"La théorie des groupes finis et continus et l'analysis situs," Mém. sci. math., fasc. 42, Gauthier-Villars, Paris, 1952.

MEMOIRS

IN THE

MATHEMATICAL SCIENCES

PUBLISHED UNDER THE PATRONAGE OF

THE PARIS ACADEMY OF SCIENCES,

AND THE ACADEMIES OF BELGRADE, BRUSSELLS, BUCHAREST, COIMBRA, KIEV, MADRID, PRAGUE, ROME, STOCKHOLM (MITTAG-LEFFLER FOUNDATION), THE MATHEMATICAL SOCIETY OF FRANCE, WITH THE COLLABORATION OF NUMEROUS SCHOLARS.

Director

Henri Villat

Member of the Institute,
Professor at the Sorbonne,
Director of the "Journal de Mathématiques pures at appliqués."

FASCICLE XLII

The theory of finite, continuous groups and analysis situs

By Élie CARTAN

Member of the Institute,
Professor on the science faculty of Paris

Translated by D. H. Delphenich

PARIS
GAUTHIER-VILLARS, PRINTER-EDITOR
LIBRARY OF THE BUREAU OF LONGITUDES OF L'ÉCOLE POLYTECHNIQUE
Quai des Grands-Augustins, 55

New printing
1952

TABLE OF CONTENTS

	Page
INTRODUCTION……	*i*
CHAPTER I – GENERALITIES ON MANIFOLDS AND ABSTRACT, CONTINUOUS GROUPS……	1
I. Manifolds. Closed and open manifolds.	1
II. Abstract finite, continuous groups.	3
III. Subgroups	5
IV. The abstract groups of order 1	6
V. Isomorphism	7
VI. Homogeneous spaces	9
CHAPTER II – LIE GROUPS……	11
I. Definition and review of some fundamental theorems	11
II. Adjoint. Generating a group by its infinitesimal transformations	15
III. The subgroups of a Lie group	16
IV. Homogeneous spaces whose fundamental group is a Lie group	18
V. Orientable and non-orientable homogeneous spaces. Volume. Metric homogeneous spaces	22
CHAPTER III – CLOSED, LIE GROUPS……	24
I. Volume of a closed group	24
II. A theorem of H. Weyl	24
III. The structure of closed groups	26
IV. Closed, semi-simple groups	28
V. Constructing the most general closed group	32
VI. Homogeneous spaces with closed fundamental group	35
CHAPTER IV – SYMMETRIC, RIEMANNIAN SPACES……	34
I. Definition and first properties	34
II. Reducible and irreducible homogeneous spaces	37
III. Closed, irreducible, symmetric spaces	40
IV. Closed, reducible, symmetric spaces	43
V. Open, irreducible, symmetric spaces	44
VI. Applications to the topology of open, simple groups	45
BIBLIOGRAPHIC INDEX……	47

INTRODUCTION

The original idea of applying the considerations of the *analysis situs* to the theory of finite, continuous groups goes back to Hurwitz in 1897 [**4**], who appealed to integrals that were applied to the entire domain of certain closed groups (viz., the linear group of a positive-definite Hermitian form and the orthogonal group) in his research on invariants. That process was used by H. Weyl in 1925 [**8**], who, thanks to some considerations of the *analysis situs*, made important progress in the theory of the linear representation of semi-simple groups, which is a theory for which E. Cartan gave a basis in 1912 by assuming the infinitesimal viewpoint of S. Lie, but with a lacuna that one cannot further manage to fill by algebraic means. From a different viewpoint, H. Poincaré [**5**, **6**, **7**], in three penetrating memoirs that were published in 1900, 1901, and 1908, showed the importance of the role that is played by the singular transformations of a group in the theory of the structure of that group, which is a role that is analogous to the one that is played by the critical points of an analytic function. Finally, we point out two memoirs of O. SCHREIER [**24** and **25**] that appeared in 1926 and 1927 on the abstract, continuous groups that were envisioned from a very general viewpoint.

In all of these papers, which remain isolated, except for the relatively recent ones by H. Weyl and O. Schreier, the finite and continuous groups are studied in their entire domain of existence, and not just, as with S. Lie, in the neighborhood of the identity transformation: They are "integral" studies, and not "local" ones. The objective of this fascicle is to review a certain number of fundamental problems that are posed in the theory of groups upon assuming the "integral" viewpoint, either by envisioning a finite, continuous group, as in Chapter I, to be a variety inside of which one defines an associative law of multiplication or composition that satisfies, at a minimum, some continuity conditions, or, as in Chapter II, by introducing some supplementary hypotheses on the analytic properties of the law of composition of the group in order to obtain what I call "Lie groups." One knows of no finite, continuous group that is not a Lie group; a fundamental theorem (no. **26**) shows that if such a group existed then it could not be isomorphic to any linear group. In the theory of Lie groups itself, we point out the insufficiency of the usual proofs of the third fundamental theorem, which prove the existence of only *a subset of a group* when one is given a system of constants c_{iks} that cannot be prolonged to form a complete group; a rigorous proof of that theorem will be summarized in Chapter II. We also point out the search for necessary and sufficient conditions that a connected or mixed subgroup g of a Lie group G must satisfy in order for g to be the largest subgroup that leaves invariant a point of a manifold that is transformed transitively by G; these conditions *are not* of an exclusively local nature. The manifolds that are capable of being transformed transitively by a Lie group are not, moreover, arbitrary from the viewpoint of the *analysis situs.*

Chapter III is dedicated to the study of closed groups, which play a very important role in applications. Chapter IV presents the principles E. Cartan's theory of Riemannian symmetric spaces, when envisioned from the viewpoint of group theory, and which have a great variety of applications to geometry and the theory of groups itself.

The theory of linear representations of closed groups, along with some applications that one can make to the theory of complete, orthogonal systems of function on a closed manifold that is transformed transitively by a closed group, is left completely aside in this fascicle, since it is far too easy to go beyond its scope, which can already be quite extensive.

CHAPTER I

GENERALITIES ON MANIFOLDS AND ABSTRACT, CONTINUOUS GROUPS

I. – Manifolds. Closed and open manifolds.

1. The notion of a manifold is suggested by those of a line and a surface that are embedded in ordinary space. We shall, at the same time, specify the generalization and limitation of that by the introduction of a number of postulates that are analogous to the ones that were stated by F. Haussdorff in his *Grundzüge der Mengenlehre* (Leipzig, 1914).

What we shall call an *n-dimensional manifold* is a set of elements – or *points* – such that one can define a system of subsets – called *neighborhoods* – that satisfy the following conditions:

A. *Each neighborhood $\mathcal{V}$ is associated with a well-defined one-to-one correspondence between the points of $\mathcal{V}$ and the points of a hypersphere Σ in n-dimensional Euclidian space.* The points of $\mathcal{V}$ that correspond to the interior point of Σ will be called *interior* to $\mathcal{V}$, while the other ones will constitute the *frontier* of $\mathcal{V}$.

B. *Any point of the manifold is interior to at least one neighborhood.*

C. *Let $\mathcal{V}$ be an arbitrary neighborhood, let Σ be the hypersphere that is associated with it, let M be an interior point of $\mathcal{V}$, let m be the corresponding point of Σ, and let s be a hypersphere with its center at m that is interior to Σ. There exists a neighborhood $\mathcal{V}$ that is interior to $\mathcal{V}$ and is such that the correspondents of all points of $\mathcal{V}$ in Σ belong to σ.*

D. *Let M be a point that belongs to the interior or to the frontier of $\mathcal{V}$, let m be its correspondent in Σ, and let $\mathcal{V}'$ be a neighborhood that contains M in its interior. There exists a hypersphere σ with its center at m such that the correspondents of all points of Σ that belong to σ in $\mathcal{V}$ will be interior to $\mathcal{V}'$.*

E. *If one is given two distinct points M and N then one can find two neighborhoods that have M and N in their interiors, respectively, and have no point in common.*

2. A point *A* of the manifold is called an *accumulation point* for an infinite set of distinct points of that manifold if any neighborhood that contains *A* in its interior contains at least one point of the set that is distinct from *A*. Any infinite set of distinct points that belongs to the same neighborhood $\mathcal{V}$ will admit at least one accumulation point that belongs to $\mathcal{V}$ (by virtue of postulates A, D and the Bolzano-Weierstrass theorem).

One says that an infinite sequence of points $A_1, A_2, \ldots, A_n, \ldots$ tends to a *limit point A* if when one is given an arbitrary neighborhood $\mathcal{V}$ that contains A in its interior, all of the points of the set are interior to $\mathcal{V}$, past a certain rank. The infinite sequence cannot tend to another limit point B (by virtue of postulate E).

From any infinite set that admits an accumulation point A, one can extract an infinite sequence of distinct points that tends to A.

It then results from postulates A, C, and D that the one-to-one correspondence that exists between the interior of a neighborhood $\mathcal{V}$ and the interior of the hypersphere Σ that it is associated with will be bicontinuous. One can thus analytically define the points that are interior to any neighborhood of an n-dimensional manifold in a unique manner by means of n coordinates, in such a way that two infinitely close points will have infinitely close coordinates.

3. A *continuous path* is a set of points that one can put into one-to-one correspondence with the numerical values of a real variable t that satisfies $0 \leq t \leq 1$, in such a way that if $t_n \to t_0$ then the sequence of points that corresponds to t_n will tend to the point that corresponds to t_0 .

The manifold is called *connected* if two arbitrary points can be linked by a continuous path. We shall consider only connected manifolds or ones that are composed of a finite or denumerably infinite number of connected manifolds.

4. Now, assume the following supplementary hypothesis:

F. *It is possible to find a finite or denumerably infinite number of neighborhoods such that any point of the manifold is interior to at least one of these neighborhoods.*

To abbreviate, we agree to say that the manifold is *covered* by the neighborhoods in question.

Arrange the neighborhoods considered into a certain order:

$$\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n, \ldots$$

We suppose that the first neighborhood $\mathcal{V}_\alpha$ in the preceding sequence for which any point that is interior to $\mathcal{V}_\alpha$ is interior to at least one of the preceding neighborhoods. We begin the process again with the new sequence that is thus obtained, and so on. *We thus arrive at a sequence of neighborhoods such that for each neighborhood $\mathcal{V}_i$ of the sequence there exists at least one interior point that is not interior to any of the neighborhoods $\mathcal{V}_1$, $\mathcal{V}_2$, …, $\mathcal{V}_{i-1}$.* Such a sequence will be called *normal*.

5. The manifolds that are capable of being covered by a *finite* normal sequence of neighborhoods are distinguished from the other ones by some characteristic properties. Indeed, consider an infinite set of points in such a manifold. There will exist at least one of the neighborhoods in the sequence – say, $\mathcal{V}_n$ – that contains an infinitude of points of the set, so (no. **2**) the given set will admit at least one accumulation point.

On the contrary, suppose that the manifold is covered by a normal sequence of a denumerable infinitude of neighborhoods. Take a point M_i in each neighborhood $\mathcal{V}_i$ that is interior to $\mathcal{V}_i$, but which is not interior to any neighborhood of an accumulation point. Indeed, such a point A will be interior to a certain neighborhood $\mathcal{V}_k$ without being interior to the preceding neighborhoods. Let $\mathcal{V}_k'$ be a neighborhood (that does not belong to the normal sequence) that is interior to $\mathcal{V}_k$ and contains A in its interior. None of the points M_{k+1}, M_{k+2}, … of the set belong to $\mathcal{V}_k'$, and therefore $\mathcal{V}_k'$ can contain only a finite number of points of the set, which contradicts the hypothesis.

We say that a manifold is open or closed according to whether one can or cannot find infinite sets of points that admit no accumulation point, respectively.

One sees that *if a closed manifold can be covered by a denumerable infinitude of neighborhoods then it can be covered by a finite number of neighborhoods* (generalization of the Heine-Borel theorem).

II. – Abstract finite, continuous groups.

6. One calls a set of *elements* an *abstract group* when one has defined an operation – called "multiplication" – on it that makes two arbitrary elements A, B that are arranged in a certain order correspond to a third element that is denoted by AB and satisfies the following conditions:

a. There exists an element 1 *(viz., the unity element) such that for any A, one will have* $1A = A1 = A$.

b. Any element A corresponds to an element A^{-1} *such that* $AA^{-1} = 1$.

c. One has:

$$(AB)\,C = A\,(BC).$$

It results from these hypotheses that one will also have $A^{-1}A = 1$. Indeed, the equality $BA = CA$ implies that $B = C$. As a result, the product $A^{-1}A = J$ will agree with 1 due to the equalities $JA^{-1} = A^{-1}AA^{-1} = A^{-1}1 = 1A^{-1}$. The equality $AB = AC$ will then also imply that $B = C$.

7. One can associate each element A of an abstract group with an *operation* – or *transformation* – namely, the one that makes the element M of the group correspond with the element $M' = AM$. That set of transformations contains the *identity transformation* $\mathcal{T}_1$. Each transformation $\mathcal{T}_A$ will correspond to an inverse transformation $\mathcal{T}_{A^{-1}}$. Finally, the resultant of the transformations $\mathcal{T}_A$ and $\mathcal{T}_B$, when performed in succession, will be the transformation:

$$M' = B\,(AM) = (BA)\,M,$$

which corresponds to the element BA. We say that the transformations $\mathcal{T}_A$ *realize* the abstract group as a group of transformations. They constitute the *parameter group* of the abstract group. The transformations $M' = MA$ define the *second parameter group*.

8. An abstract group is called *finite and continuous of order r* if its elements generate an r-dimensional manifold. Moreover, if one is given two infinite sequences of elements A_n and B_n that tend to A and B, respectively, then the infinite sequence of elements $A_n B_n$ will tend to AB. Finally, if A_n tends to 1 then A_n^{-1} will tend to 1. If $\mathcal{V}_0$ is a neighborhood of the group manifold that contains the element 1 in its interior then the set of elements $A\mathcal{V}_0$ that is obtained by multiplying A times the elements of $\mathcal{V}_0$ can be regarded as another neighborhood that contains the element A in its interior; the same thing will be true for the set of elements $\mathcal{V}_0 A$.

A finite, continuous group will be called *connected* or *mixed* according to whether its manifold is connected or composed of a finite or denumerably infinite number of connected manifolds, respectively. One of the connected families of elements that it is composed of will define a group in its own right, namely, the one that contains the element 1.

9. The manifold of an abstract, finite, continuous group always satisfies hypothesis F by itself: *It can be covered by a denumerable infinitude of neighborhoods $A_n\mathcal{V}_0$, in which $\mathcal{V}_0$ denotes any of the neighborhoods that contain the element* 1 *in its interior.*

We shall first show that any element of the group (which one can assume to be connected) can be obtained by multiplying a finite number of elements in the interior of $\mathcal{V}_0$. Indeed, join the element 1 to a given element A by a continuous path, where a variable element of the path will depend upon a parameter t that varies from 0 to 1. Let t_0 be the lower limit of the set of values of t that correspond to the elements of the path that cannot be obtained by the indicated process, and let A_0 be the corresponding element. The element A_0 itself cannot be the product of a finite number q of elements that are interior to $\mathcal{V}_0$, since for all of the values of t that are greater than t_0 and sufficiently close to t_0, one will have an element that will be the product of $q + 1$ elements that are interior to $\mathcal{V}_0$. Now, consider the neighborhood $A_0\mathcal{V}_0$. It contains elements of the curve that correspond to values of t that are less than t_0 and are also as close to t_0 as one desires, for example, an element $A_0' = A_0 s$, where s is as close to 1 as one desires; for example, close enough to 1 for s^{-1} to belong to $\mathcal{V}_0$. It will then result that the element $A_0 = A_0' s^{-1}$ is the product of a finite number of elements that are interior to $\mathcal{V}_0$, which contradicts the hypothesis.

Now, take an integer p. By hypothesis, one can put the elements of $\mathcal{V}_0$ into a continuous, one-to-one correspondence with the points of a hypersphere Σ of radius R in ordinary n-dimensional space. One can find a number ρ that enjoys the following property: If $A_1, A_2, \ldots, A_p$ are p points that are interior to Σ, and if $M_1, M_2, \ldots, M_p$ are likewise interior to Σ, but interior to the hyperspheres of radius ρ whose centers are at $A_1, A_2, \ldots, A_p$ then the product $M_1M_2\ldots M_p$ will belong to the neighborhood $A_1A_2\ldots A_p\mathcal{V}_0$. When the number ρ has been determined in that way, we can find a finite sequence of points in the interior of Σ – say, $C_1, C_2, \ldots, C_{N_\rho}$ – such that any point that is interior to S will be interior to at least one of the hyperspheres of ρ whose centers are at $C_1, C_2, \ldots, C_{N_\rho}$. It will then result that any element that is capable of being obtained by the multiplication of p elements that are interior to $\mathcal{V}_0$ will be interior to at least of the $(N_\rho)^p$ neighborhoods $C_{\alpha_1}C_{\alpha_2}\cdots C_{\alpha_p}\mathcal{V}_0$. Since that property is true for any p, one will thus arrive at a denumerable sequence of neighborhoods $A_k\mathcal{V}_0$ such that any element of the group is interior to at least one of these neighborhoods [cf., **24**, pp. 19].

III. – Subgroups.

10. A *subgroup* of an abstract group G is a group whose elements all belong to G. A subgroup might contain just a finite number of elements. If it contains an infinitude then it might or might not be continuous. In the latter case, there is yet another distinction to be made. The subgroup g is called *properly discontinuous in G* if one can find a neighborhood $\mathcal{V}_0$ in G that contains 1 in its interior and contains no element of g that is not 1. In the contrary case, the subgroup will be called *improperly discontinuous in G*. Each element of g will then be an accumulation point in G for the set of elements of g.

The subgroup g is called *closed in G* if any accumulation point in G of the set of elements of g also belongs to g. Any properly discontinuous subgroup is closed in G. In the contrary case, the subgroup will be called *open in G*.

The set of elements of a subgroup g that is closed in G and its accumulation points will define a new subgroup $\bar{g}$ that is closed in G.

If the group G is closed then the subgroups g that are closed in G will be the closed subgroups.

11. One calls the element AMA^{-1} the *transform* of an element M of the group G by an element A of that group. One says that a subgroup g is invariant under G if the transforms of the elements of g by the various elements of G again belong to g. In that case, the set of elements Ag that are obtained by multiplying a given element A of G by the various elements of g is identical to the set of elements gA. If one regards such sets as being composed of new elements then one can define an associative multiplication upon them by agreeing that the product of Ag with Bg will be ABg. These new elements define an abstract group whose unity element is g; one denotes it by the symbol G / g.

One calls the set of elements of a group that commute with all elements of the group the *center* of the group. These elements form an invariant commutative subgroup in G. The center of G will agree with G when G is commutative.

IV. – The abstract groups of order 1.

12. One can easily determine all of the finite, continuous, connected groups of order 1. Let $\mathcal{V}_0$ be a neighborhood that contains the element 1 in its interior. One can represent it by a line segment along which one takes the point that corresponds to 1 to be the origin. The abscissas x of the points of the segment vary, for example, from $-a$ to $+a$. If x and x' are the abscissas of the two points that are sufficiently close to the origin – for example, between $-b$ and $+b$ – then the product of two corresponding elements will belong to $\mathcal{V}_0$. If x'' is the abscissa of the point that represents that product then one will have a relation:

$$x'' = \varphi(x, x'),$$

in which φ is a continuous function. One easily proves that φ is an increasing function of its two arguments.

One can then find one and only one root in the interval $(0, b)$ of the successive equations:

$$\begin{aligned} &\varphi(a_1, a_1) = a, \\ &\varphi(a_n, a_n) = a_1, \\ &\dots\dots\dots\dots\dots\dots \\ &\varphi(a_n, a_n) = a_{n-1}, \\ &\dots\dots\dots\dots\dots\dots \end{aligned}$$

The numbers $a_1, a_2, \dots, a_n, \dots$ decrease while remaining positive; they thus tend to a limit $a \geq 0$. However, since one has:

$$\varphi(a, a) < \varphi(a, a_n) < \varphi(a_n, a_n) = a_{n-1},$$

it will then result that in the limit:

$$\varphi(a, a) \leq a = \varphi(0, a);$$

that will be possible only if $a = 0$.

Let S_n be the element of the parameter a_n. Assign a new parameter $p_n / 2^n$ to an element $S_n^{p_n}$. The new parameters of elements of that nature will follow in the same order as the old ones, and the multiplication of two elements whose new parameters are t and t' will give an element whose new parameter is $t + t'$. The assignment of a new parameter t extends, by continuity, to all of the elements whose old parameter x is found between 0 and a, and the multiplication formula will become:

$$t'' = t - t' \qquad\qquad (0 \le t, t', t'' \le 1).$$

If we agree to set $S_x = \Sigma_t$ then we can define Σ_n to be $(\Sigma_1)^n$ for any positive integer n, and then Σ_{n+t} to be the product $\Sigma_n\ \Sigma_t$ for t between 0 and 1. The multiplication law is extended to these new group elements. Finally, one defines Σ_{-t} to be $(\Sigma_t)^{-1}$ for positive t and again extends the multiplication law.

One is sure to obtain all of the elements of the group by this process (no. **9**), but one might possibly get each of them several times. If that is true and c is the smallest positive value of t for which Σ_c is the element 1 then the element Σ_{t+c} will be the same as the element Σ_t . The group that is obtained will then be *closed*, while in the contrary case, t can vary from $-\infty$ to $+\infty$ without the elements of the group being obtained two, and the group is *open*.

The connected groups of order 1 *are therefore all commutative; one type is open and the other is closed.*

One can add that all of the open groups are basically identical, as well as all the closed groups. Indeed, in the case of a closed group, one can always take a new parameter t' whose period is 1 instead of c.

V. – Isomorphism.

13. A group G is called *isomorphic* to a group G' if it is possible to make an element of G correspond to a well-defined element of G' in such a way that if A', B', C' are three elements of G' that satisfy $A'B' = C'$ then the three corresponding elements A, B, C of G will satisfy $AB = C$. The unity element $1'$ of G' will necessarily correspond to the unity element 1 of G.

The isomorphism is called *holohedral* if any element of G corresponds to one and only one element of G'; it is *hemihedral* in the contrary case. The elements of G' that have the unity element of G for their correspondent define an invariant subgroup of G'.

Two finite, continuous groups of the same order G and G' are called *locally isomorphic* if one can establish a one-to-one, continuous correspondence between the elements of a neighborhood $\mathcal{V}_0$ of G that contains unity in its interior and those of a neighborhood $\mathcal{V}_0'$ of G' that contains the unity element in its interior, and that correspondence will satisfy the condition that if A, B, and C are three elements of $\mathcal{V}_0$ such that $AB = C$ then the corresponding elements of $\mathcal{V}_0'$ must satisfy $A'B' = C'$.

Suppose that the manifold of one of the groups – G, for example – is *simply connected.* That means that any closed, continuous contour can be deformed in a continuous manner until it reduces to a point. Then let S be an arbitrary element of G that does not belong to $\mathcal{V}_0$. Join the unity element 1 to S by a continuous path that belongs to $(\mathcal{C})$ and take some intermediate points $S_1, S_2, \ldots, S_{p-1}$ on that path such that the elements $S_1,\ S_1^{-1}S_2,\ \ldots,\ S_{p-1}^{-1}S_p$ belong to $\mathcal{V}_0$; denote them by $s_1, s_2, \ldots, s_p$. Let $s_1', s_2', \ldots, s_p'$ be the corresponding elements of $\mathcal{V}_0'$, and consider the element $S' = s_1' s_2' \dots s_p'$ of G'. It is easy to see that if one takes another sequence of intermediate points in the path ($\mathcal{C}$) then one will always arrive at the same element S'. One finally arrives at the same element again by deforming the path ($\mathcal{C}$) that joins 1 to S sufficiently little.

Since the manifold of G is simply connected, one can thus make any element S of G correspond to a well-defined element S' of G'.

An analogous argument will show that any element S' of G' provides at least one element S of G, and one easily sees that one has an isomorphic correspondence between the two groups, so G' will be isomorphic to G.

14. If the isomorphism is not holohedral then several elements of G will correspond to the unity element $1'$ of G' that can be finite or infinite in number and which will generate a *properly-discontinuous* subgroup of G. Let 1, T_1, T_2, … be the elements of that subgroup. If the element S of G corresponds to the element S' of G' then all of the other elements of G that enjoy the same property will have the form $T_i S$, and also the form ST_j. However, the equality $T_i S = ST_j$ would demand that if S were very close to 1 then T_j would have to be equal to T_i, and therefore if one displaces them by continuity in the manifold of the group then the index j would only remain equal to the index i. The elements T_i would thus belong to the center (no. **11**) of the group G.

Hence, *if the group G' is locally isomorphic to the simply-connected group G then a properly-discontinuous subgroup of the center of G will correspond to the unity element of G'.*

15. This theorem admits a converse. Let g be a properly-discontinuous subgroup of the center of G. Take the sets $Sg = gS$ to be new elements, where S is fixed, and g successively denotes all of the elements that comprise that group. Define the multiplication of these new elements by the relation:

$$Sg \cdot S'g = SS'g.$$

One sees immediately that the new elements Sg generate a finite, continuous abstract group G' that is locally isomorphic to G and is such that the unity element of G' (namely, g) corresponds to the given subgroup g.

The search for groups that are locally-isomorphic to G thus amounts to the search for properly-discontinuous subgroups of the center of G.

For example, if G is the group of translations of the line then it will agree with its own center, and any properly-discontinuous subgroup will be defined by the powers with integer exponents of a particular translation; one will obtain the closed group of order 1.

The group of G of similitudes of the line is simply connected, and its center reduces to the identity element. Any group that is locally isomorphic to it is then integrally isomorphic.

16. Given a finite, continuous, connected, abstract group G, the problem of the search for groups that are locally isomorphic to it is then solved when G is simply-connected. In the contrary case, one can construct a simply-connected group $\bar{G}$ that is locally isomorphic to G. In order to do that [**13**, **25**], introduce new elements, each of which will be the set $[S, (\mathcal{C}),]$ of an element S of G and a continuous path $(\mathcal{C})$ that joins 1 to S. We continue to say that the two elements $[S, (\mathcal{C}),]$ and $[S', (\mathcal{C}'),]$ are identical if $S' = S$ and one can pass from $(\mathcal{C})$ to $(\mathcal{C}')$ by a continuous deformation. One will define the product of two elements $[S, (\mathcal{C}),]$ and $[S', (\mathcal{C}'),]$ by imagining an element P that moves along $(\mathcal{C})$ and considering the product SP that, when one follows it by starting with S, will describe a certain path $(\mathcal{C}'')$. The desired product will be $[S', (\mathcal{C}) + (\mathcal{C}'')]$. One easily verifies that this definition satisfies the conditions for the new elements $[S, (\mathcal{C})]$ to constitute an abstract group $\bar{G}$. That group will obviously be simply connected, and, on the other hand, it will be locally isomorphic to G. The unity element of G will corresponds to many elements of $\bar{G}$ that are closed contours in the manifold of G that are irreducible to each other. The properly-discontinuous commutative subgroup of the center of $\bar{G}$ that corresponds to the unity element of G will be the *fundamental group*, in the sense of the *analysis situs*, of the manifold of G. We shall give preference to the name of *connection group*, while reserving an entirely different significance for the expression "fundamental group," as we shall do in the following number.

VI. – Homogeneous spaces.

17. One calls a connected, n-dimensional manifold upon which a finite, continuous group operates transitively a *homogeneous space*. That can say that there exists a finite, continuous group of point-like transformations of the manifold that satisfies the following conditions:

1. *Any transformation of G makes a well-defined point M' correspond to a point M of the manifold.*

2. *Given two arbitrary points M and M' of the manifold, there exists at least one transformation of the group that takes M to M'.*

3. *If the sequence of points $M_1, M_2, \ldots, M_n, \ldots$ of the manifold tends to a limit point M, and if the sequence of transformations $S_1, S_2, \ldots, S_n, \ldots$ of the group tends to a transformation S then the point M'_n that is the transform of M_n by S_n will tend to the point M' that is the transform of M by S.*

The latter condition expresses the idea that the continuity in the manifold of the group (when considered as an abstract group) assures the continuity of the effects that are produced in the points of space.

A homogeneous space must therefore be regarded as the set of a connected manifold and a group G that operates transitively on that manifold. We say that G is the *fundamental group* of the space.

One can define the neighborhoods of a homogeneous space by the set of points that are transforms of a fixed point O by the transformations of an arbitrary neighborhood of the fundamental group. *A homogeneous space can thus be covered by a denumerable infinitude of neighborhoods.* A homogeneous space whose fundamental group is closed is obviously closed, but the converse is not true. For example, the projective line, which is a closed, one-dimensional space, is transformed transitively by the homographic group of one variable, which is open.

18. The set of transformations of G that leave invariant a particular given point O of space defines a subgroup g of G that is obviously *closed in G*, since if an infinite sequence of transformations of g tend to a transformation S of G then that transformation will leave the point O invariant; later on (no. **29**), we shall return to that important property.

It can happen that G admits transformations that leave all points of space fixed; they will necessarily belong to g and generate a subgroup γ that is *invariant in G.* In reality, the group of transformations of the space is then G / γ. If one excludes the possibility that was envisioned before then the subgroup γ will contain no subgroup that is invariant in the total group.

CHAPTER II

LIE GROUPS

I. – Definition and review of some fundamental theorems [1].

19. We say that a finite, continuous, abstract group is a *Lie group* if one can find a system of coordinates or (real) parameters $a_1, a_2, \ldots, a_r$ in a sufficiently small neighborhood $\mathcal{V}_0$ of the unity element such that the parameters c_i of the element $C = AB$ result from the multiplication of the element A with the parameters a_i by the element B whose parameters are b_i are expressed by functions:

$$c_i = \varphi_i (a, b)$$

that admit continuous partial derivatives of the first two orders.

In short, the problem of knowing whether there exist finite, continuous groups of order $r > 1$ that are mot Lie groups has never been addressed. Later on (no. **26**), we shall confirm the only precise result that one knows regarding that question.

If one is dealing with a Lie group then one can choose the group parameters in such a manner that the φ_i are *analytic* functions of their arguments [**2**]. The operations of each group of parameters (no. **7**) are, moreover, generated by linearly-independent *infinitesimal transformations* $X_1, X_2, \ldots, X_r$.

The brackets of pairs of infinitesimal transformations satisfy relations of the form:

$$(1) \qquad X_i (X_j) - X_j (X_i) \equiv (X_i\, X_j) = \sum_s c_{ijs} X_s \, .$$

The (real) constants c_{ijs} satisfy the algebraic relations:

$$(2) \qquad \sum_\rho (c_{ij\rho} c_{\rho kh} + c_{jk\rho} c_{\rho ih} + c_{ki\rho} c_{\rho jh}) = 0 \qquad (i, j, k, h = 1, 2, \ldots, r)$$

that are deduced from the *Jacobi identity:*

$$[(X_i\, X_j)\, X_k] + [(X_j\, X_k)\, X_i] + [(X_k\, X_i)\, X_j] = 0.$$

If a group of transformations other than a group of parameters *realizes* the abstract group, and if that group admits infinitesimal transformations then they will also satisfy the relation (1) with the same constants c_{ijs} .

20. One can add the following properties [**13**] to the preceding ones, which constitute the first two fundamental theorems of S. Lie: If one chooses a system of coordinates $a_1, \ldots, a_r$ in a certain neighborhood of a group then the infinitesimal element $S_a^{-1} S_{a+da}$ can be represented by the symbol $\sum \omega_k X_k$, where the Pfaff forms ω_i satisfy the relations (*Maurer-Cartan equations*):

(3) $$d\omega_s(\delta) - \delta\omega_s(d) = \sum_{i,j} c_{ijs}\,\omega_i(d)\,\omega_j(\delta).$$

Likewise, the infinitesimal transformation $S_{a+da}S_a^{-1}$ can be represented by the symbol $\sum \varpi_k X_k$, with the relations:

(4) $$d\varpi_s(\delta) - \delta\varpi_s(d) = -\sum_{i,j} c_{ijs}\,\varpi_i(d)\,\varpi_j(\delta).$$

The forms ω_s are invariant under the first group of parameters, while the forms ϖ_s are invariant under the second one.

Finally, the group can be defined by its *canonical parameters* in a sufficiently small neighborhood $\mathcal{V}_0$ of the unity element, where an operation is characterized by the parameters a_i of the infinitesimal transformation $\sum a_i X_i$ that generates it. With these canonical parameters, the forms ω_i can be obtained [**11**] by integrating the differential equations:

(5) $$\frac{d\omega_s}{dt} = da_s + \sum_{i,j} c_{ijs}\,a_i\,\omega_j$$

while assuming that they are annulled for $t = 0$, and then setting $t = 1$ in it. In these equations, one must regard the arguments a_i and da_i as constant parameters, while the ω_s are unknown functions of the independent variable t.

The forms ϖ_s can likewise be obtained by integrating the equations:

(5) $$\frac{d\varpi_s}{dt} = da_s - \sum_{i,j} c_{ijs}\,a_i\,\varpi_j\,.$$

All of these results can be regarded as classical.

21. The *third fundamental theorem* of S. Lie expresses the idea that if one has a system of constants c_{ijk} that satisfy the relations (2) then there will exist a finite, continuous group of order r whose independent infinitesimal transformations satisfy relations (1). In order to prove it, one can, for example, integrate equations (5), as we said above. By virtue of the relations (2), the Pfaff equations:

(7) $$\omega_s\,(u'; du') = \omega_s\,(u; du)$$

are *completely integrable* and give the u'_i as functions of the u_i and r parameters a_i that define a group of order r that satisfies the desired conditions; for example, one can take the parameters a_i to be the values of u'_i for $u_1 = \dots = u_r = 0$.

In reality, the preceding proof, like the other known proofs, moreover, except for the first proof of Lie that we shall discuss soon, is completely unsatisfactory. The ω_s are linear forms in du_1, du_2, …, du_r whose coefficients are *entire analytic* functions of the variable u_i, but the determinant of the coefficients of du_i is non-zero only in a certain neighborhood of the origin $u_i = 0$. Moreover, the determinant will be everywhere non-zero, which will not suffice to assure the existence of the finite transformations of the group that are valid in the entire space of the u_i . In order to convince oneself of that, it will suffice to consider the simple equation:

$$\frac{du'}{1+u'^2} = \frac{du}{1+u^2},$$

which does not provide *any* finite transformation that valid in the entire domain of existence of the real variable u.

One has thus proved definitively the existence of a set of transformations that are defined for sufficiently small values of the parameters in a sufficiently small region of the Euclidian space of the u_i, and that the product of two transformations of the set will again will belong to the set *in the case where that product is defined in the region considered.* In short, one obtains a *piece of the group* that operates upon a *piece of space.* It is necessary to prove that one can prolong that piece of space and that piece of the group in such a manner that one would obtain a manifold in which a group operates.

22. Lie's first proof, when it is valid from the *local* viewpoint, easily provides the basis for a rigorous proof. It consists of starting with r infinitesimal transformations:

$$E_s \equiv \sum_{i,j} c_{isj} e_i \frac{\partial f}{\partial e_j} \tag{8}$$

that satisfy the relations (1). For sufficiently small values of the parameters, they generate a piece of the group *in which all of the operations are valid in the entire space of the* e_i . Upon multiplying them together a finite number of times, and in all possible ways, one will obtain a group of well-defined linear transformations of the entire Euclidian space of variables e_i . The argument is valid only if the r transformations (8) are linearly independent, which demands that the infinitesimal group admit no *distinguished* infinitesimal transformation; i.e., one that commutes with all of the other ones. That will be true, for example, if the form:

$$\varphi(e) = \sum_{i,j,k,h} e_i e_j c_{ikh} c_{jhk} ,$$

which gives the sum of the squares of the roots of the *Killing equation:*

$$\left| \sum_s e_s c_{isj} - \delta_{ij}\lambda \right| = 0 \qquad \left(\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \right),$$

has a non-zero discriminant. The groups that satisfy that condition are the *simple* or *semi-simple* groups [**3**].

In the general case, one can prove the theorem directly by beginning with the case of an *integrable* group. One can choose the infinitesimal basis for an integrable group and the parameters $u_1, \dots, u_r$ in such a manner that the matrix of coefficients of $du_1, \dots, du_r$ in $\omega_1, \dots, \omega_r$ will have the form:

$$\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ * & e^{U_2} & 0 & \cdots & 0 \\ * & * & e^{U_3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & . \\ * & * & * & \cdots & e^{U_r} \end{matrix}$$

in which U_i is a linear form in the variables $u_1, \dots u_{i-1}$, and the terms that appear as asterisks are, for the i^{th} row, *entire* analytic functions of the $u_1, \dots u_{i-1}$ ([1]). The integration of equations (7) will then give entire, analytic functions of the u_i and the a_i for the u_i' that are initial values of the u_i'. One thus has directly a group whose manifold is homeomorphic to Euclidian space and which operates on a space that is homeomorphic to Euclidian space. That group is simply connected.

In the case of a non-integrable infinitesimal group G that admits a larger invariant, integrable subgroup g, one can, in order to obtain a finite group with the given infinitesimal structure, come down to the integration of a Pfaff system:

$$\omega_s (u'; du') = \alpha_{s1}(a)\ \omega_1 (u; du) + \dots + \alpha_{ss}(a)\ \omega_s (u'; du'),$$

in which the ω_s are the forms that were just in question for the integrable groups, and the $\alpha_{ij}(a)$ are the coefficients of a semi-simple, *linear* group with a known infinitesimal structure. The conclusion is the same, so the manifold of the group that is obtained will be composed of points (a, u), each of which is the set of a point a of the manifold of a semi-simple linear group and a point u of the Euclidian space of the u_i .

([1]) Meanwhile, it can happen that one has:

$$\begin{aligned} a_{ii} &= e^{U_i} \cos U_{i+1}, & a_{i,\,i+1} &= -e^{U_i} \sin U_{i+1}, \\ a_{i+1,\,i} &= e^{U_i} \sin U_{i+1}, & a_{i+1,\,i+1} &= e^{U_i} \cos U_{i+1} \end{aligned}$$

for two consecutive rows – for example, the i^{th} and the $(i + 1)^{\text{th}}$ – in which the elements a_{ij} and $a_{i+1,\,j}$ $(j < i)$ are entire functions of the $u_1, \dots, u_{i-1}$.

II. – Adjoint group. Generating a group by its infinitesimal transformations.

23. If S_a is a particular operation of a group G, and S_u is a variable operation then the equation:

$$S_{u'} = S_a S_u S_a^{-1}$$

will define an operation T_a that makes S_u go to the *transform* $S_{u'}$ *of* S_u *by* S_a . These operations T_a are *automorphisms* of the group, in the sense that if S_u and S_v have transforms of $S_{u'}$ and $S_{v'}$, resp., then the product $S_v\, S_u$ will have the transform of $S_{v'}\, S_{u'}$. Moreover, they form a group, which is the *adjoint Lie group*. In particular, they leave invariant the identity transformation and transform the infinitesimal transformations linearly amongst themselves. From this viewpoint, they constitute the *linear adjoint group* Γ of G. The infinitesimal transformations of Γ are given precisely by the formulas (8): In fact, the transformation $\sum e_i X_i$, when transformed by εX_s becomes:

$$\sum e_i X_i + \varepsilon \left(\sum e_i X_i \cdot X_s\right) = \sum_j e_j X_j + \varepsilon \sum_{i,j} c_{isj} e_i X_j \,.$$

The coefficients of the infinitesimal transformation $\sum_s e_s E_s$ are the elements of the matrix H_a :

$$H_a = \begin{pmatrix} \sum a_s c_{s11} & \sum a_s c_{s21} & \cdots & \sum a_s c_{sr1} \\ \cdots & \cdots & \cdots & \cdots \\ \sum a_s c_{s1r} & \sum a_s c_{s2r} & \cdots & \sum a_s c_{srr} \end{pmatrix}; \tag{9}$$

this matrix plays a fundamental role in the question of the generation of a group by its infinitesimal transformations.

24. In a sufficiently small neighborhood $\mathcal{V}_0$ of the identity operation, any operation of the group will admit a system of well-defined canonical parameters ($a_1, \ldots, a_r$), which are those of the infinitesimal transformation that it generates. Now, follow a continuous path in the manifold of the group that starts at the identity operation; let $S(t)$ be the operation that corresponds to point a of the path, which one can assume depends upon a parameter t. One then follows the canonical parameters that one assigns to $S(t)$, step-by-step. Indeed, if the a_i are the parameters of $S(t)$ then one can calculate the parameter $a_i + da_i$ of $S(t + dt)$ if the parameters ω_i of $[S(t)]^{-1}\, S(t + dt)$ are independent linear forms in $da_1, \ldots, da_r$. Now, integrating equations (5) will show that the ω_i are deduced from the da_i by performing the linear substitution that is represented by the matrix:

$$1 + \frac{1}{2!} H_a + \frac{1}{3!} H_a^2 + \ldots + \frac{1}{n!} H_a^{n-1} + \ldots = \frac{e^{H_a} - 1}{H_a},$$

whose determinant is annulled only if one of the characteristics roots of the matrix H_a (viz., the Killing roots) is a *non-zero* integer multiple of $2i\pi$. Consequently, *one can pursue the determination of the canonical parameters as long as one does not arrive at a transformation S whose canonical parameters (which are obtained by continuity step-by-step) give a Killing root to the matrix H_a that is a non-zero integer multiple of* $2\pi i$.

The transformations S of the group that one will conclude with are the ones for which the substitution corresponding T of the linear adjoint group will admit a characteristic root $e^{2ni\pi}$ that is equal to 1, *but which is provided from a root that is different from* 1 *by continuity*. If one knows in advance that these *singular* transformations define a manifold of dimension greater than $r - 2$ in the group manifold then one can always arrive at a non-singular transformation without encountering a singular transformation, and in turn, *the group* (or at least the set of its non-singular transformations) *will be generated completely by its infinitesimal transformations.*

In the contrary case, and contrary to what H. Poincaré [**5**] believed that he had proved, it can happen that the infinitesimal transformations generate only a portion of the group. The simplest case is provided by the group of real, unimodular, linear substitutions in two variables:

$$\left\{\begin{aligned} x' &= a\,x + b\,y, \\ y' &= a'x + b'y. \end{aligned}\right. \tag{10}$$

The substitutions for which the equation $(a - \lambda)\,(b' - \lambda) - ba' = 0$ admits two *distinct, negative* real roots ($a + b' < -2$) cannot be generated by any infinitesimal transformation of the group.

25. One can arrive at a very interesting result in the particular case of a group G of *real* linear substitutions. Any substitution of the group that cannot be generated by an infinitesimal transformation can be regarded as the product of two substitutions of the group that commute with each other, one of which is *involutive*, and the other of which is generated by an infinitesimal substitution. The proof is based upon the consideration of the group G' that is obtained by regarding the real parameters of the G as *complex*. In a more precise manner, G' is the linear group of $2r$ real parameters that are generated by infinitesimal substitutions $X_1, \ldots, X_r$ of G and by the substitutions $iX_1, iX_2, \ldots, iX_r$.

In that regard, it is important to remark that if one is given a group G of order r then *there will never exist* a group G' of order $2r$ that has G as a subgroup and is such that the $2r$ real, canonical parameters of G' are obtained by giving arbitrary *complex* values to the r canonical parameters of G. As an example, we cite the *simply-connected* Lie group G that is infinitesimally isomorphic to the homographic group of one real variable.

III. – The subgroups of a Lie group.

26. One can prove two fundamental theorems that relate to the subgroups of a Lie group.

The first theorem is the following one: *Any continuous subgroup of a Lie group is a Lie group*. More precisely, one can find two neighborhoods $\mathcal{V}_0$ and v_0 of the unity elements in the manifolds of G and g, resp., that are sufficiently small that the various operations of v_0 that are interior to $\mathcal{V}_0$ are the ones that are generated by a certain linear family of infinitesimal transformations of G. A particular case of that theorem that refers to the subgroups of the linear group in n variables was proved by J. von Neumann [**23**].

Let N and n be the order of G and g, respectively. Take a hypersphere Σ in the Euclidian space $\mathcal{E}_N$ of dimension N of canonical parameters of G that has the origin for its center and a radius R that is small enough that two distinct, interior points of Σ will represent two distinct elements of G. We agree to call the distance from the origin to the representative point of an element of G that is interior to Σ its *modulus*. Now, consider a neighborhood v_0 in g that surrounds the unity element such that all of its elements have moduli that are less than R. It is legitimate to suppose that it is represented by a hypersphere σ of radius r in Euclidian space $\mathcal{E}_N$ of dimension n, the center represents the unity element. Let $R' < R$ be the lower bound of the moduli of the elements g that are represented by frontier points of σ. Take a number $R'' < R'$. We can find a hypersphere σ' in $\mathcal{E}_N$, whose radius r' is sufficiently close to r that the elements of g that are exterior to σ' and interior to σ will all have modulus greater than R''. Finally, determine a number ε that is small enough that the product of an element g that is interior to the hypersphere σ_ε of radius ε with an element of g that interior to σ' will itself be interior to σ.

Having said that, consider an infinite sequence of points in σ that converge to the origin; let $A_1, A_2, \ldots, A_n, \ldots$ be the corresponding points of Σ. The half-lines in $\mathcal{E}_N$ that join the origin to these points will admit at least one accumulation half-line Δ; take an arbitrary point H on that half-line that is situated at a given distance $R_0 \le R''$ from the origin. If s_n is the element of g that is represented in $\mathcal{E}_N$ by A_n then determine the largest integer p_n such that σ_n and all of its powers up to $s_n^{p_n}$ inclusively have moduli that are less than R_0. If n is large enough that s_n is interior to σ_ε then the representative point of s_n^2 in the space $\mathcal{E}_N$ will be interior to σ, and it will likewise be interior to σ', since its modulus if less than R''; the same thing will be true for all of the other powers. The representative points in the space $\mathcal{E}_N$ will all be interior to the hypersphere whose radius is R_0, and they will obviously admit the point H as an accumulation point. One can then extract a partial infinite sequence of points that converges to H from that sequence. The sequence of corresponding points in $\mathcal{E}_n$, which are all interior to σ', admits at least one accumulation elements in $\mathcal{E}_n$, and since it cannot admit more than one in G, the point H must therefore represent an element of g that is interior to σ. The reasoning that was made for H is valid for all points of Δ whose modulus is less than R'', and in turn, R'. In other words, the *neighborhood* v_0 *of* g *contains all elements of modulus less than* R' *of a subgroup that is generated by an infinitesimal transformation of* G.

One easily sees then that all of the infinitesimal transformations of G that belong to g generate a Lie subgroup g' that is contained in g, and whose elements of modulus less than R' all belong to the neighborhood v_0 of g. No other element that belongs to v_0 can be interior to the hypersphere Σ of radius v_0 in g. Indeed, such an element s can be joined to the unity element of g by a continuous path that belongs to v_0, and whose elements will all be of modulus less than R. Let A be the representative point of that element that is interior to Σ. Interior to Σ, the manifold that is the locus of element sg' will be an *analytic* manifold that passes through A and has the same dimension n' as the order of g'. The planar manifold of dimension $N - n'$ that issues from the origin and is orthogonal to g' will meet that manifold at a well-defined point. One will then have a sequence of points that converges to the origin at the same time as A and is such that the lines that join the origin to these points form angles with the lines that generate g' that converge to $\pi / 2$. It then results that there exist infinitesimal transformation in g that are distinct from those of g', which is contrary to hypothesis. The theorem is then proved completely.

In particular, one deduces the following consequence: *Any finite, continuous, linear group is a Lie group*. The same thing will be true for any projective, conformal, etc., group. *Therefore, if there exists a finite, continuous group that is not a Lie group then it cannot be isomorphic to any linear group*. The question of knowing whether any Lie group is isomorphic to a linear group is, as one knows, still open.

27. The second fundamental theorem relates to subgroups that are *closed in G*. It is stated in the following manner: *If a subgroup g of a Lie group G is closed in G without being properly discontinuous then one can find a neighborhood* $\mathcal{V}_0$ *of the unity element in G that is sufficiently small that all of the elements of g that are interior to* $\mathcal{V}_0$ *will be the ones that are generated by a certain linear family of infinitesimal transformation of G.*

The proof is analogous to the preceding, but simpler, and its starting point is again the consideration of an infinite sequence of elements of g that converge to the unity element in G.

In particular, it results from the second theorem that *any improperly-discontinuous subgroup g is open in G, so the subgroup that is composed of g and its accumulation points in G will be a continuous Lie group*.

IV. – Homogeneous spaces whose fundamental group is a Lie group.

28. Among the homogeneous spaces whose fundamental group is a Lie group (viz., Lie homogeneous spaces), one finds, in particular, the Lie groups in which one or the other of the parameter groups operate transitively. These spaces are not arbitrary manifolds from the viewpoint of the *analysis situs,* as the examination of the two-dimensional case will show. The two-parameter Lie groups are either commutative or isomorphic to the group of similitudes of the line. The manifold of a commutative group is homeomorphic to either the Euclidian plane (viz., the group of translations of the plane), a cylinder of revolution, or a torus. As for the manifold of the group of similitudes of the line $x' = ax + b$ ($a > 0$), it is homeomorphic to the Euclidian plane (or to a half-plane, which is the same thing). That group will then be simply connected, and since its center reduces to the identity operation, there will exist no other group that has the same infinitesimal structure. *We thus obtain the Euclidian plane, the cylinder of revolution, and the torus as the only Lie group manifolds of order* 2.

29. Now consider an arbitrary Lie homogeneous space E that admits a connected, continuous group G for its fundamental group. The largest subgroup g that leaves invariant a particular point O of space is, as we saw (no. **18**), closed in G, and in turn (no. **27**), properly discontinuous, or even continuous, and connected or mixed. Moreover, it admits no invariant subgroup in G.

Conversely, let g be an arbitrary closed subgroup in G that admits no invariant subgroup in G. Let $r - n$ and n be the orders of g and G, respectively. If there exists a homogeneous space that is transformed transitively by G and is such that g is the largest subgroup that leaves a point O of space invariant then one can associate each point M of space with the set Sg of transformations of G that take O to M, and which are all obtained by multiplying a particular transformation S times all of the transformations of g. Then, consider the set of "elements" or "points" Sg. It defines an n-dimensional manifold that satisfies the desired conditions.

Indeed, define the neighborhood of a "point" Sg to be the set of "points" sSg, where s is an arbitrary element of a neighborhood $\mathcal{V}_0$ of the unity element in G. Choose n infinitesimal transformations $X_1, \ldots, X_n$ arbitrarily that define a basis for the group G with the $r - n$ infinitesimal transformations SgS^{-1}. Any element s is, in one and only manner, the product of a transformation t of $\mathcal{V}_0$ that is generated by $e_1 X_1 + \ldots + e_n X_n$ and a transformation of $\mathcal{V}_0$ that belongs to SgS^{-1}. One will then have:

$$s\,S\,g = t\,S\,g.$$

One can thus make any "point" of the neighborhood $\mathcal{V}_0Sg$ considered correspond to a point $(e_1, \ldots, e_n)$ of an n-dimensional Euclidian space that is interior to a hypersphere of sufficiently small radius. On the other hand, two distinct points of that hypersphere will correspond to two distinct "points" tSg. If that were not the case then no matter how small one took the neighborhood $\mathcal{V}_0$ one could find an infinite sequence of pairs of elements t_n, t'_n that converged to the unity element and were such that $t'_n S$ had the form t_nSR_n, where the element R_n belongs to g. One would then have:

$$t_n^{-1}t'_n = SR_nS^{-1},$$

in which R_n could converge to the unity element *without belonging to the immediate neighborhood of the unity element in g*. However, that would contradict the second fundamental theorem (no. 27) that relates to subgroups g that are closed in G. Postulate A is then verified. The other postulates present no difficulty, except perhaps the last one E, which is proved in the following manner: If one is given two distinct "points" Sg and $S'g$ such that one cannot find two neighborhoods of these "points" that have no point in common then one can find an infinite sequence of pairs of elements s_n, s'_n of G that converge to the unity element, and are such that:

$$s_n\, S\, g = s'_n\, S'g\,.$$

One will then have:

$$S' = s'^{-1}_n\, s_n\, S\, R_n\,,$$

in which R_n belongs to g. The element R_n will converge to $S^{-1}\,S'$, which does not belong to g, which contradicts the hypothesis that g is closed in G.

It is indeed clear that one can start with any other subgroup $S_0\;g\,S_0^{-1}$ that is *homologous* to g in G.

30. Instead of supposing that the space E is transformed transitively under a group that is holohedrally isomorphic to G, one can suppose simply that it is transformed by a group that is *infinitesimally isomorphic to G*. In that case, the subgroup g can contain elements that define a subgroup γ that is invariant in G, but not continuous. On the other hand, since γ is composed of the set of transformations of G that leave all of the points of space invariant, it is closed in G, and in turn, in g; it is thus *properly discontinuous in g*. Each of its elements is then invariant by itself in G – in other words, it belongs to the center of G. *The homogeneous spaces that are transformed transitively under G, with the possibility that there exists a non-continuous subgroup in G that leaves invariant all of the points of space, are thus associated with the various closed subgroups g in G that, like the possible subgroups that are invariant in G, admit only one properly-discontinuous subgroup of the center of G*.

If G is simply-connected then one can construct all of the homogeneous spaces that admit a group that is infinitesimally isomorphic to G for their fundamental group.

31. Suppose that the group G is simply connected. The subgroup g can be connected or mixed. In the latter case, the connected family g_0 of g that contains the unity element will be invariant under all transformations of g.

If g is connected then the homogeneous space E will be simply connected. Indeed, take a closed contour $(\mathcal{C})$ in E that starts at O and returns to it, and associate each point M of that contour by continuity with one of the transformations of G that take O to M, starting with the identity transformation. The contour $(\mathcal{C})$ will correspond to a path $(\mathcal{C}')$ in the manifold G that starts with the unity element and ends at an element of g, *which is a path one can close without leaving g*, since g is *connected*. One can deform the closed contour $(\mathcal{C}')$ thus obtained in a continuous manner in such a way that it reduces to a point. That deformation will imply a corresponding continuous deformation of the contour $(\mathcal{C})$, which can thus be reduced to a point.

Q. E. D.

If g is not connected then every connected family g_i that constitutes g will correspond to a set of closed contours in space E that are all reducible to each other by continuous deformations. In order to obtain them, one connects the unity element in the manifold of G to an element of g_i by a closed path. The elements of that path will yield transformations that will give a closed contour in E when they are applied to the point O. In a general manner, there are many closed contours in E that are not reducible to each other that give distinct, connected families in g. The *connection group* of the space E, in the sense of the *analysis situs*, is the abstract group whose elements (e_i) can be identified with the family g_i, where the product $(e_i)(e_j)$ will be equal to (e_k) if the products of the elements of g_i with the elements of g_j give elements of g_k .

32. An interesting consequence results from the preceding. No longer suppose that G is simply connected. If the homogeneous space E is simply connected then one can assert that the subgroup g of G that is associated with the space E is connected. On the contrary, if the space E is not simply connected then one can assert that either the subgroup g is not connected or that the group G is not simply connected. That is what happens, for example, in the case of the projective line when it is transformed transitively by the (connected) homographic group of one variable. The projective line is not simply connected, but the subgroup g that leaves the point $x = \infty$ invariant is the group $x' = ax + b$ $(a > 0)$, which is connected; *therefore, the homographic group is not simply connected*. One can infer the same conclusion for the unimodular linear group in two real variables, which transforms the pointed Euclidian plane (in which, one has singled out the origin) transitively, and for which the subgroup g that leaves the point (1, 0) invariant is the connected group:

$$\begin{aligned} x' &= x + ay, \\ y' &= \qquad y. \end{aligned}$$

These remarks show how the topological study of the fundamental group of a homogeneous space can be quite interesting in the topological study of that space.

33. In conclusion, we point out that the knowledge of all of the types of Lie groups in two variables will permit one to determine all of the two-dimensional Lie homogeneous spaces. We content ourselves by pointing out the result.

Any two-dimensional Lie homogeneous space is homeomorphic to one of the following spaces:

The Euclidian plane.
The cylinder of revolution.
The pointed projective plane.
The sphere.
The projective plane.
The torus.

The first three are open, while the last three are closed. One sees that *the Riemann surface of an algebraic curve of genus greater than* 1 *cannot be transformed transitively by any Lie group*. An analogous theorem, which is less restrictive as far as the nature of the group is concerned, was proved by D. van Dantzig and B.-L. van der Waerden [**26**].

V. – Orientable and non-orientable homogeneous space. Volume. Metric homogeneous spaces.

34. Let G be the fundamental group of a homogeneous space, let g be the associated subgroup, and let γ be the subgroup of the linear adjoint group that corresponds to g. Suppose that the last $r - n$ infinitesimal transformations in the infinitesimal basis for G are the ones that generate g, or at least, the connected part of g that contains the unity element. The linear substitutions of g subsume the parameters e_i of the most general infinitesimal transformation $\sum e_i X_i$ of the group, but since g obviously leaves invariant the set of transformations of g, these substitutions will transform the parameters $e_1, \ldots, e_n$ between themselves. We let $\bar{\gamma}$ denote the linear group that indicates how these n parameters are transformed.

Suppose that the determinants of the various substitutions of $\bar{\gamma}$ are always positive: The space E will then be *orientable*. Consider a parallelepiped that is constructed from n infinitely small vectors OA_i that issue from the point of origin O. Each point A_i can be obtained by applying an infinitesimal transformation $\sum_{k=1}^{n} e_k^{(i)} X_k$ to O. Arrange the n vectors into a certain order, and agree to say that the parallelepiped has a *positive* or *negative* sense according to whether the determinant $\left| e_i^{(j)} \right|$ is positive or negative, resp. The parallelepiped will be changed into another one by any transformation of g that will have the same sense as the first one, since one passes from the values $e_1^{(j)}, e_2^{(j)}, \ldots, e_n^{(j)}$ to the transformed values by a substitution of the $\bar{\gamma}$, and similarly for the indices i. One can likewise define the sense of an infinitely small parallelepiped of origin A that is different from O by moving its origin to O by a transformation of G, and *the sense will be conserved by any transformation of G*.

On the contrary, if certain substitutions of $\bar{\gamma}$ have negative determinants then the space will not be orientable.

If the subgroup g is connected then it will be clear that the determinants of the substitutions of the *connected* linear group $\bar{\gamma}$ will always be positive; the space will then be orientable. *In particular, the manifold of a group is then always orientable.*

35. The preceding considerations permit one to define the volume of an infinitely small parallelepiped of a homogeneous space if all of the linear substitutions of $\bar{\gamma}$ have determinants that are equal to 1 (viz., orientable spaces) or determinants that are equal to ± 1 (viz., non-orientable spaces). The volume thus-defined will be conserved by any transformation of G.

In particular, take the manifold of a group G, when considered to be a space that is transformed transitively by the first parameter group; g will reduce to the identity transformation here. Define the volume of the parallelepiped whose origin is at O (viz., the unity element) and is constructed from the vectors that define the infinitesimal transformations $e_1 X_1$, $e_2 X_2$, …, $e_r X_r$ to be equal to $e_1 e_2 \ldots e_r$. The volume element of the space will be [**12**, **13**]:

$$d\tau = \omega_1 \, \omega_2 \ldots \omega_r,$$

in which the right-hand side is an exterior product.

On the contrary, if one regards the manifold of the group as a space is transformed transitively by the second group of parameters then one will have a second volume element [**12**, **13**]:

$$d\tau' = \varpi_1 \, \varpi_2 \ldots \varpi_r .$$

36. If the linear group $\bar{\gamma}$ leaves invariant a positive-definite quadratic form – for example, $e_1^2 + e_2^2 + \ldots + e_n^2$ – then there will exist a Riemannian metric in the homogeneous space E that is invariant under G. Indeed, let A be a point that is infinitely close to the origin O. Call the quantity $\sqrt{e_1^2 + e_2^2 + \cdots + e_n^2}$ the *distance OA*, in which e_1, …, e_n denote the parameters of the infinitesimal transformation $e_1 X_1 + \ldots + e_n X_n$ that takes O to A. If A goes to A' under a transformation of g then one will see that the distance OA' is equal to the distance OA. One then defines the distance MN between two infinitely close points M and N by taking M to O by a transformation of G. If N then goes to A then one can set $MN = OA$. The distance that is obtained is independent of the transformation that takes M to O. It is preserved by an arbitrary transformation of G.

Analytically, if S_a and S_{a+da} are two transformations of G that take O to two infinitely close points M and N, respectively, and if $S_a^{-1} S_{a+da}$ has the symbol $\omega_1 X_1 + \ldots + \omega_r X_r$ then one will have:

$$\overline{MN}^2 = \omega_1^2 + \omega_2^2 + \ldots + \omega_n^2 .$$

In particular, the space of the group G, when considered as being transformed transitively by the first parameter group, admits an infinitude of metrics that are invariant under that group; it will suffice to take a positive-definite metric with arbitrary constant coefficients in $\omega_1, \omega_2, \ldots, \omega_n$. If one takes a finite, properly-discontinuous subgroup for g, in place of the identity transformation, then $\bar{\gamma}$ will be a finite, linear group that always leaves invariant at least one positive-definite quadratic form, and in turn, the r-dimensional space E that is associated with g will always admit at least one metric that is invariant under G.

CHAPTER III

CLOSED, LIE GROUPS

I. – Volume of a closed group.

37. We saw (no. **35**) that one can define two different volumes in the manifold of a Lie group. It is obvious that each of them will be finite if the group is closed, since the manifold can be covered by a finite number of neighborhoods, each of which has a finite volume.

On the contrary, if the group is open then one and the other of the volumes of its manifold will be infinite. Indeed, let $\mathcal{V}_0$ be a neighborhood of the unity element, and let $\mathcal{V}_0'$ be a neighborhood that is interior to $\mathcal{V}_0$ and sufficiently small that if s and s' are two arbitrary elements of $\mathcal{V}_0'$ then the element ss'^{-1} will belong to $\mathcal{V}_0$. We know that if p is an arbitrary integer then there will exist elements of G that cannot be obtained by multiplying p elements that are interior to $\mathcal{V}_0$, since otherwise the manifold of G could be covered by a finite number of neighborhoods. Let S be an element of that nature, so:

$$S = s_1\, s_2 \dots s_q \qquad (q > p);$$

we can suppose that q is the minimum number of factors that one can take in the interior of $\mathcal{V}_0$ in order to obtain S. Consider the neighborhoods:

$$\mathcal{V}_0', \quad s_1\, s_2\, \mathcal{V}_0', \qquad s_1\, s_2\, s_3\, \mathcal{V}_0', \qquad \dots;$$

one easily sees that no pair of them has an element in common. On the other hand, they all have the primary volume v', which is the volume of $\mathcal{V}_0$. One can thus find as many regions in the group manifold as one desires that all have volume v' and have no point in common.

Q. E. D.

The two volumes that one can define on the manifold of a closed group are identical.

II. – A theorem of H. Weyl.

38. There exists a fundamental theorem that is due to H. Weyl [**8**, pp. 289] for closed, linear groups G that are connected or mixed, namely, that *such a group leaves invariant at least one positive-definite Hermite form*.

First, suppose that the group G is connected and defined by the equations:

$$x_i' = \sum_k a_{ik}\, x_k \qquad (i = 1, 2, \dots, n),$$

where the coefficients a_{ik} naturally depend upon the substitution S of the group G. We denote the right-hand sides by Sx_i and their complex conjugates by $\overline{Sx_i}$. For the following integral that is taken over the entire group manifold:

$$\int (Sx_1\overline{Sx_1} + Sx_2\overline{Sx_2} + \cdots + Sx_n\overline{Sx_n})\, d\tau_S ;$$

it is a positive-definite Hermitian form $F(x_1, \ldots, x_n)$. It is invariant under G, since if one performs the particular substitution S_0 on the variables x_i then the form will become:

$$\int (SS_0x_1\overline{SS_0x_1} + \cdots + SS_0x_2\overline{SS_0x_n})\, d\tau_S .$$

Upon setting $SS_0 = S'$ and remarking that $d\tau_S = d\tau_{S'}$, if one takes $d\tau$ to be the *second* volume element then one will prove the theorem.

If the group G is mixed then it will necessarily be formed of a *finite* number of connected families; it will then suffice to define the form F by the sum of as many integrals as their families in the group.

If the closed, linear group G has real coefficients then one can substitute a positive-definite quadratic form for the Hermitian form.

39. A particular consequence of Weyl's theorem is that the coefficients a_{ij} of the substitutions of a closed, linear group are *bounded*, because if one supposes that the invariant form F is, for example:

$$F \equiv x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_n\bar{x}_n ,$$

then one will have:

(1) $$\sum_{k,i} a_{ki}\bar{a}_{ki} = 1.$$

This property can be proved directly, moreover. Call the quantity $\sqrt{\sum_{i,j} a_{ij}\bar{a}_{ij}}$ the *modulus* of a linear substitution. If the coefficients were not bounded then one could find an infinite sequence of substitutions $S_1, S_2, \ldots, S_n, \ldots$ in the group such that the modulus of each of them is greater than twice the modulus of the preceding one, and that sequence would have no accumulation element in the group.

One can add another essential property that follows from the preceding one, namely, that the roots λ of the characteristic equation of S, namely:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0,$$

are all of modulus equal to 1. Indeed, one can, by a previous change of variables, suppose that one of the equations of the substitution S is:

$$x_1' = \lambda x_1 ;$$

the substitution S_n will thus have λ_n for its coefficient, and that quantity can be bounded only if λ has modulus 1. Finally, it results from this that the determinant Δ of the substitution, which is the product of the characteristic roots, has a modulus that is equal to 1. One can say directly that each substitution S is associated with the substitution $v' = \Delta v$ that shows how S changes the volume; these substitutions can generate a closed group only if Δ has a modulus that is equal to 1. That is, moreover, the reason for the fact that there can be only kind of volume in the manifold of closed group.

40. One can attach the following important theorem to the preceding considerations: *Any bounded, linear, algebraic group is closed.* A linear group G will be called *bounded* if the coefficients of its equations are bounded, and *algebraic* if it is defined by a system of entire algebraic relations between the coefficients. The theorem is almost obvious, since if the coefficients are bounded then any infinite set of substitutions of the group will admit at least one accumulation element Σ in the group of *all* linear substitutions that act upon the given variables, and Σ will belong to the group G, since its coefficients satisfy the given entire algebraic relations.

The orthogonal group of n real variables, the linear group of a positive-definite Hermitian form, and the unimodular linear group of such a form are thus closed groups. However, *their subgroups are not all closed*, as one proves with the example of the group:

$$x' = e^{ia} x, \qquad y' = e^{mia} y,$$

in the real parameter a, while m is a real *irrational* constant; that group is not closed, and yet it leaves invariant the Hermitian form $x\bar{x} + y\bar{y}$.

III. – The structure of closed groups.

41. If a group G is closed then its adjoint group Γ will also be so. It thus leaves invariant at least one positive-define quadratic form, namely:

$$F(e) \equiv e_1^2 + e_2^2 + \ldots + e_n^2 ;$$

upon expressing the idea that the infinitesimal transformation E_i (no. **22**) of the adjoint group leaves F invariant, one will obtain the relations:

$$c_{jik} + c_{ijk} = 0.$$

One can thus choose the basis for a closed group in such a way that one has:

(2) $$c_{ijk} = c_{jki} = c_{kij} = -c_{ikj} = -c_{kji} = -c_{jik}\,.$$

One deduces that one has:

$$\sum_{j,k} c_{1jk} c_{1kj} = -\sum_{j,k} c_{1jk}^2$$

for the coefficients of e_1^2 in the form $\varphi(e)$ that was defined in no. **22**.

The form $\varphi(e)$ is therefore negative-definite or negative semi-definite. That is, moreover, a property that results immediately from the fact that it represents the sum of the squares of the characteristic roots of the infinitesimal substitutions of the adjoint group, and that these roots are purely imaginary, since otherwise the characteristic roots of the finite substitutions of the adjoint would have modulus 1.

42. Regard $e_1, e_2, \ldots, e_r$ as the rectangular coordinates of a point in an r-dimensional Euclidian space. If Γ leaves invariant a planar manifold in that space that passes through the origin then it will also leave invariant the orthogonal manifold. One can thus suppose that an infinitesimal basis for the group has been chosen in such a manner that Γ leaves separately invariant the planar manifolds that are define by the first p_1 coordinate axes, then the next p_2, then the next p_3, and so on, such that Γ leaves invariant no smaller planar manifold that is contained in one of the preceding manifolds. A simple calculation will then show that the constant c_{ijk} can be non-zero only if all three indices i, j, k belong to the first p_1 indices, or the next p_2 indices, and so on. The infinitesimal transformations of each sequence generate a group, and *the group G is* (at least, in a neighborhood of the identity element) *the direct product of a certain number of other groups $G_1, G_2, \ldots, G_h$*. That must say that any transformation of G that is sufficiently close to the identity can be regarded, in one and only one manner, as the product of a transformation of G_1, a transformation of G_2, etc., such that these h component transformations commute with each other (and each taken in a neighborhood of the identity transformation).

The *component groups $G_1, G_2, \ldots, G_h$* are *simple*, because they obviously cannot admit any continuous, invariant subgroup, so such an invariant subgroup will correspond to a planar manifold that is invariant under Γ.

43. Some of the component groups can have one parameter. First, suppose that they all enjoy that property. The constants c_{ijk} are then all zero, and one has a *closed, commutative group*. The simply-connected group with the same infinitesimal group is the group of translations of an r-dimensional Euclidian space. In order to pass from the latter to a closed group, it is necessary to determine a properly-discontinuous subgroup. One sees immediately that the closed group can always be obtained by regarding two translations as identical when their projections differ by integers. Such a group is therefore always holohedrally isomorphic to the linear group:

$$x_1' = e^{ia_1} x_1\,, \qquad x_2' = e^{ia_2} x_2\,, \qquad \ldots, \qquad x_r' = e^{ia_r} x_r$$

in the parameters $a_1, a_2, \ldots, a_r$.

Any linear group that is isomorphic to the preceding one will be reducible to the form:

$$y_1' = e^{i\sum m_{1k}a_k} y_1, \qquad y_2' = e^{i\sum m_{2k}a_k} y_2, \qquad \ldots, \qquad y_n' = e^{i\sum m_{nk}a_k} y_n,$$

in which the m_{kh} are arbitrary *integers*. In order for the isomorphism to be holohedral, it is necessary and sufficient that one can conversely express $a_1, a_2, \ldots, a_r$ in terms of linear combinations with integer coefficients in the n forms $\sum_k m_{ik} a_k$ ($i = 1, 2, \ldots, n$).

It is interesting to remark that the group G admits a continuous infinitude of *local* isomorphisms that one obtains by performing an arbitrary linear substitution on the a_i. However, such a local automorphism can be prolonged into any group only if the substitution has integer coefficients and a determinant that is equal to ± 1.

44. Any closed group is (infinitesimally) the direct product of a commutative group and another group for which the form $\varphi(e)$ is negative-definite. The groups for which the form $\varphi(e)$ has a non-zero discriminant are the simple and semi-simple groups. We shall briefly study the groups for which the form $\varphi(e)$ is definite.

IV. – Closed, semi-simple groups.

45. Let G be a connected group for which the form $\varphi(e)$ is definite:

$$\varphi(e) = \pm(e_1^2 + e_2^2 + \cdots + e_r^2).$$

The structure constants c_{ijk} then satisfy the relations (2), and in turn, the form $\varphi(e)$ will be *negative*- definite. We shall show that *the adjoint linear group* Γ *is closed.*

Indeed, consider [**9**, **21**] the set of *linear automorphisms* of G; viz., the set of linear substitutions:

$$e_i' = \sum_k a_{ik} e_k,$$

which, when performed on the parameters of an infinitesimal transformation $\sum_i e_i X_i$, will preserve the structure relations:

$$(X_i\, X_j) = \sum_k c_{ijk} X_k.$$

They are defined by the entire algebraic relations:

$$\sum_{k,h} a_{ki} c_{ijk} c_{khs} = \sum_k c_{ijk} a_{sk} \qquad (i, j, s = 1, 2, \ldots, r). \tag{3}$$

The group of linear automorphisms is therefore algebraic, and also bounded, since it leaves the form $\varphi(e)$ invariant; it is therefore closed (no. **40**). Now, the linear adjoint group Γ is a subgroup, and similarly, an invariant subgroup, since the transform by an automorphism $\mathcal{A}$ of the transformation T_a of the adjoint group:

$$(T_a) \qquad S_{a'} = S_a\, S_n\, S_n^{-1}$$

will be the transformation T_b that is provided by the element S_b of G that is the transform of S_a by the automorphism $\mathcal{A}$. The group Γ is therefore (no. **42**) (at least, infinitesimally) the direct product of Γ with a group Γ_1 that commutes with Γ; however, it is impossible for that group Γ_1 to not reduce to the identity operation, because it will give linear automorphisms that leave invariant each transformation T_a of the adjoint group, and in turn, each element of G.

The group Γ is therefore identical to Γ', or at least, it constitutes one of the connected families (which are finite in number) that Γ' is composed of. It is therefore closed.

46. We shall now show that the group G itself is closed; in order to do that, it will suffice to prove, with H. Weyl [**8**, pp. 380], that the simply-connected group of the same infinitesimal structure can cover the adjoint group only a finite number of times, or even that *there exist finite number of closed contours in the adjoint group that are not reducible to each other by continuous deformations.* That number will be that of the elements of the center of the simply-connected group with the given infinitesimal structure.

In order to prove that theorem, it is necessary to establish previously some properties of the adjoint group Γ. Suppose that one can find l independent infinitesimal transformations of Γ that commute with each other and which do not all simultaneously commute with any other infinitesimal transformation; we can suppose that these l transformations are $E_1, E_2, \ldots, E_l$. The subgroup γ of Γ that they generate is *closed*, since it is one of the connected subsets of a bounded group, and it is formed algebraically from the substitutions of Γ' that leave invariant the variables $e_1, e_2, \ldots, e_l$. On the other hand, that subgroup γ is commutative; as a result (no. **43**), the characteristics roots of its most general infinitesimal transformation $a_1 E_1 + a_2 E_2 + \ldots + a_l E_l$ have the form $\pm i\, \omega_\alpha$, in which the ω_α are linear combinations with integer coefficients of the l canonical parameters $\varphi_1, \varphi_2, \ldots, \varphi_l$ that are each defined up to 2π and depend linearly upon the, likewise canonical, parameters $a_1, a_2, \ldots, a_l$. The characteristic roots are pair-wise equal and opposite because the group γ has real coefficients.

47. Since the infinitesimal transformations of γ that are not *singular* – i.e., the ones for which, one of the quantities ω_α is annulled – are invariant under γ, they each admit ∞^{n-l} homologues in Γ, and since they depend upon l parameters, one sees that the transformations of γ and the homologues depend upon r parameters. The singular transformations of γ that are invariant under a subgroup of at least $l + 2$ parameters each admit at most ∞^{n-l-2} homologues, and since they depend upon at most $l - 1$ parameters, *the singular transformations of* Γ *will depend upon at most:*

$$r - l - 2 + (l - 1) = r - 3$$

parameters. In particular (no. **24**), it then results *all of the non-singular, finite transformations of* Γ, *and also of G, admit infinitesimal transformations as generators.* Moreover, the same thing will be true for the singular transformations that are limits of non-singular transformations, because one can always suppose that the parameters φ_1, φ_2, …, φ_l of a generating infinitesimal transformation are found between 0 and 2π – i.e., they are bounded – and under these conditions, the generating infinitesimal transformation of an infinite sequence of non-singular, finite transformations that converge to a singular, finite transformation will admit at least one accumulation point, which will provide a generating infinitesimal transformation of the singular transformation. *Any semi-simple group of the definite form* φ (*e*) *is therefore generated completely by its infinitesimal transformations.*

48. Now, consider the hyperplanes:

$$\omega_\alpha = 0, \qquad \omega_\alpha = \pm 2\pi$$

in the *l*-dimensional Euclidian space whose rectangular coordinates are the canonical parameters $a_1, …, a_l$ of a transformation of γ.

The first of them delimit a certain number of polyhedral angles (D_1), (D_2), … around the origin. The last ones, along with the first ones, delimit a certain number of polyhedra (P_1), (P_2), … in the interior of these angles that have the origin for their vertices. Describe a path ($\mathcal{C}$) in the manifold Γ that takes the unity element to an arbitrary element *T*; we can always, if needed, suppose that this path meets no singular element. Follow the parameters φ_1, φ_2, …, φ_l of the variable element along ($\mathcal{C}$) by continuity. Starting from the origin, we enter into one of the polyhedra (*P*) without ever leaving it. As a result, *any element of* Γ *is homologous to an element of* γ *that has its image in the interior or on the frontier of one of the polyhedra* (*P*). Now, suppose that the contour ($\mathcal{C}$) returns to the identity element. The interior image point of (*P*), starting from the origin, will necessarily end at one of the summits of (*P*) that corresponds to the identity transformation of Γ. (The ω_α are all multiples of 2π for the identity transformation of Γ). If the image point returns to the origin then the closed contour that it describes can be reduced to the origin by a sequence of homotheties with ratios of *k* that decrease from 1 to 0, and the closed contour ($\mathcal{C}$) can be correspondingly reduced to a point by a continuous deformation. On the contrary, if the image point interior to (*P*) goes from the origin to another summit then the contour ($\mathcal{C}$) cannot be reduced to a point by a continuous deformation, since if it could then it would always be possible to realize the reduction without ever encountering singular elements, *which only define r –* 3-

dimensional manifold, but then the image point, which does not leave (P), will always be the origin at the same summit of (P), which is absurd.

There are thus [**15**] *as many closed contours in the manifold of* Γ *that are not reducible to each other as there are summits in the polyhedron* (P) *that represent the identity transformation.* Since that number is finite, the theorem is proved.

49. The various polyhedra (P) that emanate from the origin in l-dimensional space each represent transformations of the group. In particular, two neighboring polyhedra (P) and (P_1), which are contiguous along a lateral face that issues from the origin, are mutually symmetric with respect to that face. There exists a transformation of the adjoint group that transforms the infinitesimal transformations of γ that are interior to (P) into infinitesimal transformations of γ that are interior to (P_1). All of these transformations of γ into itself generate a finite group (S). Moreover, there is no other transformation that leaves γ invariant in the adjoint group Γ; indeed, otherwise, one of those transformations T would leave invariant the polyhedral angle (D). As a result, there will exist a *non-singular* infinitesimal transformation X of γ that is invariant under T. Now, the transformation T can always be generated by an infinitesimal transformation of Γ that leaves X invariant. However, the only infinitesimal transformations of Γ that commute with X all belong to γ. The transformation T will thus leave invariant all of the transformations of g, which is absurd.

The polyhedral angle (D) thus *the fundamental region* of the finite group (S) whose generating operations are symmetric with respect to the lateral faces of (D). The number of polyhedra (P) that emanate from the origin is equal to the number of operations of (S). The number of lateral faces of (D) is equal to the rank l of the group [**16**].

50. The search for automorphisms of G can be attached to the preceding considerations. It comes down to the search for rotations and symmetries about the origin that leave invariant the figure that is defined by the polyhedral angles (D), (D_1), etc. The number of these operations is a multiple of the number of operations of (S). Its knowledge immediately gives the number of distinct, connected families into which the total group Γ of automorphisms of G is decomposed. For simple groups, that number is equal to 1, 2, or 6, as was determined by E. Cartan [**10**].

51. There exist four general classes of closed, simple groups and five exceptional groups, in addition. The general classes of groups are isomorphic to:

A. The unimodular linear group of a positive-definite Hermitian form in $l + 1$ variables. That group is simply connected, and it covers its adjoint group $l + 1$ times. For $l > 1$, it will admit two distinct families of automorphisms.

B and *D*. The orthogonal group of n real variables ($n = 2l + 1$ or $n = 2l \geq 8$). That group covers its adjoint group twice if n is even and once if n is odd. It is covered twice by the simply-connected group of the same structure. The number of its distinct, connected families of automorphisms is 1 for odd n, 6, for $n = 8$, and 2, for n even and greater than 8.

C. The linear group that leaves invariant the Hermitian form:

$$x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_{2l}\bar{x}_{2l}$$

and the exterior quadratic form:

$$[x_1\, x_2] + [x_3\, x_4] + \ldots + [x_{2l-1}\, x_{2l}].$$

That group is simply connected and covers its adjoint group twice. Its automorphisms all belong to the adjoint group.

V. – Constructing the most general closed group.

52. Return to an arbitrary, closed, connected group G. Infinitesimally (no. **42**), it is the direct product of a commutative group and several simple groups. The simply-connected group $\bar{G}$ with the same infinitesimal structure as G is thus the direct product of a group of translations $\bar{G}_0$ of order r_0 and several simply-connected, closed, simple groups $\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_h$.

In order to pass from $\bar{G}$ to G, one must (no. **14**) construct a properly-discontinuous subgroup g of the center if $\bar{G}$, which is a subgroup that will provide the unity element of G. Now, the center of $\bar{G}$ is the direct product of the centers of the component groups. On the other hand, since the center of the group $\bar{G}_0$ is $\bar{G}_0$ itself, if one denotes the centers of $\bar{G}, \bar{G}_1, \ldots, \bar{G}_h$ by $C, C_1, \ldots, C_h$ then one will have:

$$C = \bar{G}_0 \times C_1 \times \ldots \times C_h .$$

Any element of C is the product of $h + 1$ elements that are taken from $\bar{G}_0, C_1, \ldots, C_h$, respectively. It can happen that the subgroup g is the direct product of a subgroup of $\bar{G}_0$, a subgroup of C_1, …, and a subgroup of C_h. In that case, G will be the direct product of a closed, commutative group and h closed, simple groups.

In the general case, we denote the largest subgroup of $\bar{G}_0, C_1, \ldots, C_h$ that belongs to g by $g_0, g_1, \ldots, g_h$, respectively. The group $g_0 \times g_1 \times \ldots \times g_h$ is a subgroup of g. It defines a group G' that covers G an integer number of times, and that number is finite, since it is equal to at most the number of operations of $C_1 \times \ldots \times C_h$; the group G' is then closed. On the other hand, one passes from G' to G by constructing a finite subgroup of its center.

As a result, one can obtain any closed group G by starting with a closed group G' that is the direct product of a closed, commutative group and a certain number of closed, simple groups. It will suffice to take a finite subgroup g' of the center of G' to be the *unity element of G, where g′ has only the unity element in common with each of the factor groups that the center of G′ is a direct product of.* If g' does not reduce to the unity element then *G will not be* the direct product of commutative or simple groups.

53. A classical example of a closed, semi-simple group that is not the direct product of simple groups is furnished by the connected orthogonal group in four variables, which is covered twice by the direct product of two simply-connected simple groups of order 3, and which covers its adjoint group twice. Similarly, the linear group of a positive-definite Hermitian form $x_1\bar{x}_1 + \ldots + x_n\bar{x}_n$ is covered n times by the direct product of the closed commutative group:

$$x'_k = e^{i\theta} x_k \qquad (k = 1, 2, \ldots, n)$$

and unimodular linear group of the Hermitian form.

VI. – Homogeneous spaces with closed fundamental groups.

54. If G is a closed group then any homogeneous space E that is transformed transitively by G will be associated with a closed subgroup g of G. The space itself is closed. *There exists at least one Riemannian metric on the space that is invariant under the group.* That amounts to saying that the subgroup γ of the adjoint group that corresponds to g is closed, and in turn, will leave invariant a positive-definite quadratic form. That form will permit us to define the metric in the neighborhood of the origin, and in turn, in all of space (no. **36**).

Moreover, the preceding reasoning that makes the fundamental group G closed or open applies to any homogeneous space for which the group g is closed. One can easily prove that if the space E admits a metric invariant under G, and if G is the largest continuous group that leaves that metric invariant then g will be closed.

The property of a closed space that transforms transitively under a closed group G that it must admit a metric that is invariant under G is very important. One can appeal to it in order to prove the possibility of constructing a *complete* orthogonal system of functions in space by starting with linear groups that are isomorphic to G. However, that is a theory that, due to its importance, exceeds the scope of this fascicle.

CHAPTER IV

SYMMETRIC RIEMANNIAN SPACES ([1])

I. – Definition and first properties.

55. Consider a Riemannian manifold with an everywhere-regular metric on which we suppose that any infinite, bounded set of distinct points admits at least one accumulation point. (A set is called *bounded* if the distance from all of its points to a fixed point remains bounded, where the distance between two points is defined to be the lower bound of the lengths of the arcs of the curve that join the two points.)

The Riemannian manifold will be called *symmetric* if the symmetry with respect to an arbitrary point *A* of the space preserves the metric. (That *symmetry* is defined in the following manner: One can make any point *M* that is sufficiently close to *A* correspond to the point *M´* that is obtained by joining the geodesic *MA* to the prolongation of an arc *AM´* that has the same length as the arc *AM*.) The property of a Riemannian manifold being symmetric is equivalent to the following: Levi-Civita parallel transport preserves the Riemannian curvature; however, shall ignore this viewpoint completely.

Any symmetric, Riemannian manifold admits a transitive, continuous group of isometric transformations. Indeed, if *M* and *N* are two arbitrary (sufficiently close) points then one only has to join them with the geodesic *MN* and successively perform the symmetry with respect to *M* and the symmetry with respect to the midpoint *P* of *MN*: The point *M* will then be taken to *N*. That isometric transformation belongs to a continuous family of isometries that are obtained by leaving the point *M* fixed and describing a geodesic that issues form *M* to the point *N*.

If *G* is the largest connected, continuous group of isometries of the manifold then that manifold can be considered to be a homogeneous space whose fundamental group *G* is endowed with a metric that invariant under *G*. The largest subgroup *g* of *G* that leaves invariant a given point *O* of space is then *closed* (no. **54**). *We assume that G is a Lie group*.

56. Let σ be the symmetry with respect to *O*. That symmetry defines an *involutive automorphism* of group *G* that makes the displacement *S* correspond to the displacement:

$$\bar{S} = \sigma D\, \sigma^{-1} = \sigma D\, \sigma ;$$

if *S* takes *M* to *N* then $\bar{S}$ will take the symmetric point $\bar{M}$ of *M* with respect to *O* to the symmetric point $\bar{N}$ of *N* with respect to *O*.

The transformations of *g* are obviously invariant under that automorphism. On the other hand, if there are other ones then each of them must take *O* to a point that must be

---

([1]) In this chapter, we shall summarize and simplify the theories that were presented in the memoirs [**14**], [**15**], [**16**], [**17**], [**21**] of E. Cartan; *see* also [**12**] and [**13**].

its proper symmetric point with respect to O, so if they exist then they must define a family of displacements that one cannot link to g by continuity. In particular, the only infinitesimal transformation that are invariant under the automorphism are the ones that belong to g.

57. Conversely, start with a connected, continuous group G and an involutive automorphism $\mathcal{A}$ in that group such that the infinitesimal transformations that are invariant under $\mathcal{A}$ generate a closed subgroup g. We shall show that the homogeneous space $\mathcal{E}$ that is associated with g can be endowed with a symmetric Riemannian metric that is invariant under G.

The automorphism $\mathcal{A}$ effects a linear substitution of the form:

$$\begin{cases} e'_i = -e_i & (i = 1, 2, \dots, n), \\ e'_\alpha = e_\alpha & (\alpha = n+1, \dots, r) \end{cases} \tag{1}$$

in the parameters e_i of the infinitesimal transformations of G; in what follows, we shall denote the first n indices by Latin letters and the last $r - n$ ones by Greek letters.

By hypothesis, the infinitesimal transformations X_α generate a closed, continuous subgroup g. The subgroup γ of the adjoint group Γ that corresponds to g is closed. It transforms the $e_1, \dots, e_n$ amongst themselves. It then leaves invariant (no. **38**) at least one positive-definite quadratic form, namely:

$$f(e) = e_1^2 + e_2^2 + \cdots + e_n^2 .$$

58. Let O be the origin that is invariant under g. Let $\bar{S}$ be the transformation of G that is the transform of S under the automorphism $\mathcal{A}$. Finally, let Sg be the set of transformations of G that take O to a point M in space. The conjugate transformations $\bar{S}g$ define another well-defined point $\bar{M}$. One thus obtains a point-like transformation of the space $\mathcal{E}$ that leaves the point O fixed; we denote it by the symbol σ. *That transformation is isometric.* Indeed, if S_a and S_{a+da} take O to two infinitely close points M and M' then their transforms $\bar{M}$ and $\bar{M}'$ under σ will be the transformed points of O under $\bar{S}_a$ and $\bar{S}_{a+da}$, respectively. The distance MM' is obtained (no. **36**) by considering the infinitesimal transformation $S_a^{-1}S_{a+da}$ whose symbol is $\sum (\omega_i X_i + \omega_\alpha X_\alpha)$, and one will have:

$$MM' = \sqrt{\sum_i \omega_i^2} .$$

The distance $\bar{M}\bar{M}'$ is obtained, in its own right, by considering the conjugate infinitesimal transformation $\sum(-\omega_i X_i + \omega_a X_a)$. One then sees that the distance MM' is not altered by the operation σ.

The trajectories of the infinitesimal transformations $\sum e_i X_i$, when applied to the point O are obviously invariant under σ (with a different sense of traversal). The transformation σ then preserves the directions that issue from O, with a change in sense. In particular, the *geodesics* that issue from O are invariant under the isometry σ. It then results immediately that one can pass from a point M to the point $\bar{M}$ that is the transform of M under σ by performing the symmetry with respect to O, at least as long as there exists a geodesic that joins O to M.

The space $\mathcal{E}$ thus admits an isometric symmetry with respect to O.

59. The existence of an isometric symmetry σ_A with respect to an arbitrary point A in space follows immediately from the preceding. Two points will be called *symmetric* with respect to A if one can, by a displacement of G, simultaneously take A to O and the two given points to two points that are symmetric with respect to O. The symmetry σ_A will obviously be isometric.

If S_0 is one of the transformations that take O to A, and if S is one of the transformations that take O to a point M then the symmetric point of M with respect to A will be defined by the transformation:

$$S' = S_0\,\bar{S}_0^{-1}\bar{S}\,. \tag{2}$$

One immediately verifies that this point does not change if one multiplies S_0 and S by an arbitrary transformation of g.

60. We agree to say that a transformation of G is a *rotation* if it belongs to g, and that it is a *transvection* if one can generate it by means of an infinitesimal transformation $\sum e_i X_i$. We denote a rotation by the letter R and a transvection by the letter T. One has:

$$\bar{R} = R, \qquad \bar{T} = T^{-1}.$$

Let (C) be a line that links the points that are obtained by applying to the point O the transformations $T(t)$ of the one-parameter group of transvections that are generated by a given infinitesimal transvection:

$$e_1 X_1 + \ldots + e_n X_n\,.$$

We take t to be the canonical parameter of the subgroup, which we can suppose to be equal to the length of the arc that separates the point O on (C) from the point M that is the transform of O by $T(t)$.

From (2), the symmetric point to the point M whose abscissa is t on the line (C) with respect to the point A whose abscissa t_0 is given by:

$$S' = T(t_0)\,[T(-\,t_0)]^{-1}\,T(-\,t) = T(2t_0 - t);$$

it will again be a point of (C). *The trajectory is then its own symmetric image with respect to any of its points.*

Having said that, let A be a point that is close to O along the line (C). The geodesic OA is its own symmetric image with respect to A; it thus contains the point A_1 that is the symmetric image of O with respect to A, which is a point that will belong to (C). It will thus likewise contain the points A_2, A_3, … that are obtained by successively measuring out constant lengths on (C). If A approaches O indefinitely then one will get the geodesic that is tangent to (C) at O in the limit, which must contain all of the points of (C). *The geodesics that issue from O are thus the trajectories of transvections.*

61. One can add a remarkable theorem. Take two points A and A' on the geodesic (C) whose abscissas are t_0 and t_0', respectively; upon taking the symmetric point to a point S with respect to A and A' in succession, one will obtain the points:

$$T(2t_0)\,\bar{S} \qquad \text{and} \qquad T(2t_0')\,T(-\,2\,t_0)\,S = T\,(2t_0' - 2t_0)\,S.$$

The result of the two symmetries is then the transvection whose amplitude is twice the distance AA′; it does not change if one slides the arc AA' along the geodesic that carries it without changing its length or sense.

It is important to remark that *the trajectory of a one-parameter group of transvections is a geodesic only if the trajectory starts from the point O.*

II. – Reducible and irreducible homogeneous spaces.

62. Formulas (1), which define the involutive automorphisms $\mathcal{A}$, show immediately that the brackets $(X_i\,X_j)$ and $(X_\alpha\,X_\beta)$ depend upon only the X_α, while the brackets $(X_i\,X_\alpha)$ depend upon only the X_i. One thus has structure formulas of the form:

$$(3) \qquad \left\{\begin{aligned} (X_i X_j) &= \sum_\rho c_{ij\rho} X_\rho, \\ (X_i X_\alpha) &= \sum_k c_{i\alpha k} X_k, \\ (X_\alpha X_\beta) &= \sum_\rho c_{\alpha\beta\rho} X_\rho. \end{aligned}\right.$$

The subgroup γ of the adjoint group Γ that corresponds to the subgroup g transforms the e_i amongst themselves. It transforms the variables e_α amongst themselves, since g leaves invariant the linear family of infinitesimal transformations $\sum e_i\, X_i$. Since the subgroup γ is closed, it will leave invariant not only the form $f(e) \equiv e_1^2 + \ldots + e_n^2$, but also a positive-definite form such as:

(4) $$F(e) \equiv e_1^2 + \ldots + e_n^2 + e_{n+1}^2 + \ldots + e_p^2 .$$

One immediately deduces the relations:

(5) $$c_{\alpha ij} + c_{\alpha ji} = 0, \qquad c_{\alpha\beta\gamma} + c_{\alpha\gamma\beta} = 0.$$

63. Consider the form $\varphi(e)$ that relates to the group G. Since it is invariant under the automorphism $\mathfrak{A}$, one can suppose that the infinitesimal basis for the group has been chosen in such a manner that one has:

(6) $$-\varphi(e) = \sum_i \lambda_i e_i^2 + \sum_\alpha \lambda_\alpha e_\alpha^2 .$$

The coefficients λ_α are all positive; indeed, one has:

$$\lambda_\alpha = \sum_{i,j} c_{\alpha ij}^2 + \sum_{\beta,\gamma} c_{\alpha\beta\gamma}^2 .$$

If λ_α is zero then the infinitesimal transformation X_α will be *distinguished;* the subgroup g will thus contain a continuous subgroup that is invariant under G, which is impossible (no. **18**). Since the subgroup γ leaves the form $\sum \lambda_\alpha e_\alpha^2$ invariant, nothing prevents us from assuming that it is the one that one appeals to in order to define F, which amounts to assuming that the λ_α are equal to 1.

Upon now expressing the idea that the infinitesimal transformations E_i and E_α of the adjoint group leave the form $\varphi(e)$ invariant, one will get the relations:

(7) $$c_{ij\alpha} = \lambda_j \, c_{\alpha ij} = \lambda_i \, c_{\alpha ij} .$$

64. Having made these preliminaries, suppose that the coefficients λ_i are not all equal to each other. For example, separate the first n indices into two series, with the letters i, j, … being reserved for the first series, the letters i', j', …, for the second series, and suppose that the λ_i are all different from the λ_i'. The relations (7) then give:

$$c_{ii'\alpha} = c_{\alpha ii'} = 0.$$

One sees immediately that the infinitesimal transformations X_i and X_α generate a group G_1; similarly, the transformations $X_{i'}$ and X_α generate a group G_1'. Finally, the transformations X_i commute with the $X_{i'}$. Any transvection of G, in turn, will be the product of a transvection of G_1 and a transvection of G_1' that commute with each other. The group G_1 gives rise to a symmetric space $\mathcal{E}_1$ that is associated with g, and the group

G_2, to a symmetric space $\mathcal{E}_2$. Any point of $\mathcal{E}$ (that is sufficiently close to O) can be defined by a transvection $T = T_1 T_1'$. There then exist a one-to-one correspondence between the points of $\mathcal{E}$ and the pairs of points of $\mathcal{E}_1$ and $\mathcal{E}_1'$. On the other hand, the distance between two infinitely close points $T_{e,e}$ and $T_{e+de,\ e'+de'}$ of $\mathcal{E}$ is given by the consideration of the infinitesimal transformation:

$$T_{e,e'}^{-1}\, T_{e+de,e'+de'} = (T_{1e}^{-1}\, T_{1e+de})(T_{1e'}^{\prime -1}\, T'_{1e'+de'})\,.$$

If the first factor on the right-hand side has the symbol:

$$\sum \omega_i X_i + \sum \omega_\alpha X_\alpha\ ,$$

and if the second one has the symbol:

$$\sum \omega_{i'} X_{i'} + \sum \omega'_\alpha X_\alpha$$

then one will see immediately that the ds^2 of $\mathcal{E}$ is the sum of the ds^2 of $\mathcal{E}_1$ and $\mathcal{E}_1'$.

We say that *the space $\mathcal{E}$ results from the composition of symmetric spaces* $\mathcal{E}_1$ and $\mathcal{E}_1'$; it will be called *reducible*.

One will arrive at an analogous conclusion if the subgroup γ, when considered as operating on the e_i, leaves invariant a plane manifold of dimension at least n, so one can assume that $e_{i'} = 0$.

65. If the space $\mathcal{E}$ is *irreducible* then the n coefficients λ_i of the form $-\varphi(e)$ are then all equal to each other. However, there are three cases to distinguish:

1. If the λ_i are all zero then relations (7) show that the transvections commute with each other. *The space is Euclidian*, or more precisely, it is the manifold of a commutative group in which one has taken ds^2 to be a positive-definite quadratic form with constant coefficients in the differentials of the n canonical parameters. For $n = 2$, the space will be homeomorphic to the Euclidian plane, to the cylinder of revolution, or to the torus.

2. If the common value λ of the λ_i is negative then the group G will be *open* and simple or semi-simple. The space $\mathcal{E}$ will itself be open, as one can prove by appealing to the property of g being closed.

3. If the common value λ of the λ_i is positive then the group G will be *closed* and simple or semi-simple. The space $\mathcal{E}$ will likewise be closed.

66. One can deduce the open irreducible symmetric spaces of the closed spaces by a very simple procedure. Indeed, introduce the symbols:

$$Y_k = i\,X_k\,, \qquad Y_\alpha = X_\alpha\,;$$

we obtain:

$$(3')\qquad \begin{cases} (Y_i\,Y_j) = -\sum_\rho c_{ij\rho} Y_\rho\,, \\ (Y_i\,Y_\alpha) = -\sum_k c_{i\alpha\rho} Y_k\,, \\ (Y_\alpha\,Y_\beta) = -\sum_\rho c_{\alpha\beta\rho} Y_\rho\,, \end{cases}$$

which are formulas that define a new structure for the groups that admit the involutive automorphism (1). The corresponding new form – $\varphi(e)$ is obviously deduced from the preceding by changing λ into – λ. As a result, *any open, non-Euclidian, irreducible symmetric space is associated with a closed, non-Euclidian, irreducible, symmetric space, and conversely.*

The search for the irreducible, symmetric spaces is then reduced to the search for closed ones.

67. Before beginning that search, we shall prove that *if an irreducible, symmetric space is furnished by a semi-simple group G then G will be the largest continuous group of displacements of the space.*

First, observe that the brackets $(X_i\ X_j)$ must yield $r - n$ linearly-independent combinations of the X_α; indeed, otherwise these brackets would generate an invariant subgroup g' of g, as the Jacobi identity shows when it is applied to $[X_\alpha\,(X_i\,X_j)]$. Since the subgroup g is closed, it will be the direct product of g' and another subgroup g''. However, if X_ρ belongs to g'' then formulas (7) will show that since $c_{ij\rho}$ is zero, the $c_{\rho ij}$ will be zero; g'' is therefore invariant in G, which is impossible.

Having said that, suppose that there is a continuous group G' that contains G as a subgroup and leaves invariant the metric on the space. It is impossible for G' to be semi-simple, since the $(X_i\,X_j)$ would not provide all of the transformations of the new subgroup g' that leave the origin invariant then. On the other hand, since space is irreducible, it can only be Euclidian, which is contrary to hypothesis. There is thus a contradiction.

III. – Closed, irreducible, symmetric spaces.

68. Let us pass over the locally-Euclidian spaces. The group G is then closed and simple or semi-simple.

If a closed group G is simple and admits an involutive automorphism $\mathcal{A}$ that leaves invariant the transformations of a continuous subgroup g then one can easily see that g is closed. *The associated symmetric space $\mathcal{E}$ will necessarily be irreducible.* Indeed, if γ

leaves invariant a linear family $e_1 X_1 + \ldots + e_\nu X_\nu$ ($\nu < n$) then the transformations X_i and $(X_i X_j)$, where the indices i and j take the values 1, 2, …, ν, will generate an invariant subgroup of G, which is impossible. One will thus have a very large class of irreducible, symmetric spaces.

Now, suppose that G is (at least, infinitesimally) the direct product of several simple groups G_1, G_2, …, G_k . The automorphism $\mathcal{A}$ will transform G_1 into one of the component groups. Since it is involutive, it will perform an involutive permutation of the component groups. If h is greater than 2 then one can regard G as the direct product of two groups G_1', G_2', each of which is invariant under $\mathcal{A}$. The corresponding subgroup g will be the direct product of the two subgroups g_1' and g_2' of G_1' and G_2',. One sees immediately that the space $\mathcal{E}$ results from the composition of the spaces that are associated with the subgroups g_1' and g_2'; it is then reducible.

The only case in which space is irreducible with a semi-simple group G is the one in which G is the direct product of two simple groups G_1, G_2 *that are isomorphic to each other*, and in which the automorphism $\mathcal{A}$ transforms each element of G_1 into the corresponding element of G_2 .

69. In the case where the group G is semi-simple, let S_α and Σ_α denote two corresponding elements of the two component groups. The rotations, which are invariant under $\mathcal{A}$, are the transformations $S_\alpha \Sigma_\alpha$, while the transvections are the transformations $S_\alpha \Sigma_\alpha^{-1}$ that are inverse to their conjugates $S_\alpha^{-1} \Sigma_\alpha$.

Amongst all of the transformations of G that have the form:

$$S_a \Sigma_b g = S_a \Sigma_b S_a \Sigma_a$$

and which take O to a point M in space, one and only one of them belongs to G_1, namely, $S_a S_b^{-1}$; *one can thus regard the space $\mathcal{E}$ as the space of the simple group G_1* . If one applies the transformation $S_\alpha \Sigma_\alpha$ to the point S_x of the space $\mathcal{E}$ then one will obtain the transformation $S_a \Sigma_b S_x$, or rather, the set of transformations:

$$S_a \Sigma_b \; S_x g = S_a \Sigma_b \; S_b^{-1} g .$$

Let $X_1, X_2, \ldots, X_r$ denote the infinitesimal transformations of the group G_1, and let Y_1, Y_2, …, Y_r denote the corresponding transformations of G_2 . The infinitesimal rotations are:

$$U_i = X_i + Y_i ,$$

while the infinitesimal transvections are:

$$V_i = X_i - Y_i .$$

The form $-\varphi(e)$ that relates to G is the sum of the forms $-\varphi(e)$ that relate to the two groups G_1 and G_2, and each of them is the sum of the squares of the r parameters. It then results that the ds^2 of the space $\mathcal{E}_1$, when considered to be the space of G_1, is:

$$\omega_1^2 + \omega_2^2 + \cdots + \omega_r^2 .$$

Symmetry with respect to the origin replaces $S_x\, g$ with $\Sigma_x\, g$ or $S_x^{-1} g$; it is then defined by:

$$S_{x'} = S_x^{-1} .$$

The forms ω_i are changed by that operation into the forms $-\varpi_i$, which are parameters of $S_x S_{x+dx}^{-1}$; one will then hve the relation:

$$\omega_1^2 + \cdots + \omega_r^2 = \varpi_1^2 + \cdots + \varpi_r^2 .$$

There exist two remarkable families of displacements in the space $\mathcal{E}$, namely, the *left translations* $S_{x'} = S_a\, S_x$, and the *right translations* $S_{x'} = S_x\, S_b$. In the particular case in which G_1 is the group of rotations in ordinary space, the space $\mathcal{E}$ will be the three-dimensional elliptical space, which, in fact, admits two families of translations in the Clifford sense.

70. The geodesics that issue from the origin in a space of a closed, simple group are lines that represent one-parameter subgroups [**13**]. If the rank l of the group is greater than 1 then an arbitrary geodesic is not closed, but will pass infinitely close to all of the points of an l-dimensional, locally-Euclidian manifold. That manifold is represented by the polyhedron that is defined by the set of polyhedra (P) (no. **48**) that emanate from the origin in l-dimensional Euclidian space; the opposite faces of the total polyhedron must be regarded as identical [**15**]. Any point of space admits different *antipodal manifolds* [**15**].

71. Closed, irreducible symmetric spaces whose group G is simple [**17**] present analogous peculiarities. Here, the rank of the space is the maximum number l of linearly-independent, infinitesimal transvections that commute with each other. One introduces polyhedra ($\mathcal{P}$) that are analogous to the polyhedra (P), whose interior points serve to represent the transvections of the group G. If that group is simply connected, which one can always assume (no. **30**), then one can prove, as in no. **48**, that the manifold of finite transvections is simply connected.

Now, suppose that the subgroup g is connected: The space $\mathcal{E}$ will then be simply connected (no. **31**). One proves that any transformation S of G can be put into the form of the product TR of a transvection and a rotation in at least one way, which amounts to saying that any point M can be related to the origin O by at least one geodesic. No matter what the transformation S might be that takes O to M, the product $S\bar{S}^{-1} = T^2$ will always be the same; any point M will thus correspond to a well-defined transvection T^2. As a result, the space $\mathcal{E}$ will be the simply-connected covering space of the manifold of transvections, which we have seen to be simply connected, in its own right. *There thus exists a one-to-one correspondence between the points of $\mathcal{E}$ and the transvections T^2.* From the viewpoint of displacements of space, that translates into the formula:

$$T'^2 = S\,T\,\bar{S}^{-1}. \tag{8}$$

That formula shows that if one considers the manifold V *in the space of the group G* of transvections, which one can regard as the image of the space $\mathcal{E}$, then on the manifold V the displacements of that space $\mathcal{E}$ will translate into the displacements of all of the ambient group space. There is more: The metric that is induced on V by its presence in the group space is identical to the proper metric of $\mathcal{E}$. The manifold V is a *totally geodesic* [**13**] of the group space.

72. If the subgroup that leaves invariant the origin of the symmetric space is mixed then it will be composed of a connected subgroup g and a certain number of other families $\Theta_1 g$, $\Theta_2 g$, …, in which the Θ_i are conveniently-chosen transvections. Since the transvections Θ_i are finite in number, they will belong to the center of G. Conversely, any subgroup of transvections that belongs to the center of G will correspond to what one can call a *Klein form* of the simply-connected space $\mathcal{E}$; one can obtain them by regarding the points T^2, $\Theta_1^2 T^2$, $\Theta_2^2 T^2$, … of $\mathcal{E}$ as being identical. The transformations of G that yield a zero identity displacement are the transformations that belong to the center of G and which leave the origin fixed.

IV. – Closed, reducible, symmetric spaces.

73. If a closed, symmetric space $\mathcal{E}$ is *reducible* then that will say (no. **64**) that *in a neighborhood of O*, any point of $\mathcal{E}$ is in one-to-one correspondence with a pair of points of two other symmetric spaces. However, that correspondence cannot be prolonged to all of space.

Start with a certain number of closed, simply-connected, irreducible, symmetric spaces $\mathcal{E}_1$, $\mathcal{E}_2$, …, $\mathcal{E}_h$, and consider the closed, *integrally reducible*, symmetric space $\mathcal{E}$ that results from the composition of the preceding spaces. One will get Klein form that is likewise symmetric by considering the finite, Abelian subgroups in each of the simply-connected groups G_1, G_2, …, G_h that are defined by the transvections that belong to the center of the group considered. Let (c_1), (c_2), …, (c_h) be these groups, respectively. One takes an arbitrary subgroup of the group:

$$(c_1) \times (c_2) \times \ldots \times (c_h).$$

If that subgroup is the direct product of h subgroups that belong to (c_1), (c_2), …, (c_h), respectively, then the symmetric space that is obtained will be *integrally* reducible; in the contrary case, it will be only be *locally* reducible. One then has that the four-dimensional space whose points are each defined by a pair of points on two spheres will be integrally reducible, but will cease to be so if one regards two points as identical when they correspond to two pairs of points MN, $M'N'$, where M' is the antipode of M on the first sphere, and N' is the antipode of N on the second one. That space is nothing but the manifold of lines in three-dimensional elliptical space.

V. – Open, irreducible, symmetric spaces.

74. Any open, irreducible, symmetric space $\mathcal{E}$ is *associated* (no. **66**) with a closed, irreducible, symmetric space $\mathcal{E}_u$.

First, suppose that the closed space $\mathcal{E}_u$ is the space of a closed, simple group G. The infinitesimal rotations and the infinitesimal transvections of the group of displacements will be:

$$\begin{matrix} X_k + Y_k \\ X_k - Y_k \end{matrix} \qquad (k = 1, 2, \ldots, r),$$

respectively, when one introduces the infinitesimal transformations X_k and Y_k of two groups that are isomorphic to G. Set:

$$U_k = X_k + Y_k , \qquad V_k = i\,(X_k - Y_k);$$

we will have:

$$(U_i\,U_j) = -\,(V_i\,V_j) = \sum_k c_{ijk} U_k ,$$
$$(U_i\,V_j) = \quad (V_i\,U_j) = \sum_k c_{ijk} V_k .$$

These formulas define the structure of the group *with complex parameters* that is generated by the infinitesimal transformations $\sum (a_k + i\,b_k)\,U_k$. *The open space* $\mathcal{E}$ *thus has a fundamental group that consists of the simple group with complex parameters that has the same structure as G.* The involutive automorphism that gives rise to the space $\mathcal{E}$, and which makes any transformation of the complex group correspond to the conjugate imaginary transformation will change $\sum (a_k + i\,b_k)\,U_k$ into $\sum (a_k - i\,b_k)\,U_k$.

If the closed space $\mathcal{E}_u$ admits a simple group G_u for a larger group of displacements then the associated open space $\mathcal{E}$ will admit an open, simple group G *with the same complex structure,* but a different real structure for a larger group of displacements.

75. In one and the other case, in order to realize the fundamental group of the open space, take the corresponding open, linear, adjoint group Γ. The real, infinitesimal transvections of space correspond to the purely imaginary, infinitesimal transvections of the associated closed space; the characteristic roots of a finite transvection of $\mathcal{E}$ are thus all real or all positive. One easily proves that any transformation of Γ can be put into the form *TR in one and only one way*, in which *T* is a transvection, and *R* is a rotation. One and only one geodesic will then pass through any two points of space. Moreover, the canonical parameters are valid in the entire domain of transvections, in such a way that *the space* $\mathcal{E}$ *is simply connected and homeomorphic to Euclidian space*. It admits no non-simply-connected Klein form.

Two locally reducible, symmetric spaces that are reducible to several other open, symmetric spaces are integrally reducible.

Finally, we remark that the subgroup *g* of rotations is connected for an open, irreducible, symmetric space, and the same thing will be true for the closed, simply-connected space it is associated with.

76. One might wonder whether the adjoint group Γ of an arbitrarily-given, open, simple group can always be regarded as the group of displacements of an irreducible, symmetric space. The answer is in the affirmative. There is more: All of the involutive automorphisms of an open, simple group that can be generated by a symmetric space are mutually homologous in the continuous, adjoint group [**15**, **21**]. In other words, *if one is given a space with an open, simple, fundamental group then there will exist one and only one choice of generating element of the space that can make the geometry of the space Riemannian symmetric.*

In particular, the preceding theorem asserts the existence of a closed form with real parameters for any simple group with complex parameters. If one can prove that theorem, *a priori*, without, like E. Cartan, verifying it for each particular structure, or without, like H. Weyl [**8**, pp. 371], appealing to the previously-established theory of simple groups then that will permit a considerable simplification in the presentation of the theory of simple groups [**21**].

VI. – Applications to the topology of open, simple groups.

77. Let Γ be the adjoint group of an open, simple group *G* with real or complex parameters. There exists an open, irreducible, symmetric space $\mathcal{E}$ that is homeomorphic to Euclidian space that admits Γ for the group of displacements. Let *g* be the closed, connected subgroup of rotations in space. Any transformation *S* of Γ can, in one and only one manner, can be put into the form of a product *TR* of a transvection and a rotation. Any closed manifold that is traced in the manifold of Γ will correspond to the set of two closed manifolds in a one-to-one way, one of which is traced in the manifold of the closed group *g*, while the other one is traced in the space $\mathcal{E}$. The latter is reducible to a point by continuous deformations. It then results [**17**, **21**] that *the Betti numbers of the* *manifold of the open group* Γ *will be the same as those of the manifold of the closed group* g. One can add a Betti number to them that is equal to 1 and corresponds precisely to the manifold of g, which is closed in the space of Γ. The last non-zero Betti number of the manifold of an open, simple group is thus equal to 1, and it refers to refers to the closed manifolds that have the same number of dimensions as the symmetric, Riemannian space that has Γ for its group of displacements.

The first Betti number of the closed group g is, moreover, equal to 0 or to 1. In the former case, the covering group of Γ will cover Γ only a finite number of times, while in the latter case it will cover it an infinite number of times, but there will exist only one category of closed curves in Γ such that no integer multiple of the is reducible to a point by continuous deformation. The real projective group in $n \geq 2$ variables belongs to the former case, along with the complex projective group in one or more variables. The real projective group in one variable will belong to the latter case, and its manifold will be homeomorphic to the interior of a torus.

78. One sees that any progress in the topology of the closed groups will imply progress in the topology of open groups. As far as the former are concerned, if one has information about the first Betti number, and likewise the second one, then one will know almost nothing about the other Betti numbers. Nevertheless, one is certain [**20**] that the third Betti number is non-zero, at least, if the group is not commutative, since there exists a triple integral of an exact differential – namely, the invariant integral $\iiint \sum c_{ijk}\, \omega_i\, \omega_j\, \omega_k$ – that admits non-zero periods; for example, the ones that one obtains by extending the integral over the manifold of a three-dimensional, simple subgroup of a given group. That is a very important subject of research that one can say has been almost unexplored.

____

BIBLIOGRAPHIC INDEX

1. S. LIE and Fr. ENGEL – *Theorie der Transformationsgruppen* (B. G. Teubner, Leipzig; v. I, 1888; v. II, 1890; v. III, 1893).
2. F. SCHUR – "Ueber den analytischen Charakter der eine endliche continuierliche Transformationsgruppe darstellenden Funktionen," Math. Ann. **41** (1893), 509-538.
3. E. CARTAN – "Sur la structure des groupes de transformations finis et continus," Thesis, Nony, Paris, 1894.
4. A. HURWITZ – "Ueber die Erzuegung der Invarianten durch Integration," Gött. Nachr. (1897), 71-90.
5. H. POINCARÉ – "Sur les groupes continus," Comptes rendus **128** (1899), 1065-1069.
6. H. POINCARÉ – "Sur les groupes continus," Cambr. Trans. **18** (1900), 220-255.
7. H. POINCARÉ – "Sur les groupes continus," Rend. Circ. matem. Palermo **15** (1901), 321-368; **25** (1908), 81-130.
8. H. WEYL – "Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen," Math. Zeit. **23** (1925), 271-309; **24** (1925), 328-395.
9. E. CARTAN – "Les tenseurs irréductibles et les groupes linéaires simples et semi-simples," Bull. Sc. math. (2) **49** (1925), 130-152.
10. E. CARTAN – "La principe de dualité et la théorie des groupes simples et semi-simple," Bull. Sc. math. (2) **49** (1925), pp. 361-374.
11. E. CARTAN – "Sur certains systèmes différentiels dont les inconnues sont des formes de Pfaff," Comptes rendus **182**(1926), 956-958.
12. E. CARTAN and J. A. SCHOUTEN, "On the Geometry of the Group-manifold of simple and semi-simple groups," Proc. Amsterdam **29** (1926), 803-815.
13. E. CARTAN – "La géométrie des groupes de transformations," J. Math. pures et appl. **6**(1927), 1-119.
14. E. CARTAN – "Sur une classe remarquable d'espaces de Riemann," Bull. Soc. math. **54** (1926), 214-264; **55** (1927), 114-134.
15. E. CARTAN – "La géométrie des groupes simples," Annali di Mat. (4) **4** (1926/27), 209-256.
16. E. CARTAN – "Complément au Mémoire: 'Sur la géométrie des groupes simples'," Annali di Mat. (4) **5** (1928), 253-260.
17. E. CARTAN – "Sur certaines formes riemanniennes remarquable des geometries à groupe fundamental simple," Ann. École Norm. (3) **44** (1927), 345-467.
18. E. CARTAN – "La théorie des groupes et la géométrie," L'Enseignement math. **26** (127), 200-225.
19. E. CARTAN – "Sur les espaces de Riemann clos admettant un groupe continu transitif de déplacements," Comptes rendus **186** (1928), 1817-1819.
20. E. CARTAN – "Sur les nombres de Betti des espaces de groupes clos," Comptes rendus **187** (1928), 196-198.

21. E. CARTAN – "Groupes simples clos et ouverts et géométrie riemannienne," J. Math. pures et appl. **8** (1929), 1-33.
22. A. BUHL – "Aperçus modernes sur la théorie des groupes continus et finis," (Mémorial Sc. math. math., fasc. XXXIII, 1928).
23. J. von NEUMANN – "Zur Theorie der Darstellung kontinuierlicher Gruppen," Sitz. Akad. Berlin (1927), 76-90.
24. O. SCHREIER – "Abstrakte kontinuierliche Gruppen," Abh. math. Seminar Hamburg **4** (1926), 15-32.
25. O. SCHREIER – "Die Verwandschaft stetiger Gruppen im grossen," Abh. math. Seminar Hamburg **5** (1927), 233-244.
26. D. van DANTZIG and B.-L. van der WAERDEN – "Ueber metrische homogene Räume," Abh. math. Seminar Hamburg **6** (1928), 367-376.

___