial(X,Y)}{\partial(x,\xi)}, \qquad \frac{\partial(X,Y)}{\partial(\xi,\eta)} = -\,\mu\,\frac{\partial(X,Y)}{\partial(x,\xi)}, \tag{41.4}$$

so all six functional determinants (40.2) and (40.3) must then vanish, contrary to the assumption.

42. An important corollary to the lemma that we just proved consists in the fact that *at least one of the four expressions* (40.2) *must be non-zero.* Namely, if all of the determinants (40.2) are equal to zero then all of the first derivatives of X must vanish, and one must then have that the expression (40.3) vanishes, contrary to the assumption. For the determination of Ξ, H, there are now four cases to distinguish from each other, according to whether one desires that the first, second, third, or fourth functional determinant (40.2) is non-zero, resp. However, the treatment of these four cases leads to computations that are, in principle, entirely similar.

For example we assume that:

$$\frac{\partial(X,Y)}{\partial(\xi,\eta)} \neq 0. \tag{42.1}$$

One can then solve the equations:

$$x' = X(x, y, \xi, \eta), \qquad y' = Y(x, y, \xi, \eta) \tag{42.2}$$

for ξ, η, and obtain:

$$\xi = \varphi(x, y, x', y'), \qquad \eta = \psi(x, y, x', y'). \tag{42.3}$$

From the identities:

$$x' = X(x, y, \varphi, \psi), \qquad y' = Y(x, y, \varphi, \psi), \tag{42.4}$$

one can compute the first partial derivatives ψ_y and φ_x by differentiation. One finds the following equations:

$$\frac{\partial(X,Y)}{\partial(\xi,\eta)}\,\varphi_y = -\frac{\partial(X,Y)}{\partial(y,\eta)}, \qquad \frac{\partial(X,Y)}{\partial(\xi,\eta)}\,\psi_x = \frac{\partial(X,Y)}{\partial(x,\xi)}. \tag{42.5}$$

From these, it follows, with the help of (40.1) and (42.1), that:

$$\varphi_y = \psi_x. \tag{42.6}$$

The latter equation states that the functions φ and ψ can be represented as partial derivatives of a function $-E(x, y, x', y')$ in an infinite number of ways, and that one can write:

$$\xi = -E_x, \qquad \eta = -E_y. \tag{42.7}$$

We now assume that we have determined functions $\Xi(x, y, \xi, \eta)$, $\mathrm{H}(x, y, \xi, \eta)$, $\Psi(x, y, \xi, \eta)$, for which equation (33.7) will be true identically, when we substitute the given functions in place of the X, Y. If one replaces the variables ξ, η with the expressions (42.7) in these functions then one will obtain new functions $\Xi^*(x, y, x', y')$, $\mathrm{H}^*(x, y, x', y')$, $\Psi^*(x, y, x', y')$, for which the identity exists:

$$\Xi^* dx' + \mathrm{H}^* dy' + E_x\, dx + E_y\, dy = d\Psi^*. \tag{42.8}$$

From this, it now follows that:

$$E_x = \Psi^*_x, \qquad E_y = \Psi^*_y; \tag{42.9}$$

$(E - \Psi^*)$ must then be a function of x' and y' alone. However, since the function $E(x, y, x', y')$ is defined only up to an arbitrary additive function of (x', y'), we can, without loss of generality, set $E = \Psi^*$. It then follows from (42.8) that:

$$\xi' = E_{x'}, \qquad \eta' = E_{y'}, \tag{42.10}$$

and these equations determine our canonical transformation completely. Namely, one needs merely to replace the quantities x', y' in (42.10) with X, Y in order to obtain the desired functions Ξ and H.

Moreover, one remarks that the function E that we have computed is not entirely arbitrary: Namely, since it is always possible to solve the equations (42.3) for x', y', and therefore, also equations (42.7), the function E must necessarily satisfy the equation:

$$\begin{vmatrix} E_{xx'} & E_{xy'} \\ E_{yx'} & E_{yy'} \end{vmatrix} \neq 0. \tag{42.11}$$

43. The eikonals. In optics, a function $E(x, y, x', y')$ that fulfills the condition (42.11) is called an *eikonal* ([59]). If an arbitrary eikonal is given then the associated canonical transformation will be determined by equations (42.7) and (42.10). One computes the functions X, Y, Ξ, H by an elimination that is always possible, due to the validity of (42.11).

We obtain all of the canonical transformations for which (42.1) *is true in this way.*

([59]) With this terminology, we are following BRUNS (cf., footnote 18). We will strictly distinguish the eikonals from the characteristic functions of HAMILTON, a distinction that is not always made. For example, K. SCHWARZSCHILD consistently called HAMILTON's characteristic functions eikonals. (K. SCWARZSCHILD: "Untersuchungen zur geometrischen optik I, II, III," Astronom. Mitteil. d. Kgl. Sternwarte zu Göttingen, Pt. 9 – 11, 1905, pp. 1-31, 1-28, and 1-54). The two notions are occasionally confused in recent times (M. HERZBERGER: *Strahlenoptik*, Part 5, pp. 111. Berlin, Julius Springer, 1931).

Second, we assume that the first of the functional determinants (40.2) is non-zero, namely:

$$\frac{\partial(X,Y)}{\partial(x,y)} \neq 0. \tag{43.1}$$

One can then solve equations (42.2) for x, y and obtain:

$$x = \varphi(\xi,\ \eta,\ x',\ y'), \qquad y = \psi(\xi,\ \eta,\ x',\ y'). \tag{43.2}$$

One proves, by a process that this entirely similar to the one in the previous paragraphs, that here one has:

$$\varphi_\eta = \psi_\xi ,$$

and deduces from this, just as before, the existence of a function $V(\xi,\ \eta,\ x',\ y')$ for which equations (43.2) can be replaced with:

$$x = V_\xi , \qquad y = V_\eta . \tag{43.3}$$

We now remark that, by the introduction of a function:

$$\Omega(x,\ y,\ \xi,\ \eta) = \Psi(x,\ y,\ \xi,\ \eta) + x\xi + y\eta, \tag{43.4}$$

in place of (33.7), one write:

$$\Xi\ dX + \mathrm{H}\ dY + x\ d\xi + y\ d\eta = d\Omega(x,\ y,\ \xi,\ \eta). \tag{43.5}$$

By this, our problem is brought into a form that coincides with the form of the one that was posed in the previous paragraphs, up to the notation. Thus, when one introduces the independent variables ξ, η, x, y, with the help of (43.2) [or (43.3)], it will follow in the same way as it did before that one can always set:

$$\Omega(\varphi,\ \psi,\ \xi,\ \eta) = V(\xi,\ \eta,\ x',\ y') \tag{43.6}$$

$$\xi' = V_{x'} , \qquad \eta' = V_{y'} . \tag{43.7}$$

In addition, one sees that the following relation for V must be true:

$$\begin{vmatrix} V_{\xi x'} & V_{\xi y'} \\ V_{\eta x'} & V_{\eta y'} \end{vmatrix} \neq 0, \tag{43.8}$$

because, by assumption, equations (43.3) must be soluble for x' and y'.

One calls an arbitrary function $V(\xi,\ \eta,\ x',\ y')$ for which this condition (43.8) is true a *mixed eikonal*. This notation is supposed to suggest that V depends upon the two point coordinates x', y', and the two canonical direction coordinates ξ, η. One obtains the functions X, Y by solving the first two equations for x', y', the function Ξ, H by substituting these values in the right-hand side of equations (43.7), and the function Ψ will ultimately be represented by the equation:

$$\Psi = V(\xi, \eta, x', y') - x\xi - y\eta \tag{43.9}$$

with the help of (43.4) and (43.7).

44. It still remains for us to carry out a similar argument for the two cases in which the second or third functional determinant (40.2) is non-zero. These two cases go into each other when one switches x with y and ξ with η, such that it suffices to examine one of these cases. For example, we assume that:

$$\frac{\partial(X,Y)}{\partial(y,\xi)} \neq 0. \tag{44.1}$$

One can then solve equations (42.2) for y and ξ and obtain:

$$y = \varphi(x, \eta, x', y'), \qquad \xi = \psi(x, \eta, x', y'). \tag{44.2}$$

One proves, just as we did in § 42, that here we must have:

$$\varphi_x = -\psi_\eta , \tag{44.3}$$

and it follows, exactly as before, that one can describe the canonical transformation with the help of a *skew eikonal* $U(x, \eta, x', y')$.

In order to summarize the equations that will determine our canonical transformation, we remark that one can write:

$$\xi' dx' + \eta' dy' - \xi\, dx + y\, d\eta = d(\Psi + y\eta) = dU, \tag{44.4}$$

in case one sets:

$$\Psi(x, \varphi, \psi, \eta) + \varphi\eta = U(x, \eta, x', y'). \tag{44.5}$$

It then follows from this equation that:

$$\left.\begin{array}{c} \xi = -U_x \quad y = U_y \quad \xi' = U_{x'} \quad \eta' = U_{y'} \\ \begin{vmatrix} U_{xx'} & U_{xy'} \\ U_{\eta x'} & U_{\eta\eta'} \end{vmatrix} \neq 0 \\ \Psi = U(x,\eta, X, Y) - y\eta. \end{array}\right\} \tag{44.6}$$

45. We have completely solved the problem that we posed of determining all possible ray maps. Our result reads:

Any conceivable canonical transformation in two pairs of variables can always be computed, with the help of one of the eikonals E, V (one of the two skew eikonals U, resp.). It is always possible to determine a transformation when the two functions X, Y are given and the conditions of § 40 *are satisfied.*

From the theoretical standpoint, this result is completely satisfactory. However, one remarks that in the system of equations (37.3) to (37.5) the first four of them are on entirely the same footing when we replace the function pair X, Y that we based our reasoning upon with one or the other function pair Ξ, H or X, H or Ξ, Y. By either of these two combinations, one obtains four possible eikonals, at least one of which can be employed in each case. In all, one obtains *sixteen* eikonals in this way that one can summarize in the following table:

	xy	$x\eta$	ξy	$\xi\eta$
$x'y'$	E	U	U	V
$x'\eta'$	U'			
$\xi' y'$	U'			
$\xi'\eta'$	V'			W

Each of these eikonals depends upon four variables, and indeed these variables are specified in the left-most column and the top row of the table, and one finds the associated eikonal listed in the intersection of the selected row and column. From our result, one can find at least one eikonal in each row and column, by which, a given canonical transformation can be represented. However, only one of sixteen cells in the tables will correspond to a possible eikonal, when furnished with notation, because the eikonals that are taken from the remaining free cells are not used at all in practice ([60]). Namely, except for the eikonals E, U, V that we have considered up to now, in practical optics, only the eikonals U and V come into consideration, which one obtains from U and V by switching the object and image spaces, and the *angular* eikonal $W(\xi, \eta, \xi', \eta')$, for which the following formulas are valid:

$$x = W_{\xi}\,, \qquad y = W_{\xi}\,, \qquad x' = -\,W_{\xi}\,, \qquad y' = -\,W_{\eta'}\,, \tag{45.1}$$

$$\begin{vmatrix} W_{\xi\xi'} & W_{\xi\eta'} \\ W_{\eta\xi'} & W_{\eta\eta'} \end{vmatrix} \neq 0, \tag{45.2}$$

$$\Psi = W(\xi, \eta, \Xi, \mathrm{H}) + X\mathrm{X} + Y\mathrm{H} - x\xi - y\eta. \tag{45.3}$$

([60]) The remark above does not pertain to the theory of HAMILTON characteristic functions, which is used in the theory of eikonals (cf., § 64). HAMILTON employed a characteristic function S (*Mathem. Papers*, pp. 268) that depended partly on points and partly on direction coordinates in each of the optical spaces that were being mapped to each other. An eikonal that possesses this property will be indicated in, e.g., the third cell of the third row.

46. It can naturally occur – and this is the general case – that all sixteen eikonals that we spoke of are simultaneously appropriate to the description of one and the same canonical transformation. However, in this case there are also many practical grounds for preferring one or the other formal system in the problem being addressed. One obtains a hint about which choice to make by considering the limiting cases in which some of the eikonals are excluded from the outset.

For that reason, we would like to consider the various restrictions that, for instance, the function $E(x, y, x', y')$ is subject to when one or more of the first three functional determinants (40.2) vanishes identically.

First, if:

$$\frac{\partial(X,Y)}{\partial(y,\xi)} = 0 \tag{46.1}$$

then one can eliminate the variables y and ξ from equations (42.2) simultaneously , and one will obtain a relation of the form $\Phi(x, \eta, x', y') = 0$. However, since, by assumption, x, y, x', y' must be employed as the independent variables, our condition equation can be described in the form $\eta = \eta(x, x', y')$. From (42.7), it then follows that E_y is independent of y and that as a consequence the eikonal possesses the form:

$$E = \varepsilon_0(x, x', y') + y\,\varepsilon_1(x, x', y'). \tag{46.2}$$

It is self-explanatory that, conversely, the condition (46.1) will be fulfilled when E appears in the form (46.2).

The case in which the second determinant (40.2) vanishes identically can be treated in a completely similar manner.

Incidentally, it follows from this that when one has both:

$$\frac{\partial(X,Y)}{\partial(y,\xi)} \equiv 0, \qquad \frac{\partial(X,Y)}{\partial(x,\eta)} \equiv 0 \tag{46.3}$$

simultaneously the eikonal E must necessarily have the form:

$$E = \varepsilon_0(x', y') + y\,\varepsilon_1(x', y') + x\,\varepsilon_2(x', y') + xy\,\varepsilon_3(x', y'). \tag{46.4}$$

However, if we now assume that the functional determinant:

$$\frac{\partial(X,Y)}{\partial(x,y)} \equiv 0, \tag{46.5}$$

and indeed only that one, then there will exist a relation $\Phi(\xi, \eta, x', y') = 0$ that, from (42.7), can be written:

$$\Phi(-E_x, -E_y, x', y') = 0. \tag{46.6}$$

This relation is much more complicated to treat than the corresponding relations in the foregoing cases. As a result, if the *skew eikonal* $U(x, y, x', y')$ of § 44 were chosen in place of the eikonal $E(x, y, x', y')$ then it would follow from (46.5), when one observes the first equation in (44.5), that *U must be linear in x.*

We can then assert that whenever a relation of the form $\Phi(\xi, \eta, x', y') = 0$ *exists, without the need for one of the identities* (46.3) *to be in effect, a classical case is at hand in which one must employ a skew eikonal.* This may also be the case when the condition $\Phi = 0$ is fulfilled *approximately* in the neighborhood of a point,

By contrast, if not only (46.5), but also, for example, (46.1), is fulfilled then there will exist no basis for replacing the eikonal E with a skew eikonal. Namely, it possible here to pose the condition for the function (46.2) that one infers from the relation (46.6). Indeed, in this case one must demand that x and y can be eliminated from the equations:

$$-\xi = \frac{\partial \varepsilon_0}{\partial x} + y\frac{\partial \varepsilon_1}{\partial x}, \qquad -\eta = \varepsilon_1(x, x', y').$$

This is, however, the case if and only if ε_1 does not depend upon x; i.e., when one has:

$$E = \varepsilon_0(x, x', y') + y\,\varepsilon_1(x', y'). \tag{46.7}$$

It only remains for us to speak of the last case, in which the three identities (46.3) and (46.5) are simultaneously valid. E must then have the form (46.4), as well as the form (46.7); one finds that E has the form:

$$E = \varepsilon_0(x', y') + y\,\varepsilon_1(x', y') + x\,\varepsilon_2(x', y'). \tag{46.8}$$

47. Semi-telescopic, stigmatic, and telescopic maps. The eikonal (46.8) possesses a noteworthy property. From the previous formula, one must have, in fact:

$$\xi = -E_x = -\varepsilon_2(x', y'), \qquad \eta = -E_y = -\varepsilon_1(x', y'), \tag{47.1}$$

and it follows that one must also have:

$$x' = X(\xi, \eta), \qquad y' = Y(\xi, \eta). \tag{47.2}$$

The functions X, Y are independent of x, y, which one can, moreover, infer directly from the result of § 41.

If the object space is isotropic and homogeneous and the coordinates are rectangular then the previous equations will state that parallel light rays in the object space will be transformed by the passage through the instrument into a stigmatic light pencil. The ray map will then be called *semi-telescopic*.

A semi-telescopic ray map, for which the right-hand side of (47.2) is given, will also be represented by an angle eikonal (§ 45) to advantage. One likewise finds, as before, that W must have the form:

$$W = -\omega_0(\xi, \eta) - \xi' X(\xi, \eta) - \eta' Y(\xi, \eta). \tag{47.3}$$

In order for the two eikonals (46.8) and (47.3) to represent the same ray map one must have, first of all, that the two systems of equations (47.1) and (47.2) are equivalent, and secondly, one must have:

$$\omega_0(\xi, \eta) = \varepsilon_0(X(\xi, \eta), Y(\xi, \eta)). \tag{47.3}$$

One obtains this latter equation from our previous formulas when one observes (45.3) and uses the fact that $\Psi(x, y, -\varepsilon_2, -\varepsilon_1) = E$.

48. One obtains entirely similar results when one demands that the ray map should be *stigmatic*; i.e., that the points of the xy-plane and $x'y'$-plane will be mapped to each other in a one-to-one manner. One must then, in fact, have:

$$x' = X(x, y), \qquad y' = Y(x, y), \tag{48.1}$$

and from these equations, it then follows that the eikonals E, U, U' that appear in the first row and the first column of the table in § 45 are all unusable and that one therefore must use either the mixed eikonal V or the mixed eikonal V'. Since, from (48.1), a relation must exist between the quantities (x, y, x', η') and the quantities (x, y, y', ξ'), this points to a conclusion that is entirely analogous to the one that gave us (46.4), namely, that $V'(x, y, \xi', \eta')$ must necessarily be of the form:

$$V' = \omega_0(x, y) + \omega_1(x, y)\,\xi' + \omega_2(x, y)\,\eta' + \omega_3(x, y)\,\xi'\eta'. \tag{48.2}$$

However, the equations that belong to this eikonal:

$$x' = V'_{\xi'}, \qquad y' = V'_{\eta'}, \tag{48.3}$$

are equivalent to equations (48.1) if and only if one takes:

$$V' = \omega_0(x, y) + \xi' X(x, y) + \eta' Y(x, y). \tag{48.4}$$

From § 43, one must add the following relations to this:

$$\frac{\partial(X, Y)}{\partial(x, y)} \neq 0, \tag{48.5}$$

$$\xi = \frac{\partial \omega_0}{\partial x} + \xi' \frac{\partial X}{\partial x} + \eta' \frac{\partial Y}{\partial x}, \tag{48.6}$$

$$\eta = \frac{\partial \omega_0}{\partial y} + \xi' \frac{\partial X}{\partial y} + \eta' \frac{\partial Y}{\partial y}, \tag{48.7}$$

$$\Psi = -\,\omega_0(x, y), \tag{48.8}$$

in order for the canonical transformation to be completely computable.

49. A third type of ray map that is singular in the same way as the last two that were treated is the so-called *telescopic* ray map, for which the functions Ξ, Η depends upon only ξ and η, but not x and y. If the coordinates of the object and image spaces are Cartesian then this will mean that parallel rays, in turn, remain parallel when they pass through the instrument. One deduces in precisely the same way as in the previous paragraphs that of all of the usual eikonals, once again, the mixed eikonals V and V' are the only ones that are useful, and that one must set, e.g.:

$$V(\xi, h, x', y') = \Psi(\xi, \eta) + x'\,\Xi(\xi, \eta) + y'\,\mathrm{H}(\xi, \eta), \tag{49.1}$$

$$\frac{\partial(\Xi, \mathrm{H})}{\partial(\xi, \eta)} \neq 0, \tag{49.2}$$

$$x = \Psi_\xi + x'\,\Xi_\xi + y'\,\mathrm{H}_\xi, \tag{49.3}$$

$$y = \Psi_\eta + x'\,\Xi_\eta + y'\,\mathrm{H}_\eta, \tag{49.4}$$

$$\xi' = \Xi(\xi, \eta), \quad \eta' = \mathrm{H}(\xi, \eta), \tag{49.5}$$

$$\Psi = V - \xi\,V_\xi - \eta\,V_\eta\,. \tag{49.6}$$

50. Most general ray maps for which the four eikonals E, V, V', W **are not applicable.** In the conventional representations of the theory of eikonals, it is tacitly assumed that one can represent all possible ray maps (or, at least, all ray maps that are not completely trivial) by at least one of the ordinary eikonals E, V, V', and W. This is, however, a mistake: There are maps that can be described only by eikonals that are not bilinear in the variables that appear and for which none of the variable combinations that are used in E, V, V', and W can be chosen as independent variables. These eikonals are, however, not very numerous, and for that reason, we would like to present all of them, since this knowledge can be worthwhile in certain circumstances. We thus demand that between the eight variables (x, …, η') there should exist relations of the form:

$$\mathrm{K}(x, y, x', y') = 0, \qquad \Lambda(x, y, \xi', \eta') = 0, \tag{50.1}$$

$$\mathrm{M}(x, y, x', y') = 0, \qquad \mathrm{N}(\xi, \eta, \xi', \eta') = 0. \tag{50.2}$$

When one switches the coordinate pairs x', ξ' and y', η' with each other in the desired cases, one can, in the general theory, always arrive at the fact that skew eikonals $U'(x, y, x', \eta')$ can be employed for the representation of our ray maps. We compute the quantities ξ, η, y', ξ' with the help of the formula:

$$dU' = -\,\xi'\,dx' + y'\,d\eta' + \xi\,dx + \eta\,dy\,. \tag{50.3}$$

Due to the existence of the relations (50.1), one deduces from this, similar to what one did in § 46, that U' must have the form:

$$U' = A(x, y)\,x'\eta' + B(x, y)\,\eta' + C(x, y)\,x' + D(x, y). \tag{50.4}$$

The expression that corresponds to the functional determinant (42.11) has the following form here:

$$\begin{vmatrix} A_x x' + B_x & A_y x' + B_y \\ A_x \eta' + C_x & A_y \eta' + C_y \end{vmatrix};$$

since, from the general theory, this expression must not vanish, at least one of the functional determinants:

$$\frac{\partial(A,B)}{\partial(x,y)}, \qquad \frac{\partial(A,C)}{\partial(x,y)}, \qquad \frac{\partial(B,C)}{\partial(x,y)} \tag{50.5}$$

must be non-zero.

Furthermore, the identities (50.2) must now be fulfilled, which, when one uses the eikonal (50.3), will have the form:

$$M(U'_x, U'_y, x', Ax' + B) = 0, \qquad N(U'_x, U'_y, -(A\eta' + C'), \eta') = 0. \tag{50.6}$$

We partially differentiate the first of these equations with respect to x, y, η' and obtain:

$$\begin{aligned} M_\xi U'_{xx} \quad + \quad & M_\eta U'_{xy} & + M_{y'}(A_y x' + B_x) &= 0, \\ M_\xi U'_{xy} \quad + \quad & M_\eta U'_{yy} & + M_{y'}(A_y x' + B_y) &= 0, \\ M_\xi (A_x x' + B_x) + \quad & M_\eta (A_y x' + B_y) & &= 0. \end{aligned}$$

The three functions of M_ξ , M_η, M_y cannot vanish simultaneously since otherwise the variable x' could not be chosen to be one of the independent variables; from the last system of equations, one can then deduce the identity:

$$(A_y x' + B_y)^2 U'_{xx} + 2(A_y x' + B_y)(A_x x' + B_x) U'_{xy} + (A_x x' + B_x)^2 U'_{yy} = 0.$$

One obtains a second identity from the second equation (50.6). The left-hand sides of these equations represent polynomials in x' and η' whose coefficients must all vanish. In order to write down these conditions conveniently, we introduce the symbol:

$$\{\varphi\chi,\ \psi\} = \varphi_y \chi_y\ \psi_{xx} - (\varphi_y \chi_x + \varphi_x \chi_y)\ \psi_{xy} + \varphi_x \chi_x\ \psi_{yy}\,. \tag{50.7}$$

Our conditions will then be represented by the following fifteen second-order partial differential equations:

$$\{AA, A\} = 0, \quad \{AA, B\} = 0, \quad \{AA, C\} = 0, \tag{50.8}$$
$$\{AB, A\} = 0, \quad \{AC, A\} = 0, \tag{50.9}$$
$$\{BB, A\} + 2\{AB, B\} = 0, \qquad \{CC, A\} + 2\{AC, C\} = 0, \tag{50.10}$$
$$\{BB, B\} = 0, \quad \{CC, C\} = 0, \tag{50.11}$$
$$\{AA, D\} = -\,2\{AB, C\} = -\,2\{AC, B\}, \tag{50.12}$$
$$\{BB, C\} + 2\{AB, D\} = 0, \qquad \{CC, B\} + 2\{AC, D\} = 0, \tag{50.13}$$
$$\{BB, D\} = 0, \quad \{CC, D\} = 0. \tag{50.14}$$

51. We must now exhibit the most general common integral of the fifteen equations. The integration of these differential equations leads to fundamentally distinct calculations, according to whether $A(x, y)$ is constant or not. The final results generally go over to each other by an elementary transformation of the variables, as one subsequently verifies (cf., § 55).

In the case where $A(x, y)$ is variable, one can (if need be, by switching x with y) always assume that:

$$A_x \neq 0. \tag{51.1}$$

As a consequence, one can introduce A and y as independent variables, and in particular, set:

$$x = F(A, y). \tag{51.2}$$

For an arbitrary function $\Phi(x, y)$, one can then write:

$$\Phi(x, y) = \varphi(A, y), \tag{51.3}$$

and with these relations one will then have the identity:

$$\{AA, \Phi\} = \varphi_A\{AA, A\} + \varphi_{yy} A_x^2 . \tag{51.4}$$

It will then follow from equations (50.8) that one can set:

$$B = f_1(A) + y\, g_1(A), \qquad C = f_2(A) + y\, g_2(A). \tag{51.5}$$

One now has the equations:

$$\frac{\partial(A,B)}{\partial(x,y)} = A_x\, g_1, \quad \frac{\partial(A,C)}{\partial(x,y)} = A_x\, g_2, \quad \frac{\partial(B,C)}{\partial(x,y)} = A_x[g_2(f_1' + yg_1') - g_1(f_2' + yg_2')], \tag{51.6}$$

from which, one deduces that the determinants (50.5) vanishes simultaneously only when at least one of the two functions g_1 or g_2 is non-zero. We assume, e.g., that:

$$g_1(A) \neq 0. \tag{51.7}$$

If one now observes equations (50.9) and the special form (51.5) of the functions $B(x, y)$ and $C(x, y)$ then one will obtain the identities:

$$\frac{1}{A_x^2}\{AB, \Phi\} = -\varphi_{Ay}\, g_1 + \varphi_{yy}(f_1' + yg_1'), \tag{51.8}$$

$$\frac{1}{A_x^2}\{AB, \Phi\} = -\varphi_{Ay}\, g_2 + \varphi_{yy}(f_2' + yg_2'), \tag{51.9}$$

from which, it will follow that:

$$\frac{1}{A_x^2}\{AB, C\} = -g_2'g_1\,, \qquad \frac{1}{A_x^2}\{AC, B\} = -g_1'g_2\,.$$

The second equation in (50.12) can then be written $g_2'g_1 = g_1'g_2$, and one thus has the relation:

$$g_2 = r \cdot g_1, \tag{51.10}$$

in which r is a constant that can possibly be zero. Therefore, one finally has:

$$\frac{1}{A_x^2}\{AB, C\} = -\,rg_1g_1'. \tag{51.11}$$

52. From (51.8) and (51.9), one now computes:

$$\frac{1}{A_x^2}\{AB, B\} = -g_1g_1'\,, \qquad \frac{1}{A_x^2}\{AC, C\} = -g_2g_2' = -\,r^2g_1g_1', \tag{52.1}$$

such that from equations (50.10), it will follow, moreover, that:

$$\frac{1}{A_x^2}\{BB, A\} = 2\,g_1g_1'\,, \qquad \frac{1}{A_x^2}\{CC, A\} = 2\,r^2g_1g_1'. \tag{52.2}$$

These equations, when combined with our previous results, will then allow us to write:

$$\frac{1}{A_x^2}\{BB, \Phi\} = 2g_1g_1'\varphi_A + g_1^2\varphi_{AA} - 2g_1(f_1' + yg_1')\varphi_{Ay} + (f_2' + yrg_2')^2\varphi_{yy}\,, \tag{52.3}$$

$$\frac{1}{A_x^2}\{CC, \Phi\} = 2g_1g_1'\varphi_A + r^2g_1^2\varphi_{AA} - 2rg_1(f_2' + yrg_1')\varphi_{Ay} + (f_2' + yrg_1')^2\varphi_{yy}\,. \tag{52.4}$$

One immediately deduces from this that:

$$\frac{1}{A_x^2}\{BB, B\} = g_1^2(f_1'' + yg_1'')\,, \qquad \frac{1}{A_x^2}\{CC, C\} = r^2g_1^2(f_2'' + yrg_1''), \tag{52.5}$$

and from equations (50.11), which must be fulfilled for all values of y, the relations will then follow:

$$g_1'' = 0, \tag{52.6}$$

$$f_1'' = 0, \qquad r\,f_2'' = 0. \tag{52.7}$$

53. If we now set $D = \delta(A, y)$ then it will follow from (50.12), when combined with (51.4) and (51.11), that:

$$\delta_{yy} = 2r\, g_1 g_1' , \tag{53.1}$$

such that we can set:

$$D = f_3(A) + y\, g_3(A) + y^2\, r\, g_1 g_1' . \tag{53.2}$$

When we calculate $\{BB, C\}$ and $\{AB, D\}$ with these formulas, it will then follow from the first equation of (50.13) that we have the relation:

$$g_1 f_2'' + 2g_1' f_2' = 2(g_3' - r g_1' f_1') . \tag{53.3}$$

One likewise deals with the equation $\{BB, D\} = 0$ and finds, after one has taken into account that the coefficient of y^2 vanishes identically, the two conditions:

$$g_3'' = 0, \tag{53.4}$$

$$g_1 f_3'' + 2g_1' f_2' = 2 f_1'(g_3' - r\, g_1' f_1') . \tag{53.5}$$

54. Equations (52.6), (52.7), (51.10), in combination with the last three equations, allow us to calculate the six functions $f_1, \ldots, g_3$ explicitly. If we remark that in the definition of the eikonal U we can always ignore a linear function of the variables that comes about only by a translation of the coordinate origin, we can, by a suitable choice of the origin of the η, y', and ξ-axes, set:

$$g_1 = \beta(A - a), \qquad g_2 = r\, \beta(A - a), \qquad g_3 = s\, \beta(A - a), \tag{54.1}$$

$$f_1 = \alpha(A - a), \tag{54.2}$$

$$f_2 = (s - r\, \alpha)(A - a) + \frac{\mu_2}{A - a}, \tag{54.3}$$

$$f_3 = (s - r\, \alpha)(A - a) + \frac{\mu_3}{A - a}. \tag{54.4}$$

In addition, due to the second equation in (52.7) and (51.7), one must have:

$$r\, \mu_2 = 0, \qquad \beta \neq 0. \tag{54.5}$$

In order to also compute $A(x, y)$, we remark that from the method above, it will follow from:

$$\{AA, x\} = 0, \quad \{AB, x\} = 0, \quad \{BB, x\} = 0 \tag{54.6}$$

that:

$$x = f(A) + y\, g(A), \qquad g' = 0, \qquad g_1 f'' + 2g_1' f' = 0. \tag{54.7}$$

By a certain choice of origin of the x-axis, one can, for that reason, write:

$$A - a = \frac{1}{\beta(px + qy)} \qquad (p \neq 0,\ \beta \neq 0). \tag{54.8}$$

55. From these calculations, we find that the eikonal must have the form:

$$U' = a\,x'\eta' + \frac{(x'+\alpha+\beta y)(\eta'+s-r(\alpha-\beta y))}{\beta(px+qy)} + \mu_2\,\beta\,x'(px+qy). \tag{55.1}$$

By a suitable translation of the origin of the coordinates, one can ultimately arrive at the fact that the constants α and s both keep their zero values, and the desired ray map can be calculated from the skew eikonal:

$$U' = a\,x'\eta' + \frac{(x'+\beta y)(\eta'+r\beta y)}{\beta(px+qy)} + \frac{b}{p\beta}\,x'(px+qy). \tag{55.2}$$

Since one must have $p\beta \neq 0$ and $r\,b = 0$, we have two essentially different cases to consider. In the first one, $r \neq 0$ and $b = 0$, and one obtains:

$$\left.\begin{aligned}
x' &= -\beta y + \frac{px+qy}{2pr}\left(p\eta - q\xi - \sqrt{(p\eta-q\xi)^2+4pr\beta\xi}\right),\\
y' &= -a\beta y + \frac{1+a\beta(px+qy)}{2pr\beta}\left(p\eta - q\xi - \sqrt{(p\eta-q\xi)^2+4pr\beta\xi}\right),\\
\xi' &= ar\beta y - \frac{1+a\beta(px+qy)}{2p\beta}\left(p\eta - q\xi + \sqrt{(p\eta-q\xi)^2+4pr\beta\xi}\right),\\
\eta' &= -r\beta y + \frac{px+qy}{2p}\left(p\eta - q\xi + \sqrt{(p\eta-q\xi)^2+4pr\beta\xi}\right).
\end{aligned}\right\} \tag{55.3}$$

In the second case, $b \neq 0$ and $r = 0$, and one obtains:

$$\left.\begin{aligned}
x' &= \frac{\beta p(x\xi+y\eta)}{b(px+qy)-(p\eta-q\xi)},\\
y' &= \frac{a\beta p(x\xi+y\eta)+\xi+by}{b(px+qy)-(p\eta-q\xi)},\\
\xi' &= \frac{b}{p\beta}(px+qy) - \frac{1+a\beta(px+qy)}{p\beta}(p\eta-q\xi),\\
\eta' &= \frac{1}{p}(px+qy)(p\eta-q\xi).
\end{aligned}\right\} \tag{55.4}$$

In fact, in the case where r and b vanish simultaneously both of these formulas will coincide.

It is, moreover, very easy to verify that four relations of the form (50.1) and (50.2) must always exist for all of these systems of formulas. For instance, one sees immediately that from the first two equations of (53.3) that x and y can be simultaneously eliminated as well as ξ and η, and furthermore, that the last two equations in (55.3) possess similar properties. This same result follows for equations (55.4), once, from the equations:

$$y' - ax' = \frac{x' + \beta y}{\beta(px + qy)} = \frac{bx' - \beta\xi}{\beta(p\eta - q\xi)},$$

and then, however, from the comparison of the last equations of (55.4) with the relation:

$$-p\beta(\xi' + a\eta') = b(px + qy) + (p\eta - q\xi).$$

The case in which $A(x, y)$ is a constant can be treated by the same means. Since the latter of the determinants (50.5) must be non-zero, one can take, e.g., the quantities B and y to be independent variables. Of the fifteen differential equations at the end of § 50, nine of them are fulfilled identically, and the remaining ones lead to the final formulas that essentially arise from the system of formulas (55.3) or (55.4) when one switches the variable pairs xy and $x'y'$ with $\xi\eta$ and $\xi'\eta'$, resp. One then obtains all of the ray maps that can be generated without the help of any of the eikonals E, V, V', W from these formulas by entirely elementary transformations or permutations of coordinates.

56. Rotationally-symmetric systems. For the applications, the most important ray systems are the ones that that are rotationally-symmetric. We understand this to mean: If one replaces the variables x, y, ξ, η in the right-hand side of equations (33.1) with:

$$\left.\begin{aligned} \bar{x} &= x\cos\vartheta - y\sin\vartheta, \quad & \bar{y} &= x\sin\vartheta + y\cos\vartheta, \\ \bar{\xi} &= \xi\cos\vartheta - \eta\sin\vartheta, \quad & \bar{\eta} &= \xi\sin\vartheta + \eta\cos\vartheta, \end{aligned}\right\} \tag{56.1}$$

and if one denotes the new values of the functions X, … by $\bar{x}'$, $\bar{y}'$, $\bar{\xi}'$, $\bar{\eta}'$ then these values shall be connected with the previous ones by the equations:

$$\left.\begin{aligned} \bar{x}' &= x'\cos\vartheta - y\sin\vartheta, \quad & \bar{y}' &= y'\sin\vartheta + y'\cos\vartheta, \\ \bar{\xi}' &= \xi'\cos\vartheta - \eta'\sin\vartheta, \quad & \bar{\eta}' &= \xi'\sin\vartheta + \eta'\cos\vartheta. \end{aligned}\right\} \tag{56.2}$$

If the points of the object (image, resp.) space are established by rectangular coordinates t, x, y (t', x', y', resp.) then this requirement will says that under a rotation of the object space around the t-axis and a rotation of the image space around the t'-axis the ray map will remain invariant when the rotational angle ϑ is the same in both cases.

We now assume that the ray map is calculated with the help of an eikonal $E(x, y, x', y')$. The equations:

$$\left.\begin{aligned} \xi &= -E_x(\bar{x}, \bar{y}, \bar{x}', \bar{y}'), \quad & \eta &= -E_y(\bar{x}, \bar{y}, \bar{x}', \bar{y}'), \\ \xi' &= \quad E_{x'}(\bar{x}, \bar{y}, \bar{x}', \bar{y}'), \quad & \eta' &= \quad E_{y'}(\bar{x}, \bar{y}, \bar{x}', \bar{y}'), \end{aligned}\right\} \tag{56.3}$$

must then be fulfilled for all values of ϑ when one replaces the quantities $\bar{x}, \ldots, \bar{\eta}'$ on the right-hand sides of (56.1) and (56.2). We now consider the first partial derivatives with respect to x of the function:

$$\left.\begin{aligned}&\Omega(x, y, x', y', \vartheta)\\ &= E(x\cos\vartheta - y\sin\vartheta, x\sin\vartheta + y\cos\vartheta, x'\cos\vartheta - y'\sin\vartheta, x'\sin\vartheta + y'\cos\vartheta)\end{aligned}\right\} \tag{56.4}$$

and obtain, upon consideration of (56.3) and (56.1):

$$\Omega_x = -\cos\vartheta\,\bar{\xi} - \sin\vartheta\,\bar{\eta} = -\xi.$$

One then has $\Omega_x = E_x(x, y, x', y')$, and one verifies the equations $\Omega_y = E_y$, $\Omega_{x'} = E_{x'}$, and $\Omega_{y'} = E_{y'}$. However, from this, it follows that:

$$\Omega_x(x, y, x', y', \vartheta) = E_x(x, y, x', y') + f(\vartheta). \tag{56.5}$$

If one differentiates this latter equation with respect to ϑ and then sets $\vartheta = 0$ then what will follow, when one then employs the notation $f'(0) = \lambda$, is the first-order partial differential equation:

$$-y\,E_x + x\,E_y - y'\,E_{x'} + x'\,E_{y'} = \lambda, \tag{56.6}$$

which is then the condition that the eikonal of a rotationally-symmetric system must satisfy. A particular integral of this partial differential equation is λ arctan y/x; moreover, particular integrals of the homogeneous differential equation (56.6) for $l = 0$ are the functions:

$$2a = x^2 + y^2, \quad b = xx' + yy', \quad 2c = x'^2 + y'^2. \tag{56.7}$$

Thus, it ultimately follows from the theory of linear, first-order partial differential equations ([61]) that the eikonal E must have the following form here:

$$E = \mathcal{E}(a, b, c) + \lambda \arctan\frac{y}{x}. \tag{56.8}$$

One subsequently verifies that, conversely, every eikonal that possesses the form (56.8) will generate a rotationally-symmetric ray map.

57. This result provokes several remarks:

First, if E is a *single-valued* function of the variables (x, y) in the neighborhood of the point $x = y = 0$ then one must necessarily take $\lambda = 0$.

Second, we assume that E is developable in a convergent TAYLOR series in a neighborhood of that point and can be written:

$$E = P_1(x, y, x', y') + P_2(x, y, x', y') + \ldots, \tag{57.1}$$

([61]) *Variationsrechnung*, § 22.

in which $P_n(x, y, x', y')$ means a homogeneous polynomial of n^{th} degree in the four variables. One proves that any P_n must be a solution of the partial differential equation:

$$-y\frac{\partial P_n}{\partial x}+x\frac{\partial P_n}{\partial y}-y'\frac{\partial P_n}{\partial x'}+x'\frac{\partial P_n}{\partial y'}=0; \tag{57.2}$$

i.e., it must satisfy the differential equation (56.6) with $\lambda = 0$.

One further remarks that the expression:

$$d = xy' - yx' = \sqrt{4ac-b^2} \tag{57.3}$$

is a solution of the homogeneous differential equation (56.6). One then proves that the function P_n can be represented by polynomials in the four expressions a, b, c, and d. Due to the identity $d^2 = 4ac - b^2$, one can, in turn, demand that P_n should be linear in d.

It is now very easy to write down the right-side of (57.1): The polynomial P_1, P_2, P_3, …, vanishes identically, P_2 is a linear, homogeneous expression in a, b, c, d, the polynomial P_4 is quadratic in a, b, c, d, and if one so desires then the term in d^2 can be suppressed, etc.

Proof. The proof of the properties of the polynomials $P_n(x, y, x', y')$ that were just described rests on certain results of formal algebra.

First, the validity of the equation (57.2) will be verified, when one substitutes the development (57.1) for E in the left-hand side of (56.6) and then develops the resulting expression in homogeneous polynomials. Any one of the polynomials must then vanish, and the n^{th}-degree polynomial in the developments considered will coincide with the left-hand side of (57.2).

The second assertion – viz., that the polynomial P_n also can be written as a polynomial in the expressions a, b, c, d – is proved most simply when one introduces complex variables. We set:

$$z = x + iy, \qquad \bar{z} = x - iy, \qquad z' = x' + iy', \qquad \bar{z}' = x' - iy', \tag{57.4}$$

where i means the imaginary unit, and calculate P_n as homogeneous, n^{th}-degree polynomials $Q(z, \bar{z}, z', \bar{z}')$. The condition (57.2) can be replaced by another one, namely, that Q must be a solution to the partial differential equation:

$$zQ_z + z'Q_{z'} = \bar{z}Q_{\bar{z}} + \bar{z}'Q_{\bar{z}'}\,. \tag{57.5}$$

This equation possesses the particular solutions:

$$\alpha = z\bar{z}\,, \qquad \beta = z\bar{z}'\,, \qquad \gamma = z'\bar{z}\,, \qquad \delta = z'\bar{z}'\,. \tag{57.6}$$

We deduce the following relations from these formulas:

$$\bar{z} = \frac{\alpha}{z}, \qquad \bar{z}' = \frac{\beta}{z}, \qquad z' = \frac{\gamma z}{\alpha}, \tag{57.7}$$

and introduce these values into Q. In this way, we obtain the equation:

$$Q = \sum_{m=-p}^{q} A_m z^m, \tag{57.8}$$

in which p and q are positive whole numbers and the A_m mean rational functions of α, β, and γ. Now, since Q must be a solution of the partial differential equation (57.5), one must have:

$$\sum_{m=-p}^{q} m\, A_m z^m \equiv 0,$$

from which, it will follow that the right-hand side of (57.8) consists of only one term, which is independent of z. The polynomial Q can therefore be represented as a polynomial in α, β, and γ that is divided by a power of α. One must then show that it can be represented as a polynomial in α, β, γ, and δ.

This result follows by induction on the following argument: We assume that a polynomial in z, $\bar{z}$, z', $\bar{z}'$ can be represented by an expression of the form:

$$\frac{\tilde{\omega}(\beta,\gamma,\delta)}{\alpha}, \tag{57.9}$$

in which $\tilde{\omega}$ means a polynomial. Now, since $\tilde{\omega}$, after substituting the value (57.6), is divisible by z, as well as by $\bar{z}$, $\tilde{\omega}(\beta,\gamma,\delta)$ must be divisible by β, as well as by γ. It is then also divisible by $bg = ad$, and one can also write the expression (57.9) as a polynomial $\tilde{\omega}_1(\beta,\gamma,\delta)$ in any case.

After we have represented Q as a polynomial in α, β, γ, and δ, we revert to our original variables, with the help of the formulas:

$$\alpha = 2a, \qquad \beta = b - id, \qquad \gamma = b + id, \qquad \delta = 2c, \tag{57.10}$$

and ultimately obtain the desired representation of P_n as a polynomial in α, β, γ, and δ.

58. One obtains results for the mixed eikonals V, V', and the angle eikonal W that are completely analogous to the ones for E.

For instance, by rotational symmetry, the mixed eikonal V has the form:

$$\left.\begin{aligned} V &= \mathcal{V}(a,b,c) + \lambda \arctan\frac{\eta}{\xi} \\ 2a = \xi^2 + \eta^2 \quad b &= \xi x' + \eta y' \quad 2c = x'^2 + y'^2, \end{aligned}\right\} \tag{58.1}$$

while one obtains the following formulas for the angular eikonal W:

$$\left.\begin{aligned} W = \mathcal{W}(a,b,c) + \lambda \arctan\frac{\eta}{\xi} \\ 2a = \xi^2 + \eta^2 \quad b = \xi\xi' + \eta\eta' \quad 2c = \xi'^2 + \eta'^2. \end{aligned}\right\} \tag{58.2}$$

59. Semi-telescopic, stigmatic, and telescopic ray maps can be rotationally symmetric.

For example, should the mixed eikonal V' of § 48 represent a stigmatic rotationally-symmetric ray map then one will find that V' must have the form:

$$\left.\begin{aligned} V' = \omega_0(a) + (x\xi' + y\eta')\omega_1(a) + (x\eta' - y\xi')\omega_2(a) + \lambda \arctan\frac{y}{x} \\ a = \tfrac{1}{2}(x^2 + y^2). \end{aligned}\right\} \tag{59.1}$$

This yields:

$$x' = x\,\omega_1 - y\,\omega_2, \qquad y' = y\,\omega_1 + x\,\omega_2, \tag{59.2}$$

$$\left.\begin{aligned} \xi = \xi'\omega_1 + \eta'\omega_2 - \lambda\frac{y}{2a} + x\left[\frac{d\omega_0}{da} + (x\xi' + y\eta')\frac{d\omega_1}{da} + (x\eta' - y\xi')\frac{d\omega_2}{da}\right], \\ \eta = \eta'\omega_1 - \xi'\omega_2 + \lambda\frac{y}{2a} + y\left[\frac{d\omega_0}{da} + (x\xi' + y\eta')\frac{d\omega_1}{da} + (x\eta' - y\xi')\frac{d\omega_2}{da}\right]. \end{aligned}\right\} \tag{59.3}$$

Also, when $\lambda \neq 0$ this ray map will be *single-valued* inside of a circular ring:

$$0 < r_0^2 \leq x^2 + y2^2 \leq r_1^2 \ . \tag{59.4}$$

However, since, from (48.8), one must set:

$$\Psi = -\,\omega_0(a) - \lambda \arctan\frac{y}{x} \tag{59.5}$$

here, and since the function Ψ is *many-valued* in the circular ring (59.5), it will be impossible to realize the ray system that is provoked by our eikonal through a rotationally-symmetric system lens when one does not have $\lambda = 0$.

60. For rotationally-symmetric systems, it can also be useful to employ *skew eikonals*. In order to derive the condition of rotational symmetry without extensive calculations, we remark that due to equations (42.7) and (42.10), the condition (56.6) may also be written:

$$y\xi - x\eta - y'\xi' + x'\eta' = \lambda. \tag{60.1}$$

For the skew eikonal $U'(x, y, x', \eta')$, however, one has the equations:

$$\xi = U'_x, \qquad \eta = U'_y, \qquad \xi' = -U'_{x'}, \qquad y' = U'_{\eta'}, \tag{60.2}$$

and this eikonal will then generate a rotationally-symmetric ray map if and only if a solution exists to the partial differential equation:

$$yU'_x - xU'_y + U'_{x'}U'_{\eta'} + x'\eta' = \lambda. \tag{60.3}$$

It is not necessary to integrate this differential equation in general; indeed, we need only to consider the case for which the eikonals E and V' cannot be employed, and thus U' must have the form (50.4). However, is we substitute this value of U' in (60.3) then we will obtain the following conditions for the functions A, B, C, and D:

$$x\,A_y - y\,A_x = A^2 + 1, \tag{60.4}$$
$$x\,B_y - y\,B_x = AB, \tag{60.5}$$
$$x\,C_y - y\,C_x = AC, \tag{60.6}$$
$$x\,D_y - y\,D_x = BC - \lambda. \tag{60.7}$$

If we now write $A = \tan \varphi$ then (60.4) will go to:

$$x\,\varphi_y - y\,\varphi_x = 1, \tag{60.8}$$

whose general solution can be written:

$$\left.\begin{aligned} &\varphi = \arctan\frac{y}{x} - \arctan\alpha(a), \\ &a = \tfrac{1}{2}(x^2 + y^2). \end{aligned}\right\} \tag{60.9}$$

It follows from this that:

$$A = \frac{y - x\cdot\alpha(a)}{x + y\cdot\alpha(a)}. \tag{60.10}$$

If we next set:

$$u = \frac{1}{x + y\alpha}, \qquad v = \frac{y}{x + y\alpha} \tag{60.11}$$

then we will find by differentiation that:

$$\left.\begin{aligned} &xu_y - yu_x = u\cdot A, \\ &xv_y - yv_x = \frac{2a}{(x + y\alpha)^2}, \end{aligned}\right\} \tag{60.12}$$

and we will then easily obtain the general solutions for the equations (60.5) to (60.7):

$$B = \frac{\beta(a)}{x + y\alpha(\alpha)}, \qquad C = \frac{\gamma(a)}{x + y\alpha(\alpha)}, \tag{60.13}$$

$$D = \frac{\beta(a)\gamma(a)}{2a}\frac{y}{x+y\alpha(\alpha)} + \delta(a) - \lambda \arctan \frac{y}{x}. \tag{60.14}$$

In these equations, the four functions $\alpha(a)$, $\beta(a)$, $\gamma(a)$, $\delta(a)$ are arbitrary functions of $a = (x^2 + y^2) / 2$.

61. It is easy to recognize that the arbitrary functions $a, \ldots, d$ that enter into the last formulas cannot be chosen in such a way that the eikonal U' takes on one of the forms that were specified in § 55. *It follows from this that any rotationally-symmetric ray map is always representable by at least one of the four eikonals E, V, V′, or W.* This assertion, which was repeated over and over again, was still never proved up to now.

By comparison, one can easily give examples for which three of the conventional eikonals – e.g., the eikonals E, V, and V' – do not come under consideration. One will obtain one such ray map when one chooses α and β to be constant in the formulas above, and sets $\gamma = \delta = \lambda = 0$. One can then always choose the coordinates such that $\alpha = 0$. Consequently, the eikonal will have the form:

$$U' = \frac{y}{x}x'\eta' + \frac{\beta}{x}\eta' \tag{61.1}$$

and will yield the ray map:

$$x' = -\frac{\beta\eta}{x\xi + y\eta}, \qquad y' = \frac{\beta\eta}{x\xi + y\eta}, \tag{61.2}$$

$$\xi' = \frac{y}{\beta}(x\xi + y\eta), \qquad \eta' = -\frac{x}{\beta}(x\xi + y\eta). \tag{61.3}$$

From these equations, it follows that:

$$xy' - yx' = \beta, \qquad \xi x' + \eta y' = 0, \qquad x\xi' + y\eta' = 0,$$

and these relations show that all of the eikonals E, V, V' must remain outside of consideration here. From our result, if one would like to employ one of the four useful eikonals then the ray map can be computed only with the help of an angle eikonal W; this says that one must set:

$$W = 2\sqrt{\beta(\eta\xi' - \xi\eta')}. \tag{61.4}$$

The rotationally-symmetric ray map that is represented by equations (61.2) and (61.3) possesses many remarkable geometric properties. In addition, it yields one of the simplest examples of a ray map for which the invariance of the LAGRANGE bracket exists without it having to be constructible by optical media. This is connected with the fact that the rays for which the expressions $(x\xi + y\eta)$ or $(x'\xi' + y'\eta')$ vanish in one of the spaces cannot be associated with any ray in the other space. In case the object and image spaces are homogeneous and isotropic, these singular rays will define quadratic line complexes that include the rotational axis.

Chapter IV.

Coupled optical spaces.

62. Representation of a ray map in three-dimensional space. We would now like to associate the *individual line elements* of the object and image space with each other, and indeed, in such a way that for a ray map under which the LAGRANGE brackets remain invariant (§ 31), the line elements of two associated rays of the object and image space should correspond to each other.

In order to exhibit such an association of line elements, we return to the arguments at the beginning of the previous chapter and the notations of § 31. The mutually associated rays will be represented by the parameters a_j, b_j, and a_i', b_i', while the association itself will be defined by equations (11.5). From § 33, a function $\psi(a_j, b_j)$ must then exist for which the relation:

$$b_1' \, da_1' + b_2' \, da_2' = b_1 \, da_1 + b_2 \, da_2 + d\psi \tag{62.1}$$

exists identically.

One can establish the association of line elements with mutually-corresponding rays by a relation of the form:

$$t' = \tau(t, a_j, b_j), \tag{62.2}$$

in which τ means an otherwise arbitrary function that satisfies the condition:

$$\frac{\partial \tau(t, a_j, b_j)}{\partial t} \neq 0. \tag{62.3}$$

We further assume that we find ourselves in a coordinate domain (t, a_j, b_j) in which equations (31.1) are valid and are soluble for the a_j, b_j, such that we can write:

$$a_j = a_j(t, x_k, y_k), \qquad b_j = b_j(t, x_k, y_k). \tag{62.4}$$

By substituting these values in (62.2), we obtain a function:

$$t' = t'(t, x_k, y_k), \tag{62.5}$$

and by substituting the same functions in (31.5), we obtain further relations:

$$a_i' = a_i'(t, x_k, y_k), \qquad b_i' = b_i'(t, x_k, y_k) \tag{62.6}$$

that we will employ along with (62.5), in their own right, in order to calculate the relations:

$$x_i' = x_i'(t, x_k, y_k), \qquad y_i' = y_i'(t, x_k, y_k) \tag{62.7}$$

from (31.3).

Equations (62.5) and (62.7) then represent the associations (transformations, resp.) between the line elements that we would like to examine.

Since equations (31.5) represent a canonical transformation, it will always be soluble for the a_j , b_j , and it will follow from that and (62.3) that we can solve equations (62.5) and (62.7) for t, x_k, y_k , such that the functional determinant will also satisfy:

$$\frac{\partial(t', x_i', y_i')}{\partial(t, x_k, y_k)} \neq 0. \tag{62.8}$$

The identity (17.4) now exists for the functions ξ_j, η_j that enter into (31.1), but with the difference that due to the initial conditions (31.2), the term that is multiplied by $d\tau$ drops out, such that we can write:

$$-H(t, \xi_j, \eta_j)\,dt + \eta_j\,d\xi_j = d\Omega + b_i\,da_i\,. \tag{62.9}$$

We likewise find that:

$$-H'(t', \xi_j', \eta_j')\,dt' + \eta_i'\,d\xi_i' = d\Omega' + b_i'\,da_i' \tag{62.10}$$

for the functions ξ_i', η_i' that enter into (31.3). If we then observe (62.1) and calculate the function:

$$\Psi(t, x_k, y_k) = \Omega'(t', a_j', b_j') - \Omega(t, a_j, b_j) + \psi(a_j, b_j)\,, \tag{62.11}$$

with the help of the previous equations, then it will follow that the transformation that is defined by equations (62.5) and (62.7) must always satisfy the condition:

$$-H'(t', x_i', y_i')\,dt' + y_i'\,dx_i' = -H(t, x_k, y_k)\,dt + y_j\,dx_j + d\Psi. \tag{62.12}$$

63. Extended canonical transformations. It is remarkable that the last relation can be employed in order to characterize an association of line elements of the kind that we just presented. In order to show this, we start from any one-to-one association of line elements that is defined by equations of the form (62.5) and (62.7) and assume that (62.12) is also fulfilled. We calculate the expressions on the right-hand sides of (62.5) and (62.7) as functions of t, a_j, b_j, with the help of equations (31.1), and get:

$$t' = \tau(t, a_j, b_j), \qquad x_i' = f_i(t, a_j, b_j), \qquad y_i' = g_i(t, a_j, b_j). \tag{62.13}$$

We further calculate the quantities a_k', b_k' as functions of t', x_i', y_i' using (31.3); by substituting the functions (63.1) in the expressions thus obtained, we can now write:

$$a_k' = \alpha_k(t, a_j, b_j), \qquad b_k' = \beta_k(t, a_j, b_j). \tag{63.2}$$

Finally, we remark that after introducing the function:

$$\bar{\psi}(t, a_j, b_j) = \Psi(t, \xi_j, \eta_j) + \Omega(t, a_j, b_j) - \Omega'(\tau, \alpha_j, \beta_j), \tag{63.3}$$

equation (62.12) will be equivalent to the relation:

$$\beta_k \, d\alpha_k = b_i \, da_i + d\bar{\psi}\,, \tag{63.4}$$

due to the existence of (62.9) and (62.10). If we then succeed in showing that the functions $\alpha_k(t, a_j, b_j)$, $\beta_k(t, a_j, b_j)$, and $\bar{\psi}(t, a_j, b_j)$ do not depend upon t then equations (63.2) will show that our association of line elements represents a ray map, and the relation (63.4) will teach us, in addition, that the LAGRANGE bracket will remain invariant under this ray map (§ 33).

In order to prove this, we replace the variable t in α_k, β_k, and $\bar{\psi}$ with a new variable a_0 and introduce three new variables a_0', b_0, b_0', which should be coupled by the equations:

$$a_0' = a_0, \qquad b_0' = b_0\,. \tag{63.5}$$

The system of equations that consists of equations (63.2) and (63.5) then represents a transformation of three pairs of variables a_i, b_i, for which one can write:

$$b_0' \, da_0' + b_1' \, da_1' + b_2' \, da_2' = b_0 \, da_0 + b_1 \, da_1 + b_2 \, da_2 + d\bar{\psi}\,, \tag{63.6}$$

instead of (63.4), and which is canonical, for that reason.

However, the properties of POISSON brackets that we derived in § 37 are also true for canonical transformations with arbitrarily many pairs of variables (cf., *Variationsrechnung*, chap. 6, esp. § 92).

In particular, the relations:

$$(b_0', a_1') = 0, \quad (b_0', a_2') = 0, \quad (b_0', b_1') = 0, \quad (b_0', b_2') = 0 \tag{63.7}$$

must then exist. On the other hand, if $F(a_0, \ldots, b_2)$ means an arbitrary function of our six variables then, due to equations (63.5), one will have:

$$(b_0', F) = \frac{\partial F}{\partial a_0}\,. \tag{63.8}$$

As a result, equations (63.7) state that the four functions α_k, β_k are independent of t, and it then follows immediately from (63.4) that $\bar{\psi}$ will also possess the same property. With that, however, our assertion is proved completely.

A transformation between the line elements of two spaces for which the condition (62.12) exists shall be called an *extended canonical transformation*; we would then like to say of the two spaces that they are *optically coupled.*

It is self-explanatory that these concepts are transitive: If an optical space $\mathfrak{R}$ is coupled with a space $\mathfrak{R}'$, and correspondingly, the space $\mathfrak{R}'$ is coupled with a space $\mathfrak{R}''$ then a coupling of the spaces $\mathfrak{R}$ and $\mathfrak{R}''$ will be defined by the composed transformation that links $\mathfrak{R}$ to $\mathfrak{R}''$.

64. HAMILTON's characteristic function. The statement of formula (62.12) and its application to various problems was the guiding principle for the great discoveries of Sir W. R. HAMILTON in geometric optics. In his papers, HAMILTON replaced the function that we have called Ψ with another one for which the independent variables were chosen in such a way that these functions could be employed as generating functions for the transformation formulas. In particular, if one sets:

$$\Psi(t, x_k, y_k) = V(t', x'_i, t, x_j) \tag{64.1}$$

then one will get:

$$H(t, x_j, y_j) = V_i, \qquad y_j = -V_{x_j}, \tag{64.2}$$

$$H'(t', x'_i, y'_i) = -V_{t'}, \qquad y'_i = V_{x'_j}. \tag{64.3}$$

The similarity of these formulas with the ones that we became acquainted with in the theory of the eikonal is immediately apparent. In fact, one can just as well employ the function V, which HAMILTON called a *characteristic function*, as the eikonal E for many problems. HAMILTON also discovered other characteristic functions that correspond to the mixed eikonal and the angle eikonal. The parallelism between both theories is explained by the fact that the ideas of HAMILTON influenced the genesis of the theory in the previous chapter. Certainly, this happens unconsciously in an indirect and disguised way, but for that reason this influence was not less emphatic (cf., Introduction).

Just the same, the implementation of the HAMILTONian apparatus is unnecessarily complicated. Not only does its characteristic function depend upon more variables than the corresponding eikonal, but the great advantage that the theory of the previous chapter enjoys, which consists of the fact that the theory is completely independent of the form of the HAMILTONian functions H and H' (cf., § 32), is lost here. By contrast, the functions H and H' enter explicitly, since equations (64.2) and (64.3) teach us that the characteristic function V must satisfy both partial differential equations:

$$V_t - H(t, x_j, -V_{x_j}) = 0, \qquad V_{t'} + H'(t', x'_j, V_{x'_j}) = 0 \tag{64.4}$$

simultaneously.

In exchange, the presentation of the formulas for a coupling of the two optical spaces is somewhat simpler than before. One needs only to add another equation of the form:

$$t' = t'(t, x_j, y_j)$$

to equations (64.2) and (64.3) in order to arrive at such a coupling. The choice of the latter function is arbitrary to a large extent; one must observe only that the condition (62.8) is verified.

65. Canonical sliding transformations. One obtains the simplest extended canonical transformations (§ 63) when one lets the two spaces of the t, x_i and t', x'_i coincide and associates any ray with itself under the ray map. The line element t, x_i, y_i

will then be simply displaced along the light on which it lies. As a result, these special canonical translations shall be called *canonical sliding transformations*.

In order to obtain such a canonical sliding transformation, we calculate the inverse functions from the general solutions:

$$x_i = \xi_i\,(t, a_j, b_j), \qquad y_i = \eta_i\,(t, a_j, b_j) \tag{65.1}$$

of the canonical differential equations, which will give:

$$a_j = \varphi_j\,(t, x_i, y_i), \qquad b_j = \psi_j\,(t, x_i, y_i)\ . \tag{65.2}$$

The sliding of the line elements along the different rays will then be represented by equation (62.2) with fixed a_j, b_j . If one now calculates the function:

$$\chi(t, x_i, y_i) = \tau\,(t, \varphi_j\,(t, x_i, y_i), \psi_j\,(t, y_i, x_i)) \tag{65.3}$$

then the system of equations:

$$t' = \chi(t, x_i, y_i), \tag{65.4}$$

$$x_i' = \xi_i\,(\chi, \varphi_j, \psi_j), \qquad y_i' = \xi_i\,(\chi, \varphi_j, \psi_j), \tag{65.5}$$

when regarded as functions of t, x_j , y_j in the right-hand side of equations (65.5), will represent the desired sliding transformation.

There are infinitely many extended canonical transformations for one and the same ray map, which one will obtain from one of them when one composes the original coupling of the object and image space with an arbitrary sliding transformation in one of those spaces.

One can imprint special properties on the coupling of the spaces by a suitable choice of sliding transformation, and therein lies the advantage that the introduction of extended canonical transformations offers.

One can, e.g., succeed in making the function Ψ in formula (62.12) constant for the coupling of the two space by invoking a sliding transformation. The coupling will then be represented by an ordinary LIE contact transformation. Another special coupling that is important for the purposes of geometric optics is the *tangential* coupling, which we would now like to describe.

66. Unions of elements. Tangential coupling. Let any extended canonical transformation (62.12) be defined by formulas (62.5) and (62.7). The variables t, x_j, y_j should be regarded as arbitrary functions of two parameters u, v; one can then calculate the t, x_i', y_i' as functions of those parameters. For any function $f(u, v)$ of those parameters, we introduce the notations:

$$df = f_u\,du, \qquad \delta f = f_v\,dv, \qquad \delta df = f_{uv}\,du\,dv = d\delta f. \tag{66.1}$$

If we partially differentiate (62.12) with respect to v, in which the differential d is regarded as derivation with respect to u, in the sense of (66.1), then we will get, with the notations (66.1):

$$\begin{aligned} &- \delta H' \, dt' - H' \, \delta dt' + \delta y_i' \, dx_i' + y_i' \, \delta dx_i' \\ &= - \delta H \, dt - H \, \delta dt + \delta y_j \, dx_j + y_j \, \delta dx_j + \delta d\Psi. \end{aligned}$$

If we switch the symbols δ and d in this, subtract the equation thus obtained from the previous one, and observe the last of relations (66.1) then following relation will arise:

$$dH' \, \delta t' - \delta H' \, dt' + \delta y_i' \, dx_i' - dy_i' \, \delta x_i' = dH \, \delta t - \delta H \, dt + \delta y_j \, dx_j - dy_j \, \delta x_j . \tag{66.2}$$

When one develops dH and δH, the right-hand side of this equation can be written:

$$(H_{x_i} dx_i + H_{y_i} dy_i) \, \delta t + \delta y_i (dx_i - H_{y_i} dt) + \delta x_i (dy_i + H_{x_i} dt) \, ; \tag{66.3}$$

on the other hand, one has:

$$H_{x_i} dx_i + H_{y_i} dy_i = H_{x_i} (dx_i - H_{y_i} dt) + H_{y_i} (dy_i + H_{x_i} dt) \, . \tag{66.4}$$

If one replaces the left-hand side of (66.3) with the right-hand side of (66.4), and one transforms the left-hand side of (66.2) in the same way then one will finally get:

$$\left.\begin{aligned} &(\delta y_i' + H_{x_i'}' \delta t')(dx_i' - H_{y_i'}' dt') - (\delta x_i' - H_{y_i'}' \delta t')(dy_i' + H_{x_i'}' dt') \\ &= (\delta y_i + H_{x_i} \delta t)(dx_i - H_{y_i} dt) - (\delta x_i - H_{y_i} \delta t)(dy_i + H_{x_i} dt). \end{aligned}\right\} \tag{66.5}$$

A large number of relations can be deduced from this formula in which Ψ no longer appears. For example, if one replaces the parameter v – which was left completely arbitrary, up to now – with y_i' and x_i', in succession, then one will get:

$$dx_i' - H_{y_i'}' dt' = \left(\frac{\partial y_j}{\partial y_i'} + H_{x_j} \frac{\partial t}{\partial y_i'} \right) (dx_j - H_{y_j} dt) - \left(\frac{\partial x_j}{\partial y_i'} - H_{y_j} \frac{\partial t}{\partial y_i'} \right) (dy_j + H_{x_j} dt) , \tag{66.6}$$

$$dy_i' + H_{x_i'}' dt' = - \left(\frac{\partial y_j}{\partial x_i'} + H_{x_j} \frac{\partial t}{\partial x_i'} \right) (dx_j - H_{y_j} dt) + \left(\frac{\partial x_j}{\partial x_i'} - H_{y_j} \frac{\partial t}{\partial x_i'} \right) (dy_j + H_{x_j} dt) . \tag{66.7}$$

If one takes u to be equal to y_j in (66.6) then it will follow from these equations that:

$$\frac{\partial x_j'}{\partial y_i} - H_{y_j'}' \frac{\partial t'}{\partial y_i} = - \frac{\partial x_j}{\partial y_i'} + H_{y_j} \frac{\partial t}{\partial y_i'} . \tag{66.8}$$

67. According to S. LIE, a family of line elements $t(u)$, $x_i(u)$, $y_i(u)$ that depends upon a parameter u is called a *union of elements* when one has:

$$\frac{\partial x_i}{\partial u} = H_{y_i}(t, x_j, y_j)\frac{\partial t}{\partial u}. \tag{67.1}$$

We would like to exhibit all unions of elements in the object space that again go to unions of elements. For this, we must require that the equations:

$$dx_i' - H_{y_i'}' dt' = 0 \tag{67.2}$$

must be verified simultaneously with the equations:

$$dx_i - H_{y_i} dt = 0. \tag{67.3}$$

However, with this assumption, it will follow from (66.6) that:

$$\left(\frac{\partial x_j}{\partial y_i'} - H_{y_j}\frac{\partial t}{\partial y_i'}\right)(dy_j + H_{x_j} dt) = 0 \qquad (i = 1, 2). \tag{67.4}$$

If the determinant:

$$\left|\frac{\partial x_j}{\partial y_i'} - H_{y_j}\frac{\partial t}{\partial y_i'}\right| \neq 0 \tag{67.5}$$

then our requirement will be equivalent to the simultaneous validity of equations (67.3) with the following equations:

$$dy_j + H_{x_j} dt = 0 \qquad (j = 1, 2). \tag{67.6}$$

In this case, which is the general one, the individual union of elements that is once more mapped to a union of elements is the light ray itself. The fact that they will be associated with each other by the coupling is due to the fact that we required that at the beginning of the entire investigation. One can then read off this property of the extended canonical transformations directly from equations (66.6) and (66.7).

68. We would now like to consider the singular case, for which the left-hand side of (67.5) does not vanish identically. Due to the relation (66.8), this condition can be replaced with:

$$\left|\frac{\partial x_j'}{\partial y_i} - H_{y_j'}'\frac{\partial t'}{\partial y_i}\right| \neq 0, \tag{68.1}$$

from which, one recognizes that the condition possesses a simple geometric meaning. Namely, if one considers the totality of all line elements in object space that go through a fixed point (t^0, x_i^0) then one will obtain the corresponding line element in image space from the equations:

$$t' = t'(t^0, x_j^0, y_j), \qquad x_i' = x_i'(t^0, x_j^0, y_j), \tag{68.2}$$

$$y_i' = y_i'(t^0, x_j^0, y_j). \tag{68.3}$$

This line element goes through the points of a surface that will be represented by the equations (68.2) with the help of the parameters y_j. The direction of this line element in image space will be given by the vector with the components:

$$1, \quad H'_{y_1'}, \quad H'_{y_2'}, \tag{68.4}$$

while, on the other hand, the normal to the surface (68.2) can be described by three determinants of the matrix:

$$\left| \begin{matrix} \dfrac{\partial t'}{\partial y_1} & \dfrac{\partial x_1'}{\partial y_1} & \dfrac{\partial x_2'}{\partial y_1} \\ \dfrac{\partial t'}{\partial y_2} & \dfrac{\partial x_1'}{\partial y_2} & \dfrac{\partial x_2'}{\partial y_2} \end{matrix} \right| .$$

Since we can also write equation (68.1) as:

$$\left| \begin{matrix} \dfrac{\partial t'}{\partial y_1} & \dfrac{\partial x_1'}{\partial y_1} & \dfrac{\partial x_2'}{\partial y_1} \\ \dfrac{\partial t'}{\partial y_2} & \dfrac{\partial x_1'}{\partial y_2} & \dfrac{\partial x_2'}{\partial y_2} \\ 1 & H'_{y_1'} & H'_{y_2'} \end{matrix} \right| = 0, \tag{68.5}$$

this condition says that the directions of the line elements in the image space lie in the tangential plane to the surface (68.2).

Now, any time that the stigmatic light pencil of the object space with the center (t, x_i) is transformed into a congruence of light rays in image space, one can transform the coupling into another one for which (68.1) is fulfilled by a canonical sliding transformation. We would like to call such couplings of optical spaces *tangential couplings.* Since, as we have seen, the two conditions (68.1) and (67.5) are equivalent to each other, the inverse transformation must have precisely the same geometric property as the transformation itself under a tangential coupling, moreover.

In addition to the light rays, there are other unions of elements in object space that go to unions of elements in image space under tangential couplings. A simple example of such a union of elements is defined by the envelopes of the light rays of the pencil that possesses the focal surface (68.2). The most general union of elements that again goes to a union of elements is closely connected with the theory of optical images of a surface, which C. W. OSEEN has recently made known ([62]).

69. Absolute optical instruments. An optical instrument is called *absolute* when all stigmatic light pencils in object space are again taken to stigmatic light pencils in image space. This shall be true for at least the rays that lie in the field of the instrument; i.e., the ones that go through the instrument.

The rays that go through a point t, x_i in object space must then be converted into rays that go through a point:

$$t' = t'(t, x_j), \qquad x'_i = x'_i(t, x_j) . \tag{69.1}$$

Therefore, the two optical spaces will be mapped to each other *point-wise* for a absolute instrument.

The map (69.1) is, however, not arbitrary. Namely, one is dealing with any extended canonical transformation by which the ray map that is generated by the absolute instrument is represented. The elements that lie in the field of the instrument and go through the point t, x_j will be transformed into line elements in image space that lie on rays that go through the point (69.1). If we invoke a suitable canonical sliding transformation then we will obtain a new extended canonical transformation under which the line elements through t, x_j will be transformed into line elements through t', x'_i directly.

This is a tangential canonical transformation (§ 68) that will be represented when one add two more equations of the form:

$$y'_i = y'_i(t, x_j, y_j) \qquad\qquad (i = 1, 2) \tag{69.2}$$

to equations (69.1). One can calculate the functions on the right-hand side of (69.2) immediately, and indeed in two different ways, according to whether one uses the fact that the transformation in question is a point transformation or the fact that it is an extended canonical transformation. However, the two calculations must naturally lead to the same result.

For the first kind of calculation, one remarks that the mutually corresponding line elements t, x_j, $\dot{x}_j$, and t', x'_i, dx'_i / dt' can be obtained directly from the stigmatic map (69.1). Now, since one must have:

$$\dot{x}_j = H_{y_j}, \qquad \frac{dx'_i}{dt'} = H'_{y'_j} \tag{69.3}$$

for these line elements, one can write:

([62]) OSEEN, C. W., "Une méthode nouvelle de l'optique géométrique," Kungl. Svenska Vetenskapsakademiens Handlingar (3) Bd. 15, no. 6 (1936).

$$\frac{\partial x_i'}{\partial t} + \frac{\partial x_i'}{\partial x_j} H_{y_j} = H_{y_i'}' \left(\frac{\partial t'}{\partial t} + \frac{\partial t'}{\partial x_j} H_{y_j} \right) \qquad (i = 1, 2); \tag{69.4}$$

these are two equations from which one can get (69.2).

For the second kind of calculation, one starts from the fact that the formula:

$$- H' dt' + y_i' \, dx_i' = - H \, dt + y_j \, dx_j + d\Psi, \tag{69.5}$$

by which the coupling of the two optical spaces is represented, must be fulfilled identically when one substitutes (69.1) and (69.2). Since the right-hand sides of equations (69.1) do not contain the canonical direction coordinates, Ψ will be a *function of position*, and the relation (69.3) will be equivalent to the equations:

$$- H + \Psi_t = - H' \frac{\partial t'}{\partial t} + y_i' \frac{\partial x_i'}{\partial t}, \tag{69.6}$$

$$y_j + \Psi_{x_j} = - H' \frac{\partial t'}{\partial x_j} + y_i' \frac{\partial x_i'}{\partial x_j} \quad (i = 1, 2). \tag{69.7}$$

One can, in turn, calculate (69.2) from the two equations (69.7), and the values of y_i' that are obtained in this way must yield an identity when they are substituted into (69.6).

One can verify the fact that the result is the same in both cases in the following way: By differentiating (69.6) and (69.7) with respect to y_i', one will obtain:

$$\left.\begin{aligned} -H_{y_j} \frac{\partial y_j}{\partial y_i'} &= -H_{y_j'}' \frac{\partial t'}{\partial t} + \frac{\partial x_i'}{\partial t}, \\ \frac{\partial y_j}{\partial y_i'} &= -H_{y_j'}' \frac{\partial t'}{\partial x_j} + \frac{\partial x_i'}{\partial x_j}, \end{aligned}\right\} \tag{69.8}$$

and (69.4) will follow by combining these last two equations.

We now remark that, from § 10, one can write:

$$\left.\begin{aligned} -H' dt' + y_i' \, dx_i' &= L'\left(t', x_i', \frac{dx_i'}{dt'}\right) dt', \\ -H \, dt + y_j \, dx_j &= L(t, x_j, \dot{x}_j) \, dt, \end{aligned}\right\} \tag{69.9}$$

such that one will get the following from the relation (69.5):

$$\int_{\gamma'} L'\left(t', x_i', \frac{dx_i'}{dt'}\right) dt' = \int_\gamma L\left(t, x_j, \dot{x}_j\right) dt + \int_\gamma d\Psi. \tag{69.10}$$

The latter equation says that the difference between the optical lengths of two curve segments γ and γ' that correspond to each other by means of the stigmatic map is equal to the difference between the values of $\Psi(t, x_j)$ at the endpoints of the curve γ. *This difference is then independent of the form of the curves γ, γ' and depends upon only the positions of their endpoints.*

70. If the basic function $L(t, x_j, \dot{x}_j)$ of the object space is prescribed then the basic function $L'(t', x'_i, dx'_i / dt)$ in the image space cannot be chosen arbitrarily if a stigmatic optical coupling of the two spaces is to be at all possible.

In fact, equation (69.10) says that the relation:

$$L'\left(t', x'_i, \frac{dx'_i}{dt'}\right) dt' = L(t, x_j, \dot{x}_j) + \Psi_t \, dt + \Psi_{x_j} dx_j \tag{70.1}$$

must be fulfilled identically when one replaces the variables t', x'_i with the expressions (69.1) and correspondingly calculates dt' and dx'_i by means of the equations:

$$dt' = \frac{\partial t'}{\partial t} dt + \frac{\partial t'}{\partial x_j} dx_j , \qquad dx'_i = \frac{\partial x'_i}{\partial t} dt + \frac{\partial x'}{\partial x_j} dx_j . \tag{70.2}$$

Therefore, the quantities must naturally be chosen in such a way that the light ray of the line element lies in the field of the instrument, in addition. However, it follows from this that $L'(t', x'_i, dx'_i / dt)$ must have a very special form.

Moreover, this shows that the function Ψ must be subject to restrictions that one can already determine when one knows the functions L and L', but not the stigmatic coupling of the two optical spaces. In particular, we would like to show that Ψ must always be constant when the two optical spaces are isotropic or crystalline.

We next assume that the two optical spaces are isotropic, but not necessarily homogeneous, fix the point t, x_j, and let the direction of the light ray vary. One can then write equation (70.1) in the form:

$$n'(t', x'_i)\sqrt{dt'^2 + dx_1'^2 + dx_2'^2} = n(t, x_j)\sqrt{dt^2 + dx_1^2 + dx_2^2} + \Psi_i \, dt + \Psi_{x_j} dx_j . \tag{70.3}$$

After we have substituted the value (70.2) in this equation, we denote the variables dt, dx_i by ξ_0, ξ_1, ξ_2, for the sake of greater symmetry. After dividing by n, the last equation will then have the form:

$$\sqrt{A} = \sqrt{B} + C, \tag{70.4}$$

where

$$A = a_{ij} \, \xi_i \, \xi_j , \quad B = \xi_0^2 + \xi_1^2 + \xi_2^2 , \qquad C = p_0 \, \xi_0 + p_1 \, \xi_1 + p_2 \, \xi_2 . \tag{70.5}$$

The relations:

$$A = B + C^2 + 2C\sqrt{B}$$

$$(A - B - C^2)^2 = 4C^2B. \tag{70.6}$$

now follow from (70.4) by successive squaring.

By assumption, the relation (70.3) should only be assumed for line elements that lie in the field of the instrument. Thus, it will only be required that (70.6) be true in a small region of the space of ξ_0, ξ_1, ξ_2. However, since polynomials are on both sides of this equation, it will already follow from this assumption that the corresponding coefficients of these polynomials must agree. On the left-hand side of (70.6), one finds the square of an entire rational function. If C is not identically zero then $A - B - C^2$ must be divisible by C, and by carrying out that division, B must also appear as the square of a rational function. Since that is not the case, C must vanish identically, and A must be equal to B.

It follows from this that the first derivatives of Ψ *must vanish for any point of the space of* t, x_j, *and consequently,* Ψ *must be constant.*

71. The corresponding calculations will become very complicated for the case in which the two media are crystalline. However, one can also derive the desired result here by using an argument from the general theory of functions. After introducing the homogeneous variables ξ_i, equation (70.1) will assume the form:

$$\Phi'(\xi_0, \xi_1, \xi_2) = \Phi(\xi_0, \xi_1, \xi_2) + p_0\,\xi_0 + p_1\,\xi_1 + p_2\,\xi_2\,; \tag{71.1}$$

in this, the functions $\Phi(\xi_i)$, $\Phi'(\xi_i)$ are positive homogeneous of order one ([63]) in the ξ_i, and of the equations:

$$\Phi(\xi_0, \xi_1, \xi_2) = 1, \qquad \Phi'(\xi_0, \xi_1, \xi_2) = 1, \tag{71.2}$$

the first one represents the (rotated in some way) FRESNEL ray surface in object space and the second represents an affine transformation of the FRESNEL ray surface in image space. In addition, on the sphere:

$$\xi_0^2 + \xi_1^2 + \xi_2^2 = 1, \tag{71.3}$$

the functions $\Phi(\xi_i)$, $\Phi'(\xi_i)$ are analytic functions of position that can be singular only at finitely many points P^* that correspond to the conical points of the two FRESNEL surfaces. Now, by assumption, equation (71.1) is fulfilled identically on a small patch σ on the sphere (74.3), and one can always choose σ to be small enough that, along with all points P of σ, the opposite points $\bar{P}$ to P will also be different from the singular points P_k^*.

We now link a point P of σ with its opposite point $\bar{P}$ by an analytic curve γ that lies on the sphere (71.3). From the principle of analytic continuation, equation (71.1) must then be fulfilled along all of that curve. Now, Φ and Φ' have the same value at the two endpoints of γ, while the values of the linear form will be equal and opposite, unless one has:

$$p_0\,\xi_0 + p_1\,\xi_1 + p_2\,\xi_2 = 0. \tag{71.4}$$

([63]) *Variationsrechnung*, § 249.

Since this latter equation must be true for all points of σ, it will necessarily follow that:

$$p_0 = p_1 = p_2 = 0, \tag{71.5}$$

and this is precisely the result that we would like to prove.

72. The identical vanishing of the total differential $d\Psi$ in equation (69.4) has the consequence that for any curve γ in the object space that is transformed into a curve γ' in the image space, the relation:

$$\int_{\gamma'} L'\left(t', x_i', \frac{dx_i'}{dt'}\right) dt' = \int_{\gamma} L\left(t, x_i, \frac{dx_i}{dt}\right) dt \tag{72.1}$$

must be true. However, this says that the optical lengths of the two corresponding curves must be equal to each other *in such a way that an absolute instrument can either increase or decrease them* ([64]).

One observes that, whereas the relation (69.4) should be valid only for line elements that lie in the field of the instrument, equation (72.1) is true *for any entirely arbitrary curve*, since the equation $L'\,dt' = L\,dt$ is true for all pairs of line elements that are related to each other by equation (69.1).

The theorem that we proved has a long history. It was proved in 1858 by MAXWELL ([65]) for isotropic and homogeneous media, but generally only in the first approximation – i.e., for small objects. Later, it was found implicitly in the investigations of BRUNS for just those media, and was explicitly discussed for the first time by F. KLEIN and was proved by a very original method ([66]).

73. The MAXWELLian fisheye. For the case in which the object space, as well as the image space, are isotropic, but not necessarily homogeneous, from § 70, the quadratic form A must be identical with B. However, this will be the case if and only if the transformation (70.2) of the line elements is orthogonal, which is equivalent to demanding that the map (69.1) of the object space to the image space must be conformal. From a celebrated theorem of LIOUVILLE ([67]), in contrast to the planar conformal maps, which depend upon infinitely many constants, there is only a restricted class of conformal maps of three-dimensional space. It can always be represented as a sequence of transformations through reciprocal radii of at most five spheres. It follows from this that the circle and lines in object space will be transformed into curves in image space that will always be either circles or lines. MAXWELL treated the simplest case of such a ray map (when one ignores a reflection) on occasion ([68]). In the study of the spherical lenses

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([64]) This theorem is not true for GAUSSian optics (cf., § 98).

([65]) MAXWELL, J. C.: "On the general laws of optical instruments," Quart. J. of pure and applied Mathem. **2** (1858), 233-244; Sci. Pap., v. 1, pp. 271-285.

([66]) KLEIN, F.: "Räumliche Kollineationen bei optischen Instrumenten," Z. Math. u. Physik, **46** (1901), 376-382; Gesammelte Abh. (cf. footnote 15), Bd. II, pp. 607-612.

([67]) MONGE, G.: *Application de l'Analyse à la Géométrie*, 5th ed., revised, corrected, and annotated by Liouville. Note 6, pp. 609. Paris, 1850.

([68]) MAXWELL, J. C.: "Solutions of Problems," Cambr. and Dubl. Math. J. **8** (1854), 188-193; Sci. Pap., s. 1, pp. 74-79.

in the eyes of fish, he established that the index of refraction n in the lens is independent of position, and indeed in the following way. If one denote the distance from a point of the lens in the eye to its center and n denotes the index of refraction at the point in question then the equation:

$$n = \frac{2ab}{b^2 + r^2} \tag{73.1}$$

will be fulfilled, in which a and b mean positive constants. Now, MAXWELL imagined the entire space to be occupied by a medium whose index of refraction obeyed the rule (73.1), and discovered that under the propagation of light a stigmatic map of the space into itself arose. The light rays themselves are then circular (or rectilinear). One confirms this result most simply when one remarks that in the equation:

$$\left.\begin{aligned} d\sigma &= \frac{2ab}{b^2 + r^2}\sqrt{dx^2 + dy^2 + dz^2}, \\ r^2 &= x^2 + y^2 + z^2, \end{aligned}\right\} \tag{73.2}$$

the differential $d\sigma$, which defines the optical length of a line element in the interior of the MAXWELLian fisheye, can also be interpreted as a line element in a three-dimensional boundary of a four-dimensional sphere of radius a that is projected stereographically onto a space of x, y, z that should be found at a distance of b from the center of the projection.

Namely, if one denotes the rectangular coordinates of the point in four-dimensional space by ξ, η, ζ, and τ then the transformation formulas for the stereographic projection will read:

$$\xi = \frac{2abx}{b^2 + r^2}, \quad \eta = \frac{2aby}{b^2 + r^2}, \quad \zeta = \frac{2abz}{b^2 + r^2}, \quad \tau = a\frac{b^2 - r^2}{b^2 + r^2}. \tag{73.3}$$

One further calculates the relations:

$$\xi^2 + \eta^2 + \zeta^2 + \tau^2 = a^2, \tag{73.4}$$

$$x = \frac{b\xi}{a + \tau}, \qquad y = \frac{b\eta}{a + \tau}, \qquad z = \frac{b\zeta}{a + \tau}, \qquad r^2 = \frac{b^2(a - \tau)}{a + \tau} \tag{73.5}$$

from these equations.

By differentiating one or the other of the systems of formulas (73.3), (73.5), it will follow further that:

$$d\sigma^2 = d\xi^2 + d\eta^2 + d\zeta^2 + d\tau^2 = \frac{4a^2b^2(dx^2 + dy^2 + dz^2)}{(b^2 + r^2)^2}. \tag{73.6}$$

If one let x, y, z (x', y', z', resp.) denote the stereographic projections of two opposite points on the four-dimensional sphere with the coordinates ξ, η, ζ ($-\xi$, $-\eta$, $-\zeta$, resp.) then one must write:

$$x' = -\frac{b^2 x}{r^2}, \quad y' = -\frac{b^2 y}{r^2}, \quad z' = -\frac{b^2 z}{r^2}, \quad r' = \frac{b^2}{r^2}. \tag{73.7}$$

The great circles of the sphere will be determined by the intersection of two hyperplanes:

$$A_k\,\xi + B_k\,\eta + C_k\,\zeta + b \cdot D_k\,\tau = 0 \qquad (k = 1, 2), \tag{73.8}$$

and their projections onto the space of x, y, z will satisfy the equations:

$$D_k\,(x^2 + y^2 + z^2 - b^2) - 2A_k\,x - 2B_k\,y - 2G_k\,z = 0 \qquad (k = 1, 2). \tag{73.9}$$

The light rays now coincide with the images (73.9) of the great circles of our four-dimensional sphere. These images are, however, the circles (or lines) in the space of x, y, z that contain two diametrically opposite points of the outer surface of the three-dimensional sphere:

$$x^2 + y^2 + z^2 = b^2. \tag{73.10}$$

They will be characterized by the facts that their planes will contain the coordinate origin O and that the power of the point O relative to any of these circles is always equal to $-b^2$. Thus, if A is a point of space that is different from the center O of the fisheye then any light ray through A will be circular and will contain a fixed point A_1 that lies on the extension of the line segment AO and is determined by the relation $AO \times OA_1 = b^2$. The fisheye is then a absolute optical instrument that maps the point A to the point A_1. These two points correspond to diametrically opposite points of the four-dimensional sphere. Here, one can verify the theorem of the previous paragraph with no further calculation. Here, in fact, the equality of the optically lengths of corresponding curves will follow immediately from the fact that the spherical length of two diametrically opposite curve segments is the same for both curves ([69]).

74. Stigmatic maps of surfaces that lie tangentially to the field of the instrument. We would like to say that a curve lies tangentially to the field of the instrument when the light rays that contact that curve go through the instrument. A two-dimensional surface patch $\mathfrak{F}$ that contains at least one pencil of curves that lie tangentially will be described similarly. We now assume that a surface patch:

$$x_j = \varphi_j\,(t, u) \qquad (j = 1, 2) \tag{74.1}$$

that lies tangentially to the field of the instrument is mapped stigmatically. One can then choose equations (62.5) and (62.7) for the coupling of the object space with the image space, possibly after performing a canonical sliding transformation, in such a way that after substituting the values (74.1) for the x_j, the three functions $t'(t, x_j, y_j)$ and $x'_j(t, x_j, y_j)$

([69]) CARATHÉODORY, C.: "Über den Zusammenhang der absoluten optischen Instrumente mit einem Satze der Variationsrechnung," S.-B. Bayer. Akad. Wiss. Math.-naturwiss. (1926), 1-18. One finds a generalization of the MAXWELLian fisheye in W. LENZ: "Zur Theorie der optischen Abbildingen," Sommerfeld-Festschrift, pp. 198-207, edited by P. DEBYE. Leipzig, Hirzel, 1928.

will be independent of the y_j. It then follows from (62.12) that after substituting the value (74.1) for the x_j in the function $\Psi(t, x_j, y_j)$, it must likewise follow that it is independent of y_j, and one proves in a similar way to the one in § 70 (or in § 71) that two corresponding curve segments on the surface $\mathfrak{F}$ in the object space and on the surface $\mathfrak{F}'$ in the image space must have equal optical lengths, and that the two surface patches $\mathfrak{F}$ and $\mathfrak{F}'$ *can then be optically unrolled (abwickelt) from each other*.

75. This last result seems to contradict the results of § 48. In that paragraph, we were able to choose the mutually stigmatically related surfaces *in a completely arbitrary way*. The contradiction is resolved when one shows: If the mutually stigmatically mapped surfaces $\mathfrak{F}$ and $\mathfrak{F}'$ cannot be optically unrolled from each other then they also cannot lie tangentially to the field of the instrument.

We assume, e.g., that the two media are isotropic at two corresponding points P and P' of the mutually stigmatically related surfaces, such that one can write:

$$H = -\sqrt{n^2 - y_1^2 - y_2^2}\,, \qquad H' = -\sqrt{n'^2 - y_1'^2 - y_2'^2}\,, \tag{75.1}$$

when one employs rectangular coordinate axes. We let p, q, r denote the components of a unit vector that coincides with the tangent to the light ray at the point P with respect to the axes x_1, x_2, and t, and let p', q', r' denote the components of the corresponding vector in image space. One then has, first, the equations:

$$p^2 + q^2 + r^2 = 1, \qquad p'^2 + q'^2 + r'^2 = 1, \tag{75.2}$$

and secondly, from:

$$\frac{p}{r} = \frac{y_1}{\sqrt{n^2 - y_1^2 - y_2^2}}\,, \qquad \frac{q}{r} = \frac{y_2}{\sqrt{n^2 - y_1^2 - y_2^2}}\,, \quad \dots$$

one calculates the relations:

$$y_1 = n\,p, \qquad y_2 = n\,q, \qquad H = -\,n\,r, \tag{75.3}$$

$$y_1' = n'\,p', \qquad y_2' = n'\,q', \qquad H' = -\,n'\,r'. \tag{75.4}$$

If one chooses the t and t' axes to be parallel to the normal to the surfaces $\mathfrak{F}$ and $\mathfrak{F}'$, resp., at the points P and P', resp. then equations will exist between the y_j and the y_i' that are completely analogous to equations (48.6) and (48.7), and can be written:

$$y_1 = \frac{\partial \omega_0}{\partial x_1} + y_1' \frac{\partial x_1'}{\partial x_1} + y_2' \frac{\partial x_2'}{\partial x_1}\,, \tag{75.5}$$

$$y_2 = \frac{\partial \omega_0}{\partial x_2} + y_1' \frac{\partial x_1'}{\partial x_2} + y_2' \frac{\partial x_2'}{\partial x_2}\,. \tag{75.6}$$

Now, one can always rotate the coordinate axes around the t (t', resp.) axis in such a way that the quantities $\partial x_2' / \partial x_1$ and $\partial x_1' / \partial x_2$ vanish at the points considered. It will then follow from the last equations that one can write:

$$\alpha p' = p + a, \quad \beta q' = q + b. \tag{75.7}$$

However, from (75.2), one will have:

$$p^2 + q^2 \le 1, \quad p'^2 + q'^2 \le 1, \tag{75.8}$$

such that from (75.7), one will also have:

$$\frac{(p+a)^2}{\alpha^2} + \frac{(q+a)^2}{\beta^2} \le 1. \tag{75.9}$$

Now, in order for light rays to go through the point P at all, the ellipse (75.9) must have common interior points with the circle $p^2 + q^2 \le 1$, and the isolated rays that lie tangentially to the field of the instrument must correspond to the common points of the boundaries of these two surface patches. Therefore, if the one surface patch, along with its boundary, lies completely within the interior of the other one then there will be no isolated rays of that kind. In general, the circle and the ellipse will intersect, and there will be a *finite* number of rays, which can be at most *four*, that lie tangentially to the field of the instrument. Finally, *infinitely many* rays can also have this property. The latter can only occur when one has:

$$a = b = 0, \qquad a^2 = b^2 = 1; \tag{75.10}$$

i.e., when the ellipse (75.9) coincides with the unit circle.

One now remarks that the coefficients a and b will vanish only when the derivatives of ω_0 at the point P are equal to zero. If the condition (75.10) is then fulfilled, not only at the point P itself, but also in a neighborhood of that point, then ω_0 must be constant, which agrees with the result of § 70. One further remarks that the coefficients α and β will represent the expansion ratio of the two line elements in the surfaces $\mathfrak{F}$ and $\mathfrak{F}'$ that coincide with the axes at the points P and P', resp., when one measures their lengths as light path lengths. If $\alpha = \beta$ then, as is known, this expansion ratio must be the same for *all* directions. Since the surfaces $\mathfrak{F}$ and $\mathfrak{F}'$ can be optically unrolled from each other in the case (75.10), we can then deduce a new proof of the result in § 74 from the second of equations (75.10).

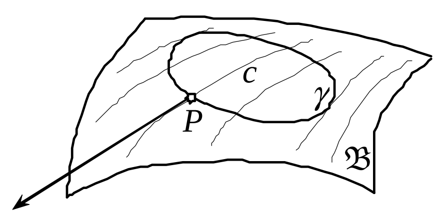

Figure 8.

76. The map of the focal surfaces of ray congruences. We consider two associated ray congruences in two optically coupled spaces that possess real, non-decomposable focal surfaces. The ray map will then be described completely when we give the focal surface, the families of curves on each of them that are enveloped by the rays of the congruences in question, and finally the association of the points of the two focal surfaces with each other that will be generated by the ray map.

If we prescribe all of these data arbitrarily then the integral invariants of §§ 26 and 27 will not necessarily preserve their values when one goes from the object space to the image space, and we must exhibit the condition that will express the conservation of the invariance. To that end, we consider (cf., Fig. 8) a closed curve γ on the one focal surface $\mathfrak{B}$ that goes through the enveloping c of the congruence of light rays. The totality of light rays of our congruence that meet the curve γ defines a tubular surface for which the invariant J can be obtained by the construction that was described at the end of § 27 (cf., Fig. 5, pp. 35). For the case in which the optical medium is isotropic and homogeneous, this invariant will have a very intuitive meaning. Namely, if we assume that γ possesses the form of a curvilinear rectangle for which the two adjacent sides coincide with path curves that belong to the family of curves c, while the two opposite sides are defined by orthogonal trajectories of the family of curves c then we will see, with no further ado, that we must have:

$$J = n\,(s' - s) \tag{76.1}$$

when one denotes the lengths of the latter sides by s' and s. In fact, the orthogonal trajectories consist of the generators of the tubular ruled surface considered as evolutes of the sides of the rectangle that coincide with the curve segments of the family c and of curves that are parallel to the remaining sides of the rectangle.

This geometric interpretation will allow us to characterize the function under the double integral in (26.3) by geometric determining pieces, as well. Namely, to calculate the difference $(ds' - ds)$ between the lengths of the sides of an elementary rectangle of the same kind as the one just considered and obtain a figure for which the lengths ds' and ds of the sides and the surface area $d\omega$ of the rectangle remain unchanged up to quantities of third (fourth, resp.) order. From Fig. 9 below, we now have:

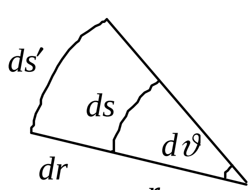

Figure 9.

$$ds = r\,d\vartheta, \qquad ds' = (r + dr)\,d\vartheta, \qquad d\omega = dr \cdot ds, \tag{76.2}$$

in the plane, from which it will follow that:

$$ds' - ds = dr\,d\vartheta = \frac{1}{r}\,d\omega. \tag{76.3}$$

However, $1 / r$ is equal to the curvature of the projected curve, so it is equal to the geodetic curvature k_g of the original curve c. In place of (76.1), we can then write:

$$J = n \iint k_g\,d\omega. \tag{76.4}$$

77. We let $\mathfrak{B}'$ denote the focal surface of the given ray congruence in image space, and let c^* denote the curve in $\mathfrak{B}'$ that is enveloped by rays of the congruence. Should the integral invariant (76.4) be preserved, then the relation:

$$n' k_g^* \, d\omega' = n \, k_g \, d\omega \tag{77.1}$$

would have to exist at corresponding points P and P' of the focal surfaces $\mathfrak{B}$ and $\mathfrak{B}'$.

One remarks that an expression appears on each side of the last equation whose numerical value does not change when one modifies the unit of length. If one chooses this unit of length to be equal to the distance that the light moves through each medium in a given time then one will have $n = n'$, and equations (77.1) will say that *the ratio* $k_g^* : k_g$ *of the geodetic curvatures of the curves* c *and* c^* *at corresponding points of the focal surfaces is equal to the dilatation of area* $d\omega : d\omega'$ *that is induced by the map of the two focal surfaces to each other.*

This theorem expresses the requirement of the conservation of the integral invariant J (§ 25), or – what amounts to the same thing – the conservation of the LAGRANGE bracket in the event that one puts the focal surfaces at the center of consideration.

If one of the two ray congruences is a normal congruence then the integral invariant J must vanish identically, and for that reason we will have $k_g = k_g^* = 0$. The envelopes of the rays of the congruence are, in this case, geodetic lines on the focal surface. Conversely, the two ray congruences will always be normal congruences when the family of curves c consists of geodetic lines on the focal surface in object space; the curves c^* must also be geodetic lines on the focal surface in image space then.

78. This result can be generalized: Completely similar theorems are valid when the optical spaces in question are either homogeneous or isotropic. The formulas that we presented have especial practical significance, but also in the ordinary case of homogeneous, isotropic media, where they permit the introduction of arbitrary curvilinear coordinates.

We consider a surface:

$$t = t(s, u), \qquad x_i = x_i(s, u) \qquad (i = 1, 2) \tag{78.1}$$

in the space of (t, x_1, x_2) that depends upon the parameters s and u. The curves on these surfaces should be established by equations of the form:

$$u = u(s) \tag{78.2}$$

(and thus not in a parameter representation). Now, if $H(t, x_j, y_j)$ is the HAMILTONian function of the optical spaces in question then the line elements s, u, du / ds of the curve (78.2), when one interprets them as spatial line elements, will correspond to certain values of the conjugate variables y_j that one can calculate from the equations:

$$\left(\frac{\partial x_i}{\partial s}+\frac{\partial x_i}{\partial u}\frac{du}{ds}\right)-H_{y_i}\left(\frac{\partial t}{\partial s}+\frac{\partial t}{\partial u}\frac{du}{ds}\right)=0 \qquad (i = 1, 2). \tag{78.3}$$

However, instead of considering the y_j to be functions of s, u, du / ds in equations (78.3), we try to introduce a new variable v and determine y_1, y_2, du / ds, as well as another function K, as functions of u, s, and v. In order to do that, we establish the three functions $y_1(s, u, v)$, $y_2(s, u, v)$, and $K(s, u, v)$, which should satisfy, along with equations (78.3), the identity:

$$-H\,dt + y_i\,dx_i = -K\,dt + v\,du, \tag{78.4}$$

which is equivalent to the two equations:

$$-H(t, x_j, y_j)\frac{\partial t}{\partial u}+y_i\frac{\partial x_i}{\partial u} = v, \tag{78.5}$$

$$+H(t, x_j, y_j)\frac{\partial t}{\partial s}-y_i\frac{\partial x_i}{\partial s} = K. \tag{78.6}$$

One can calculate y_1, y_2, and du / ds as functions of (s, u, v) from (78.3) and (78.5) and then obtain $K(s, u, v)$ with the help of (78.6). This latter function $K(s, u, v)$ can be regarded as the HAMILTONian function of a variational problem that is *coupled* with given problem on the surface (78.1). One refers to it as the variational problem that is *induced* on the surface by the original problem ([70]).

If follows from the relation (78.4) that the calculations of § 66 can be carried over here when one replaces t', x', y' with s, u, v, resp., and writes K, instead of H'. In particular, when one thinks of t and x_i as independent of v, it will follow from (66.6) that:

$$du - K_v\,ds = \frac{\partial y_j}{\partial v}(dx_j - H_{y_j}dt)\,;$$

with consideration given to (78.3), one will then have:

$$\frac{du}{ds} = K_v\,. \tag{78.7}$$

In an entirely similar way, if will follow from (66.7) that:

$$dv + K_u\,ds = \left(\frac{\partial x_j}{\partial u}-H_{y_j}\frac{\partial t}{\partial u}\right)(dy_j + H_{x_j}dt)\,, \tag{78.8}$$

which is a relation that can also be written:

([70]) *Variationsrechnung*, § 342.

$$\frac{dv}{ds} + K_u = \frac{\partial(t, x_j)}{\partial(s, u)} \left(\frac{dy_j}{dt} + H_{x_j} \right), \tag{78.9}$$

due to (78.3).

79. In order to now define a family of curves c on the surface (78.1), it will suffice to take:

$$v = \varphi(s, u). \tag{79.1}$$

A closed curve γ on the same surface corresponds to a closed curve γ^* in the su-plane, and, as a result of the relation (78.4), POINCARÉ's relative integral invariant J for the light rays that contact the curve c at the points of γ can be written:

$$J = \int_{\gamma^*} -K(s, u, \varphi(s, u))\, ds + \varphi(s, u)\, du\,. \tag{79.2}$$

If one now transforms this boundary integral into a double integral then one will get:

$$J = \iint_{G^*} ((K_u + K_v \varphi_u) + \varphi_s)\, du\, ds\,. \tag{79.3}$$

Due to equation (78.7), the function under the integral can be written:

$$\frac{d\varphi}{ds} + Ku(s, u, \varphi(s, u)); \tag{79.4}$$

it will then have the same form as the left-hand side of (78.9). In the special case of § 76, it is self-explanatory that this function will have the same geometric interpretation as in (76.4).

We now consider a second variational problem in the space of t', x'_i of a surface $\mathfrak{B}'$ whose one-to-one map on (78.1) will be established in such a way that we can represent $\mathfrak{B}'$ by the equations:

$$t' = t'(s, u), \qquad x'_i = x'_i(s, u) \tag{79.5}$$

and establish that points of the surfaces (78.1) and (79.5) should correspond to each other when they belong to the same values of the parameters s, u. Now, a variational problem will be induced on the surface $\mathfrak{B}'$ whose HAMILTONian function can be calculated in a manner that is entirely similar to the previous one with the introduction of a new variable v'; it will be denoted by $K'(s, u, v')$. Furthermore, we determine a family of curves c^* on $\mathfrak{B}'$ by the equation:

$$v' = \varphi^*(s, u), \tag{79.5)[sic}$$

and consider the congruence of light ray that possesses $\mathfrak{B}'$ as its focal surface and the curves c^* as its envelope. The condition that is analogous to the condition (77.1), which says that the two ray congruences that possess $\mathfrak{B}$ ($\mathfrak{B}'$, resp.) as focal surfaces will be optically coupled, will be expressed by the equation:

$$\varphi_s + K_v(s, u, \varphi)\, \varphi_u + K_u(s, u, \varphi) = \varphi_s^* + K'_{v'}(s, u, \varphi^*)\, \varphi_u^* + K'_u(s, u, \varphi^*)\,. \tag{79.6}$$

80. The latter condition gives us the possibility of treating a large number of problems that are connected with the optical coupling of ray congruences.

One can, e.g., prescribe the map of the two focal surfaces to each other and the family of curves c^*; the right-hand side of (79.6) will then be a known function of s, u that we denote by $-\partial f / \partial u$, whereas the function $\varphi(s, u)$ will still be undetermined. The condition (79.6) then says that any family of curves c that defines a family of extremals of the variational problem on the focal surface $\mathfrak{B}$ with the HAMILTONian function:

$$K(s, u, v) + f(s, u) \tag{80.1}$$

will define the envelope of a ray congruence that is optically coupled with the given ray congruence in the space of t', x'. In order to determine the ray map with the given data completely, one can, e.g., give the directions of those rays at the points of a curve segment on the surface $\mathfrak{B}$, because the family of extremals of the problem (80.1) is established by that.

If two congruences of light rays possess the same focal surface, and each of them contact that focal surface along a family of curves that can be interpreted as a family of extremals of the variational problem with the HAMILTONian function (80.1), then one will obtain an optical map of these ray congruences to each other when one associates each two rays that contact their common focal surface at the same point with each other.

One can also determine those curves of the focal surface $\mathfrak{B}$ that are transformed into curves by the *map* of $\mathfrak{B}$ to $\mathfrak{B}'$, for which, the relation (79.6) is valid. In general, these curves must be solutions of a second-order, ordinary differential equation. However, there are also extreme cases for which no single curve of that kind will exist and other ones for which *any* curve of $\mathfrak{B}$ will possess the required property.

One obtains an example of the latter kind when one bends the focal surface $\mathfrak{B}$ with the homogeneous and isotropic propagation of light and carries all pencils of light rays whose centers lie at a point of $\mathfrak{B}$ rigidly under the bending of the surface ([71]).

([71]) Cf., CARATHÉODORY, C.: "Bemerkungen zu den Strahlenabbilungen der geometrischen Optik," Math. Ann. **114** (1937), 187-193.

Chapter V.

The map in the first approximation.

81. The formulas for the accessory problem. If the two optical spaces are isotropic and homogeneous then the HAMILTONian functions for the propagation of light will have the form:

$$H = -\sqrt{n^2 - y_1^2 - y_2^2}\,, \qquad H' = -\sqrt{n'^2 - y_1'^2 - y_2'^2}\,. \tag{81.1}$$

We would like investigate the ray map in the near vicinity of two arbitrary corresponding rays. Due to the isotropy and homogeneity of the space, it is no restriction to let the two corresponding rays coincide with the t (t', resp.) axis. Using a method that was developed in the calculus of variations for the theory of the second variation and in mechanics for the theory of small oscillations, we replace the HAMILTONian functions (81.1) with functions H, H' of the so-called "accessory problem"; we obtain these latter functions when we develop H and H' in powers of y_j, y_j', and keep only the lowest powers. One must then write:

$$\mathsf{H} = -n + \frac{y_1^2 + y_2^2}{2n}, \qquad \mathsf{H}' = -n' + \frac{y_1'^2 + y_2'^2}{2n'}, \tag{81.2}$$

and the basic functions of the corresponding variational problems will read:

$$\Lambda = n\left(1 + \frac{\dot{x}_1^2 + \dot{x}_2^2}{2}\right), \qquad \Lambda' = n'\left(1 + \frac{\dot{x}_1'^2 + \dot{x}_2'^2}{2}\right). \tag{81.3}$$

As a result, the light rays will have the equations:

$$x_i = u_i + y_i\,\frac{t}{n}, \qquad x_i' = u_i' + y_i'\,\frac{t'}{n'}. \tag{81.4}$$

The ray map for the original problem shall likewise be replaced with another one that will be obtained from a similar consideration. It is defined when we let u_i', y_i' be linear, homogeneous functions of u_j, y_j that shall satisfy the condition that the expression:

$$y_i'\,du_i' - y_j\,du_j$$

is a complete differential.

Remark. The interpretation of the formulas for the linear ray map can come about in two fundamentally different ways.

For the first of these interpretations, one starts with families of light rays in the object and images spaces *that are coupled to each other by the original problem that was posed,* and which depend upon a parameter α. The value $\alpha = 0$ shall be associated with the basic rays; i.e., the t (t', resp.) axis. The rays of such a family will be represented by the functions $x_j(t, \alpha)$, $y_j(\alpha)$, $x_i'(t', \alpha)$, $y_i'(\alpha)$, their initial elements in the planes $t = t_0$ and $t' = t_0'$ will be represented by functions $u_j(\alpha)$, $u_i'(\alpha)$, and the ray map in question will be represented by equations that will perhaps read as follows:

$$u_i'(\alpha) = A_i\,(u_j(\alpha), y_j(a)), \qquad y_i'(\alpha) = B_i\,(u_j(\alpha), y_j(a))\,.$$

One develops all of these functions and equations in powers of α and remarks that for sufficiently small values of this parameter the coupling of the two optical spaces in the neighborhood of the basic rays can be represented by the linear terms in this power series approximately.

The first interpretation of the formula is the one that the physicists have discussed the most, and it will be employed as a rule.

For the second interpretation, which will be applied in what follows, we consider linear couplings between two space *whose optical properties are characterized, not by the original HAMILTONian functions* (81.1), *but by the HAMILTONian functions* (81.2). We assume that this should be the case on all of space and study the mapping rule, without considering the original map in any way. This has the advantage that we do not have an approximation problem before us, but an ordinary, optical problem, to which, all of our previous methods and results can be applied without restriction.

After this "osculating" problem has been examined for its own sake, one can, if one needs to, employ the fact that the two maps deviate from each other only slightly in the vicinity of the basic rays.

82. One will obtain the linear ray maps that we have just discussed most quickly when one employs the theory of the eikonal. Since the partial derivatives of the eikonal should all be linear and homogeneous, the eikonal itself must be a quadratic form in the four variables upon which it depends. It would be computationally advantageous if one could treat all possible cases with the angle eikonal alone, since in that case a displacement of the starting point along the t (t', resp.) axis would give rise to a very simple transformation of the eikonal. One must then write:

$$\left.\begin{aligned} 2W = (a_{11}y_1^2 + 2a_{12}y_1y_2 + a_{22}y_2^2) + (\alpha_{11}y_1'^2 + 2\alpha_{12}y_1'y_2' + \alpha_{22}y_2'^2) \\ + 2p_{11}y_1y_1' + 2p_{12}y_1y_2' + 2p_{21}y_2y_1' + 2p_{22}y_2y_2', \end{aligned}\right\} \tag{82.1}$$

and must calculate the ray map from:

$$u_i'\,dy_i' - u_j\,dy_j = dW. \tag{82.2}$$

One will then find that:

$$\left.\begin{aligned} u_1' &= \alpha_{11}y_1' + \alpha_{12}y_2' + p_{11}y_1 + p_{21}y_2, \\ u_2' &= \alpha_{12}y_1' + \alpha_{22}y_2' + p_{12}y_1 + p_{22}y_2, \end{aligned}\right\} \tag{82.3}$$

and two similar relations for the u_j that we will not, however, require in what follows.

The eikonal W is, however, useful only when no relation exists between the four variables y_i', y_j, and it can very well happen that such a relation is actually present.

One now remarks that a comparison of (81.4) with (82.3) will yield the equations:

$$x_i' = \left(\alpha_{11} + \frac{t'}{n'}\right) y_1' + \alpha_{12} y_2' + p_{11}\, y_1 + p_{21}\, y_2\,,$$

$$y_i' = \alpha_{12} y_1' + \left(\alpha_{22} + \frac{t'}{n'}\right) y_2' + p_{12}\, y_1 + p_{22}\, y_2\,.$$

Here, one can always give the variable t' a value for which these latter equations are soluble for the y_i', from which, one easily concludes that in all cases for which the angle eikonal is useful (after a possible displacement of the starting point along the t'-axis), the mixed eikonal, which depends upon the y_j and the u_i', can also be employed. One can reach precisely the same conclusion when one starts with one of the skew eikonals, and one sees that in order to avoid distinguishing between the cases one would do best to perform our calculations with only that mixed eikonal from the outset. Due to the results of § 45, we are guaranteed that we can represent all possible linear ray maps with the help of this eikonal.

83. We must then discuss all cases in which the map is derived from the identity:

$$y_i'\, du_i' + u_j\, dy_j = dV, \tag{83.1}$$

where the eikonal V reads:

$$\left.\begin{aligned} 2V = (a y_1^2 + 2b y_1 y_2 + c y_2^2) + (\alpha u_1'^2 + 2\beta u_1' u_2' + \gamma u_2'^2) \\ + 2p y_1 u_1' + 2q y_1 u_2' + 2r y_2 u' + 2s y_2 u_2'. \end{aligned}\right\} \tag{83.2}$$

One can, with no loss of generality, simplify the form of the eikonal when one gives special positions to the coordinate axes. Namely, if one sets:

$$y_1 = \bar{y}_1 \cos\vartheta - \bar{y}_2 \sin\vartheta, \qquad y_2 = \bar{y}_1 \sin\vartheta + \bar{y}_2 \cos\vartheta, \tag{83.3}$$

$$u_1' = \bar{u}_1' \cos\vartheta - \bar{u}_2' \sin\vartheta, \qquad u_2' = \bar{u}_1' \sin\vartheta + \bar{u}_2' \cos\vartheta \tag{83.4}$$

then one can choose the angles ϑ and φ in such a way that after calculating the coefficients in the new variables the relations:

$$b = 0, \quad \beta = 0, \quad a \geq c, \quad \alpha \geq \gamma \tag{83.5}$$

will exist.

In special cases, one can push the simplification even further. For example, if $b = 0$, along with $a = c$, is, by chance, true from the outset then the angle ϑ will be undetermined, and one can employ the rotation (83.3) around the t-axis in order to obtain a relationship between the coefficients p, q, r, s by which the geometric properties of the ray map will emerge more quickly. In particular, if one denotes the values of the new coefficients by $\tilde{p}$, … then one will have:

$$2(\bar{p}\,\bar{r} + \bar{q}\,\bar{s}) = 2pr \cos 2\vartheta - (p^2 + q^2 - r^2 - s^2) \sin 2\vartheta.$$

If $b = 0$ and $a = c$ then can always assume that $pr + qs = 0$.
If $\beta = 0$ and $\alpha = c$ then one can likewise always assume that $pr + qs = 0$.

However, instead of this, one can, if one desires, also arrive simply at the fact that $q = 0$ in each of these cases. Finally, if, along with $b = 0$ and $\beta = 0$, one simultaneously has $a = c$ and $\alpha = \gamma$ from the outset then one can always arrive at the fact that $q = 0$ and $r = 0$ by a suitable choice of the angles ϑ and φ in equations (83.3) and (83.4).

We now return to the general case. From the previously-developed theory, the functional determinant (43.8), which possesses the constant value $ps - qr$ here, must always be non-zero. However, this expression will change sign when one performs the coordinate transformation:

$$\bar{t}' = t', \qquad \bar{x}_1' = x_1', \qquad \bar{x}_2' = -x_2'.$$

One can always assume from the outset that the coordinates are chosen in such a way that:

$$ps - qr > 0. \tag{83.6}$$

From (83.1) and (83.2), our ray map will be established by the following formulas, with consideration given to (83.5):

$$\left.\begin{aligned} y_1' &= \alpha u_1' + p y_1 + r y_2, \\ y_2' &= \gamma u_2' + q y_1 + s y_2, \end{aligned}\right\} \tag{83.7}$$

$$\left.\begin{aligned} u_1 &= \alpha y_1 + p u_1' + q u_2', \\ u_2 &= \gamma y_2 + r u_1' + s u_2'. \end{aligned}\right\} \tag{83.8}$$

In conclusion, we would like to derive the condition for a ray map to be rotationally symmetric. From § 60, the equation:

$$u_2\, y_1 - u_1\, y_2 - u_2'\, y_1' + u_1'\, y_2' = \lambda$$

must be satisfied identically in order for this to be true. By replacing the values (83.7), (83.8), one finds that $\lambda = 0$, and that one must have:

$$a = c, \qquad \alpha = \gamma, \qquad q + r = 0, \qquad p - s = 0.$$

From the remark above, one can then choose the mutual positions of the two coordinate systems in such a way that $q = r = 0$, in addition.

84. Coupling of the spaces. From the remark at the end of § 32, all of these formulas are valid for the accessory problem of a ray map for which the HAMILTONian functions H and H' are completely arbitrary. The formulas that will be written from now on, which are obtained by comparing relations (81.4) with (83.7) and (83.8), are, however, valid only under the assumption that the accessory problem possesses the HAMILTONian function (81.2) and the basic function (81.3). These formulas read:

$$\left.\begin{aligned} x_1 &= \left(a + \frac{t}{n}\right) y_1 + p u_1' + q u_2', \\ x_2 &= \left(c + \frac{t}{n}\right) y_2 + r u_1' + s u_2', \end{aligned}\right\} \tag{84.1}$$

$$\left.\begin{aligned} x_1' &= \left(1 + \alpha \frac{t}{n}\right) u_1' + (p y_1 + r y_2)\frac{t'}{n'}, \\ x_2' &= \left(1 + \gamma \frac{t}{n}\right) u_2' + (q y_1 + s y_2)\frac{t'}{n'}. \end{aligned}\right\} \tag{84.2}$$

For any arbitrary choice of the four parameters y_j, u_i', these equations represent two rays that will be associated with each other by our map. If one adds yet another arbitrary relation of the form:

$$t' = t'(t, y_j, u_i') \tag{84.3}$$

to these equations then one will obtain a coupling of the spaces in question to which the theory of Chapter IV is applicable.

In what we have been doing up to now, the t and t' axes played a special role. This assumption is, however, only apparent: Namely, with the help of an almost trivial artifice, *any* two rays:

$$x_i = U_i + V_i \frac{t}{n}, \qquad x_i' = U_i' + V_i' \frac{t'}{n'} \tag{84.4}$$

that are associated by the ray map in question can appear in place of the two axes; i.e., ones for which the relations (83.7), (83.8) are fulfilled when one replaces u_i with U_i, u_i' with U_i', y_i with V_i, and y_i' with V_i'. In order to show this, we consider a collineation of the space t, x_i to a space t, ξ_i, and a collineation of the space t', ξ_i' that is defined by the equations:

$$x_i = U_i + V_i \frac{t}{n} + \xi_i, \qquad x_i' = U_i' + V_i' \frac{t'}{n'} + \xi_i'. \tag{84.5}$$

Two rays:

$$x_i = u_i + y_i \frac{t}{n}, \qquad x_i' = u_i' + y_i' \frac{t'}{n'}, \tag{84.6}$$

will be transformed into the lines:

$$\xi_i = \bar{u}_i + \eta_i \frac{t}{n}, \qquad \xi_i' = \bar{u}_i' + \eta_i' \frac{t'}{n'} \tag{84.7}$$

by these collineations. Now, the relations:

$$\bar{u}_i = u_i - U_i, \quad \eta_i = y_i - V_i, \quad \bar{u}_i' = u_i' - U_i', \quad \eta_i' = y_i' - V_i' \tag{84.8}$$

exist between the coefficients, and we see that since equations (83.7) and (83.8) are fulfilled for u_i, y_i, u_i', y_i', as well as for U_i, V_i, U_i', V_i', they must also be satisfied for $\bar{u}_i$, η_i, $\bar{u}_i'$, η_i'. They then say that the two rays (84.6) correspond to each other.

An important application of this remark is the following: In some situations, one can easily recognize that there is at least one stigmatic pencil of light with the center t_0, $x_1 = x_2 = 0$ that is transformed into a stigmatic pencil of light with the center t_0', $x_1' = x_2' = 0$. If this is the case then, due to the last result, one can conclude that *any* point of the plane $t = t_0$ will be mapped stigmatically to a point in the plane $t' = t_0'$.

Later on, we will see that the rotationally symmetric systems are the only ones for which *any* stigmatic pencil of light will be taken to another stigmatic one. It follows from our remark above that any cone of light rays that meets the planes t = const. in circles will be mapped onto a cone of light rays that possesses a similar property.

85. The images of stigmatic pencils of light. From equations (84.4), we get:

$$py_1 + ry_2 = \frac{px_1}{\frac{t}{n}+a} + \frac{rx_2}{\frac{t}{n}+c} - \frac{p^2u_1' + pqu_2'}{\frac{t}{n}+a} - \frac{r^2u_1' + rsu_2'}{\frac{t}{n}+c}; \tag{85.1}$$

once we have also calculated $qy_1 + sy_2$ in a similar way, we can write the quantities x_i' as functions of t, x_j, and t', u_i'. To abbreviate, we introduce the notations:

$$\left.\begin{aligned} A &= \alpha - \frac{p^2}{\frac{t}{n}+a} - \frac{r^2}{\frac{t}{n}+c}, \\ B &= \beta - \frac{pq}{\frac{t}{n}+a} - \frac{rz}{\frac{t}{n}+c}, \\ C &= \gamma - \frac{q^2}{\frac{t}{n}+a} - \frac{s^2}{\frac{t}{n}+c}. \end{aligned}\right\} \tag{85.2}$$

Since the expressions A, B, C have the dimension of reciprocal lengths, it is, in addition, preferable to set:

$$\frac{t'}{n'} = \frac{1}{z'}. \tag{85.3}$$

We then get:

$$\left.\begin{aligned} z'x_1' &= (z' + A)u_1' + Bu_2' + \frac{px_1}{\dfrac{t}{n}+a} + \frac{rx_1}{\dfrac{t}{n}+c}, \\ z'x_2' &= Bu_1' + (z'+C)u_2' + \frac{qx_1}{\dfrac{t}{n}+a} + \frac{sx_1}{\dfrac{t}{n}+c}, \end{aligned}\right\} \tag{85.4}$$

$$\left.\begin{aligned} y_1' &= Au_1' + Bu_2' + \frac{px_1}{\dfrac{t}{n}+a} + \frac{rx_1}{\dfrac{t}{n}+c}, \\ y_2' &= Bu_1' + Cu_2' + \frac{qx_1}{\dfrac{t}{n}+a} + \frac{sx_1}{\dfrac{t}{n}+c}. \end{aligned}\right\} \tag{85.5}$$

If we fix the point t, x_j in formulas (85.4) and let the u_i' vary arbitrarily then they will represent those ray congruences onto which a stigmatic pencil of rays with the center t, x_j will be mapped. These ray congruences consists of the totality of all lines that cut two real rectilinear focal lines. Namely, there are values of z' for which the coefficients of u_1' in the two equations (85.4) will be proportional to the coefficients of u_2'. These values will be determined by the roots of the quadratic equation:

$$(z' + A)(z' + C) - B^2 = 0, \tag{85.6}$$

which one can write explicitly as:

$$z_i' = \frac{-(A+C) \pm \sqrt{(A-C)^2 + 4B^2}}{2}, \tag{85.7}$$

which are then always real.

Each of the two focal rays will then yield a line of intersection that is the intersection of the plane:

$$\frac{t'}{n'} = \frac{1}{z_i'} \qquad (i = 1, 2) \tag{85.8}$$

with another plane that one obtains by eliminating the u_i' from the two equations (85.4), in which one has replaced z' with the chosen root z_i' of equation (85.6).

We let φ_1 and φ_2 denote the angles that the two focal rays subtend with the x_1-axis, so we can then write:

$$\tan \varphi_i = \frac{B}{z'_i + A} = \frac{z'_i + C}{B}. \tag{85.9}$$

It will follow from this that:

$$\tan 2\varphi_i = \frac{2B}{(z'_i + A)\left(1 - \dfrac{z'_i + C}{z'_i + A}\right)} = \frac{2B}{A - C}; \tag{85.10}$$

one therefore always has:

$$\tan 2\varphi_1 = \tan 2\varphi_2 , \tag{85.11}$$

i.e., the two focal rays must be perpendicular to each other.

The only exception is defined by the case in which one simultaneously has $B = 0$ and $A = C$ for certain values of t; this case will be treated thoroughly below (§ 93).

One observes that the ray congruence that is described here, even though it is the image of a stigmatic pencil of light, is not a normal congruence in the ordinary sense of the word. That rests upon the fact that we have replaced the isotropic, homogeneous media from which we started with other ones for which the propagation of light will be described by the basis function (81.3). There are therefore field-like ray congruences for this accessory variational problem, and one can, in particular, easily show that any linear ray congruence whose focal rays are perpendicular to each other and lie in the plane t' = const. will be transversally (in the sense of the variational problem of § 81) intersected by a family of surfaces that satisfy the differential equation:

$$S_{t'} + \frac{1}{2n'}(S_{x'_1}^2 + S_{x'_2}^2) = n'. \tag{85.12}$$

86. In formula (85.2), we set:

$$\frac{t}{n} = z, \tag{86.1}$$

such that z (unlike z') has the dimension of length; if we now develop equation (85.6) then we will get the relation:

$$\left.\begin{array}{c}(z'+\alpha)(z'+\gamma)(z+a)(z+c) - p^2(z'+\gamma)(z+c) - q^2(z'+\alpha)(z+c) \\ -r^2(z'+\gamma)(z+a) - s^2(z'+\alpha)(z+a) + (ps - rq)^2 = 0.\end{array}\right\} \tag{86.2}$$

In a projective plane with the coordinates z, z', this equation will represent a fourth-order curve with a double point at the points at infinity on the z and z' axes. It always has the form of a "double hyperbola" (cf., Fig. 10), even when a third, finite, double point is present. However, this can occur only when $B = 0$ and $C = A$ are true simultaneously for a certain value of t. In this case, the curve will have the form in Fig. 11 when one ignores some further exceptional cases, and is unicursal, as is suggested in the same figure by arrows. It is known that one can then represent this curve analytically when one makes z

and z' rational functions of a parameter λ. One finds these rational functions most quickly in the following way: In order for the curve (86.2) to possess a third double point, a relation must exist between the coefficients of our transformation that has the effect that one can factor out a complete square from the function under the square root in (85.7) and put it in front of the square root. Only a quadratic function of z now remains under the root, and one can therefore simultaneously write z itself and the square root as rational functions of a parameter λ in a known way, by means of which, z' can also be expressed rationally.

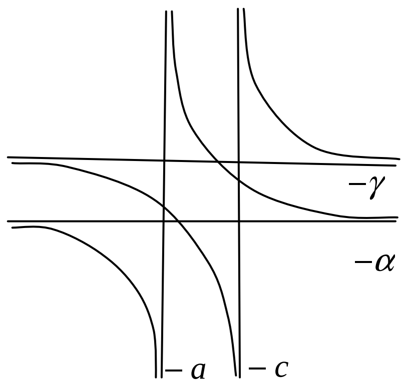

Figure 10.

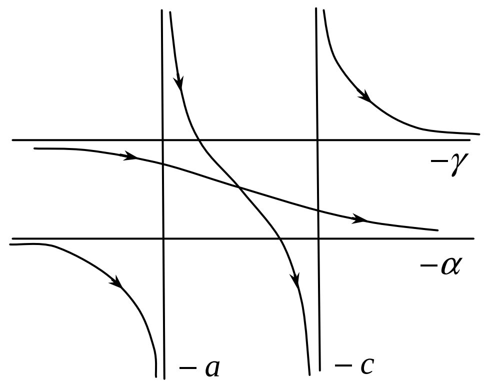

Figure 11

87. Calculating the invariants. It was the contribution of A. GULLSTRAND to have discovered that an already sufficiently-precise classification of our linear maps could be deduced from the study of the behavior of the function:

$$\tan 2\varphi = \frac{2B}{A-C} = -2\,\frac{pq(z+c)+rs(z+a)}{(\alpha-\gamma)(z+a)(z+c)-(p^2-q^2)(z+c)-(r^2-s^2)(z+a)}. \tag{87.1}$$

Here, one actually deals with the classification of the pencil of the two quadratic forms:

$$Q_1 = (pq+rs)\,\zeta_1\,\zeta_2 + (pqc+rsa)\,\zeta_2^2, \tag{87.2}$$

$$\left.\begin{aligned} Q_2 = (\alpha-\gamma)\zeta_2^2 &+ [(\alpha-\gamma)(a+c)-(p^2-q^2)-(r^2-s^2)]\zeta_1\zeta_2 \\ &+ [(\alpha-\gamma)ac-(p^2-q^2)c-(r^2-s^2)a]\zeta_2^2 \end{aligned}\right\} \tag{87.3}$$

that one obtains when one replaces the quantity z with the homogeneous coordinates $\zeta_1 : \zeta_2$ in the numerator and the denominator of (87.1).

One thus starts with the fact that there is always one pair of points (they can possibly coincide or also be imaginary) that are simultaneously conjugate for the two quadratic forms Q_1 and Q_2. These two points determine an involution whose double points they are and is expressed by the equation:

$$(\alpha-\gamma)[pq(z+c)(z_0+c)+rs(z+a)(z_0+a)]-(a-c)(ps-qr)(pr+qs)=0. \tag{87.4}$$

One obtains the double points themselves when one sets the expression:

$$\psi = (\alpha - \gamma)[pq(z + c)^2 + rs(z + a)^2] - (a - c)(ps - qr)(pr + qs) \tag{87.5}$$

equal to zero. The discriminant of this latter quadratic function is equal to the product of $-(\alpha - \gamma)(a - c)$ with the function:

$$\Phi = (\alpha - \gamma)(a - c)\, pqrs - (ps - qr)(pq + rs)(pr + qs). \tag{87.6}$$

Since we have always taken $a \geq c$, and with $a = c$, always $pr + qs = 0$, as well (§ 83), we see that the double points of the involution will be imaginary when $\Phi > 0$, they will be real when $\Phi < 0$, and for $\Phi = 0$ they coincide. One remarks, moreover, that the condition $\Phi = 0$ is also necessary and sufficient for the expression under the square root in (85.7) to vanish for a certain value of z, and therefore also for the fourth-order curve (86.2) to possess a third double point.

One obtains a higher singularity when the quadratic forms Q_1 and Q_2 are both complete squares that vanish for the same values of $\zeta_1 : \zeta_2$. This singularity will be expressed by the equations:

$$\alpha = \gamma, \qquad pq + rs = 0, \qquad p^2 + r^2 = q^2 + s^2. \tag{87.7}$$

Moreover, it will be shown that for the ray map this singularity is completely equivalent to the other one for which all coefficients of Q_1 or all coefficients of Q_2 vanish but the numerator and denominator on the right-hand side of (87.1) are identically zero. Finally, the last one can, however, occur, and the ray map will be rotationally symmetric for a suitable choice of coordinates.

88. For the case in which one has:

$$\Phi = 0, \tag{88.1}$$

the numerator and denominator of the expression on the right-hand side of (87.1) have a common factor. By cancelling this common factor, one gets:

$$\tan 2\varphi = \frac{-2(pq + rs)^2}{(\alpha - \gamma)[pq(z + a) + rs(z + c)] - (pq + rs)(p^2 - q^2 + r^2 - s^2)}. \tag{88.2}$$

If one eliminates $(\alpha - \gamma)$ from this with the help of (88.1) then one will get:

$$\tan 2\varphi = \frac{2(a - c)\, pqrs}{pq(s^2 - r^2)(z + a) + rs(p^2 - q^2)(z + c)}. \tag{88.3}$$

The expression (88.2) can also be employed for $a = c$, while the latter one can be employed only when $a > c$ (cf., § 92).

89. The quadratic function ψ that we presented in § 92 has a remarkable geometric interpretation that is connected with the form of the ray map. Namely, if one differentiates equation (87.1) with respect to z then when one multiplies both sides of $\cos^2 2\varphi$ one will get:

$$n\frac{d\varphi}{dt} = \frac{d\varphi}{dz} = \frac{\psi}{(A-C)^2+4B^2)(z+a)^2(z+c)^2}, \tag{89.1}$$

in which the function ψ on the right-hand side is defined by equation (87.5)

One now remarks that any stigmatic pencil of rays whose center lies on the t-axis itself will be mapped to a ray congruence in the image space that contains two mutually perpendicular pencils of rays whose centers lie on the t'-axis. One calculates these two pencils of rays when one sets:

$$u_1' = -\rho B, \qquad u_2' = \rho(z_1'+A) \tag{89.2}$$

in the formulas of § 85 one time and:

$$u_1' = -\sigma B, \qquad u_2' = \rho(z_2'+A) \tag{89.3}$$

the other time. In this, ρ and σ are variable proportionality factors, and z_1', z_2' are the roots of equation (85.6) such that one will have, in addition:

$$z_1'+z_2' = -(A+C), \qquad z_1'z_2' = AC - B^2. \tag{89.4}$$

The rays of the two pencils that were just considered are images of rays of the stigmatic pencils of light that one starts from. One obtains the directions of these latter two from equation (84.1), in which one must set $x_1 = x_2 = 0$, by the formula:

$$\left.\begin{aligned} y_1 &= -\frac{pu_1'+qu_2'}{z+a}, & y_2 &= -\frac{ru_1'+su_2'}{z+c}, \\ \bar{y}_1 &= -\frac{p\bar{u}_1'+q\bar{u}_2'}{z+a} & \bar{y}_2 &= -\frac{r\bar{u}_1'+s\bar{u}_2'}{z+c}, \end{aligned}\right\} \tag{89.5}$$

in which one must substitute the values of u_i', $\bar{u}_i'$ from (89.2) and (89.3), resp. It follows from this that any of the pencils of rays above must be the image of a pencil of rays in object space, but these two pencils will lie in planes that do not necessarily need to be perpendicular to each other. Namely, if one lets Θ denote the angle that they define between them then one will have:

$$\pm \cot\Theta = \frac{y_1\bar{y}_1 + y_2\bar{y}_2}{y_2\bar{y}_1 - y_1\bar{y}_2}. \tag{89.6}$$

By replacing the values above, one finds that:

$$\pm \cot \Theta = \frac{\psi}{(ps-qr)(z+a)(z+c)\sqrt{(A-C)^2+4B^2}}, \tag{89.7}$$

and by comparing this with (89.1), one will get:

$$(ps - qr)^2 \cot^2 \Theta = \psi \cdot n \, \frac{d\varphi}{dt}. \tag{89.8}$$

The function ψ can thus be expressed in a very simple way with the help of Θ *and* $d\varphi$ / *dt*.

One calls the points of the t-axis for which the angle that was just introduced is $\Theta = \pi / 2$ *orthogonal points.* Equation (89.7) shows that orthogonal points are present in the general case only when the equation $\psi = 0$ has real roots, and equation (89.1) teaches us that these will be the points for which one also has $d\varphi / dt = 0$.

90. Up to now, we have considered the images of stigmatic pencils of light. However, one can, without introducing very many new calculations, consider the analogous problem that one obtains when one exchanges the two optical spaces. Those ray congruences in object space shall then be exhibited that are mapped to stigmatic pencils in the image space. In order to do this, we must solve equations (84.2) for the u_i' and substitute the values thus found in (84.1). The desired ray congruences will then be represented with the help of the parameters y_i . Not only are the results that one obtains completely analogous to the previous ones, but most of the formulas do not need to be recalculated at all. One obtains them from the older ones when one first switches z with z', and then α, γ with a, c, resp., and finally q with r. However, one must not forget that z and z' do not possess mutually corresponding geometric interpretations here, since $z = t / n$ and $1 / z' = t' / n'$. Therefore, some formulas will become considerably more complicated. For example, equation (87.4) gets a corresponding condition for the pairs of coupled points on the t'-axis that takes the form:

$$\left.\begin{aligned} &\frac{t'}{n'}\cdot\frac{t_0'}{n'}[(a-c)(pr\gamma^2+qs\alpha^2)-(\alpha-\gamma)(ps-qr)(pr+rs) \\ &\quad+\left(\frac{t'}{n'}+\frac{t_0'}{n_0'}\right)(a-c)(pry+qs\alpha)+(a-c)(pr+qs)=0. \end{aligned}\right\} \tag{90.1}$$

However, the asymmetry does not extend to the function Φ, which remains invariant under the exchanges that were just given.

91. Twisted and re-twisted systems. We first examine the case in which $\Phi \neq 0$, and remark that, from § 83, we can assume that $\alpha > \gamma$ and $a > c$. One could then take $pq + rs = 0$, along with $\alpha = \gamma$, and one would then, in fact, have $\Phi = 0$.

For $\Phi > 0$, the expression $pq + rs \neq 0$, since for $pq + rs = 0$ the function Φ, from (87.6), must assume the sign of $pqrs$, which is necessarily negative. The function ψ takes the sign of $(\alpha - \gamma)(pq + rs)$ for all values of z. From (89.1), $d\varphi / dt$ is also always a sign, and since $\tan 2\varphi$ tends to zero when $|z|$ becomes infinitely large, and in addition, possesses precisely one zero point for:

$$z = -\frac{pqc + rsa}{pq + rs}, \tag{91.1}$$

the angle 2φ must vary from 0 to 2π (-2π, resp.) when t describes the t-axis; the angle φ itself increases monotonically from zero to π for $\psi > 0$ and decreases monotonically in the interval from zero to $-\pi$.

Following GULLSTRAND, the ray map is then called *twisted* in the case of $\Phi > 0$.

For $\Phi < 0$, it is no longer necessary that one have $pq + rs \neq 0$. However, we would next like to assume – and this is the general case – that this inequality is also fulfilled here. One then has $\tan 2\varphi = 0$ at the point (91.1) again, but, from (89.1), $d\varphi / dt$ will always has the same sign as φ, and will therefore have a sign at this point that is opposite to the one that it assumes for large values of $|t|$. If this latter sign is – e.g. – positive then one will easily find that the following is true: When the center of the stigmatic pencil describes the t-axis, the angle φ will increase from zero to a positive maximum, which is $< \pi / 2$, in any event, and then it will decrease to a negative minimum $> -\pi / 2$, in order to ultimately increase monotonically and converge to zero.

The system is then called *re-twisted*. The points of the t-axis for which $d\varphi / dt = 0$, and therefore one will have $\varphi = 0$ simultaneously, are the two orthogonal points of § 89. The oscillation of the angle φ between its maximum and its minimum is always smaller than π.

The previously-excluded special case:

$$pq + rs = 0, \quad a > c, \quad \alpha > \gamma$$

also corresponds to a re-twisted system. Namely, a root of $\psi = 0$ will then lie at infinity, and the maximum or the minimum of φ will be attained for $t = \infty$.

We shall refrain from a closer study of the twisted and re-twisted systems. For such a study, one must distinguish a series of special sub-classes and investigate them. For example, the case in which identities exist between the components y_j and y_i' of the directions must be examined precisely; in fact, the latter will occur when α or γ (but not both of them) vanish.

92. Semi-twisted systems. We now assume that the invariant Φ that is defined by (87.6) vanishes without the function $\tan 2\varphi$ being a constant.

We first remark that one must certainly have:

$$pqrs \neq 0. \tag{92.1}$$

Namely, if, e.g., $q = 0$ then, from (87.6), one must have:

$$\Phi = -p^2 s^2 r^2 = 0, \tag{92.2}$$

and from (83.6), $ps \neq 0$. One must then necessarily also have $r = 0$, and $\tan 2\varphi$ would be constant, namely, zero.

Secondly, one cannot simultaneously have:

$$pq + rs = 0, \qquad pr + qs = 0. \tag{92.3}$$

By adding these two equations, one will get $(p + s)(q + r) = 0$, one will therefore have either $q = -r$, $p = s$ or $q = r$, $p = -s$. In addition, since $\Phi = 0$, one must have either $\alpha = \gamma$ or $a = c$. However, for each of these assumptions, one will have that $\tan 2\varphi$ is constant.

Thirdly, one verifies similarly that $\tan 2\varphi$ is constant when one has either:

$$pq + rs = 0, \qquad a = c \tag{92.4}$$

simultaneously or:

$$pq + rs = 0, \qquad \alpha = \gamma \tag{92.5}$$

simultaneously.

If $\Phi = 0$, without φ being constant, then only the three following possibilities will remain:

$$pq + rs \neq 0, \quad (pr + qs) \neq 0, \quad a > c, \quad \alpha > \gamma, \tag{92.6}$$
$$pq + rs = 0, \quad pr + qs \neq 0, \quad a > c, \quad \alpha = \gamma, \tag{92.7}$$
$$pq + rs \neq 0, \quad pr + qs = 0, \quad a = c, \quad \alpha > \gamma. \tag{92.8}$$

In each of these three cases, $\tan 2\varphi$ can be represented by at least one of the two equivalent formulas (88.2) and (88.3).

It follows from this that, as for twisted systems, which rotate in *one* sense about the t-axis and a plane that goes through one of the focal lines when the center of the stigmatic pencil of light describes the t-axis here, the angle 2φ will, however, only increase or decrease by π.

The angle φ itself thus changes by $\pi / 2$, which is precisely one-half of the change that occurs for twisted systems, and for that reason the system is called *semi-twisted.*

93. For semi-twisted systems, the point:

$$\frac{t}{n} = z = -\frac{pqc + rsa}{pq + rs} \tag{93.1}$$

plays a special role. Namely, one simultaneously has $B = 0$ and $A = C$ (§ 87) for that point. The numerator and denominator on the right-hand side of (87.1) vanish. Finally, equation (85.6) has a double root:

$$z' = -A = -\alpha + \frac{(pq+rs)(ps-qr)}{(a-c)qs} = -\gamma - \frac{(pq+rs)(ps-qr)}{(a-c)pr}. \tag{93.2}$$

If one substitutes these values for z and z' in (85.4) then one will find that:

$$\left.\begin{aligned}[(pq+rs)(ps-qr)-\alpha(a-c)qs]\,x_1' &= (pq+rs)(sx_1-qx_2),\\ [(pq+rs)(ps-qr)+\gamma(a-c)pr]\,x_2' &= (pq+rs)(-rx_1+px_2).\end{aligned}\right\} \tag{93.3}$$

It follows from the latter equations that the map of all stigmatic pencils whose centers lie in the plane (93.1) is *stigmatic*. However, at the same time, we have also obtained formulas by which one characterizes the map of both planes that are transformed to each other by the corresponding stigmatic pencils of light.

We are now also in a position to understand how the twisted and semi-twisted ray maps can be taken to each other continuously, despite the fact that the apparently erratic oscillation of φ will be reduced from π to $\pi / 2$. Namely, if the invariant $\Phi > 0$ and converges continuously to zero then there will be a pair of coupled points – i.e., points that will be transformed to each other by the involution (87.4) – that will converge to one and the same point. The angle φ increases by $\pi / 2$ when the center of the stigmatic pencil describes the small interval that links the two coupled points. However, since the two focal lines are mutually perpendicular, the figure that consists of two focal lines will go to another one under the traversal of a small interval that differs only unnoticeably from the first one. In the limit, it is no longer possible to establish whether the rotation around the angle $\pi / 2$ has or has not taken place.

The connection between semi-twisted and re-twisted system can be explained in an entirely similar way when one employs the fact that the difference between the maximum and the minimum of φ converges to $\pi / 2$ in the limit.

94. There are three different kinds of semi-twisted systems, according to whether the planes that are mapped to each other for corresponding stigmatic pencils of light are both finite, or one of them, or finally all of them, lie at infinity.

We obtain the semi-twisted systems of the first kind from equations (83.7) and (83.8) when we demand that the conditions $u_1' = u_2' = 0$ must imply the further conditions $u_1 = u_2 = 0$. We must then write:

$$a = 0, \quad c = 0, \tag{94.1}$$

and from § 83, we can then always choose the coordinates in such a way that:

$$pr + qs = 0. \tag{94.2}$$

From (87.1), we now have:

$$\tan 2\varphi = -\frac{2(pq+rs)}{(\alpha-\gamma)z-(p^2-q^2)-(r^2-s^2)}, \tag{94.3}$$

and this function of z is non-constant only when:

$$pq + rs \neq 0, \qquad \alpha > \gamma. \tag{94.4}$$

The semi-twisted systems of the third kind are treated in a completely similar way to the telescopic ones. One finds the conditions:

$$\alpha = \gamma = 0, \qquad pq + rs = 0, \tag{94.5}$$
$$a > c, \qquad pq + rs \neq 0, \tag{94.6}$$

The semi-twisted, telescopic systems are less complicated to calculate. We must demand that it should follow from $u_1 = u_2 = 0$ that $y_1' = y_2' = 0$. From (83.8), one finds that for $u_1 = u_2 = 0$:

$$(ps - qr)\ u_1' = -as\, y_1 + cq\, y_2, \qquad (ps - qr)\ u_2' = -ar\, y_1 + cp\, y_2. \tag{94.7}$$

If one substitutes these quantities in the right-hand side of equations (83.7) and demands that the coefficients of y_1 and y_2 must vanish then one will get:

$$\left.\begin{aligned} -\alpha as + (ps - qr)p = 0, \quad \alpha cq + (ps - qr)r = 0, \\ \gamma ar + (ps - qr)q = 0, \quad -\gamma cp + (ps - qr)s = 0. \end{aligned}\right\} \tag{94.8}$$

It first follows that all of the quantities α, γ, a, and c must be non-zero, and then also that equations (94.8) are compatible with each other only when:

$$pqc + rsa = 0. \tag{94.9}$$

Since, from (92.1), the condition $pqrs \neq 0$ is true, one can then write:

$$a = \lambda pq, \qquad c = \lambda rs, \qquad \alpha = \frac{ps - qr}{\lambda sq}, \qquad \gamma = -\frac{ps - qr}{\lambda pr}, \tag{94.10}$$

in which λ means a non-zero parameter. One ultimately finds that, from (87.1), the function $\tan 2\varphi$ is non-constant if and only if one simultaneously has:

$$pq + rs \neq 0, \qquad pqc + rsa = 0. \tag{94.11}$$

The latter equations are equivalent to the following ones:

$$a \neq c, \qquad \alpha \neq \gamma. \tag{94.12}$$

95. Orthogonal systems. We now consider the case in which φ possesses a constant, well-defined value for all values of t.

If $\alpha > \gamma$ then, from (87.1), the function $\tan 2\varphi$ will be constant only if it vanishes identically. This yields the conditions:

$$pq + rs = 0, \qquad pqc + rsa = 0, \tag{95.1}$$

from which one concludes that one must have either:

$$pq = 0, \qquad rs = 0 \tag{95.2}$$

or $a = c$. However, by the second assumption, the conditions (95.2) must be fulfilled in any case. In the case $a = c$, one can always choose the coordinates such that $q = 0$ is true from the outset (§ 83). Equation (87.1) will then take on the form:

$$\tan 2\varphi = \frac{-2rs}{(\alpha-\gamma)(z+a)-(p^2+r^2-s^2)},$$

and $\tan 2\varphi$ will then be constant only when $rs = 0$.

Secondly, if $\alpha = \gamma$ then one can again assume that $q = 0$, and one will have:

$$\tan 2\varphi = \frac{2rs(z+a)}{p^2(z+c)+(r^2-s^2)(z+a)}. \tag{95.3}$$

However, from (83.6) it will follow here that $ps \neq 0$, and the right-hand side of (95.3) will, as a result, be constant only when either $rs = 0$ or $a = c$. In the latter case, one can, from § 83, again choose the coordinates such that $q = 0$, $r = 0$. In both cases, equations (95.2) will then be fulfilled here, as well.

Consequently, the condition for $\tan 2\varphi$ to be constant can always be written in the form (95.2). Since $ps - qr \neq 0$, one can, if one so desires, write this condition in the form:

$$q = 0, \quad r = 0, \tag{95.4}$$

after a possible rotation of one of the coordinate systems through 90°. However, one can then no longer assume that one simultaneously has $\alpha \geq \gamma$ and $a \geq c$.

Equations (84.1) and (84.2) now have the form:

$$\left.\begin{aligned} x_1 &= \left(a+\frac{t}{n}\right)y_1 + pu_1', & x_2 &= \left(c+\frac{t}{n}\right)y_2 + su_2', \\ x_1' &= \left(1+\alpha\frac{t'}{n'}\right)u_1' + py_1\frac{t'}{n'}, & x_2' &= \left(1+\gamma\frac{t'}{n'}\right)u_2' + sy_2\frac{t'}{n'}, \end{aligned}\right\} \tag{95.5}$$

from which, it follows that the ray map possesses two symmetry planes in this case. In addition, from (87.5), the function ψ is identically zero, from which it will follow (§ 89) that *all* points on the t-axis are orthogonal points. Equations (85.2) have the form:

$$A = \alpha - \frac{p^2}{z+a}, \qquad B = 0, \qquad C = \gamma - \frac{s^2}{z+c} \tag{95.6}$$

here, which can be written rationally as solutions of equation (85.6), since the fourth-order curve in § 86 decomposes into a product of two hyperbolas.

96. Since B vanishes identically, the condition for the map to be stigmatic will reduce to $A = C$, which can also be written as:

$$(\alpha - \gamma)(z + a)(z + c) - p^2 (z + c) + s^2 (z + a) = 0. \tag{96.1}$$

The discriminant of this quadratic equation reads:

$$\Psi = ((\alpha - \gamma)(a - c) - (p + s)^2)((\alpha - \gamma)(a - c) - (p - s)^2). \tag{96.2}$$

If $\Psi < 0$ then both roots of equation (96.1) are imaginary, and there is no stigmatic point. If $\Psi = 0$ then there is a stigmatic point, which is counted twice, and for $\Psi > 0$ a pair of stigmatic points is present.

For the case in which two mutually corresponding stigmatic points lie at finite points, one can choose the coordinate origin in such a way that a pair coincides with the points $t = 0$, $t' = 0$. For this to be true, one must take:

$$a = 0, \qquad c = 0, \tag{96.3}$$

and the second point-pair then possesses the abscissas:

$$\frac{t}{n} = \frac{p^2 - s^2}{\alpha - \gamma}, \qquad \frac{t'}{n'} = \frac{p^2 - s^2}{p^2\gamma - s^2\alpha}. \tag{96.4}$$

The map of the stigmatic planes in the case of the first pair of stigmatic points considered will be represented by the formulas:

$$x'_1 = \frac{x_1}{p}, \qquad x'_2 = \frac{x_2}{s}, \qquad t' = t = 0. \tag{96.5}$$

The condition for the point-pair (96.4) to coincide with the first one is expressed by:

$$p^2 = s^2, \qquad \alpha \neq \gamma; \tag{96.6}$$

the map (96.5) is then rotationally symmetric.

If the ray map is semi-telescopic for a pair of corresponding stigmatic planes then, perhaps, $A = C = 0$ must follow from $z = 0$. From (95.6), this will yield the condition:

$$\alpha = \frac{p^2}{a}, \qquad \gamma = \frac{s^2}{c}, \tag{96.7}$$

and in place of the relations (96.5), we must now write:

$$y'_1 = \frac{p}{a} x_1, \qquad y'_2 = \frac{s}{c} x_2, \qquad \frac{1}{t'} = t = 0. \tag{96.8}$$

The condition for the coefficient of z in (96.1) to vanish, along with the constant term, with the values (96.7) reads:

$$p^2\,c^2 - s^2\,a^2 = 0. \tag{96.9}$$

It once more follows from this that the existence of a double root of equation (96.1) can be expressed by the rotational symmetry of the map (96.8).

Finally, if the ray map for the one pair of associated stigmatic points is telescopic then one must write:

$$\alpha = \gamma = 0, \qquad y_1' = p\,y_1, \qquad y_2' = s\,y_2\,,$$

and the rotational symmetry will have the same meaning as before.

97. GAUSSian systems. The reasoning of the previous paragraph loses its meaning when all coefficients vanish in equation (96.1). In order for this to be the case, one must have:

$$\alpha = \gamma, \quad a = c, \quad p^2 = s^2, \tag{97.1}$$

and – possibly after one performs a reflection in one of the coordinate planes – one can indeed always arrange that the last of equations (97.1) is replaced by:

$$p = s. \tag{97.2}$$

From § 83, the ray map is then itself (not just the point map in the stigmatic planes of the previous paragraphs) *rotationally symmetric*, and we have the classical case before us that GAUSS first investigated in a celebrated treatise ([72]). However, what is much more important than this rotational symmetry here are the facts that *any* stigmatic pencil of rays will go to a stigmatic pencil of rays, and that the two optical spaces will be mapped collinearly to each other.

Our discussion shows further that the converse of this result is also true here: *If a linear ray map has the property that any stigmatic pencil of rays is again transformed to a stigmatic pencil of rays – i.e., when the instrument is absolute in the sense of § 69 – then the linear map must be rotationally-symmetric – i.e., it must be a* GAUSS*ian ray map.*

Here, equations (83.7) and (83.8) have the simple form:

$$y_i' = \alpha u_i' + p y_i\,, \qquad u_i = a y_i + p u_i' \qquad (i = 1, 2). \tag{97.3}$$

Moreover, with the notations of § 85, one will have:

$$A = C = \alpha - \frac{p^2}{\dfrac{t}{n} + a}, \quad B = 0, \tag{97.4}$$

and in place of equation (85.6), one can now write:

([72]) GAUSS, C. F.: "Dioptrische Untersuchungen," Abh. Ges. Wiss. Göttingen **1** (1843), 1-34. Werke, Bd. 5, pp. 243-276.

$$(z' + a)(z + a) = p^2. \tag{97.5}$$

For the collinear relationship between the two optical spaces, one thus obtains from (85.6) and (85.4) that:

$$\frac{t'}{n'} = \frac{\frac{t}{n}+a}{p^2-\alpha\left(\frac{t}{n}+a\right)}, \qquad x_i' = \frac{p x_i}{p^2-\alpha\left(\frac{t}{n}+a\right)}, \tag{97.6}$$

and one will have the equations:

$$y_i' = \frac{\alpha}{p}\left(x_i - \frac{t}{n} y_i\right) + p\, y_i\,, \tag{97.7}$$

in addition. All of these formulas can be simplified in a well-known way when one translates the origins of the t and t'-axes.

98. Since we have a absolute instrument before us with the GAUSSian map, the theorem of § 69 is true. One then satisfies a relation that has the form:

$$n'\left(1+\frac{1}{2}\left(\frac{dx_1'}{dt'}\right)^2+\frac{1}{2}\left(\frac{dx_2'}{dt'}\right)^2\right)dt' - n\left(1+\frac{1}{2}\left(\frac{dx_1}{dt'}\right)^2+\frac{1}{2}\left(\frac{dx_2}{dt'}\right)^2\right)dt = d\Psi, \tag{98.1}$$

in which Ψ is a function of t, x_1, x_2, and which must be fulfilled identically when t' and the x_i' are replaced with the values (97.6).

In order to calculate the function Ψ, one remarks that, from (81.2) and (81.4), one will have:

$$-\mathrm{H}\, dt + y_1\, dx_1 + y_2\, dx_2 = d\Omega + y_1\, du_1 + y_2\, du_2\,, \tag{98.2}$$

in which one must take:

$$\Omega = nt + \frac{y_1^2+y_2^2}{2}\cdot\frac{t}{n}; \tag{98.3}$$

an analogous formula is true for the second optical space. Moreover, it follows from equations (97.3) that:

$$y_i'\, du_i' - y_i\, du_i = d\mathrm{X}, \tag{98.4}$$

with:

$$\mathrm{X} = \frac{\alpha}{2}(u_1'^2 + u_2'^2) - \frac{a}{2}(y_1^2 + y_2^2)\,. \tag{98.5}$$

One then obtains Ψ by using the Ansatz:

$$\Psi = \Omega' - \Omega + \mathrm{X}. \tag{98.6}$$

One then expresses the right-hand side as a function of t', t, x_i, and y_i . The coefficients of y_i then vanish when the first equation in (97.6) exists between t' and t, and one then gets:

$$2\Psi = 2(n't' - nt) + \alpha\left(\alpha\frac{t'}{n'}+1\right)\frac{x_1^2 + x_2^2}{p^2}, \tag{98.7}$$

which can also be written as:

$$2\Psi = \frac{2\left(\dfrac{t}{n}+a\right)(n'^2 + n\alpha t) - 2np^2 t + \alpha(x_1^2 + x_2^2)}{p^2 - \alpha\left(\dfrac{t}{n}+a\right)}. \tag{98.8}$$

The difference between the optical lengths of two arbitrary curves that are mapped to each other by the transformation (97.6) depends upon only the endpoints of the curve when one calculates these lengths from the basic functions (81.3). For example, if the endpoints of the curve in object space lie on one and the same second-order surface Ψ = const. then both curves will have the same optical length.

The reason why the way that we reached the conclusions in § 71 cannot be applied here is to be found in the fact that curves that lie in a plane t = const. must have an infinite optical length.

99. Concluding remarks. In practice, only rotationally-symmetric instruments are constructed. It would then seem that the consideration of dioptric systems that are not rotationally symmetric is completely superfluous. However, that is not the case. If one would study the ray map in the vicinity of a ray that does coincide with the rotational axis of the instrument itself then one would already have to consider orthogonal systems when the ray cuts the axis. However, if the ray were skew to the rotational axis then one would indeed be dealing with a general system.

A second remark that justifies our rigorous treatment of the theory in the first approximation is the following one: If one investigates the ray map in homogeneous, isotropic media then every stigmatic pencil of rays in object space will correspond to a ray congruence with two real focal surfaces. A line element through the center of the stigmatic pencil will correspond to a line element on each of these focal surfaces, and in fact, these last two line elements will lie upon one and the same ray of the congruence, namely, the ray in image space that is established by the line element in object space. If one goes to the accessory problem then its position can then be calculated by means of equation (85.6).

If one now lets the line element in object slide along a ray then the values of the coefficients α, γ, a, c, p, q, r, s will not change; as a consequence, if one selects a specific ray in object space then one cannot, however, provisionally establish the center of the stigmatic pencil on it, although on the associated ray image the respective contact points of the focal surfaces for an arbitrary position of the center of the stigmatic object ray pencil can already be always calculated when one has collected enough data to determine the fourth-order curve in § 86.

One will not wonder why such legal details must be present for the arrangement of the focal surfaces when one considers that this result is entirely analogous to the well-known theorem that says that the tangential elements along a generator of an arbitrary ruled surface are uniquely determined everywhere when one knows them for three positions of the generators.