"Les espaces à connexione conforme," *Ann. Soc. Pol. Math.*, v. 2, (1923), 171-221.

Spaces with conformal connections

By

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The goal of the present memoir is the study of the essential properties of what I call *spaces with conformal connections.* It is a sequel to a very important memoir that is dedicated to the theory of spaces with affine connections and spaces with metric connections that will appear next in the Annales de l'Ecole Normale Supérieure; it may be read independently of the latter with no loss of rigor. Either of them constitute the development of some Notes that I have published throughout the last year in the Comptes Rendus de l'Académie des Sciences de Paris [1]). The notion of a space with an affine connection is not absolutely new, since it has been considered, at least in the special case of what I call spaces without torsion, by various authors, especially H. Weyl and Eddington, who have been led to such a notion in their remarkable work on the general theory of relativity [2]). On the other hand, the notion of a space with a conformal connection is essentially new, since the viewpoint that was taken by the preceding authors in order to edify the notion of a space with affine connection seems to exclude any generalization of that type.

This memoir is divided into two chapters. The first one is occupied with the properties of spaces with conformal connections in their own right. The second chapter is occupied with the properties of manifolds that are embedded in a given space with a conformal connection. In that regard, it poses problems that are analogous to the ones that arise in the study of the properties of surfaces vis-à-vis the group of conformal transformations of that space; in particular, I will cite the problem of the conformal representation or first order deformation of hypersurfaces, to which I have dedicated a memoir several years ago in the case of a space of dimension more than four [3]).

In the present memoir, I will prove that in a four-dimensional space with conformal connection the hypersurfaces that admit a conformal representation on other distinct hypersurfaces are exceptional and depend upon three arbitrary functions of two arguments; this is a result that I have previously announced in the case of a four-dimensional conformal space [4]). As an application, I will give several indications on the nature of the surfaces that admit a second order deformation in a *normal* three-dimensional space; these surfaces generalize the isothermal surfaces in ordinary space.

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[1] *Comptes Rendus Paris*, t. 174, pp. 487, 593, 784, 857, 1104; see, especially, "Sur les espaces à connexione conforme," pp. 857-860.

[2] H. Weyl: *Raum, Zeit, Materie*, 4th edition; A.S. Eddington, *Espace, Temps et Gravitation*.

[3] E. Cartan, "La déformation des hypersurfaces dans l'espace conforme réel à $n \geq 5$," *Bull. Soc. Math. de France*, t. XLV, pp. 57-120 (1917).

[4] E. Cartan, "Sur le problème général de la déformation," *C.R. Congrès de Strasbourg*, pp. 397-406 (1921).

In this memoir, I employ the notations and methods of the memoirs that I have already dedicated to Differential Geometry [1]).

First Chapter.

Definition and properties of manifolds with conformal connections.

Conformal space, conformal transformations.

1. Consider an n-dimensional space that is referred to a rectangular coordinate system. Any hypersphere may be defined by an equation of the form:

$$x_0(X_1^2 + \cdots + X_n^2) - 2x_1X_1 - 2x_2X_2 - \ldots - 2x_nX_n - 2x_{n+1} = 0,$$

with a system of $n+2$ homogeneous coordinates $x_0, x_1, \ldots, x_{n+1}$. The condition that the hypersphere have a null radius is expressed by the relation:

$$\Phi ; x_1^2 + \cdots + x_n^2 + 2x_0x_{n+1} = 0.$$

Conformal transformations may be defined analytically defined by linear substitutions that act on the variables $x_0, x_1, \ldots, x_{n+1}$ and leave the form Φ invariant; they therefore preserve the hypersphere of null radius, and because of that they may be considered to be point transformations, when each point of the space is associated with the hypersphere of null radius whose center is at that point. *Conformal geometry* has the goal of studying the properties of figures that remain invariant under an arbitrary conformal transformation; we therefore say that it has the object of describing a theory of *conformal spaces.*

An arbitrary set of $n+2$ coordinates $(x_0, x_1, \ldots, x_{n+1})$ that are not all null will define what we will call, in a general sense, a hypersphere, and that we will denote, for the sake of abbreviation, by a letter such as X. The symbols X and mX, in which m is an arbitrary numerical coefficient, will thus denote two distinct hyperspheres, although they are identical from a geometrical viewpoint.

A unit hypersphere will be defined by the condition that Φ has the value 1 for all of its coordinates. In a general manner, we will call the expression:

$$XY = \frac{1}{2}\sum y_i \frac{\partial\Phi}{\partial x_i} = \frac{1}{2}\sum x_i \frac{\partial\Phi}{\partial y_i}$$

the *scalar product* of two hyperspheres X and Y; this scalar product remains invariant under a conformal transformation. A unit hypersphere is characterized by the condition

[1] The reader may especially consult the memoir cited above (note 1) for the methods that are employed, and also my *Leçons sur les Invariants intégraux*, Paris, Hermann, 1922, especially chapters VI and VII. One may also confer the summary of properties that were used for Pfaff systems in involution in a memoir in the *Bulletin de la Soc. Math. de France*, t. XLVIII (1920), chap. III, pp. 136, et seq.

that its scalar square is equal to 1. Two orthogonal hyperspheres are characterized by the property that their scalar product is null.

The scalar square of a hypersphere-point – or, more simply, a point – is null; a hypersphere passes through a point when the scalar product of that point and the hypersphere is null.

2. One may find a system of (n+2)-spherical coordinates in an infinitude of ways. It suffices to consider n mutually orthogonal unit hyperspheres $A_1, \dots, A_n$ and two points A_0 and A_{n+1} that are common to these hyperspheres, when one subjects the two indeterminate factors that enter into these coordinates to the condition that the scalar product $A_0 A_{n+1}$ is equal to 1. One will thus have:

$$A_1^2 = \dots = A_n^2 = A_0 A_{n+1} = 1,$$

and all of the other scalar products that relate to pairs of the n+2 hyperspheres, whether distinct or identical, are null.

Any hypersphere X may then be put into the form:

$$X = \bar{x}_0 A_0 + \bar{x}_1 A_1 + \cdots + \bar{x}_{n+1} A_{n+1}$$

in one and only one manner, and one will have:

$$X^2 = \bar{x}_0^2 + \cdots + \bar{x}_n^2 + 2\bar{x}_0 \bar{x}_{n+1};$$

the formulas that permit us to pass from the old coordinates x_i to the new ones $\bar{x}_i$ thus define a conformal transformation. The initial system of coordinates may be regarded as a particular system of n+2-spherical coordinates in which the coordinates of the hypersphere are the origin of the coordinates for A_0, the n coordinate hyperplanes for $A_1, \dots, A_n$, and the hyperplane at infinity for A_{n+1} (which is regarded as a hypersphere of null radius). To abbreviate, we refer to the set of the n+2 hyperspheres $A_0, A_1, \dots, A_{n+1}$ as a *frame*.

3. Consider a moving frame that depends upon one or more parameters. It is obvious that one will have the formulas:

$$(1) \qquad \begin{cases} dA_0 = \omega_0^0 A_0 + \omega_0^1 A_1 + \cdots + \omega_0^{n+1} A_{n+1} \\ dA_1 = \omega_1^0 A_0 + \omega_1^1 A_1 + \cdots + \omega_1^{n+1} A_{n+1} \\ \quad \vdots \\ dA_{n+1} = \omega_{n+1}^0 A_0 + \omega_{n+1}^1 A_1 + \cdots + \omega_{n+1}^{n+1} A_{n+1}, \end{cases}$$

in which the ω_i^j are linear with respect to the differentials of the parameters.

These $(n+2)^2$ expressions ω_i^j are not arbitrary, because one must obviously have:

$$A_i\, dA_j + A_j\, dA_i = 0,$$

for any values of the two indices i, j, distinct or not; this amounts to saying that all of the scalar products A_iA_j are constant. One easily deduces the following relations, which are fundamental:

$$(2) \qquad \begin{cases} \omega_0^{n+1} = \omega_{n+1}^0 = 0, & \omega_{n+1}^{n+1} = -\omega_0^0, \\ \omega_0^i = -\omega_i^{n+1}, & \omega_{n+1}^i = -\omega_i^0, \qquad (i = 1, 2, \cdots, n) \\ \omega_i^i = 0, & \omega_i^j + \omega_j^i = 0, \qquad (i, j = 1, 2, \cdots, n). \end{cases}$$

What thus remain are the expressions for the independent ω_i^j:

$$\omega_0^0 = -\omega_{n+1}^{n+1}, \quad \omega_0^i = -\omega_i^{n+1}, \quad \omega_i^0 = -\omega_{n+1}^i, \quad \omega_i^j = -\omega_j^i;$$

they number $\frac{(n+1)(n+2)}{2}$. Moreover, the maximum number may be attained since the most general frame depends upon precisely $\frac{(n+1)(n+2)}{2}$ parameters.

4. It is obvious that the passage from a given frame to an infinitely close frame may be obtained by an infinitesimal conformal transformation, and that the quantities ω_i^j may be regarded as the *components* of that conformal transformation. It is of interest for one to study the geometric properties of the $\frac{(n+1)(n+2)}{2}$ infinitesimal conformal transformations that correspond to the $\frac{(n+1)(n+2)}{2}$ distinct components ω_i^j. They fall into three different types.

First, take the transformation whose only non-zero component is $\omega_0^0 = -\omega_{n+1}^{n+1}$. It gives:

$$dA_0 = e\, A_0, \qquad dA_{n+1} = -\, e\, A_{n+1}, \qquad dA_i = 0;$$

the hypersphere $\sum x_i\, A_i$ is then transformed into the hypersphere:

$$\sum (x_i + dx_i)\, A_i,$$

with:

$$dx_0 = e\, x_0, \qquad dx_{n+1} = -\, e\, x_{n+1}, \qquad dx_i = 0.$$

If the point A_{n+1} is at infinity then one sees that the point whose Cartesian coordinates are $\frac{x_i}{x_0}$ is transformed into the points whose coordinates are $(1 - e)\ \frac{x_i}{x_0}$; one has thus effected a *homothety* with center A_0. In the general case, we will have what we call a *conformal homothety* with centers A_0 and A_{n+1}; since all of the circles that pass through the two centers of the homothety remain invariant, a point M of one of these circles will be transformed into an infinitely close point M' of the same circle, with the condition that the anharmonic ratio of the four (geometric) points M, M', A_0, A_{n+1} must be equal to a given fixed ratio.

A second type of infinitesimal conformal transformation is obtained by giving one of the components ω_0^i or ω_i^0 a value that is different from zero, all other components being null. For example, take $\omega_0^1 = -\omega_1^{n+1} = e$. One will have:

$$dA_0 = e\, dA_1, \quad dA_{n+1} = 0, \qquad dA_1 = -\, e\, A_{n+1}, \qquad dA_2 = \ldots = dA_n = 0,$$

and, as a result:

$$dx_0 = 0, \qquad dx_1 = e\, x_0, \qquad dx_2 = \ldots = dx_n = 0, \qquad dx_{n+1} = -\, e\, x_1.$$

If the point A_{n+1} is at infinity then the point whose Cartesian coordinates are $\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}$ is transformed into the point whose coordinates are $\frac{x_1}{x_0} + e, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}$; one has effected a *translation*. In the general case, we say that the translation is an *elation* (?) of center A_{n+1}. There exists a family of circles that pass through A_{n+1} that are all tangent to each other at that point, and each circle of that family remains invariant under the transformation in such a manner that its points are transformed into each other in a *parabolic* manner; the hyperspheres that are orthogonal to these circles and pass through A_{n+1} (here, the hypersphere $A_1 + \rho A_{n+1}$) are transformed into themselves.

One will likewise have an *elation* of center A_0 if one gives $\omega_1^0 = -\omega_{n+1}^1$ a value e that is different from zero.

Finally, a third type of infinitesimal conformal transformation is obtained by giving, for example, $\omega_1^2 = -\omega_2^1$, a value e that is different from zero. In this case, one has:

$$dA_0 = 0, \qquad dA_{n+1} = 0, \qquad dA_1 = e\, A_2, \qquad dA_2 = -\, e\, A_1, \quad dA_3 = \ldots = dA_n = 0,$$

and, as a result:

$$dx_0 = dx_{n+1} = 0, \qquad dx_1 = -\, e\, x_2, \quad dx_2 = e\, x_1, \qquad dx_3 = \ldots = dx_n = 0.$$

All of the points (which consist of A_0 and A_{n+1}) that are common to the hyperspheres A_1 and A_2 remain fixed, while the hyperspheres of the sheaf $\lambda A_1 + \mu A_2$ are interchanged among themselves in such a manner that each makes a fixed (infinitesimal) angle with its transform. One thus has a *rotation* that has the hypercircle $[A_1A_2]$ that is the intersection of A_1 and A_2 for its axis. Under this rotation, each point M describes an arc of the circle that is normal to the hypersphere $\lambda A_1 + \mu A_2$ that passes through that point.

If $n = 3$ and if the circle $[A_1A_2]$ is reduced to a line then one obtains an ordinary rotation with that line for its axis.

5. In summary, if one is given two points A and B of the conformal space then any infinitesimal conformal transformation may be decomposed in one and only one way into:

1. An *elation* with center B.
2. An *elation* with center A.
3. A *homothety* with centers A and B.
4. A *rotation* around a hypercircle that passes through A and B.

If the point B is at infinity then the elation with center B becomes a translation and the homothety with centers A and B becomes an ordinary homothety with center A. The elation with center A, the homothety, and the rotation leave the point A invariant. Moreover, the elation with center A and the homothety changes any circle that passes through A into a tangent circle; in other words, it leaves all of the directions that issue from A invariant. Finally, the homothety is the one and only transformation that leaves all of the circles that pass through A and B invariant.

We remark that when one is given the point A the decomposition of an infinitesimal conformal transformation into two other ones, one of which fixes the point A_0, may not be carried out in an *invariant* manner, as would apply to displacements in Euclidian space. Indeed, in the latter space the decomposition into a *translation* and a rotation around A has an absolute significance; this is no longer true in conformal space, in which the translation is replaced with an elation whose center is arbitrary.

The structure equations of conformal space.

6. They may be obtained by saying that the integrals:

$$\int dA_0,\ \int\ dA_1,\ \ldots, \int dA_{n+1},$$

are null when taken over an arbitrary closed contour.

One thus obtains the relations:

$$(\omega_i^j)' = [\omega_i^0 \omega_0^j] + [\omega_i^1 \omega_1^j] + \cdots [\omega_i^{n+1} \omega_{n+1}^j],$$

in which $(\omega_i^j)'$ denotes the (bilinear covariant) *exterior derivative* of ω_i^j.

If one preserves only the independent expressions ω_0^i, ω_i, ω_0^0, ω_i^j then one obtains:

$$(3)\qquad \begin{cases} (\omega_0^0)' = [\omega_0^1 \omega_1^0] + \cdots + [\omega_0^n \omega_n^0], \\ (\omega_0^i)' = [\omega_0^0 \omega_0^i] + \cdots + \sum_{k=1}^{n} [\omega_0^k \omega_k^i], \\ (\omega_i^j)' = [\omega_0^i \omega_j^0] - [\omega_0^j \omega_i^0] + \sum_{k=1}^{n} [\omega_i^k \omega_k^j], \\ (\omega_i^0)' = [\omega_i^0 \omega_0^0] + \sum_{k=1}^{n} [\omega_i^k \omega_k^0]. \end{cases}$$

These are the desired structure equations.

Manifolds with conformal connections.

7. Consider an n-dimensional manifold and attach an n-dimensional conformal space to each point P of this manifold that we frame by means of a system of $n+2$ hyperspheres $A_0, A_1, \ldots, A_{n+1}$, the first of which will be identified (as far its position is concerned) with the point P itself. The given manifold will be said to have a *conformal connection* if we give it a law (which is arbitrary, moreover) that permits us to refer (in a conformal manner) the conformal space that is attached to an infinitely close point P'. If we call $A'_0, A'_1, \cdots, A'_{n+1}$ the frame that is attached to the point P' then this says that we have associated the hyperspheres $A'_0, A'_1, \cdots, A'_{n+1}$ in the conformal space that is attached to the point P' with the hyperspheres:

$$A_0 + \omega_0^0 A_0 + \omega_0^1 A_1 + \cdots + \omega_0^n A_n ,$$
$$A_1 + \omega_1^0 A_0 + \omega_1^2 A_2 + \cdots + \omega_1^n A_n - \omega_0^1 A_{n+1} ,$$
$$\cdots$$
$$A_{n+1} - \omega_1^0 A_1 - \omega_2^0 A_2 - \cdots - \omega_0^0 A_{n+1}$$

in the conformal space that is attached to P. To abbreviate, we write:

$$dA_0 = \omega_0^0 A_0 + \sum \omega_0^i A_i ,$$
$$dA_i = \omega_i^0 A_0 - \omega_0^i A_{n+1} + \sum \omega_i^k A_k ,$$
$$dA_{n+1} = -\omega_0^0 A_{n+1} - \sum \omega_i^0 A_i ,$$

and we say that the frame that is attached to P' is deduced from the frame that is attached to P by an infinitesimal conformal transformation whose components $\omega_0^0, \omega_0^i, \omega_i^0, \omega_i^j$ are linear expressions in the differentials:

1. of the parameters $u_1, \ldots, u_n$ that define the position of a point P on the given manifold.
2. of the parameters v_i that depend at each point of the manifold on the frame attached to that point.

If that frame is chosen in the most general manner possible then it depends upon $\dfrac{n^2+n+2}{2}$ parameters. It is essential to remark that the n components $\omega_0^1, \omega_0^2, \cdots, \omega_0^n$ depend upon only the differentials of the parameters u, since if one leaves the parameters u fixed then the point P does not change, the point A_0 no longer changes in *position*, and dA_0 may depend only upon A_0. From now on, we write A instead of A_0 and ω^i instead of ω_0^i.

Finally, we say that the expressions $\omega_0^0, \omega^i, \omega_i^0$ are the *components of the conformal connection* of the manifold. They naturally depend upon the frame that is attached to each point of the manifold; if one changes this frame then it is modified according to formulas that are not difficult to write down.

One may remark that the scalar square $(dA)^2$, which is equal to:

$$(\omega^1)^2 + (\omega^2)^2 + \ldots + (\omega^n)^2 ,$$

represents, up to a factor, the square of the distance between two points P and P', a distance that results from the conformal connection on the manifold.

The structure equations of manifolds with conformal connections.

8. If one considers a closed infinitesimal contour on such a manifold then when the integrals $\int dA_i$ are taken around that closed contour they make no sense in themselves, since there is no absolute conformal space with respect to which one may frame the conformal spaces that are attached to the various points of the manifold. However, if one frames the conformal spaces that are attached to the points of the contour, *which is assumed to be infinitesimal*, with respect to the conformal space that is attached to a fixed point Q that is infinitely close to the contour then the preceding integrals take on a sense, and one obtains:

$$\int dA_i = \iint \sum A_k \left[(\omega_i^k)' - \sum \omega_i^h \omega_h^k \right] = \iint \sum_{k=0}^{n+1} \Omega_i^k A_k .$$

One thus arrives at a system of $\frac{(n+1)(n+2)}{2}$ differential forms of second degree $\Omega_0^0, \Omega^i, \Omega_i^0, \Omega_i^j$ that are defined by the formulas:

$$(4) \qquad \begin{cases} (\omega_0^0)' = \sum_{k=1}^{n} [\omega^k \omega_k^0] + \Omega_0^0, \\ (\omega^i)' = [\omega_0^0 \omega^i] + \sum_{k=1}^{n} [\omega^k \omega_k^i] + \Omega^i, \\ (\omega_i^j)' = [\omega^i \omega_j^0] - [\omega^j \omega_i^0] + \sum_{k=1}^{n} [\omega_i^k \omega_k^j] + \Omega_i^j, \\ (\omega_i^0)' = [\omega_i^0 \omega_0^0] + \sum_{k=1}^{n} [\omega_i^k \omega_k^0] + \Omega_i^0. \end{cases}$$

We may take a little more general viewpoint. At each point of the manifold attach a hypersphere:

$$X = x^0 A_0 + x^1 A_1 + \ldots + x^{n+1} A_{n+1} ,$$

according some arbitrary law, and calculate the integral $\int dX$ when taken over an infinitesimal closed contour. We have:

$$dX = \sum dx^i A_i + \sum x^i dA_i,$$

from which, upon taking the exterior derivative and remarking that dA_i is not an exact differential, we have:

$$(dX)' = -\sum [dx^i\, dA_i] + \sum [dx^i\, dA_i] + \sum x^i (dA_i)' = \sum x^i \Omega_i^k A_k .$$

One thus has:

$$\int dX = \iint \sum x^i \Omega_i^k A_k .$$

Since the contour is given, the differential element in the right-hand side depends only upon the components x^i of the hypersphere X at a point of the contour.

In other words, to any infinitesimal closed contour that starts at one point P in the manifold and returns to it there is associated an infinitesimal conformal displacement that transforms any hypersphere X of the conformal space that is attached to P into an infinitely close hypersphere $X + \Delta X$; this displacement is therefore defined analytically by the formulas:

(5) $$\Delta x^k = \sum_{i=0}^{n+1} \Omega_i^k x^i \qquad (k = 0, 1, \ldots, n+1).$$

The components Ω_i^j of this infinitesimal displacement – by their very nature – are double integral elements that involve only the differentials du_i of the parameters that define the position of a point on the given manifold; they may not involve the differentials of the other parameters, which may possibly depend upon the frame that is attached to each point of the manifold. In other words, one has expressions of the form:

$$\Omega_i^j = \sum_{k,l}^{1,\ldots,n} A_{ikl}^j [\omega^k \omega^l],$$

in which the A_{ikl}^j may depend upon all of the parameters u_i and v_i.

The infinitesimal conformal displacement that is associated to any infinitesimal contour that is traced out on the given manifold defines the *curvature* of the manifold, and equations (4) are the structure equations of the manifold with the given conformal connection.

9. One may present things in a manner that has a more intuitive sense.

Consider a path on the manifold that joins two points P and Q. One may recursively associate the conformal space that is attached to a point M of this path to the conformal space that is attached to the point of departure. For this, it suffices to remark that if one is given a frame at each point M then the components ω_i^j may be put into the form $p_i^j dt$ when one displaces along the path considered; t denotes parameter that defines the position of a point M on the path and p_i^j denotes a chosen function of t. One will then have, upon displacing along the path:

(6)
$$\begin{cases} \dfrac{dA_0}{dt} = p_0^0 A_0 + \sum p^i A_i, \\ \dfrac{dA_i}{dt} = -p_i^0 A_0 - p^i A_{n+1} + \sum p_i^k A_k, \\ \dfrac{dA_{n+1}}{dt} = -p_0^0 A_{n+1} - \sum p_i^0 A_i. \end{cases}$$

One may regard these equations as linear ordinary differential equations in the unknowns $A_0, \ldots, A_{n+1}$; in the general solution of these equations, which are assumed to be integrable, one will replace the initial values of the unkown functions by the hyperspheres that define the chosen frame at P, and one will therefore have the frame that is attached to an arbitrary point M, which is referred to the frame that is attached to the point P. In other words, one will refer the conformal space that is attached to M to the conformal space that is attached to P.

One may proceed further in the following manner:

The conformal connection permits us to identify the hyperspheres of the space that is attached to a point of the manifold with the hyperspheres of the space that is attached to an infinitely close point. This identification is obtained by the symbolic identity:

$$d(\sum x^i A_i) = 0,$$

or:

(7)
$$dx^i + \sum_{k=0}^{n+1} x^k \omega_k^i = 0 \qquad (i = 0, 1, \ldots, n+1).$$

If one replaces ω_k^i with $p_k^i dt$ in these relations then one will have a system of linear differential equations whose integration will allow us to know what hypersphere in the conformal space attached to M is identified with a given hypersphere in the conformal space that is attached to P. These formulas are of the form:

(7′)
$$x^i = \sum_{k=0}^{n+1} a_k^i (x^k)_0 \qquad (i = 0, 1, \ldots, n+1),$$

in which the a_k^i are given functions of t, the $(x^k)_0$ are the coordinates of a hypersphere in the conformal space that is attached to P, and the x^i are the coordinates of the corresponding hypersphere in the conformal space that is attached to M.

The linear substitution that is defined by the preceding formulas obviously preserves the quadratic form:

$$\Phi = 2x^0 x^{n+1} + (x^1)^2 + \ldots + (x^n)^2;$$

it has the same analytical form as the one that defines a change of frame in the conformal space.

Having said this, consider a closed contour that starts at the point P and returns to it; the position parameter t on this contour starts with the value 0 and ends, we assume, with the value l. If one recursively refers the conformal space attached to P to the conformal space that is attached to a point M of the contour then one will have a correspondence, which is defined by formula $(7')$, between the hyperspheres that are attached to P and the hyperspheres that are attached to M.

When the point M returns to P *this correspondence will not be an identity in general* because it will define a certain (infinitesimal) conformal displacement that is applied to the hyperspheres that are attached to P and naturally depends upon the closed contour in question.

In the particular case in which the closed contour is infinitesimal, one may prove that this displacement is defined by the formulas:

$$\Delta x^i + \sum x^k\, \Omega_k^i = 0 \tag{8}$$ [1]).

One may further say that if one attaches a frame to each point of the manifold then the passage from the frame that is attached at a point M to the frame that is attached at an infinitely close point M' is effected by an infinitesimal conformal displacement. If we start from a given initial frame *in a conformal space, properly speaking,* and effect successive infinitesimal conformal displacements relative to the various infinitesimal segments MM' of the contour then one will not return to the conformal space E when M arrives again at P, the initial frame, on the manifold.

The point of the conformal space E that has the coordinates $(x^0, x^1, \ldots, x^{n+1})$ with respect to the initial frame will indeed have the coordinates:

$$x^i + \Delta x^i = x^i - \sum x^k \Omega_k^i \qquad (i = 0, 1, \ldots, n+1)$$

with respect to the final frame. In other words, *the final frame may be deduced from the initial frame by the infinitesimal conformal displacement of the components* Ω_i^j.

The theorem of the conservation of curvature.

10. The exterior differentiation of formulas (4) gives the relations:

$$\left\{\begin{aligned}
&(\Omega_0^0)' + \sum_{k=1}^n [\omega_k^0 \Omega^k] - \sum_{k=1}^n [\omega^k \Omega_k^0] = 0,\\
&(\Omega^i)' + [\omega^i \Omega_0^0] - [\omega_0^0 \Omega^i] + \sum_{k=1}^n [\omega_k^i \Omega^k] - \sum_{k=1}^n [\omega^k \Omega_k^i] = 0,\\
&(\Omega_i^j)' + [\omega_j^0 \Omega^j] - [\omega_i^0 \Omega^j] - [\omega^i \Omega_j^0] + [\omega^j \Omega_i^0] + \sum_{k=1}^n [\omega_k^j \Omega_i^k] - [\omega_i^k \Omega_k^j] = 0,\\
&(\Omega_i^0)' + [\omega_0^0 \Omega_i^0] - [\omega_i^0 \Omega_0^0] + \sum_{k=1}^n [\omega_k^0 \Omega_i^k] - \sum_{k=1}^n [\omega_i^k \Omega_k^j] = 0,
\end{aligned}\right. \tag{9}$$

[1]) There is no contradiction between these formulas and formula (5); the problems treated are not the same, although there is naturally a very close relationship between them.

which may be interpreted geometrically and give us the theorem of the conservation of curvature.

Classification of manifold with conformal connections.

11. The curvature of a manifold with a conformal connection translates into the statement that for any infinitesimal closed contour that starts at a point and returns to it with an infinitesimal conformal displacement, one may already distinguish certain important categories of manifold with conformal connections.

I. First, suppose that the conformal displacement that is associated with any infinitesimal closed contour is null. In this case, if we are given two arbitrary points of the manifold, P and Q, then one may refer the conformal space that is attached to Q to the conformal space that is attached to P in a manner that is independent of the path taken from P to Q. In particular, one may give the variable point Q the coordinates u^i that are the (n+2)-spherical coordinates of the hypersphere-point in the conformal space that is attached to P that corresponds to the hypersphere-point Q in the conformal space that is attached to Q. The manifold is then basically itself a conformal space, although it is referred to different frames depending upon the point of the manifold in question.

II. A more general category is defined by the manifolds for which the infinitesimal displacement associated with an arbitrary infinitesimal closed contour with origin P *leaves the point P invariant, along with all of the directions that issue from P.* This amounts to saying that this conformal displacement reduces to an elation with center A_0 and a homothety with centers A_0 and A_{n+1}. One thus has:

$$\Omega^i = 0, \qquad \Omega_i^j = 0.$$

Formulas (9) then give the relations:

$$(10) \qquad \begin{cases} [\omega^i \Omega_0^0] = 0, & (i = 1, \ldots, n) \\ [\omega^i \Omega_j^0] - [\omega^j \Omega_i^0] = 0, & (i, j = 1, 2, \ldots, n). \end{cases}$$

Suppose $n \geq 3$. The first n relations show that *the form* Ω_0^0 *is identically null;* in order words, the displacement that is associated with an infinitesimal closed contour reduces to an elation with center A_0 (or P). Formulas (9) then allow us to write:

$$\sum_{i=1}^{n} [\omega^i \Omega_i^0] = 0.$$

From the latter relations (10) one deduces:

$$[\omega^j \omega^i \Omega_i^0] = 0,$$

which shows that the form $[\omega^i \Omega_i^0]$ contains each of the forms $\omega^1, \omega^2, \ldots, \omega^n$ as a factor.

Therefore, if n > 3 then the form $[\omega^i \Omega_i^0]$ is identically null; in other words, one has:

$$\Omega_i^0 = [\omega^i \tilde{\omega}_i],$$

if we introduce n conveniently chosen linear forms $\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n$.

If we substitute this in the latter relations (10) then we obtain:

$$[\omega^i \omega^j (\tilde{\omega}_i + \tilde{\omega}_j)] = 0.$$

From this, one deduces that $\tilde{\omega}_i$ depends only upon $\omega_i, \omega_j, \omega_k$, if j and k are two arbitrary values for the $n - 1$ distinct indices of i. Since one supposes that $n\ \$\ 4$, this is possible only if $\tilde{\omega}_i$ is proportional to ω_i; however, Ω_i^0 is then identically null.

As a result, *if $n\ \$\ 4$ then, outside of conformal space, properly speaking, there exists no manifold with a conformal connection for which the conformal displacement that is associated with an arbitrary infinitesimal closed contour with origin P leaves the point P and all of the directions that issue from P invariant.*

On the contrary, if $n = 3$ then such manifolds do exist; one easily verifies that if one forms the expressions that one develops from Ω_i^0:

$$\begin{aligned}
\Omega_1^0 &= a_{123}[\omega^2 \omega^3] + a_{131}[\omega^3 \omega^1] + a_{112}[\omega^1 \omega^2], \\
\Omega_2^0 &= a_{223}[\omega^2 \omega^3] + a_{231}[\omega^3 \omega^1] + a_{212}[\omega^1 \omega^2], \\
\Omega_0^0 &= a_{323}[\omega^2 \omega^3] + a_{331}[\omega^3 \omega^1] + a_{312}[\omega^1 \omega^2]
\end{aligned}$$

then the matrix of coefficients of the right-hand side is symmetric and the sum of the elements on the principal diagonal is null.

III. One obtains a category of manifolds that is even more general than the preceding one when one supposes that the conformal displacement that is associated with an infinitesimal closed contour that starts from an arbitrary point P leaves the point P invariant. One then says that the manifold is *without torsion*.

This property is characterized by the relations:

$$\Omega^1 = 0, \quad \Omega^2 = 0, \ldots, \Omega^n = 0.$$

One then has the relations:

$$[\omega^i \Omega_0^0] - \sum_{k=1}^{n} [\omega^k \Omega_k^i] = 0, \qquad (i = 1, 2, \ldots, n)$$

between Ω_0^0 and Ω_i^j. A particular case is the one for which one also has $\Omega_0^0 = 0$. This hypothesis expresses the fact that if one makes an inversion of the point *P* at infinity then the conformal displacement that is associated with an infinitesimal closed contour that issues from *P* reduces to a Euclidian displacement (which preserves lengths). In this case, one has the relations:

$$\sum_{k=1}^{n}[\omega^k \Omega_k^i] = 0, \qquad (i = 1, 2, \ldots, n)$$

$$\sum_{k=1}^{n}[\omega^k \Omega_k^0] = 0.$$

The curvature tensor.

12. The coefficients of the forms Ω^i, $\Omega_0^0, \Omega_i^0, \Omega_i^j$ constitute a *tensor*, in the sense that if one effects a change of frame in the conformal space that is attached to a point *P* of the manifold then these coefficients are subjected to a linear substitution (the totality of which forms a group).

The coefficients of the forms Ω^i also form a tensor in their own right because the annihilation of all of these coefficients gives an intrinsic property of the manifold that is independent of the particular choice of frame. The same is true for:

The coefficients of the n+1 forms Ω^i, Ω_0^0,

The coefficients of the $\frac{n(n+1)}{2}$ forms Ω^i, Ω_i^j,

The coefficients of the $\frac{n^2+n+2}{2}$ forms Ω^i, Ω_0^0, Ω_i^j.

We will show later on that there also exist other remarkable tensors that are formed from the components of the curvature tensor. Here, there exists a fundamental difference compared to the manifolds with affine connections, namely, the curvature tensor may not be decomposed into *irreducible* tensors.

Isomorphism of manifolds with conformal connections.

13. Two *n*-dimensional manifolds with conformal connections are called *isomorphic* when one may establish a point-wise correspondence between these two manifolds such that if the frame that is attached to each point of the first one is chosen arbitrarily then one may chose choose the frame that is attached to the points of the second one in such a manner that the components of the conformal connections on the two manifolds are all equal. It is obvious that the intrinsic geometric properties of two isomorphic manifolds are identical; they basically define the same manifold with a conformal connection. One may consider a less complete isomorphism that one may call a *meromorphic isomorphism*. Imagine two *n*-dimensional manifolds with conformal connections, between which one has established a point-wise correspondence. Suppose that one has likewise established a (conformal) correspondence between the points of the two conformal spaces that are attached to two corresponding points, A and B, of the two manifolds, which amounts to saying that to any frame that is attached to A one associates a frame that is attached to the corresponding point B. Imagine further that one has brought the two frames (R) and (S) that are attached to the two particular corresponding points A and B into coincidence. In order to pass from the frames that are attached to two infinitely close corresponding points A' and B' one must give two infinitesimal conformal displacements to the frames (R) and (S). In the case of a holomorphic isomorphism these two displacements are identical, but in the case of a meromorphic isomorphism they no longer are, and the infinitesimal displacement of one of the frames *relative* to the displacement of the other is constrained to belong to a given linear sheaf of infinitesimal conformal transformation; however, in order for this statement to correspond to an essential geometric property, it is necessary that this sheaf must remain invariant under the conformal displacements that fix the point A. For example, this sheaf might consist precisely of the displacements that fix the point A, or perhaps from two of these displacements that leave all of the directions that issue from A invariant.

To each type of linear sheaf that is invariant under subgroup γ of conformal displacements that fix the point A there will correspond a definite type of meromorphic isomorphism. Denote the components of the conformal connections on the two manifolds by ω_i^j and $\bar{\omega}_i^j$, and let δ be the symbol for the differentiation that acts only on the arbitrary parameters that depend upon the choice of frames at the two given corresponding points. If we write:

$$\omega_i^j(\delta) = e_i^j$$

then we will see that one has:

$$e^i = \bar{e}^i = 0.$$

When one displaces on the two manifolds along two corresponding paths one will have, by hypothesis, a certain number of relations of the form:

$$\sum a_i \bar{\omega}^i + a_0^0 \bar{\omega}_0^0 + \sum a_i^j \bar{\omega}_i^j + \sum a_0^i \bar{\omega}_0^i = \sum a_i \omega^i + a_0^0 \omega_0^0 + \sum a_i^j \omega_i^j + \sum a_0^i \omega_0^i , \tag{11}$$

which have constant coefficients, and these relations must remain invariant under any change of frame, provided that this change is the same for both manifolds. Now, by virtue of formulas (4), one has:

$$\delta\omega^j = e_0^0 \omega^i - \sum_{k=1}^{n} e_k^i \omega^k ,$$

$$\delta\omega_0^0 = -\sum_{k=1}^{n} e_k^0 \omega^k + de_0^0 ,$$

$$\delta\omega_i^j = e_i^0 \omega^j - e_j^0 \omega^i + \sum_{k=1}^{n} [e_i^k \omega_k^j - e_k^0 \omega_i^j] + de_i^j ,$$

$$\delta\omega_i^0 = e_i^0 \omega_0^0 - e_0^0 \omega_i^0 + \sum_{k=1}^{n} [e_i^k \omega_k^0 - e_k^0 \omega_i^k] + de_i^0 .$$

One obtains similar formulas by replacing the ω_i^j with ϖ_i^j. From this, one deduces:

$$(12)\qquad \begin{cases} \delta(\varpi^i - \omega^i) = e_0^0(\varpi^i - \omega^i) - \sum_{k=1}^{n} e_k^i(\varpi^k - \omega^k), \\ \delta(\varpi_0^0 - \omega_0^0) = -\sum_{i=1}^{n} e_i^0(\varpi^i - \omega^i), \\ \delta(\varpi_i^j - \omega_i^j) = e_i^0(\varpi^j - \omega^j) - e_j^0(\varpi^i - \omega^i) + \sum_{k=1}^{n}[e_i^k(\varpi_k^j - \omega_k^j) - e_k^j(\varpi_i^k - \omega_i^k)], \\ \delta(\varpi_i^0 - \omega_i^0) = e_i^0(\varpi_0^0 - \omega_0^0) - e_0^0(\varpi_i^0 - \omega_i^0) + \sum_{k=1}^{n}[e_i^k(\varpi_k^0 - \omega_k^0) - e_k^0(\varpi_i^k - \omega_i^k)]. \end{cases}$$

All that remains for us to do is to express that the relations (11) must remain invariant when one subjects the $\varpi_i^j - \omega_i^j$ to variations that are given by formulas (12), regardless of the quantities e_0^0, e_i^j, e_i^0. A simple calculation shows that if $n \neq 3$ then relations (11) can have five possible forms:

1. $\varpi^i = \omega^j$ $\qquad (i = 1, 2, \ldots, n)$,
2. $\varpi^i = \omega^j,\ \varpi_0^0 = \omega_0^0$ $\qquad (i = 1, \ldots, n)$,
3. $\varpi^i = \omega^j,\ \varpi_i^j = \omega_i^j$ $\qquad (i, j = 1, \ldots, n)$,
4. $\varpi^i = \omega^j,\ \varpi_i^j = \omega_i^j,\ \varpi_0^0 = \omega_0^0$ $\qquad (i, j = 1, \ldots, n)$,
5. $\varpi^i = \omega^j,\ \varpi_i^j = \omega_i^j,\ \varpi_0^0 = \omega_0^0,\ \varpi_i^0 = \omega_i^0$ $\qquad (i, j = 1, \ldots, n)$.

The fifth case corresponds to the holomorphic isomorphism. As for the first one, it reduces to the second one; in other words, if one is given a pointwise correspondence between the two manifolds then one may define a correspondence between the two frames that are attached to the corresponding points in such a manner that one has $\varpi^i = \omega^j$, and one may also define this correspondence of frames in such a manner that one *also has* $\varpi_0^0 = \omega_0^0$. Indeed, suppose that the frames are chosen in a certain manner on the first manifold and that they are chosen on the second one in such a manner that one has:

$$\varpi^i = \omega^j.$$

One will then have a relation of the form:

$$\varpi_0^0 = \omega_0^0 + \rho_1\omega^1 + \rho_2\omega^2 + \ldots + \rho_n\omega^n.$$

Let $\bar{A}, \bar{A}_1, \ldots, \bar{A}_{n+1}$ be the hyperspheres that define the frame that is attached to the point $\bar{A}$ of the second manifold; now, take another frame:

$$\bar{A}' = \bar{A}\,, \qquad \bar{A}'_i = \bar{A}_i + \rho_i \bar{A}\,, \qquad \bar{A}'_{n+1} = \bar{A}_{n+1} - \sum_{i=1}^{n} \rho_i \bar{A}_i - \tfrac{1}{2}(\rho_1^2 + \cdots + \rho_n^2)\bar{A}\,.$$

We will have:

$$d\bar{A}' = d\bar{A} = \bar{\omega}_0^0 \bar{A} + \sum \omega^i A_i = (\bar{\omega}_0^0 - \sum \rho_i \bar{\omega}^i) + \sum \bar{\omega}^i \bar{A}'_i\,.$$

One sees, moreover, that with this new choice of frame the relations:

$$\bar{\omega}^i = \omega^i$$

remain invariant, but they must be combined with the relation:

$$\bar{\omega}_0^0 = \omega_0^0\,.$$

14. From this, *there exist three types of meromorphic isomorphism between two manifolds with conformal connections,* which respectively translate into the relations:

$$\begin{aligned}
&\bar{\omega}^i = \omega^i, \qquad && \bar{\omega}_0^0 = \omega_0^0\,, && \\
&\bar{\omega}^i = \omega^i, \qquad && \bar{\omega}_i^j = \omega_i^j\,, && \\
&\bar{\omega}^i = \omega^i, \qquad && \bar{\omega}_i^j = \omega_i^j\,, \qquad && \bar{\omega}_0^0 = \omega_0^0\,.
\end{aligned}$$

If the points A and $\bar{A}$ of the two manifolds that one has brought into coincidence are mapped to infinity under an inversion then the relative motion of the two frames (R) and (S) amounts to:

an arbitrary Euclidian displacement for the first type of meromorphic isomorphism,

an arbitrary homothety (or a translation) for the second type,

a translation for the third type.

One see that in the three cases the linear sheaf of infinitesimal conformal displacements generates a group [1]).

The condition of meromorphic isomorphism of the first type may be stated in a geometrically simple form:

In order for two manifolds to be meromorphically isomorphic of the first type, it is necessary and sufficient that one must establish a pointwise correspondence between them that preserves angles, i.e., that preserves the equation:

$$ds^2 = 0.$$

The condition is obviously necessary, since the ds^2 of a manifold is:

$$(\omega^1)^2 + (\omega^2)^2 + \ldots + (\omega^n)^2.$$

[1]) This conclusion may be false for manifolds with a metric or Euclidian connection.

It is also sufficient since if, given an arbitrary choice of frame attached to the second manifold, one has:

$$(\bar{\varpi}^1)^2 + (\bar{\varpi}^2)^2 + \cdots + (\bar{\varpi}^n)^2 = r[(\omega^1)^2 + (\omega^2)^2 + \ldots + (\omega^n)^2]$$

then one will obviously modify this frame in such a manner that one satisfies the relations:

$$\bar{\varpi}^i = \omega^i.$$

Finally, we remark that for a meromorphic isomorphism of the third type there is a pointwise correspondence that not only has a conformal representation, but also preserves torsion (for two arbitrary corresponding surface elements). This results from the fact that the formulas:

$$\bar{\varpi}^i = \omega^i, \quad \bar{\varpi}_i^j = \omega_i^j, \qquad \bar{\varpi}_0^0 = \omega_0^0$$

give, by exterior differentiation:

$$\bar{\Omega}^i = \Omega^i.$$

15. If two manifolds with conformal connections are without torsion then the three types of meromorphic isomorphism reduce to just one. The relations:

$$\bar{\varpi}^i = \omega^i, \qquad \bar{\varpi}_0^0 = \omega_0^0$$

imply, in effect, that:

$$\sum_{k=1}^{n} [\omega^k (\bar{\varpi}_k^i - \omega_k^i)] = 0,$$

from which:

$$\bar{\varpi}_i^j = \omega_i^j;$$

likewise, the relations:

$$\bar{\varpi}^i = \omega^i, \qquad \bar{\varpi}_i^j = \omega_i^j,$$

imply that:

$$[(\bar{\varpi}_0^0 - \omega_0^0)\omega^i] = 0,$$

from which one has:

$$\bar{\varpi}_0^0 = \omega_0^0.$$

Normal manifolds with conformal connections.

16. Among all of the manifolds with conformal connections that are meromorphically isomorphic of the first type to a given manifold there is one that is distinguished from all of the other ones by some characteristic properties.

Start with an arbitrarily given Monge equation:

$$\sum g_{ik}\, du_i\, du_k = 0,$$

and consider an arbitrary manifold for which this equation defines isotropic directions. If we decompose the left-hand side of that equation into a sum of n squares:

$$\sum g_{ik}\, du_i\, du_k = (\omega^1)^2 + (\omega^2)^2 + \ldots + (\omega^n)^2$$

then we will have the right, without making any restrictive hypothesis on the conformal connection on the manifold, to suppose that the ω^i are the components of the conformal connection that we previously denoted by the same notation, and we will also have the right to suppose that ω_0^0. The frame that is attached to each point of the manifold, which is assumed to have a given conformal connection, is then completely determined.

It is now easy to show that there exists one and only one system of forms ω such that one has:

$$(\omega^i)' = \sum_{k=1}^{n} [\omega^k \omega_k^i] \qquad (i = 1, \ldots, n);$$

this choice of forms amounts to attributing null torsion to the manifold.

Having thus chosen the ω_i^0, it is possible to annul Ω_0^0; indeed, the formula:

$$(\omega_0^0)' = 0 = \sum_{i=1}^{n} [\omega^i \omega_i^0]$$

is compatible with an infinitude of different choices for the components ω_i^0. One has, in a general manner:

$$\omega_i^0 = \sum_{k=1}^{n} \lambda_{ik}\, \omega^k \qquad (i = 1, 2, \ldots, n),$$

in which the coefficients λ_{ij} are subject to only the symmetry condition:

$$\lambda_{ij} = \lambda_{ji}\,.$$

Finally, consider the formula:

$$(\omega_i^j)' = [\omega^i \omega_j^0] - [\omega^j \omega_i^0] + \sum_{k=1}^{n} [\omega_i^k \omega_k^j] + \Omega_i^j\,.$$

From it, one deduces that:

$$\Omega_i^j = (\omega_i^j)' - \sum_{k=1}^{n} [\omega_i^k \omega_k^j] - \sum_{k=1}^{n} \lambda_{ik} [\omega^j \omega^k] + \sum_{k=1}^{n} \lambda_{jk} [\omega^i \omega^k]\,.$$

If one sets:

$$\Omega_i^j = \sum_{k,l}^{1,\ldots,n} A_{ikl}^j [\omega^k \omega^l],$$

then one sees that one has:

$$A_{iik}^{j} = a_{iik}^{j} - \lambda_{jk} \qquad (k \text{ ? } i, j)$$

$$A_{ijk}^{j} = a_{ijk}^{j} + \lambda_{ik} \qquad (k \text{ ? } i, j)$$

$$A_{jij}^{i} = a_{iij}^{j} - \lambda_{ii} - \lambda_{jj}$$

$$A_{ikl}^{j} = a_{ikl}^{j} \qquad (k \text{ ? } i, j),$$

in which the a_{ikl}^{j} denote well defined coefficients. As a result, one will have:

$$\sum_{k=1}^{n} A_{iik}^{k} = \sum_{k=1}^{n} a_{iik}^{k} - (n-1)\lambda_{ii} - \sum_{h=1}^{n} \lambda_{hh} \qquad (i = 1, \ldots, n)$$

$$\sum_{k=1}^{n} A_{ijk}^{k} = \sum_{k=1}^{n} a_{ijk}^{k} - (n-2)\lambda_{ij} \qquad (i \text{ ? } j = 1, \ldots, n).$$

From this, one immediately deduces that it is possible, in one and only one manner, to choose the coefficients λ_{ij} in such a way that one annuls the left-hand sides of the preceding equation. Due to the symmetry of the λ_{ij}, this meanwhile necessitates the proof of the equalities:

$$A_{ijk}^{k} = A_{jik}^{k} \qquad (i \text{ ? } j \text{ ? } k).$$

Now, these equalities follow from the identity:

$$\sum_{\rho=1}^{n} [\omega^{\rho} \Omega_{\rho}^{k}] = 0 \qquad (k = 1, \ldots, n),$$

which is itself a consequence of the fact that the forms Ω^{i} and Ω_0^0 are null. Upon equating all of the terms in $[\omega^i \omega^j \omega^k]$ in this identity to zero one obtains precisely:

$$A_{ijk}^{k} - A_{jik}^{k} = 0.$$

17. If one starts with a given Monge equation, it is thus possible to define a conformal connection that satisfies the relations:

$$\Omega^{i} = 0, \qquad \Omega_0^0 = 0, \qquad \sum_{k=1}^{n} A_{ijk}^{k} = 0.$$

However, the proof has been carried out by making certain hypotheses on the frame that is attached to each point of the manifold. One must now prove that the preceding relations persist for any other choice of frame. The property of the Ω^{i} and Ω_0^0 being null obviously has an intrinsic significance, as we have seen above.

In order to show that the latter relations have an invariant significance, imagine that we have chosen the most general frame possible at each point, and let δ denote the symbol for differentiation of only the parameters v_i, which depend upon the choice of frame, but not on the parameters u_i, which localize the point of the manifold.

The formulas:

$$\Omega_i'^j - [\omega^i \Omega_j^0] + [\omega^j \Omega_i^0] + [\Omega_i^k \omega_k^j] - [\omega_i^k \Omega_k^j] = 0$$

show us that one has:

$$\delta\Omega_i^j = \sum_{k=1}^{n} (e_i^k \Omega_k^j - e_k^j \Omega_i^k),$$

if we write e_i^j in place of $\omega_i^j(\delta)$.

Likewise, the formulas:

$$(\omega^i)' = [\omega_0^0 \omega^i] + \sum_{k=1}^{n} [\omega^k \omega_k^i]$$

show us that one has:

$$\delta\omega^i = e_0^0 \omega^i - \sum_{k=1}^{n} e_k^i \omega^k .$$

The relations thus obtained give us the infinitesimal variations that the forms ω^j, Ω_i^j are subjected to under an infinitesimal change of frame. One thus obtains, without difficulty:

$$\delta A_{ikl}^j = -2e_0^0 A_{ikl}^j + \sum_{\rho=1}^{n} (e_i^\rho A_{\rho kl}^j - e_\rho^j A_{ikl}^\rho + e_k^\rho A_{i\rho l}^j + e_\rho^\rho A_{ikl}^j),$$

and as a result, after some simplifications:

$$\delta \sum_{k=1}^{n} A_{ijk}^k = -2e_0^0 \sum_{k=1}^{n} A_{ijk}^k + \sum_{\rho=1}^{n} e_i^\rho \sum_{k=1}^{n} A_{\rho jk}^k + \sum_{\rho=1}^{n} e_j^\rho \sum_{k=1}^{n} A_{i\rho k}^k .$$

One thus sees that the relations:

$$\sum_{k=1}^{n} A_{ijk}^k = 0$$

remain invariant under any change of frame [1]).

We have therefore proved precisely that *if one is given an n-dimensional numerical manifold then among all of the conformal connections that are attributed to that manifold and are compatible with a Monge equation:*

$$\sum g_{ik}\, du_i\, du_k = 0,$$

[1]) In the case $n = 4$, the space-time for which the ds^2 is $(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 + (\omega^4)^2$ enjoys the property that its Einstein tensor is null.

which is given a priori in order to define the isotropic tangents on that manifold, there exists one and only one conformal connection that satisfies the characteristic relations:

$$\Omega^i = 0, \qquad \Omega_0^0 = 0, \qquad \sum_{k=1}^{n} A_{ijk}^k = 0.$$

We agree to say that the manifold is endowed with a *normal* conformal connection; we also say that we are dealing with a normal manifold with a conformal connection.

Normal manifolds with conformal connections, which are completely defined by the equation that is obtained by annulling their ds^2, play the same role in the theory of manifolds with conformal connections that Riemannian manifolds do in the theory of manifolds with metric connections.

Three-dimensional normal manifolds.

18. The case of three-dimensional normal manifolds is particularly interesting. One has:

$$\begin{aligned}
\Omega_2^3 &= A_{223}^3[\omega^2\omega^3] + A_{231}^3[\omega^3\omega^1] + A_{212}^3[\omega^1\omega^2] \\
\Omega_3^1 &= A_{323}^1[\omega^2\omega^3] + A_{331}^1[\omega^3\omega^1] + A_{312}^1[\omega^1\omega^2] \\
\Omega_1^2 &= A_{123}^2[\omega^2\omega^3] + A_{131}^2[\omega^3\omega^1] + A_{112}^2[\omega^1\omega^2].
\end{aligned}$$

The following sets of relations exist between these three forms:

1. The relations:

$$\begin{aligned}
[\omega^2\Omega_2^1] + [\omega^3\Omega_3^1] &= 0, \\
[\omega^1\Omega_1^2] + [\omega^3\Omega_3^2] &= 0, \\
[\omega^1\Omega_1^3] + [\omega^2\Omega_2^3] &= 0,
\end{aligned}$$

whenever Ω^1, Ω^2, Ω^3, Ω_0^0 are null; they give:

$$\begin{aligned}
A_{131}^2 &= A_{312}^1, \\
A_{212}^3 &= A_{123}^2, \\
A_{323}^1 &= A_{231}^3.
\end{aligned}$$

2. The relations:

$$A_{112}^2 + A_{331}^1 = 0, \qquad A_{223}^3 + A_{112}^2 = 0, \qquad A_{331}^1 + A_{223}^3 = 0,$$
$$A_{231}^1 = 0, \qquad A_{312}^2 = 0, \qquad A_{123}^3 = 0,$$

whenever the manifold is normal. These relations imply the nullity of all of the coefficients. As a result, the forms Ω_i^j are all null.

In other words, *the infinitesimal displacement that is associated with an infinitesimal closed contour that starts at a point P and returns to it leaves the point P invariant along with all of the directions that issue from P;* it is an elation with center *P*.

The fact that Ω^1, Ω^2, $\Omega^3, \Omega_0^0, \Omega_2^3, \Omega_3^1, \Omega_1^2$ are null implies the following relations between $\Omega_1^0, \Omega_2^0, \Omega_3^0$:

$$[\omega^1\Omega_1^0] + [\omega^2\Omega_2^0] + [\omega^3\Omega_3^0] = 0,$$
$$[\omega^2\Omega_3^0] - [\omega^3\Omega_2^0] = 0,$$
$$[\omega^1\Omega_2^0] - [\omega^2\Omega_1^0] = 0.$$

From this, one immediately deduces the formulas:

$$\Omega_1^0 = \alpha_1[\omega^2\omega^3] + \beta_3[\omega^3\omega^1] + \beta_2[\omega^1\omega^2],$$
$$\Omega_2^0 = \beta_3[\omega^2\omega^3] + \alpha_2[\omega^3\omega^1] + \beta_1[\omega^1\omega^2],$$
$$\Omega_3^0 = \beta_2[\omega^2\omega^3] + \beta_1[\omega^3\omega^1] + \alpha_3[\omega^1\omega^2],$$

in which the coefficients α_1, α_2, α_3 have a null sum.

19. Consider a parallelogram that has the point *P* for one of its vertices, and the two (infinitesimal) sides that issue from *P* have the projections:

$$(x^1, x^2, x^3) \qquad \text{and} \qquad (y^1, y^2, y^3).$$

The elation that is associated with a contour (or face) of this parallelogram has the components:

$$\alpha_1(x^2y^3 - x^3y^2) + \beta_3(x^3y^1 - x^1y^3) + \beta_2(x^1y^2 - x^2y^1),$$
$$\beta_3(x^2y^3 - x^3y^2) + \alpha_2(x^3y^1 - x^1y^3) + \beta_1(x^1y^2 - x^2y^1),$$
$$\beta_2(x^2y^3 - x^3y^2) + \beta_1(x^3y^1 - x^1y^3) + \alpha_3(x^1y^2 - x^2y^1).$$

To abbreviate, we set:

$$u_1 = x^2y^3 - x^3y^2, \qquad u_2 = x^3y^1 - x^1y^3, \qquad u_3 = x^1y^2 - x^2y^1;$$

the quantities u_1, u_2, u_3 may be regarded as the direction parameters of the plane of the face. One sees that the elation that is associated with this face has an axis [1]) whose direction has the direction parameters

$$\alpha_1u_1 + \beta_3u_2 + \alpha_2u_3, \quad \beta_3u_1 + \alpha_2u_1 + \beta_1u_3, \quad \beta_2u_1 + \beta_1u_2 + \alpha_3u_3;$$

this is the conjugate direction to the plane of the face with respect to the cone (*C*) that has the tangential equation:

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[1]) This must say that the elation leaves invariant all of the circles that pass through *P* and are tangent to that diection at *P*.

$$\alpha_1 u_1^2 + \alpha_2 u_2^2 + \alpha_3 u_3^2 + 2\beta_1 u_2 u_3 + 2\beta_2 u_3 u_1 + 2\beta_3 u_1 u_3 = 0.$$

The cone (*C*) thus has an invariant significance. The relation:

$$\alpha_1 + \alpha_2 + \alpha_3 = 0$$

shows that *it is capable of circumscribing a tri-rectangular triad.*

From the point-wise viewpoint, this cone defines the directions that issue from *P* such that the *elations* that admit one of these directions for their axis are associated with a face that contains that same direction. As a particular case, the cone (*C*) may be reduced to three (rectangular) lines or to an (isotropic) double line. It would be interesting to know the degree of generality of the normal manifolds for which this latter circumstance presents itself. In the general case, the elations that are associated with the closed contours that start from a point *P* and return to it admit arbitrary directions for their axes. In the case where the cone (*C*) reduces to two rectangular lines, the axis of the elation is always situated in the plane of these two lines. Finally, when the cone (*C*) reduces to an (isotropic) double line, all of the elations have that line for their axis.