

“La méthode du repère mobile, la théorie des groupes continues et les espaces généralisés,”
Exposés de géométrie, t. V, Hermann, Paris, 1935.

THE METHOD OF MOVING FRAMES, THE THEORY OF CONTINUOUS GROUPS AND GENERALIZED SPACES

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TABLE OF CONTENTS

	Page
INTRODUCTION.....	1
I. – The method of moving frames.....	2
II. – Application of the method of moving frames to the study of minimal curves.....	3
III. – The theory of plane curves in affine geometry.....	10
IV. – The method of reduced equations.....	14
V. – The notion of frame in a geometry with a given group.....	18
VI. – The Darboux-Maurer-Cartan structure equations.....	24
VII. – General method of moving frames.....	29
VIII. – The method of moving frames, based upon the structure equations.....	34
IX. – The structure equations, imagined as an organizing principle for space...	38
X. – Generalized spaces.....	42

INTRODUCTION

The following pages constitute the development of five conference talks that were made in Moscow from 16 to 20 June 1930 at the invitation of the Moscow Mathematical Institute. They were translated and published in Russian in 1933. It does not seem pointless to me to insert them in the collection of *Exposés de géométrie*, in the hopes of submitting them for the approval of a larger number of geometers. Some theorems that required technical knowledge from the theory of partial differential equations have been stated without proof.

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I

1. – All of the geometry that has been known in the study of curves and surface since G. Darboux has been derived by the use of a moving trihedron that is attached to the different points of the curve or surface according to some intrinsic law ⁽¹⁾. In the case of a curve, the trihedron, which is known by the name of the *Frenet trihedron*, has the tangent, principal binormal, and binormal to the curve for its axes. Darboux also used tri-rectangular trihedra that were subject to only the condition that the third axis must be normal to the surface. The use of the moving trihedron is also well-indicated in a certain number of other geometric theories – for example, in the theory of triple orthogonal systems.

If one would indeed like to reflect upon the profound reasons for the fecundity of the method of the moving trihedron then one must first remark that the field of application of that method is *differential* geometry; it is of no use in the theory of algebraic curves when they are considered *qua* algebraic curves. The method is adapted to only the problems that appeal exclusively to the infinitesimal properties of a curve or surface. However, in that domain of differential geometry, the success of the method is due to two reasons:

1. The trihedron that is attached to a given point of a curve or surface constitutes the simplest reference system for the study of the infinitesimal properties of the curve or the surface in the neighborhood of a point.

2. The curve (or surface) is determined completely, *up to a displacement in space*, by the knowledge of the components with respect to the moving trihedron of the infinitesimal displacement of that frame when one passes from one point of the curve (or surface) to an infinitely-close point.

2. – The second reason has the character of simple convenience – one might say, esthetics – in the sense it imposes no rigorous condition upon the choice of frame. One even imagines that for some questions, one might find that it would be more convenient to take a non-rectangular trihedron whose form can even vary from one point to another. Be that as it may, the trihedron must be determined by those differential properties of the curve or surface that present themselves at the outset – i.e., the ones that involve the differential elements of low order. In the case of a curve or surface, they are essentially the differential elements of the first two orders that determine Darboux's moving trihedron.

3. – The second reason that was pointed out above is based upon the following well-known theorem:

If one has two families that contain tri-rectangular trihedra, and if one can establish a bijective correspondence between the trihedra of those two families such that the

⁽¹⁾ See G. DARBOUX's *Leçons sur la théorie des surfaces*, especially v. I and II. See also RIBAUCCOUR, *Mémoire sur la théorie générale des surfaces courbes*, J. de Math. (4) 7 (1891).

relative components of the infinitesimal displacement of a trihedron of the first family are equal to the relative components of the infinitesimal displacement of a trihedron of the second family then there will exist a well-defined displacement that simultaneously brings the trihedra of the first family into coincidence with the corresponding trihedra of the second family.

The *relative* components of the infinitesimal displacement of a moving tri-rectangular trihedron are the six components with respect to the axes of that trihedron of the infinitesimal translation that is suffered by the origin of the trihedron and the supplementary infinitesimal rotation that is suffered by the trihedron.

The theorem does not demand, in full rigor, that one has to make the trihedra tri-rectangular, but it does demand that, at the very least, the trihedra must all be equal to each other. The six quantities that define the infinitesimal displacement of that trihedron analytically with respect to the axes of the moving trihedron will have a more or less complicated significance, but the theorem will persist nonetheless. On the contrary, if one utilizes trihedra of variable form then one can even define the passage from one trihedron to an infinitely-close trihedron analytically, but one must then introduce variations of the form of the trihedron – i.e., parasitic elements with no relationship to the properties of the curve or surface under study.

4. – The preceding considerations show that the method of the moving frame must satisfy the following conditions if it is to preserve its scope:

1. *The trihedron that is attached to each point of the manifold under study must be determined in an intrinsic manner by the first-order differential elements of the manifold.*
2. *The various trihedra must be rectangular, or at least, they must be equal to each other.*

To say that the trihedron is determined in an intrinsic manner is to say that if one makes the determination of the trihedra at two homologous points of the two equal manifolds according to the chosen law then the two trihedra that are obtained will be brought into coincidence by the displacement that brings the two manifolds into coincidence.

As for the condition of convenience that was envisioned above, it will be a question of type. From the purely logically viewpoint, nothing will prevent one from, for example, substituting any other trihedron – tri-rectangular or not – that is invariably coupled with a skew curve for the Frenet frame.

II.

5. – In the classical applications of the method of moving frames, the choice of the trihedron was indicated by itself without the geometer feeling the slightest hesitation. However, even without leaving Euclidian geometry, there are some cases in which that

would no longer be true. For example, consider a minimal curve in complex Euclidian geometry. The Frenet trihedron (which one can consider to be composed of three unitary vectors that are carried by the tangent, the principal normal, and the binormal) can no longer exist in that case, since any vector that is carried by the tangent will have length zero. True, one can choose a trihedron whose *coordinate vectors* $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are such that the first and last ones have zero length and a scalar product that is equal to 1, while the second one has length 1 and is perpendicular to the first two. One can then take the vector \mathbf{e}_1 to be tangent to the curve, but one sees no reason – at least, to begin with – to take \mathbf{e}_1 to be such a vector, rather than any other tangent to the curve. (They are, moreover, all equal to each other.) On the other hand, since the normal plane to the curve coincides with the osculating plane, there will exist no other apparent reason to make that normal play the role of principal normal, rather than any other role. One then knows neither how to choose the vector \mathbf{e}_1 nor how to choose the vector \mathbf{e}_2 .

That example indeed shows the legitimacy of the following problem:

Is it possible to attach a well-defined trihedron that is always equal to itself to each point of a minimal curve in an intrinsic manner?

More generally:

Is the method of the moving trihedron itself susceptible to being generalized to all questions of differential geometry?

6. – Before we address that general question, we remark that there certainly exist cases in which the intrinsic determination of a trihedron that attached to a variable point of a curve or surface is impossible. It will suffice to contemplate the case of a (non-isotropic) straight line. The first axis of the Frenet trihedron will be well-defined, but there will be no reason to choose the second axis to be this or that perpendicular to the line. We remark that in that case (and one will see the importance of this remark later on), the impossibility is due to the nature of things itself. Indeed, there exists a group of displacements that simultaneously leave all of the points of the line invariant. In order for the determination of the trihedron to be logically impossible, it would likewise suffice that the manifold in question should be invariant under a group of displacements, provided that there should be an infinitude of displacements of that group that leave an arbitrary given point M of that manifold fixed, because the intrinsic determination of the trihedron that is attached to M (if that is possible) will be given by that trihedron, as well as all of the ones that are deduced from it by displacement. For example, that is why the intrinsic determination of a tri-rectangular trihedron that is attached to a point M of a sphere will be impossible: viz., there is an infinitude of displacements that leave the sphere invariant and leave the point M fixed.

7. – Let us leave that case aside. (Later on, we shall see that it is the only one in which the intrinsic determination of the trihedron is impossible.) Before examining the case of a minimal curve, we recall the classical case of an ordinary skew curve and

analyze (if somewhat pedantically) the process by which one might be led to the Frenet frame, while forsaking all geometric intuition, as much as possible.

First, attach an entire family of tri-rectangular trihedra – namely, the ones that have the point M for their origins – to a given point M of a skew curve, which is assumed to be defined by means of one parameter t ; we call them the *frames of order zero*. They depend upon three parameters u_1, u_2, u_3 , which we call *secondary parameters* of order zero. If one varies the point M then the zero-order trihedra will depend upon four parameters: viz., u_1, u_2, u_3 , and t . Let $\omega_1, \omega_2, \omega_3$ denote the components along the moving axes of the translation that is suffered by the origin M of the trihedron, and let $\omega_{23}, \omega_{31}, \omega_{12}$ denote the relative components of the instantaneous rotation that is experienced by the trihedron. It is clear that the three components $\omega_1, \omega_2, \omega_3$, which will be zero when the point M remains fixed, have the form:

$$\omega_1 = p_1 dt, \quad \omega_2 = p_2 dt, \quad \omega_3 = p_3 dt.$$

As for the components $\omega_{23}, \omega_{31}, \omega_{12}$ of the rotation, they involve not only dt , but also the differentials of the second parameters du_1, du_2, du_3 . One likewise easily sees that they are linearly independent with respect to those three differentials.

The coefficients p_1, p_2, p_3 obviously depend upon the numerical values of the secondary parameters. One can then establish two relations between those parameters and t such that the ratios ω_2 / ω_1 and ω_3 / ω_1 are annulled; geometrically, that amounts to choosing the first axis of the trihedron to be tangent to the curve. Frames that satisfy that condition are called *first-order frames*, and if one confines oneself to those frames then one will have:

$$\omega_2 = 0, \quad \omega_3 = 0.$$

The first-order frames no longer depend upon more than one parameter, and the components $\omega_{23}, \omega_{31}, \omega_{12}$ of the infinitesimal rotation will be related by two relations when one sets $dt = 0$. Geometrically, when one fixes the point M , the first-order frame will only be susceptible to a rotation around its first axis, in such a way that the components ω_{31}, ω_{12} of the rotation with respect to the last two axes will be annulled. Upon varying the point M , it will then result that:

$$\omega_{31} = p_{31} dt, \quad \omega_{12} = p_{12} dt.$$

The ratios:

$$\frac{\omega_{31}}{\omega_1} = \frac{p_{31}}{p_1}, \quad \frac{\omega_{12}}{\omega_1} = \frac{p_{12}}{p_1}$$

now depend upon the first-order secondary parameter, and one can choose that parameter in such a manner as to annul one of those ratios – for example, p_{31} . One then arrives at a second-order frame that is perfectly well-defined. The expression ω_1 is the elementary arc length ds , while the ratio p_{12} / p_1 is the curvature of the curve; the component $\omega_{23} = p_{23} dt$ then gives the torsion p_{23} / p_1 .

One sees that, by successive restrictions, one passes from zero-order frames that depend upon three parameters to trihedra of order 1 that depend upon one parameter to a definite trihedron of order 2, which is the Frenet frame.

8. – We now go on to the case of a minimal curve. For reasons of convenience, we shall utilize a trihedron that is composed of three vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ that are attached to a point M of the curve and satisfy the conditions:

$$(1) \quad (\mathbf{e}_1)^2 = (\mathbf{e}_3)^2 = \mathbf{e}_1 \mathbf{e}_2 = \mathbf{e}_2 \mathbf{e}_3 = 0, \quad (\mathbf{e}_2)^2 = \mathbf{e}_1 \mathbf{e}_3 = 1.$$

The relative components of the infinitesimal displacement of the trihedron are the coefficients that are introduced into the formulas:

$$\begin{aligned} d\mathbf{M} &= \omega^1 \mathbf{e}_1 + \omega^2 \mathbf{e}_2 + \omega^3 \mathbf{e}_3, \\ d\mathbf{e}_i &= \omega_i^1 \mathbf{e}_1 + \omega_i^2 \mathbf{e}_2 + \omega_i^3 \mathbf{e}_3. \end{aligned}$$

The nine components ω_i^j are not independent. Upon differentiating the relations (1), one will easily find that:

$$(2) \quad \begin{cases} \omega_1^3 = \omega_3^1 = \omega_2^2 = 0, \\ \omega_1^2 + \omega_2^3 = \omega_2^1 + \omega_3^2 = \omega_1^1 + \omega_3^3 = 0. \end{cases}$$

We keep the six quantities $\omega^1, \omega^2, \omega^3, \omega_1^1, \omega_1^2, \omega_3^2$ for the components of the infinitesimal displacement.

We can economize immediately by skipping the zero-order frames and beginning with the trihedra of order 1, for which \mathbf{e}_1 is an arbitrary vector that is tangent to the curve. Those trihedra depend upon two secondary parameters, one of which serves to fix to the vector \mathbf{e}_1 itself, and the other of which fixes the direction of the vector \mathbf{e}_2 that is perpendicular to \mathbf{e}_1 . One first has:

$$\omega^2 = \omega^3 = 0.$$

Of the four components that remain, two of them will be annulled if one fixes the point M – namely, ω^1 and ω_1^2 . Indeed, the differential $d\mathbf{e}_1$ of the vector \mathbf{e}_1 must be a vector that is carried along the tangent that is fixed to the curve and must have any component that is parallel to \mathbf{e}_2 . As for the other two components ω_1^1 and ω_3^2 , they depend linearly upon the differentials of the two secondary parameters. Upon varying the principal parameter t and the two secondary parameters, one will then have:

$$\omega^1 = p^1 dt, \quad \omega_1^2 = p_1^2 dt.$$

The coefficients p^1 and p_1^2 depend upon secondary parameters *a priori*. In order to see the manner in which they depend, replace the frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ with another first-order trihedron $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3)$. One easily has:

$$\boldsymbol{\eta}_1 = \lambda \mathbf{e}_1, \quad \boldsymbol{\eta}_2 = \mathbf{e}_2 + \lambda \mathbf{e}_1, \quad \boldsymbol{\eta}_3 = \frac{1}{\lambda} \mathbf{e}_3 - \frac{\mu}{\lambda} \mathbf{e}_2 - \frac{\mu^2}{2\lambda} \mathbf{e}_1.$$

The new values of the components ω^1 and ω_1^2 are then calculated with no difficulty, and one will find that:

$$\bar{\omega}^1 = \frac{1}{\lambda} \omega^1, \quad \bar{\omega}_1^2 = \lambda \omega_1^2,$$

so:

$$\bar{p}^1 = \frac{1}{\lambda} p^1, \quad \bar{p}_1^2 = \lambda p_1^2.$$

One can then dispose of λ in such a manner that the ratio p_1^2 / p_1 reduces to a fixed numerical value – for example, 1. One will then obtain a family of second-order trihedra that depend upon only one secondary parameter, and for which one will have:

$$\omega^2 = 0, \quad \omega^3 = 0, \quad \omega_1^2 = \omega^1.$$

If one now varies the second-order trihedron while fixing the point M then the two components ω_1^1 and ω_3^2 will no longer be independent; they will be coupled by one relation. One will get it easily upon remarking that the vector \mathbf{e}_1 is now fixed, so its differential is zero, and as a result, the component ω_1^1 will be annulled with dt . Upon varying M , one will then have:

$$\omega_1^1 = p_1^1 dt.$$

In order to see how p_1^1 depends upon the secondary parameters of the second-order trihedron, replace the trihedron $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ with the trihedron $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3)$:

$$\boldsymbol{\eta}_1 = \mathbf{e}_1, \quad \boldsymbol{\eta}_2 = \mathbf{e}_2 + \mu \mathbf{e}_1, \quad \boldsymbol{\eta}_3 = \mathbf{e}_3 - \mu \mathbf{e}_2 - \frac{1}{2} \mu^2 \mathbf{e}_1.$$

Upon remarking that ω_1^1 is nothing but the scalar product $\mathbf{e}_3 d\mathbf{e}_1$, one will find that:

$$\bar{\omega}_1^2 = \omega_1^1 - \mu \omega_1^2 = (p_1^1 - \mu p_1) dt.$$

One can then dispose of μ in such a manner as to annul p_1^1 . One will then arrive at a well-defined third-order trihedron, for which one will have:

$$\omega^2 = \omega^3 = \omega_1^2 - \omega^1 = \omega_1^1 = 0.$$

The component ω_3^2 will then have the form $k \omega^1$, and the coefficient k will be a *differential invariant* of the curve; it is its *pseudo-curvature*.

Furthermore, the expression ω^1 , which will be indeterminate when one appeals to first-order trihedra, will take on a well-defined value when one arrives at second-order trihedra. It is the elementary *pseudo-arc length* $d\sigma$ of the minimal curve.

One can now write the Frenet formulas of the minimal curves, namely:

$$(3) \quad \left\{ \begin{array}{l} \frac{dM}{d\sigma} = \mathbf{e}_1, \\ \frac{d\mathbf{e}_1}{d\sigma} = \mathbf{e}_2, \\ \frac{d\mathbf{e}_2}{d\sigma} = -k \mathbf{e}_1 - \mathbf{e}_2, \\ \frac{d\mathbf{e}_3}{d\sigma} = k \mathbf{e}_2. \end{array} \right.$$

Two minimal curves are equal if the pseudo-curvature is the same function of the pseudo-arc length σ for those two curves; or rather, if $dk / d\sigma$ is the same function of k .

9. – We remark that along the path that we have chosen, we have made an implicit hypothesis, namely, that we have reduced the ratio:

$$\frac{\bar{p}_1^2}{p_1} = \lambda^2 \frac{p_1^2}{p_1}$$

to 1, which assumes that the coefficient p_1^2 is not zero; i.e., geometrically, that one is dealing with a straight line. If the given line is a straight line then one will have no means of differentiating one of the other first-order trihedra, and one will no longer have any means of defining a *natural parameter* σ on the straight line, which is geometrically obvious, moreover.

In addition, we remark that in the passage from first-order trihedra to trihedra of higher order, we appeal to geometric considerations in order to predict what will be the new components of the infinitesimal displacement that should no longer depend upon differentials of the secondary parameters. Although the method that we appeal to will give the form of the Frenet equations – *which is, moreover, the only interesting thing in theoretical research* – it will give us neither the expression for $d\sigma$ nor the expression for k explicitly.

One can obviate these various inconveniences by explicitly calculating the components of the infinitesimal displacement of a trihedron of order zero from the outset. We shall do that in the case of a minimal curve.

10. – With Weierstrass ⁽²⁾, define a minimal curve by the following equations in rectangular coordinates:

$$(4) \quad x = \int \frac{1-t^2}{2} \mathcal{F}(t) dt, \quad y = \int i \frac{1+t^2}{2} \mathcal{F}(t) dt, \quad z = \int t \mathcal{F}(t) dt,$$

where x , y , and z denote the rectangular coordinates of a point of the curve, and $\mathcal{F}(t)$ is an analytic function of t . Let α_i , β_i , γ_i denote the projections of the vector \mathbf{e}_i onto the fixed rectangular axes:

$$\begin{aligned} \omega^1 &= \mathbf{e}_3 dM = \alpha_3 dx + \beta_3 dy + \gamma_3 dz, \\ \omega^2 &= \mathbf{e}_2 dM = \alpha_2 dx + \beta_2 dy + \gamma_2 dz, \\ \omega^3 &= \mathbf{e}_1 dM = \alpha_1 dx + \beta_1 dy + \gamma_1 dz, \end{aligned}$$

$$\begin{aligned} \omega_1^1 &= \mathbf{e}_3 d\mathbf{e}_1 = \alpha_3 d\alpha_1 + \beta_3 d\beta_1 + \gamma_3 d\gamma_1, \\ \omega_1^2 &= \mathbf{e}_2 d\mathbf{e}_1 = \alpha_2 d\alpha_1 + \beta_2 d\beta_1 + \gamma_2 d\gamma_1, \\ \omega_1^3 &= -\mathbf{e}_3 d\mathbf{e}_1 = -\alpha_3 d\alpha_2 - \beta_3 d\beta_2 - \gamma_3 d\gamma_2. \end{aligned}$$

In order to define a first-order trihedron, one can take:

$$\begin{aligned} \alpha_1 &= \lambda \frac{1-t^2}{2}, & \beta_1 &= i\lambda \frac{1+t^2}{2}, & \gamma_1 &= \lambda t, \\ \alpha_2 &= -t + \mu \frac{1-t^2}{2}, & \beta_2 &= it + i\mu \frac{1+t^2}{2}, & \gamma_2 &= 1 + \mu t, \\ \alpha_3 &= \rho \frac{1-u^2}{2}, & \beta_3 &= i\rho \frac{1+u^2}{2}, & \gamma_3 &= \rho u, \end{aligned}$$

with the conditions:

$$u - t = \frac{2}{\mu}, \quad \lambda \rho = -\frac{1}{2}\mu^2.$$

Calculation will then give:

$$\begin{aligned} \omega^1 &= \frac{1}{\lambda} \mathcal{F}(t) dt, & \omega^2 &= 0, & \omega^3 &= 0, \\ \omega_1^1 &= -\mu dt, & \omega_1^2 &= \lambda dt, & \omega_1^3 &= 2\rho \frac{d\mu}{\mu^2} - \rho dt. \end{aligned}$$

One then sees that the ratio:

⁽²⁾ WEIERSTRASS, “Ueber die Flächen deren mittlere Krümmung überall gleich Null ist,” Sitzungsber. Berlin (1866).

$$\frac{\omega_1^2}{\omega^1} = \frac{\lambda^2}{\mathcal{F}(t)}$$

can be reduced to 1 by setting:

$$\lambda^2 = \mathcal{F}(t).$$

One will then have:

$$(5) \quad \begin{aligned} \omega^1 &= d\sigma = \sqrt{\mathcal{F}(t)} dt, \\ \omega_1^1 &= \left(\frac{1}{2} \frac{\mathcal{F}'(t)}{\mathcal{F}(t)} - \mu \right) dt, \end{aligned}$$

and one can reduce ω_1^1 to zero by setting:

$$\mu = \frac{1}{2} \frac{\mathcal{F}'(t)}{\mathcal{F}(t)}, \quad \text{hence, } \rho = -\frac{1}{8} \frac{\mathcal{F}''}{\lambda \mathcal{F}^2}.$$

One will finally have:

$$\omega_3^2 = \frac{5\mathcal{F}'^2 - 4\mathcal{F}\mathcal{F}''}{8\lambda\mathcal{F}^2} dt = \frac{5\mathcal{F}'^2 - 4\mathcal{F}\mathcal{F}''}{8\mathcal{F}^3} d\sigma,$$

which will give the curvature:

$$(6) \quad k = \frac{5\mathcal{F}'^2 - 4\mathcal{F}\mathcal{F}''}{8\mathcal{F}^3}.$$

One sees that $d\sigma$ is defined only up to a sign, but that k is defined rationally in terms of the derivatives of $\mathcal{F}(t)$ the first two orders. The two vectors \mathbf{e}_1 and \mathbf{e}_3 are likewise defined up to sign. (The second one is determined completely when one fixes the first one.) As for the vector \mathbf{e}_2 , it is defined unambiguously.

Once the Frenet trihedron has been determined, it will be up to the geometer to specify the geometric significance of the vectors \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 that determine it, as well as that of the pseudo-arc length and the curvature. However, it is clear that the previous knowledge of the Frenet formulas is a powerful aid in that purely-geometric study, and that it will suggest some interpretations that one could not have imagined *a priori*.

III

11. – Before commencing with a systematic study of the method of moving frames, we shall examine some further examples that go beyond the scope of Euclidian geometry, properly speaking.

One can propose to study the properties of curves that are not only independent of their particular position in space, but which do not change under a homothety. From that viewpoint, two similar figures must be regarded as equal. It is clear that two tri-rectangular trihedra that are constructed from three vectors of the same length must be

regarded as equal, even if the common length of the vectors of the first trihedron is not the same as that of the vectors of the second one. The passage from one tri-rectangular trihedron to another one that is infinitely close will then be accomplished by an infinitesimal translation and a rotation, accompanied by a homothety whose ratio is infinitely close to 1. One will need seven quantities, instead of six, in order to define that new type of displacement analytically, and one will have the formulas:

$$dM = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3 ,$$

$$d\mathbf{e}_1 = \omega \mathbf{e}_1 + \omega_{12} \mathbf{e}_2 + \omega_{13} \mathbf{e}_3 ,$$

$$d\mathbf{e}_2 = \omega_{21} \mathbf{e}_1 + \omega \mathbf{e}_2 + \omega_{23} \mathbf{e}_3 ,$$

$$d\mathbf{e}_3 = \omega_{31} \mathbf{e}_1 + \omega_{32} \mathbf{e}_2 + \omega \mathbf{e}_3 ,$$

in which $\omega_1, \omega_2, \omega_3$ are the components of the translation, $\omega_{23} = -\omega_{32}, \omega_{31} = -\omega_{13}, \omega_{12} = -\omega_{21}$ are the components of the rotation, and $1 + \omega$ is the homothety ratio.

If one is given a skew curve then one cannot perceive immediately what the vector \mathbf{e}_1 will be that one agrees to choose to be tangent to the curve, and in fact, if one sticks to the first-order elements of the curve then there will be no reason to choose one vector rather than another. Meanwhile, there is a unit of length that one can attach to each point of the curve in an intrinsic manner, namely, the length of the radius of curvature that relates to that point; the desired trihedron will then be determined completely by geometric considerations. Upon denoting the unitary vectors of the ordinary Frenet trihedron by $\mathbf{T}, \mathbf{N}, \mathbf{B}$, and denoting the curvature and torsion by $1 / \rho$ and $1 / \tau$, resp., one will then take:

$$\mathbf{e}_1 = \rho \mathbf{T}, \quad \mathbf{e}_2 = \rho \mathbf{N}, \quad \mathbf{e}_3 = \rho \mathbf{B},$$

and the Frenet formulas will become:

$$dM = \frac{ds}{\rho} \mathbf{e}_1 ,$$

$$d\mathbf{e}_1 = \frac{d\rho}{\rho} \mathbf{e}_1 + \frac{ds}{\rho} \mathbf{e}_2 ,$$

$$d\mathbf{e}_2 = -\frac{ds}{\rho} \mathbf{e}_1 + \frac{d\rho}{\rho} \mathbf{e}_2 + \frac{ds}{\tau} \mathbf{e}_3 ,$$

$$d\mathbf{e}_3 = -\frac{ds}{\tau} \mathbf{e}_2 + \frac{d\rho}{\rho} \mathbf{e}_3 .$$

The new natural parameter is defined by:

$$(7) \quad ds = \frac{ds}{\rho} ,$$

and one has:

$$(8) \quad \left\{ \begin{array}{l} \frac{dM}{d\sigma} = \mathbf{e}_1, \\ \frac{d\mathbf{e}_1}{d\sigma} = k\mathbf{e}_1 + \mathbf{e}_2, \\ \frac{d\mathbf{e}_2}{d\sigma} = -\mathbf{e}_1 + k\mathbf{e}_2 + h\mathbf{e}_3, \\ \frac{d\mathbf{e}_3}{d\sigma} = -h\mathbf{e}_2 + k\mathbf{e}_3, \end{array} \right.$$

with the two *fundamental invariants*:

$$(9) \quad k = \frac{d\rho}{ds}, \quad h = \frac{\rho}{\tau}.$$

Two curves for which k and h are the same functions of s are *equal* in the sense of the geometry of similitudes; i.e., they are *similar*, in the common sense of the word.

12. – Now take an example that is even more estranged from ordinary Euclidian geometry. We propose to study the properties of a plane curve that are invariant under a unimodular affine transformation – i.e., by a transformation that is defined by formulas of the form:

$$\begin{aligned} x' &= a x + b y + c, \\ y' &= a' x + b' y + c' \end{aligned}$$

in a Cartesian coordinate system, with the condition that:

$$ab' - ba' = 1.$$

The systems of reference – or *frames* – that must replace the tri-rectangular trihedra are the Cartesian coordinate systems that are defined by two vectors $\mathbf{e}_1, \mathbf{e}_2$ that are subject to the single condition that the parallelogram that is constructed from those two vectors must have a given area, which will be the unit of area.

We once more give the name of *displacement* to an affine, unimodular transformation. One will have formulas:

$$(10) \quad \left\{ \begin{array}{l} dM = \omega^1 \mathbf{e}_1 + \omega^2 \mathbf{e}_2, \\ d\mathbf{e}_1 = \omega_1^1 \mathbf{e}_1 + \omega_1^2 \mathbf{e}_2, \\ d\mathbf{e}_2 = \omega_2^1 \mathbf{e}_1 + \omega_2^2 \mathbf{e}_2 \end{array} \right.$$

for an infinitesimal displacement of the frame, whose origin is assumed to be M , but the condition that relates to the area of the parallelogram ($\mathbf{e}_1, \mathbf{e}_2$) will give:

$$(11) \quad \omega_1^1 + \omega_2^2 = 0.$$

The infinitesimal transformation in question will then have five components $\omega^1, \omega^2, \omega_1^1, \omega_2^2$ with respect to the moving frame.

One can prove that if one has two families of moving frames that correspond with equality of the components ω^i and ω_i^j then one can pass from one of the families to the other by a well-defined unimodular affine transformation.

Having said that, suppose that a fixed reference system has been chosen. Let x and y be the coordinates of a point of a curve that is assumed to be planar, to simplify, and is given by its equation $y = f(x)$. Attach a *first-order frame* to each point such that the vector \mathbf{e}_1 is tangent to the curve. The components of the two vectors \mathbf{e}_1 and \mathbf{e}_2 will then have the form:

$$\begin{aligned} \text{For } \mathbf{e}_1: & \quad \alpha, \quad \alpha y', \\ \text{For } \mathbf{e}_2: & \quad \beta, \quad \beta y' + \frac{1}{\alpha}. \end{aligned}$$

The calculation of the components ω^i and ω_i^j will be achieved with no difficulty upon appealing to equations (10), and that will give:

$$\begin{aligned} \omega^2 &= 0, & \omega^1 &= \frac{dx}{\alpha}, & \omega_1^2 &= \alpha^2 y'' dx, \\ \omega_1^1 &= \frac{d\alpha}{\alpha} - \alpha\beta y'' dx, & \omega_2^1 &= \frac{d\beta}{\alpha} + \beta \frac{d\alpha}{\alpha^2} - \beta^2 y'' dx. \end{aligned}$$

We will get second-order frames upon equating the ratio ω_1^2 / ω^1 to 1 (which assumes that $y'' \neq 0$). Those frames depend upon just one parameter β ; one will have:

$$\alpha = y''^{-1/3}.$$

Moreover, since ω^1 is determined perfectly, one can define the differential $d\sigma$ of affine arc length:

$$(12) \quad ds = y''^{1/3} dx.$$

For second-order frames, the component ω_1^1 does not depend upon the differential of the single secondary parameter β ; one has:

$$\omega_1^1 = \left(-\frac{1}{3} \frac{y'''}{y''} - \beta y''^{2/3} \right) dx.$$

One will get a third-order frame by annulling ω_1^1 , which will give:

$$\beta = -\frac{1}{3} y''^{-5/3} y''' = \frac{1}{2} (y''^{-2/3})'.$$

For the third-order frame, which is the desired well-defined frame, upon setting:

$$y''^{-2/3} = z,$$

one will get:

$$\omega_2^1 = \frac{1}{2} z^{-1/2} z'' dx = \frac{1}{2} z'' d\sigma = k d\sigma.$$

The formulas for the Frenet frames are then:

$$(13) \quad \begin{cases} \frac{dM}{d\sigma} = \mathbf{e}_1, \\ \frac{d\mathbf{e}_1}{d\sigma} = \mathbf{e}_2, \\ \frac{d\mathbf{e}_2}{d\sigma} = k \mathbf{e}_1. \end{cases}$$

The differential invariant k is the *affine curvature*. The conics are characterized by the property of having constant affine curvature, which conforms to the differential equation of the conics:

$$(y''^{-2/3})''' = 0,$$

which is due to Monge, up to form.

In reality, there are three possible choices for the frame: If j denotes an arbitrary cube root of unity then one can replace:

$$d\sigma, \quad \mathbf{e}_1, \quad \mathbf{e}_2, \quad k$$

with

$$j d\sigma, \quad j^2 \mathbf{e}_1, \quad j \mathbf{e}_2, \quad jk,$$

respectively.

In order for two curves to be *equal* – i.e., to differ only by a unimodular affine transformation – it is necessary and sufficient that either k^3 must have the same constant value for the two curves or that $dk / d\sigma$ must be the same function of k^3 for the two curves.

IV

13. – We shall now point out another method (which is very fast in certain cases) for obtaining explicitly the differential invariants and moving frame that is attached to the variety in question, at least in order to obtain the Frenet formulas. That new method utilizes the method of *reduced equations*, which was developed systematically by Tresse

(³) in order to look for differential invariants. We shall be content to present in two particular examples.

The first of them is the one that we just treated in the preceding number. We shall attach a Cartesian coordinate system to a point M of a plane curve whose origin is M in such a way that the equation of the curve in the neighborhood of the point M will be as simple as possible. We restrict the coordinate vectors \mathbf{e}_1 and \mathbf{e}_2 by the condition that the parallelogram that they determine has an area that is equal to 1. Upon choosing the first axis to be tangent to the curve, the equation of the curve, which is assumed to be analytic, will have the form:

$$y = \frac{1}{2}a_2 x^2 + \frac{1}{6}a_3 x^3 + \dots$$

One can multiply x and y by two factors λ and $1/\lambda$, resp., which corresponds to a permissible change of coordinates, in such a manner as to give the value 1 to the coefficient a_2 . Upon setting, in turn:

$$x = X + \mu Y, \quad y = Y,$$

which is also permissible, one will easily confirm that one can annul the coefficient a_3 . One will then arrive at the *reduced equation*, which we can write in the form:

$$(14) \quad y = \frac{1}{2} x^2 - \frac{1}{8} k x^4 + \dots$$

If we consider a point M' that is infinitely close to M then the abscissa x of that point will be a quantity that is coupled to the arc length of the curve MM' in an intrinsic manner; we can call it the elementary *affine arc length*. Upon taking a point on the curve to be the origin, one can attribute an *affine curvilinear abscissa* σ to each point M of the curve, and the coefficients k, \dots of the reduced equation will be well-defined functions of the abscissa σ of the corresponding point.

Having said that, one will obviously have, upon displacing along the curve:

$$\begin{aligned} \frac{dM}{d\sigma} &= \mathbf{e}_1, \\ \frac{d\mathbf{e}_1}{d\sigma} &= p_1^1 \mathbf{e}_1 + p_1^2 \mathbf{e}_2, \\ \frac{d\mathbf{e}_2}{d\sigma} &= p_2^1 \mathbf{e}_1 - p_1^1 \mathbf{e}_2. \end{aligned}$$

Everything comes down to determining the unknown coefficients p_1^1, p_1^2, p_2^1 .

In order to do that, consider a *fixed* point P on the curve. Let x, y denote its coordinates relative to the frame that is attached to the point M whose abscissa is σ ; they are functions of σ that satisfy certain differential equations that are easy to construct.

(³) A. TRESSE, "Sur les invariants différentiels des groupes continus de transformations," Acta Math. **18** (1894).

Indeed, it suffices to express the idea that the point $M + x \mathbf{e}_1 + y \mathbf{e}_2$ is fixed, which will give:

$$(15) \quad \begin{cases} \frac{dx}{d\sigma} + 1 + p_1^1 x + p_2^1 y = 0, \\ \frac{dy}{d\sigma} + p_1^2 x - p_1^1 y = 0. \end{cases}$$

On the other hand, the coordinates x and y will satisfy the relation:

$$(14) \quad y = \frac{1}{2}x^2 - \frac{1}{8}k(\sigma)x^4 + \dots = f(x, \sigma),$$

for any σ . By differentiation and the use of formulas (15), one will deduce from this that:

$$(16) \quad -p_1^2 x + p_1^1 f(x, \sigma) = f'_x [-p_1^1 - p_2^1 x - f(x, \sigma)] + f'_\sigma$$

That relation must be true for any σ , but also for any point P that one fixes along the curve; it is then an *identity* in x and σ . We shall then equate the coefficients of the different powers of x in the two sides of the identity (16), which is written, upon developing:

$$(16') \quad -p_1^2 x + p_1^1 \left(\frac{1}{2}x^2 - \frac{1}{8}kx^2 + \dots \right) \\ = \left(x - \frac{1}{2}kx^3 + \dots \right) \left(-1 - p_1^1 x - \frac{1}{2}p_2^1 x^2 + \dots \right) - \frac{1}{8}k'x^4 + \dots$$

One will find, successively, that:

$$\begin{aligned} -p_1^2 &= -1, \\ \frac{1}{2}p_1^1 &= -p_1^1, \\ 0 &= -\frac{1}{2}p_2^1 + \frac{1}{2}k, \end{aligned}$$

so

$$p_1^2 = 1, \quad p_1^1 = 0, \quad p_2^1 = k.$$

One will then find the Frenet formulas (13) that were obtained already. However, upon pursuing the identifications, one will get the coefficients of the development of y as a function of x as functions of the affine curvature and its successive derivatives. That indeed confirms what we have said already, namely, that the curve is determined completely, up to a unimodular affinity, by the knowledge of k as a function of x ⁽⁴⁾.

14. – We shall further apply the same method to a problem of Euclidian geometry. We shall consider an analytic, imaginary surface that is not a minimal developable, but whose second fundamental form has one and only one common factor with the first

⁽⁴⁾ For the same method applied to plane projective geometry, see, E. CARTAN, “Sur un problème du calcul des variations,” Recueil Soc. Math. Moscou **34** (1927).

fundamental form, in such a way that the surface admits a double family of lines of minimal curvature. Here, we use the same trihedra as in the theory of minimal curves, with the isotropic vectors \mathbf{e}_1 and \mathbf{e}_2 tangent to the surface, and the unitary vector \mathbf{e}_3 normal to it. The reduced equation of the surface will have the form:

$$y = \frac{1}{2}ax^2 + bxz + \dots$$

If one considers a point M' that is infinitely close to M then the infinitely-small quantities x and z will be linked intrinsically to the pair of those two points; they will be the components ω^1 and ω^2 of the infinitesimal displacement of the moving trihedron. As in no. 8, one then sets:

$$(18) \quad \begin{cases} dM = \omega^1 \mathbf{e}_1 + \omega^2 \mathbf{e}_2, \\ d\mathbf{e}_1 = \omega_1^1 \mathbf{e}_1 + \omega_1^2 \mathbf{e}_2, \\ d\mathbf{e}_2 = -\omega_2^2 \mathbf{e}_1 - \omega_1^2 \mathbf{e}_2, \\ d\mathbf{e}_3 = \omega_3^2 \mathbf{e}_1 - \omega_1^1 \mathbf{e}_2. \end{cases}$$

The coordinates x, y, z of a fixed point P on the surface, when referred to the trihedron whose origin is M , will satisfy equation (17), where k, α, β, \dots will depend upon M and the differential equations that express the idea that the point P is fixed, namely:

$$(19) \quad \begin{cases} dx + \omega^1 + x \omega_1^1 - y \omega_3^2 = 0, \\ dy + x \omega_1^2 + z \omega_3^2 = 0, \\ dz + \omega^3 - y \omega_1^2 - z \omega_1^1 = 0. \end{cases}$$

As in the preceding example, one will deduce from this that:

$$\begin{aligned} -x \omega_1^2 - z \omega_3^2 &= \left(x + kz + \frac{1}{2}\alpha x^2 + \beta xz + \frac{1}{2}\gamma z^2 + \dots\right) \left[-\omega^1 - x \omega_1^1 + \left(\frac{1}{2}x^2 + kxz + \dots\right) \omega_3^2\right] \\ &\quad + \left(kx + \frac{1}{2}\beta x^2 + \gamma xz + \frac{1}{2}\delta z^2 + \dots\right) \left[-\omega^3 + z \omega_1^1 + \left(\frac{1}{2}x^2 + kxz + \dots\right) \omega_1^2\right] \\ &\quad + xz dk + \frac{1}{6}x^3 d\alpha + \frac{1}{2}x^2 z d\beta + \frac{1}{2}xz^2 d\gamma + \frac{1}{6}z^3 d\delta + \dots \end{aligned}$$

By identification, one will find, successively, that:

$$-x \omega_1^2 - z \omega_3^2 = -(x + kz) \omega^1 - kx \omega^3,$$

$$0 = -x(x + kz) \omega_1^1 - \left(\frac{1}{2}\alpha x^2 + \beta xz + \frac{1}{2}\gamma z^2\right) \omega^1 + kxz \omega_1^1 - \left(\frac{1}{2}\beta x^2 + \gamma xz + \frac{1}{2}\delta z^2\right) + xz dk,$$

so

$$(20) \quad \begin{cases} \omega_1^2 = \omega^1 + k\omega^2, & \omega_3^2 = k\omega^1, & \omega_1^1 = -\frac{1}{2}\alpha\omega^1 - \frac{1}{2}\beta\omega^2, \\ \gamma = \delta = 0, & dk = \beta\omega^1. \end{cases}$$

One sees that the third-order terms in the development of y are not arbitrary. That amounts to saying that the lines of minimal curvature – which are, at the same time, the asymptotic lines – are straight lines. Indeed, one will obtain that upon setting $\omega^1 = 0$. The vector \mathbf{e}_3 is tangent to it, and upon displacing along one of those lines, one will see that $d\mathbf{e}_2$ is again tangent to that line, which proves that the line has a fixed tangent.

The (second-order) differential invariant k is the principal curvature that is unique to the surface, whose total curvature is k^2 . One can deduce all of the other differential invariants of the surface from the differential invariants α and β upon setting:

$$\begin{aligned} d\alpha &= \alpha_1 \omega^1 + \alpha_2 \omega^2, \\ d\beta &= \beta_1 \omega^1 + \beta_2 \omega^2, \\ \dots & \quad \dots \quad \dots \end{aligned}$$

The coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2, \dots$ are the differential invariants that one seeks.

Despite the results that we have obtained by this method, we also see that something essential is missing from it. For example, we do not know whether there are any necessary relations between the differential invariants that were found. If possible, we give the relation that must certainly exist between α, β , and k , etc. arbitrarily. In order to solve all of these problems, one must take into account some compatibility conditions that must be satisfied by the components of an infinitesimal displacement of a moving trihedron that depends upon *several* parameters. We will soon recover those conditions in a form that is completely general, and as we shall see, it contains the entire essence of differential geometry within it.

For the moment, we shall be content to remark that the expressions ω^1 and ω^2 are not exact differentials, in general, and that one cannot attach a system of two natural parameters σ_1 and σ_2 to the surface that are analogous to the natural parameter σ that is introduced whenever we are dealing with a curve.

V

15. – It is now time to begin the general theory of the moving frame by appealing to the fundamental principles of the theory of continuous groups. From F. Klein⁽⁵⁾, any continuous group G in n variables x_1, x_2, \dots, x_n corresponds to a geometry in the space of n dimensions that has the goal of studying the figures that are invariant under the transformations of that group. Thus, elementary geometry corresponds to the group of displacements, affine geometry corresponds to the group of affine transformations, and projective geometry corresponds to the group of projective transformations. The group G is sometimes called the *fundamental group* of the geometry, or the fundamental group of the space in which one studies the properties of figures that are invariant under the group.

In the sequel, we shall assume that the group is *finite* – i.e., that the general transformation depends upon a finite number of parameters. We also suppose, to simplify, that the transformed variables x'_1, x'_2, \dots, x'_n are analytic functions of the

⁽⁵⁾ F. Klein, “Vergleichende Betrachtungen über neuere geometrische Forschungen” (which is known by the name of the Erlanger Programm), 1872.

original variables x_1, x_2, \dots, x_n and the parameters a_1, a_2, \dots, a_r . Finally, we assume that the group is *transitive*, which signifies that there always exists at least one transformation of the group that will bring two arbitrarily-given points in space into coincidence. By extension, we give the name of *displacements* to the transformations of the group and let S_a denote the displacement whose parameters are (a_1, a_2, \dots, a_r) .

16. – The first question that we must pose is the following one: *What should replace the notion of a tri-rectangular trihedron in ordinary geometry in an arbitrary Klein geometry?* Recall that the set of tri-rectangular trihedra enjoys the following property: *Two arbitrary tri-rectangular trihedra can be brought into coincidence by one and only one displacement.* We shall indeed see that this is the property that plays the essential role in the application of the method of moving trihedra.

Having said that, we say that a family of figures constitutes a system of frames *if two arbitrary figures of the family can be brought into coincidence by one and only one map of the group G .*

It is clear that if one can find a particular figure (R_0) such that any transformation that is not identical to G will transform (R_0) into a figure that is distinct from (R_0) then the set of the figure (R_0) and the figures (R_a) that one can deduce from it by the various transformations S_a of G will constitute a system of frames. Indeed, in order to take (R_a) to (R_b) , it will be sufficient to apply, in succession, the transformation S_a^{-1} that takes (R_a) to (R_0) and then the transformation S_b that takes (R_0) to (R_b) . The resulting transformation:

$$S_c = S_b S_a^{-1}$$

will again be a transformation S_c of G , and that transformation S_c will take (R_a) to (R_b) . Conversely, if $S_{c'}$ takes (R_a) to (R_b) then the transformation $S_b^{-1} S_{c'} S_a$ will successively take (R_0) to (R_a) , and then to (R_b) , and then to (R_c) . One then has the identity transformation, which gives:

$$S_{c'} S_a = S_b, \quad S_{c'} = S_b S_a^{-1} = S_c.$$

The search for a system of frames then amounts to the search for a particular figure (R_0) that is not invariant under transformation of G that is distinct from the identity transformation.

17. – There are some cases in which frames present themselves in an entirely natural way. For example, in the general affine geometry of n dimension, it is natural to take a frame to be the figure that is composed of a point and n vectors that issue from that point, but are not situated in the same hyperplane. In n -dimensional projective geometry, the figure that is composed of $n + 2$ points can likewise serve as a frame. However, it is more convenient to take the figure that is composed of $n + 1$ *analytic* points, where an analytic point is the set of $n + 1$ numbers that are not all zero. If one is given $n + 1$

linearly-independent analytic points A_1, A_2, \dots, A_{n+1} then any analytic point can be put into the form:

$$M = x_1 A_1 + x_2 A_2 + \dots + x_{n+1} A_{n+1}$$

in one and only one way, and the coefficients x_1, x_2, \dots, x_{n+1} can be regarded as the homogeneous coordinates of the geometric point that one can associate with M and all analytic points of the form λM . However, one must remark that, as a frame, properly speaking, the ordered set of $n + 1$ analytic points $(A_1, A_2, \dots, A_{n+1})$ must not be regarded as distinct from the set of $n + 1$ analytic points $(mA_1, mA_2, \dots, mA_{n+1})$.

18. – Let us return to an arbitrary group G . If the group is simply-transitive – i.e., if there exists just one transformation of the group that takes two arbitrarily-given points into coincidence – then the points of the space will constitute a set of frames.

Suppose that the group is not simply transitive, which amounts to saying that the order r of the group is greater than the number of variables n . Start with an arbitrary point A_0 . By hypothesis, there exists an infinitude of transformations of G that leave that point fixed; they define a subgroup g_1 with $r_1 = r - n$ parameters. There certainly exist points that are not invariant under g_1 ; let B_0 be one of those points. The transformations of g_1 that leave the point B_0 fixed define a subgroup of order $r_2 < r_1$. If r_1 is positive then there will certainly exist points that are not invariant under g_2 ; let C_0 be one of those points. The transformations of g_2 that leave the point C_0 fixed will define a subgroup g_3 of order $r_3 < r_2$. If r_3 is zero – i.e., if g_3 reduces to the identity transformation – then the figure that is defined by the three points A_0, B_0, C_0 can serve as the initial frame (R_0) . Otherwise, one continues with the same procedure, which will certainly have a conclusion, since the orders of the successive subgroups g_1, g_2, \dots , must constantly decrease.

One can then always find a system of frames, each of which is defined by a finite number of points that are arranged in a certain order.

It is pointless to remark that there exists an infinitude of other possible systems of frames. One can also observe that in Euclidian geometry, the tri-rectangular trihedron can be assimilated to a figure that is composed of four points, namely, the origin of the trihedron and the extremities of three unitary vectors that carried by the axes.

19. – In Euclidian geometry, the tri-rectangular trihedron will serve as a coordinate system. The same thing will be true in the general case. Indeed, associate a particular frame (R_0) , which we shall call the *initial* frame, with the initially-given coordinate system x_1, x_2, \dots, x_n . Let (R_a) be an arbitrary frame, let M be an arbitrary point in space, and let M' be the point that is the transform of M under the displacement S_a^{-1} that takes (R_a) to (R_0) . We agree to say that the initial coordinates of M' are the *coordinates of M relative to the frame (R_a)* . We see that the figure that is defined by a frame (R) and a point M and the figure that is defined by a frame (R') and a point N are *equal* if the coordinates of M , when referred to the frame (R) are equal to the coordinates of N , when referred to the frame (R') .

If we utilize coordinates $\xi_1, \xi_2, \dots, \xi_n$ relative to the frame (R_0) , instead of the initial coordinates x_1, x_2, \dots, x_n , then the equations that define the transformations of the group G analytically will have exactly the same form as before. Indeed, let S_b be an arbitrary transformation of G , and let M_b be the point that is the transform of M by S_b . On the other hand, let M' and M'_b be the points that are the transforms of M and M_b , resp. by S_a^{-1} . The initial coordinates of M' and M'_b are the coordinates x of M and M_b . Now, one passes from M' to M'_b by the successive transformations S_a, S_b, S_a^{-1} . One then passes analytically from the coordinates ξ of M to the coordinates ξ' of a transformed point M_b by the transformation $S_a^{-1} S_b S_a$, which is a transformation of the group. However, it should be remarked that the parameters that figure in the equations of that transformation ($\xi_i \rightarrow \xi'_i$) are not the parameters b_i of S_b , but the parameters of the transformation $S_a^{-1} S_b S_a$, which are called the *transform of S_b by S_a* .

It is then indispensable to distinguish between S_a , when considered to be a *geometric* operation, and S_a , when considered to be an *analytic* operation. The *geometric transformation S_a* is represented analytically by the *analytic transformation S_a* only if one adopts the initial coordinate system that is attached to the frame (R_0) .

20. – We now arrive at the infinitesimal displacement that brings two infinitely-close frames (R_a) and (R_{a+da}) into coincidence. One passes from (R_a) to (R_{a+da}) by the *geometric* transformation $S_{a+da} S_a^{-1}$, which is the resultant of the displacement S_a^{-1} that takes (R_a) to (R_0) and the displacement S_{a+da} that takes (R_0) to (R_{a+da}) . However, if one would like to express that infinitesimal displacement analytically *with the aid of coordinates relative to the frame (R_a)* then one must first displace the figure that is defined by the two frames in such a manner as to take (R_a) to (R_0) . If (R_ε) is the position that (R_{a+da}) must occupy then the infinitesimal displacement in question can be expressed analytically by the *analytic* transformation S_ε . Now, one passes from (R_0) to (R_ε) by the successive displacements S_{a+da} and S_a^{-1} . *As a result, the parameters that relate to the infinitesimal displacement that takes (R_a) to (R_{a+da}) will be the parameters of the infinitesimal analytic transformation $S_\varepsilon = S_a^{-1} S_{a+da}$.*

If we suppose that the identity transformation corresponds to zero values of the parameters then the relative components ε_i of the infinitesimal displacement of the frame will be infinitely-small quantities that will be manifested linearly with respect to the da_i with coefficients that are functions of a_i ; we denote them by the notation $\omega_i(a; da)$:

$$(21) \quad \omega_i(a; da) = \alpha_{i1}(a) da_1 + \alpha_{i2}(a) da_2 + \dots + \alpha_{ir}(a) da_r .$$

When viewed from a certain angle, the preceding result constitutes the *first fundamental theorem* of the theory of Lie groups⁽⁶⁾. In fact, it expresses the idea that since the transformations S_a , which are assumed to depend upon r parameters, define a

⁽⁶⁾ See S. LIE, *Theorie der Transformationsgruppen*, with the collaboration of F. Engel (Leipzig, Berlin, reprinted in 1930).

group, the infinitesimal transformation $S_a^{-1} S_{a+da}$ depend upon only r parameters ω ($a; da$), and not $2r$ of them. That theorem admits a converse, but we shall pass over it here.

21. – It is easy to get the expressions ω if one knows the finite equations:

$$x'_i = f_i(x, a) \quad (i = 1, 2, \dots, n)$$

of the group G . One can first define the infinitesimal transformations of the group by giving infinitely-small values ε_i to the parameters a_i , which will give:

$$x'_i = x_i + \sum_k \left(\frac{\partial f_i}{\partial a_k} \right)_0 \varepsilon_k = x_i + \sum_k \varepsilon_k \xi_{ki}(x),$$

upon neglecting infinitesimals that are second-order with respect to the ε_i . Having said that, in order to obtain the *analytic* transformation $S_a^{-1} S_{a+da}$, we see that it is the resultant of the two transformations:

$$x'_i = f_i(x; a + da)$$

and

$$x'_i = f_i(x''; a),$$

in which the x''_i are the variables that are the transforms of x_i by the desired transformation. Upon setting:

$$x''_i = x_i + \delta x_i$$

and neglecting the second-order infinitesimals, one will find immediately that:

$$(22) \quad \sum_k \frac{\partial f_i}{\partial a_k} da_k = \sum_j \frac{\partial f_i}{\partial x_j} \delta x_j.$$

When these equations are solved for δx_j , the resulting equations must have the form:

$$\delta x_i = \sum_k \omega_k(a; da) \xi_{ki}(x).$$

Take the example of the group of Euclidian displacements. The equations of a displacement, in rectangular coordinates, are:

$$\begin{aligned} x' &= x_0 + \alpha x + \beta y + \gamma z, \\ y' &= y_0 + \alpha' x + \beta' y + \gamma' z, \\ z' &= z_0 + \alpha'' x + \beta'' y + \gamma'' z. \end{aligned}$$

There are six independent infinitesimal transformations here, which one can express by the formulas:

$$\begin{aligned}\delta x &= \varepsilon_1 + \varepsilon_5 z - \varepsilon_6 y, \\ \delta y &= \varepsilon_2 + \varepsilon_6 x - \varepsilon_4 z, \\ \delta z &= \varepsilon_3 + \varepsilon_4 y - \varepsilon_5 x.\end{aligned}$$

Having said that, we must solve the equations (22), which are written:

$$\begin{aligned}\alpha dx + \beta dy + \gamma dz &= dx_0 + x d\alpha + y d\beta + z d\gamma, \\ \alpha' dx + \beta' dy + \gamma' dz &= dy_0 + x d\alpha' + y d\beta' + z d\gamma', \\ \alpha'' dx + \beta'' dy + \gamma'' dz &= dz_0 + x d\alpha'' + y d\beta'' + z d\gamma'',\end{aligned}$$

here. Upon appealing to the relations between the nine direction cosines, they will give:

$$\begin{aligned}\delta x &= (\alpha dx_0 + \alpha' dy_0 + \alpha'' dz_0) + (\alpha d\gamma + \alpha' d\gamma' + \alpha'' d\gamma'') z - (\beta d\alpha + \beta' d\alpha' + \beta'' d\alpha'') y, \\ \delta y &= (\beta dx_0 + \beta' dy_0 + \beta'' dz_0) + (\beta d\alpha + \beta' d\alpha' + \beta'' d\alpha'') x - (\gamma d\beta + \gamma' d\beta' + \gamma'' d\beta'') z, \\ \delta z &= (\gamma dx_0 + \gamma' dy_0 + \gamma'' dz_0) + (\gamma d\beta + \gamma' d\beta' + \gamma'' d\beta'') y - (\alpha d\gamma + \alpha' d\gamma' + \alpha'' d\gamma'') x.\end{aligned}$$

One will indeed find equations of the predicted form with:

$$\begin{aligned}\omega_1 &= \alpha dx_0 + \alpha' dy_0 + \alpha'' dz_0, \\ \omega_2 &= \beta dx_0 + \beta' dy_0 + \beta'' dz_0, \\ \omega_3 &= \gamma dx_0 + \gamma' dy_0 + \gamma'' dz_0, \\ \omega_4 &= \gamma d\beta + \gamma' d\beta' + \gamma'' d\beta'' = -(\beta d\gamma + \beta' d\gamma' + \beta'' d\gamma''), \\ \omega_5 &= \alpha d\gamma + \alpha' d\gamma' + \alpha'' d\gamma'' = -(\gamma d\alpha + \gamma' d\alpha' + \gamma'' d\alpha''), \\ \omega_6 &= \beta d\alpha + \beta' d\alpha' + \beta'' d\alpha'' = -(\alpha d\beta + \alpha' d\beta' + \alpha'' d\beta'').\end{aligned}$$

We make the important remark here that the r relative components ω_i of the infinitesimal displacement of the frame are defined only up to a linear substitution with *constant* coefficients. One easily shows, moreover, that they will be linearly independent if at least r parameters of the group are essential.

22. – We shall now make an essential remark: Suppose that one has performed the same displacement S_c on the two frames (R_a) and (R_{a+da}) . *The relative components of the infinitesimal displacement that brings the two infinitely-close frames into coincidence are not changed.* That is easy to verify analytically, because when S_a is replaced with $S_c S_a$ and S_{a+da} with $S_c S_{a+da}$, the transformation $S_a^{-1} S_{a+da}$ will be replaced with:

$$(S_c S_a)^{-1} (S_c S_{a+da}) = S_a^{-1} S_c^{-1} S_c S_{a+da} = S_a^{-1} S_{a+da};$$

hence, it does not change.

The converse is fundamental. Suppose that one has two continuous families of frames (R_u) and (R_v) that depend upon the same number p of parameters u_1, u_2, \dots, u_p and v_1, v_2, \dots, v_p . Suppose that one can establish a bijective correspondence between the frames of those two families such that the relative components $\omega(u; du)$ of the infinitesimal displacement of a frame of the first family become equal to the analogous components $\omega(v; dv)$ relative to the second family. I then say that there exists a displacement that simultaneously brings all of the frames of the first family into coincidence with the corresponding frames of the second one.

The proof is simple. Indeed, let S_u and S_v denote the displacements that take (R_0) to (R_u) and (R_v) , resp. One has:

$$S_u^{-1} S_{u+du} = S_v^{-1} S_{v+dv}$$

or

$$S_v S_u^{-1} = S_{v+dv} S_{u+du}^{-1}.$$

An easy argument then shows that the transformation $S_v S_u^{-1}$ is a *fixed* transformation S_c of G . One will then have, in turn:

$$S_v = S_c S_u,$$

which proves that the frame (R_v) is deduced from the frame (R_u) by the fixed displacement S_c .

We then have the generalization of the fundamental theorem of the method of the moving trihedron.

VI

23. – We now arrive at the compatibility conditions that must be satisfied by the relative components of the infinitesimal displacement of a moving frame that depends upon several parameters. They are well-known in the theory of surfaces when they are studied with the aid of a moving trihedron that depends upon two parameters.

We first preface the search for those conditions with a remark of a very simple analytical order. Since the forms $\omega_1, \omega_2, \dots, \omega_r$ are linearly independent with respect to da_1, da_2, \dots, da_r , one can conversely express any linear differential form in da_1, da_2, \dots, da_r as a linear form in $\omega_1, \omega_2, \dots, \omega_r$.

Having said that, first consider the set of all frames in space that depend upon the r parameters a_i . Consider two mutually-interchangeable symbols for differentiation d and δ , and the expressions:

$$d\omega_s(a; da) - \delta\omega_s(a; da),$$

which are nothing but the bilinear covariants of the forms ω_s ; as one knows, they are bilinear expressions that are alternating with respect to the series of variables da_i and δa_i :

$$d\omega_s(\delta) - \delta\omega_s(d) = \sum_{i,j} \left(\frac{\partial \alpha_{sj}}{\partial a_i} - \frac{\partial \alpha_{si}}{\partial a_j} \right) da_i da_j.$$

If we now replace the variables da_i with the variables $\omega_i(d)$ and replace the variables $\delta\alpha_i$ with the variables $\omega_i(\delta)$ then we will again obtain a bilinear form that is alternating in the new series of variables, in such a way that one can write:

$$(23) \quad d\omega_s(\delta) - \delta\omega_s(d) = \sum_{i,j} c_{ijs} \omega_i(d) \omega_j(\delta) \quad (c_{ijs} + c_{jis} = 0),$$

the c_{ijs} are functions of a_1, a_2, \dots, a_r that are determined *a priori*.

In regard to these coefficients c_{ijs} , we have the following fundamental theorem, which is nothing but the *second fundamental theorem of the theory of groups*, when viewed from a certain angle:

The coefficients c_{ijs} in relations (23) are absolute constants.

The proof is immediate. Perform the same fixed displacement S_c on the various frames; we have already observed that it will not change the relative components ω_i of the infinitesimal displacement of the frame. The left-hand sides of formulas (23), as well as the quantities $\omega_i(d)$ and $\omega_j(\delta)$ in the right-hand sides, will not change then. That will be possible only if the coefficients c_{ijs} have the same numerical values for the frame (R_a) that they do for the frame (R_a') that is the resultant of (R_a) under the displacement (S_c) . However, since one can always pass from an arbitrary frame (R_a) to an arbitrary frame (R_a') by a convenient displacement, that will demand that the coefficients c_{ijs} must be absolute constants.

These constants bear the name of *structure constants* ⁽⁷⁾ of the group, and equations (23) are the *structure equations* of the group.

One can write them in a less condensed form by taking d to be the symbol of differentiation with respect to one of the parameters a_k and taking δ to be the symbol for differentiation with respect to another parameter a_h . Upon introducing the functions α_{si} in formulas (21), they will then become:

$$(24) \quad \frac{\partial \alpha_{sh}}{\partial a_k} - \frac{\partial \alpha_{sk}}{\partial a_h} = \sum_{i,j} c_{ijs} \alpha_{ik} \alpha_{jh}.$$

If one gives all possible values to the indices s, k, h then one will get the desired compatibility conditions in explicit form.

In the opposite sense, one can write the structure equations (23) in an even more condensed form. Indeed, we remark that upon combining the two terms in the right-hand side that correspond to the same *combination* (i, j) of the two indices i and j , one will get:

⁽⁷⁾ These constants are introduced in a completely different way in Lie's theory. If one lets $\varepsilon_1 X_1 f + \dots + \varepsilon_r X_r f$ denote the general infinitesimal transformation of the group then one will have:

$$(X_i X_j) = \sum_s c_{ijs} X_s f.$$

$$(25) \quad d\omega_s(\delta) - \delta\omega_s(d) = \sum_{i < j} c_{ijs} \begin{vmatrix} \omega_i(d) & \omega_j(d) \\ \omega_i(\delta) & \omega_j(\delta) \end{vmatrix},$$

which can be written in the following symbolic:

$$(25') \quad \omega'_s = \sum_{i < j} c_{ijs} [\omega_i \omega_j].$$

One remarks that the symbol $[\omega_i \omega_j]$ takes the place of a determinant, and that as a result the symbol $[\omega_j \omega_i]$ must be regarded as equal and opposite to $[\omega_i \omega_j]$ ⁽⁸⁾.

24. – We have proved, *a priori*, the existence of the structure equations, with constant coefficients c_{ijs} . It is quite clear that there exists only one system of relations of that nature. In Euclidian geometry, one can obtain them essentially without having to construct the expression ω , as we did in no. **21**. It suffices to start with the equations:

$$dM = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3,$$

$$d\mathbf{e}_1 = \omega_{12} \mathbf{e}_2 - \omega_{31} \mathbf{e}_3,$$

$$d\mathbf{e}_2 = \omega_{23} \mathbf{e}_3 - \omega_{12} \mathbf{e}_1,$$

$$d\mathbf{e}_3 = \omega_{31} \mathbf{e}_1 - \omega_{23} \mathbf{e}_2,$$

in which ω_{23} , ω_{31} , ω_{12} are the scalar products $\mathbf{e}_3 d\mathbf{e}_2$, $\mathbf{e}_3 d\mathbf{e}_1$, $\mathbf{e}_1 d\mathbf{e}_3$, resp., and are, in turn, identical to the components that we have denoted by ω_4 , ω_5 , ω_6 , resp.

If we express the idea that $d\delta M$, $d\delta\mathbf{e}_1$, $d\delta\mathbf{e}_2$, $d\delta\mathbf{e}_3$ are equal to δdM , $\delta d\mathbf{e}_1$, $\delta d\mathbf{e}_2$, $\delta d\mathbf{e}_3$, respectively, then we will obtain the desired structure equations precisely upon equating the coefficients of \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 in both cases. That will give:

⁽⁸⁾ For the exterior calculus and its applications, see: E. CARTAN, “Sur certaines expressions différentielles et problème de Pfaff,” Ann. Ec. Norm. **16** (1899). “Sur l’intégration des systèmes d’équations aux différentielles totales,” Ann. Ec. Norm. **18** (1901). *Leçons sur les invariants intégraux*, Paris, Hermann, 1922. E. GOURSAT, *Leçons sur le problème de Pfaff*, Paris, Hermann, 1922.

$$(26) \quad \left\{ \begin{array}{l} d\omega_1(\delta) - \delta\omega_1(d) = \begin{vmatrix} \omega_3(d) & \omega_{31}(d) \\ \omega_3(\delta) & \omega_{31}(\delta) \end{vmatrix} - \begin{vmatrix} \omega_2(d) & \omega_{12}(d) \\ \omega_2(\delta) & \omega_{12}(\delta) \end{vmatrix}, \\ d\omega_2(\delta) - \delta\omega_2(d) = \begin{vmatrix} \omega_1(d) & \omega_{12}(d) \\ \omega_1(\delta) & \omega_{12}(\delta) \end{vmatrix} - \begin{vmatrix} \omega_3(d) & \omega_{23}(d) \\ \omega_3(\delta) & \omega_{23}(\delta) \end{vmatrix}, \\ d\omega_3(\delta) - \delta\omega_3(d) = \begin{vmatrix} \omega_2(d) & \omega_{23}(d) \\ \omega_2(\delta) & \omega_{23}(\delta) \end{vmatrix} - \begin{vmatrix} \omega_1(d) & \omega_{31}(d) \\ \omega_1(\delta) & \omega_{31}(\delta) \end{vmatrix}, \\ d\omega_{23}(\delta) - \delta\omega_{23}(d) = \begin{vmatrix} \omega_{12}(d) & \omega_{31}(d) \\ \omega_{12}(\delta) & \omega_{31}(\delta) \end{vmatrix}, \\ d\omega_{31}(\delta) - \delta\omega_{31}(d) = \begin{vmatrix} \omega_{23}(d) & \omega_{12}(d) \\ \omega_{23}(\delta) & \omega_{12}(\delta) \end{vmatrix}, \\ d\omega_{12}(\delta) - \delta\omega_{12}(d) = \begin{vmatrix} \omega_{31}(d) & \omega_{23}(d) \\ \omega_{31}(\delta) & \omega_{23}(\delta) \end{vmatrix}. \end{array} \right.$$

One recovers Darboux's classical formulas in the case of two-parameter motions by setting:

$$(27) \quad \left\{ \begin{array}{l} \omega_1 = \xi du + \xi_1 dv, \\ \omega_2 = \eta du + \eta_1 dv, \\ \omega_3 = \zeta du + \zeta_1 dv, \\ \omega_{23} = p du + p_1 dv, \\ \omega_{31} = q du + q_1 dv, \\ \omega_{12} = r du + r_1 dv, \end{array} \right.$$

and upon taking d and δ to be the symbols of the differentiation with respect to u and v , resp. One will then get:

$$(27) \quad \left\{ \begin{array}{l} \frac{\partial \xi_1}{\partial u} - \frac{\partial \xi}{\partial v} = \zeta q_1 - q \zeta_1 - \eta r_1 + r \eta_1, \\ \frac{\partial \eta_1}{\partial u} - \frac{\partial \eta}{\partial v} = \xi r_1 - r \xi_1 - \zeta p_1 + p \eta \zeta_1, \\ \frac{\partial \zeta_1}{\partial u} - \frac{\partial \zeta}{\partial v} = \eta p_1 - p \eta_1 - \xi q_1 + q \xi_1, \\ \frac{\partial p_1}{\partial u} - \frac{\partial p}{\partial v} = r q_1 - q r_1, \\ \frac{\partial q_1}{\partial u} - \frac{\partial q}{\partial v} = p r_1 - r p_1, \\ \frac{\partial r_1}{\partial u} - \frac{\partial r}{\partial v} = q p_1 - p q_1. \end{array} \right.$$

In the theory of groups, equations (24) are known by the name of the *Maurer equations*; Buhl proposed to call them the *Maurer-Cartan equations*. One would be justified in not stopping along that beautiful path and calling them the *Darboux-Maurer-Cartan equations*.

25. – One obtains some immediate generalizations of equations (26) in affine geometry. Upon introducing a frame that is composed of a point M and n vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ that issue from that point, one will have:

$$\begin{aligned} dM &= \sum_i \omega^i \mathbf{e}_i, \\ d\mathbf{e}_i &= \sum_h \omega_i^h \mathbf{e}_h. \end{aligned}$$

The same process that was just employed in Euclidian geometry will give the structure equations of the general affine group here in the form:

$$(29) \quad \left\{ \begin{aligned} d\omega^i(\delta) - \delta\omega^i(d) &= \sum_k \begin{vmatrix} \omega^k(d) & \omega_k^i(d) \\ \omega^k(\delta) & \omega_k^i(\delta) \end{vmatrix}, \\ d\omega_i^j(\delta) - \delta\omega_i^j(d) &= \sum_k \begin{vmatrix} \omega_i^k(d) & \omega_k^j(d) \\ \omega_i^k(\delta) & \omega_k^j(\delta) \end{vmatrix}, \end{aligned} \right.$$

or in an even more condensed form:

$$(30) \quad \left\{ \begin{aligned} (\omega^j)' &= \sum_k [\omega^k \omega_k^j], \\ (\omega_i^j)' &= \sum_k [\omega_i^k \omega_k^j]. \end{aligned} \right.$$

Similarly, the structure equations of the n -dimensional projective group are:

$$(31) \quad (\omega^j)' = \sum_k [\omega_i^k \omega_k^j] \quad (i, j = 1, 2, \dots, n+1),$$

in which the components of the infinitesimal displacement are the quantities ω^j ($i \neq j$) and $\omega^i - \omega_{n+1}^i$; there are $n(n+2)$ of them, which is the number of parameters in the projective group.

26. – Before passing on to the applications, we conclude with an important theorem that explains the significance of the structure equations:

If one is given r differential forms $\omega_i(u; du)$ that are constructed from an arbitrary number p of variables u_1, \dots, u_p and their differentials du_1, \dots, du_p , and if these forms satisfy the structure equations (23) then one can make each system of values u_i correspond to a frame (R_u) in such a way that the relative components of the infinitesimal displacement of that frame are precisely the given forms ω_i ⁽⁹⁾.

We shall prove that theorem, in order to not have to enter into the theory of completely-integrable systems of partial differential equations. We shall remark only that the determination of the frames (R_a) is possible in an infinitude of ways, because if one has one solution to the problem then one will obviously get another one by performing the same arbitrary displacement S_c on all of the frames (R_a) . The general solution to the problem will then depend upon r arbitrary constants.

That theorem contains Darboux's theorem as a special case, namely, that there always exists a two-parameter motion for a trihedron for which the components of the infinitesimal displacement are given quantities that satisfy equations (28). In particular, one sees that the Codazzi equations of the theory of surfaces are only one particular application of the structure equations of the group of Euclidian displacements.

One can demand to know whether one can choose the structure constants c_{ijs} of a group arbitrarily. The response is negative: There exist algebraic relations that those constants must satisfy in order for one to be able to find r linearly-independent forms ω_i that satisfy the structure equations. We shall not enter into an examination of that question, which is the point of the *third fundamental theorem* of the theory of Lie groups.

VII

27. – We are now in a position to present the method of moving frames in full generality. We shall consider a p -dimensional manifold V in an n -dimensional space that is endowed with a fundamental group G , to each point of which we shall try to attach a frame in an intrinsic manner. To simplify, we suppose that $p = 3$.

First of all, we remark that we can associate each frame (R_a) with a point that we shall call the *origin* of the frame according to a well-defined law. If the frame is composed of a finite number of points A, B, C , as in no. **18**, then we can, for example, associate the point A with that frame. If one then considers the family of frames that have a common origin A then it can be identified with the point A , in some sense, and conversely. The frames of that family obviously depend upon $r - n$ parameters, and when one passes from one of those frames to another that is infinitely-close to it, the relative components ω_i of the infinitesimal displacement that it experiences will be linked by n linear relations, since the ω_i depend upon only $r - n$ differentials. *Those relations will have constant coefficients.* Indeed, suppose that when they are solved for n of the components ω_i they take the form:

$$\omega_1 = \lambda_{11} \omega_{r+1} + \dots + \lambda_{1,r-n} \omega_r,$$

.....

⁽⁹⁾ E. CARTAN, *Leçons sur la géométrie des espaces de Riemann*, Paris, Gauthier-Villars, 1929. nos. 46-49.

$$\omega_n = \lambda_{n1} \omega_{n+1} + \dots + \lambda_{n,r-n} \omega_r.$$

If one performs an arbitrary displacement on the two infinitely-close frames in question then the quantities ω_i will not change. As a result, the coefficients λ_{ij} will depend upon neither the point A nor the two frames considered; they will then be absolute constants. Since the ω_i are defined only up to a linear substitution with constant coefficients, *one can then suppose that one has:*

$$(32) \quad \omega_1 = \omega_2 = \dots = \omega_n = 0$$

for any infinitesimal displacement of a frame that leaves its origin fixed.

Equations (32) can be considered to be the *differential equations of the points* of space. They are completely integrable, since their general solution depends upon n arbitrary constants. One can remark that if one lets the Latin letters i, j, \dots denote the indices $1, 2, \dots, n$ and lets the Greek letters denote the indices $n + 1, \dots, r$ then the constants $c_{\alpha\beta i}$ will all be zero. Indeed, the family of frames with a given origin, like any family of frames, will satisfy the structure equations (23). As a result, if d and δ denote two elementary variations within the family then the $\omega_i(d)$ and $\omega_i(\delta)$ will all be zero, and in turn, $d\omega_i(\delta) - \delta\omega_i(d)$, as well. It results from this that one will have:

$$\sum_{\alpha, \beta} c_{\alpha\beta i} \omega_\alpha(d) \omega_\beta(\delta) = 0,$$

and that will be true for any $\omega_\alpha(d)$ and $\omega_\beta(\delta)$; that is what must be proved.

The infinitesimal transformations whose first n parameters are zero are the ones that leave the origin of the frame (R_0) fixed. If one modifies the law that associates a point to each frame then the subgroup that is generated by the last $r - n$ infinitesimal transformations of the group will change in the subgroup that leaves fixed the new point that is associated with (R_0); those two subgroups will be *homologous* in the group G .

28. – We now arrive at the method of moving frames. Attach the family of frames whose origin is M to each point M of a given manifold V . Those frames, which we call *zero-order frames*, depend upon $r - n$ secondary parameters. Since $\omega_1, \dots, \omega_n$ are annulled when the point M remains fixed, when M displaces on V , they will be linear combinations of the differentials of the three *principal* parameters t_1, t_2, t_3 that fix the position of a point on V . One will then have $n - 3$ linear relations between the ω_i , which are relations that we can write in the form:

$$(33) \quad \omega_i = a_{i1} \omega_1 + a_{i2} \omega_2 + a_{i3} \omega_3 \quad (i = 2, \dots, n).$$

The coefficients a_{ij} can depend upon both t_1, t_2, t_3 and the secondary parameters. One disposes of the latter in such a manner as to give fixed numerical values to the greatest possible number of coefficients. The other coefficients will take on values that will be well-defined functions of t_1, t_2, t_3 , and which will constitute *first-order differential*

invariants of the manifold. The frames that are obtained, in particular, as was said, from the secondary parameters constitute the family of *first-order frames*.

Let $\rho_1 \leq r - n$ be the number of parameters that the first-order frames depend upon. If ρ_1 is effectively less than $r - n$ then the relative components $\omega_{n+1}, \dots, \omega_r$ of the infinitesimal displacement of a first-order frame will no longer be linearly independent when one fixes the point M , but must be linked by $r - n - \rho_1$ linearly-independent relations. *The coefficients of those relations are well-defined functions of the first-order differential invariants.*

Indeed, if one utilizes only first-order frames then the coefficients of formulas (33) will all be either constants or functions of the first-order differential invariants. Therefore, they will not depend upon first-order secondary parameters. Then apply the structure equations (23), where the symbol δ refers to an elementary variation of the first-order frame that leaves the point M fixed, while the symbol d refers to an arbitrary variation. Put e_i in place of $\omega_i(\delta)$, to abbreviate, and ω_i in place of $\omega_i(d)$. Since e_1, \dots, e_n are all zero, one will have:

$$\delta\omega = \sum_{\substack{k \leq n \\ \alpha > n}} c_{\alpha\beta i} e_\alpha \omega_k;$$

as a result, upon remarking that $\delta a_{ij} = 0$:

$$\sum_{\substack{k \leq n \\ \alpha > n}} (c_{\alpha k i} - a_{i1} c_{\alpha k 1} - a_{i2} c_{\alpha k 2} - a_{i3} c_{\alpha k 3}) e_\alpha \omega_k = 0,$$

or, upon replacing the ω_k with their values:

$$\begin{aligned} \sum_{\alpha, k} (c_{\alpha k i} - a_{i1} c_{\alpha k 1} - a_{i2} c_{\alpha k 2} - a_{i3} c_{\alpha k 3}) a_{k1} e_\alpha &= 0, \\ \sum_{\alpha, k} (c_{\alpha k i} - a_{i1} c_{\alpha k 1} - a_{i2} c_{\alpha k 2} - a_{i3} c_{\alpha k 3}) a_{k2} e_\alpha &= 0, \\ \sum_{\alpha, k} (c_{\alpha k i} - a_{i1} c_{\alpha k 1} - a_{i2} c_{\alpha k 2} - a_{i3} c_{\alpha k 3}) a_{k3} e_\alpha &= 0. \end{aligned}$$

When one successively sets $i = 4, 5, \dots, n$, these equations will give relations that exist between the e_α – i.e., between the ω_α when one leaves the point M fixed. One will then see that the coefficients depend upon only the a_{ij} – i.e., upon first-order differential invariants of V .

29. – Suppose that upon performing, if needed, a linear substitution with coefficients that are functions of first-order differential invariants, the relations that exist between $\omega_{n+1}, \dots, \omega_r$ when one leaves the point M fixed are:

$$\omega_{n+1} = \dots = \omega_{n_1} = 0 \quad (n_1 - n = r - n - \rho_1).$$

Upon varying the point M on V , one will then have:

$$(34) \quad \omega_{n+1} = a_{n+i,1} \omega_1 + a_{n+i,2} \omega_2 + a_{n+i,3} \omega_3 \quad (i = 1, 2, \dots, n_1 - n),$$

where the coefficients on the right-hand side depend upon principal parameters and first-order second parameters.

One will then have two means of distinguishing certain first-order frames from the other ones and arriving at the family of second-order frames.

1. One seeks to reduce the largest-possible number of the coefficients $a_{n+i,j}$ to fixed numerical values.

2. If there are first-order differential invariants that are mutually-independent – for example, two of them I and J – then one forms the differentials dI and dJ , which one expresses linearly in terms of $\omega_1, \omega_2, \omega_3$:

$$(35) \quad \begin{cases} dI = I_1 \omega_1 + I_2 \omega_2 + I_3 \omega_3, \\ dJ = J_1 \omega_1 + J_2 \omega_2 + J_3 \omega_3. \end{cases}$$

One then establishes relations between the first-order secondary parameters and the principal parameters that permit one to reduce the largest possible number of coefficients that enter into formulas (35) to fixed numerical values. The first-order frames whose parameters satisfy those relations constitute the family of *second-order frames*. As for the coefficients of equations (34) and (35) that have not been reduced to fixed numerical values, they will take on values that are either constant or functions of only the principal parameters t_1, t_2, t_3 ; they will be *second-order differential invariants*.

The components $\omega_{t+1}, \dots, \omega_t$ of the infinitesimal displacement of a second-order frame will be linked by $r - n - \rho_2$ relations when one fixes the point M , where ρ_2 is the number of second-order secondary parameters. One proves, as we just did, that the coefficients of those relations are functions of the differential invariants of the first two orders. One then passes from second-order frames to the third-order frames as one did in order to pass from first-order frames to second-order ones. Nevertheless, one uses relations such as (35) only if there exists a second-order differential invariant that is not a function of the two invariants I and J that were used already. One continues in that way indefinitely.

30. – Suppose that the frames of order p have been obtained, and:

1. The frames of order $p + 1$ coincide with those of order p .
2. The differential invariants of order $p + 1$ are functions of the differential invariants of order less than or equal to p .

One then shows that since M and M' are two points of the manifold for which all of the differential invariants of order less than or equal to p have the same numerical values, and (R) and (R') are, on the other hand, two frames of order p with their origins at M and M' , respectively, the displacement that takes (R) to (R') will leave the manifold V invariant. It is then clear that it is impossible to distinguish one frame of order p from the

other. The manifold admits a group of displacements g whose order is equal to the number of secondary parameters of order p , plus the difference between 3 and the number of independent differential invariants. In particular, if there exist three independent differential invariants of order less than or equal to p then any displacement of the group g will leave all of the points of the manifold fixed ⁽¹⁰⁾.

If the situation that was just envisioned never presents itself then one will arrive at the complete determination of the frame that is attached to each point of the manifold. Indeed, one will stop in the successive specialization of the frames only if, at a given moment, the frames of order $p + 1$ coincide with those of order p , since a differential invariant of order at least $p + 1$ must be independent of the ones that were obtained already. However, since one cannot have more than three independent differential invariants, that situation can present itself only a finite number of times. As a result of that, the number of secondary parameters of frames of increasing order will always decrease until it is annulled.

31. – Suppose that one is dealing with the general case, and that one arrives at a well-defined frame, which we assume to have order q . The differential invariants of order less than or equal to q are the *fundamental invariants*.

In order to recognize whether two manifolds V and V' are equal, one constructs the differential invariants of order $q + 1$. If they are functions of the fundamental differential invariants for the manifold V then it will be necessary and sufficient that the same thing must be true for V' and that the relations that exist between the differential invariants of order $\leq q + 1$ must be the same for both manifolds.

If at least one of the differential invariants of order $q + 1$ is independent of the ones of lower order then one will construct the differential invariants of order $q + 2$. If that produces no invariant that is independent of the preceding ones then it will be necessary and sufficient that the relations that exist between the differential invariants of order $\leq q + 2$ must be the same for the two manifolds.

One then continues until the differential invariants of a certain order $q + h$ are all functions of the invariants of lower order, which will surely happen after a finite number of operations.

We remark that in the examples that were treated previously, the differential invariants will never present themselves once the frame is determined completely.

32. – We give an example of the role that is played by the differential invariants before the definitive specialization of the frame.

Take G to be the three-parameter group of translations and homotheties in the plane. Take the frame to be the figure that is composed of a point M and two rectangular vectors \mathbf{e}_1 and \mathbf{e}_2 with the same length and fixed directions. For an infinitesimal displacement of the frame, one will have:

⁽¹⁰⁾ A very general example of a manifold in which it is impossible to determine a frame intrinsically is provided in projective geometry by the ruled surfaces that admit two rectilinear directrices. If one regards those surfaces as loci of lines then there will obviously exist a group of ∞^1 homographic transformations that leave the two directrices fixed, *as well as each line that they meet*.

$$\begin{aligned}dM &= \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2, \\d\mathbf{e}_1 &= \omega \mathbf{e}_1, \\d\mathbf{e}_2 &= \omega \mathbf{e}_2.\end{aligned}$$

Let a curve be $y = f(x)$ then. Let:

$$\begin{aligned}\lambda, & 0, \\0, & \lambda\end{aligned}$$

be the projections of \mathbf{e}_1 and \mathbf{e}_2 onto the fixed axes. The zero-order frames are those with their origin at $M(x, y)$; they depend upon one secondary parameter λ . One will have, moreover:

$$\omega_1 = \frac{dx}{\lambda}, \quad \omega_2 = \frac{y'dx}{\lambda}, \quad \omega = \frac{d\lambda}{\lambda}.$$

Since the ratio $\omega_2 / \omega_1 = y'$ is independent of λ , the frames of order 1 will be the same as those of order zero, and one will have one first-order differential invariant, namely, y' .

In order to get second-order frames, one forms:

$$dy' = \lambda y'' \cdot \frac{dx}{\lambda} = \lambda y'' \omega_1.$$

Since the coefficient $\lambda y''$ depends upon λ , one can arrange λ in such a manner as to make it equal to 1. One will then have a well-defined second-order frame.

The expression $\omega = d\lambda / \lambda$ will then give:

$$\omega = -\frac{y'''}{y''} dx = -\frac{y'''}{y''^2} \omega_1.$$

As a result, the curve will be determined, up to a transformation of the group G , by the relation that exists between the third-order differential invariant y''' / y''^2 and the differential invariant y' .

We have assumed that the path makes $y'' \neq 0$. If we have $y'' = 0$ then we cannot distinguish between one first-order frame and another; the curve will be a straight line and will admit a two-parameter subgroup of G .

VIII

33. – The general method of moving frames that was presented in the preceding numbers supposes that the manifold V is essentially given, and the successive calculations were performed completely. However, as we have pointed out already, that is absurd in theoretical research, where one only wishes to arrive at the form of the Frenet formulas, while anticipating the various cases that can present themselves. From that

standpoint, the structure equations will play a fundamental role, *even in the theory of curves*.

We return to the problem that was treated in no. **28** and following. One attaches the frames whose origin is M to each point M of the three-dimensional manifold. For any infinitesimal displacement of such a frame, one will have the relations (33):

$$(33) \quad \omega_i = a_{i1} \omega_1 + a_{i2} \omega_2 + a_{i3} \omega_3 \quad (i = 4, \dots, n).$$

One must then envision the manner by which the coefficients a_{ij} depend upon the secondary parameters and what numerical values one can reduce them to. In order to do that, imagine an elementary variation of the zero-order frame, which is represented by the symbol δ , *while the point M remains fixed*. From the structure equations, and upon denoting the elementary variation that the coefficient a_{ij} is subjected to by δa_{ij} , one will have:

$$\delta a_{i1} \omega_1 + \delta a_{i2} \omega_2 + \delta a_{i3} \omega_3 = \delta \omega_i - a_{i1} \delta \omega_1 + a_{i2} \delta \omega_2 + a_{i3} \delta \omega_3,$$

so:

$$\begin{aligned} \delta a_{i1} &= \sum_{\alpha=n+1}^r e_\alpha \sum_{k=4}^n (c_{\alpha k i} - a_{i1} c_{\alpha k 1} - a_{i2} c_{\alpha k 2} - a_{i3} c_{\alpha k 3}) a_{k1}, \\ \delta a_{i2} &= \sum_{\alpha=n+1}^r e_\alpha \sum_{k=4}^n (c_{\alpha k i} - a_{i1} c_{\alpha k 1} - a_{i2} c_{\alpha k 2} - a_{i3} c_{\alpha k 3}) a_{k2}, \\ \delta a_{i3} &= \sum_{\alpha=n+1}^r e_\alpha \sum_{k=4}^n (c_{\alpha k i} - a_{i1} c_{\alpha k 1} - a_{i2} c_{\alpha k 2} - a_{i3} c_{\alpha k 3}) a_{k3}. \end{aligned}$$

The right-hand sides are then the elementary variations that are suffered by the coefficients a_{i1} , a_{i2} , a_{i3} under the action of a group, for which, one knows the infinitesimal transformations; furthermore, that group will be a homographic group. If one knows how to turn the infinitesimal transformations into finite transformations, *which is simple in the present applications*, then one will come down to the search for the numerical values to which one can reduce the a_{ij} by a transformation of that group⁽¹⁾. For example, one knows that for the orthogonal group of three variables, one can reduce two of the variables to 0, while the third one will have a well-defined value, unless one is in the complex domain, in which case, it can happen that one can reduce the three variables to the values 1, i , 0, as long as the three variables are not all zero, moreover.

Thanks to the structure equations, it will then be theoretically possible to predict the various irreducible cases among them that can present themselves and to deduce the nature of the first-order frames in each case. The same method will serve for the passage from frames of arbitrary order to the frames of immediately-higher order.

34. – We now apply what we just said to the study of plane curves in unimodular affine geometry. With the notations of no. **12**, and denoting the components of the

⁽¹⁾ It is, moreover, theoretical pointless to deal with the finite transformations of the group. S. Lie has found a method that permits one to find a representative point of each family of mutually-homologous points when one has been given only the infinitesimal transformations.

infinitesimal displacement of the frame by $\omega^1, \omega^2, \omega^3, \omega_1^1, \omega_1^2, \omega_2^1$, one will have the structure equations:

$$(36) \quad \left\{ \begin{array}{l} d\omega^1(\delta) - \delta\omega^1(d) = \begin{vmatrix} \omega^1(d) & \omega_1^1(d) \\ \omega^1(\delta) & \omega_1^1(\delta) \end{vmatrix} + \begin{vmatrix} \omega^2(d) & \omega_2^1(d) \\ \omega^2(\delta) & \omega_2^1(\delta) \end{vmatrix}, \\ d\omega^2(\delta) - \delta\omega^2(d) = \begin{vmatrix} \omega^1(d) & \omega_1^2(d) \\ \omega^1(\delta) & \omega_1^2(\delta) \end{vmatrix} - \begin{vmatrix} \omega^2(d) & \omega_1^1(d) \\ \omega^2(\delta) & \omega_1^1(\delta) \end{vmatrix}, \\ d\omega_1^1(\delta) - \delta\omega_1^1(d) = \begin{vmatrix} \omega_1^2(d) & \omega_2^2(d) \\ \omega_1^2(\delta) & \omega_1^2(\delta) \end{vmatrix}, \\ d\omega_1^2(\delta) - \delta\omega_1^2(d) = 2 \begin{vmatrix} \omega_1^1(d) & \omega_1^2(d) \\ \omega_1^1(\delta) & \omega_1^2(\delta) \end{vmatrix}, \\ d\omega_2^1(\delta) - \delta\omega_2^1(d) = 2 \begin{vmatrix} \omega_2^1(d) & \omega_1^1(d) \\ \omega_2^1(\delta) & \omega_1^1(\delta) \end{vmatrix}. \end{array} \right.$$

One can immediately start with first-order frames that satisfy $\omega^2 = 0$. Upon remarking that $\omega^1(d) = \omega^2(d) = 0$, one will then have:

$$\delta\omega^2 = -e_1^2 \omega^1,$$

so:

$$e_1^2 = 0;$$

as a result, upon varying the point of the curve, one will get a relation of the form:

$$\omega_1^2 = \alpha \omega^1.$$

On the other hand, for a variation of the first-order frame that fixes the point M ($e^1 = e^2 = e_1^2 = 0$), one will have:

$$\delta\omega_1^2 = 2 e_1^1 \omega_1^2, \quad \delta\omega^1 = -e_1^1 \omega^1,$$

so:

$$\delta\alpha = 3e_1^1 \alpha.$$

That relation proves that under an infinitesimal variation of the first-order frame, the coefficient α will be multiplied by the constant $1 + 3e_1^1$, which is infinitely-close to 1; as a result, for a finite variation, it will be multiplied by an arbitrary constant. Two cases are then possible:

1. Either $\alpha = 0$ (viz., the case of a straight line), and one can then no longer distinguish one first-order frame from the others.

2. Or $\alpha \neq 0$, and one can then choose a partial family amongst the first-order frames such that α becomes equal to 1; those are the second-order frames.

When the coefficient α reduces to 1, $\delta\alpha$ will then be zero for a variation of a second-order frame, and one will have $e_1^1 = 0$, which implies that:

$$\omega_1^1 = \beta \omega^1.$$

Now, vary the second-order frame, while the point M remains fixed (so $e^1 = e^2 = e_1^2 = e_1^1 = 0$). One will have:

$$\delta\omega_1^1 = -e_2^1 \omega_1^2 = -e_2^1 \omega^1, \quad \delta\omega^1 = 0,$$

so

$$\delta\beta = -e_2^1.$$

As a result, the coefficient β will be increased by the infinitely-small quantity $-e_2^1$; it will then be increased by an arbitrary finite quantity under a finite variation of the frame. One can then arrange that it should be zero. The frame will then be determined perfectly (third-order frame), and one will have:

$$\omega_2^1 = k\omega^1,$$

in which k is the first differential invariant (affine curvature), which has order four.

One can recover the results that were obtained before by two other procedures (nos. **12** and **13**).

35. – The preceding considerations, when bolstered by some examples that we have used to illustrate them with, show that *the structure equations of the group G contain everything that one can know about the differential geometry of a space that is endowed with the fundamental group G* , with the single condition that one must know the linear relations with constant coefficients between the ω_i that define the points of space. The classification of curves, surfaces, and all sorts of point manifolds is achieved by starting with the structure equations without the logical necessity of any geometric intuition.

One knows that in projective geometry, the study of *ruled* spaces – i.e., ones that are considered to be generated by lines – is developed in parallel with the study of point spaces. Naturally, in all geometry in the Klein sense, one can take any set of figures that enjoy the following two characteristic properties to be the generating elements of space:

1. The figures considered transform amongst themselves transitively under the fundamental group G .

2. There exists no transformation of the group G that leaves all of the figures of the set fixed.

The method of moving frames can be applied without modification when one replaces points with other generating elements. For example, take three-dimensional Euclidian

geometry. If we take the straight line to be the generating element then it will suffice to associate each frame with a well-defined line that plays the role of the *origin line* for that frame; for example, it might be the z -axis of the moving tri-rectangular trihedron. When the frame varies in such a manner that its origin line remains fixed, we know that there are four linear relations with constant coefficients between the components ω_1 , ω_2 , ω_3 , ω_{23} , ω_{31} , ω_{12} of the infinitesimal displacement of the frame. Upon remarking that the point M is displaced along the z -axis while that axis remain fixed, one will find that, in fact:

$$\omega_1 = \omega_2 = \omega_{23} = \omega_{31} = 0.$$

These are the four differential equations of the lines in space, or rather, the equations that define the families of frames that have the same origin line. The components ω_1 , ω_2 , ω_{23} , ω_{31} are the ones that the same role in the application of the method of moving frames as the role that was played by the n components $\omega_1, \omega_2, \dots, \omega_n$ in the presentation in para. VII.

In a general manner, any choice of generating element will correspond to a *completely integrable system* of total differential equations that establish linear relations in constant coefficients between $\omega_1, \omega_2, \dots, \omega_n$. The converse is easy to prove⁽¹²⁾.

The preceding, even more than before, the role that played by the structure equations of the group G .

IX

36. – One can contemplate the structure equations from yet another viewpoint. Imagine an arbitrary system of curvilinear coordinates u_1, u_2, u_3 in ordinary space. Suppose that a well-defined tri-rectangular trihedron is attached to each point in space. If we know the six relative components $\omega_1, \omega_2, \omega_3, \omega_{23}, \omega_{31}, \omega_{12}$ of the infinitesimal displacement of that trihedron as linear functions of the du_1, du_2, du_3 then we can construct all of Euclidian space by a differential route, with the necessary and sufficient condition that the six given forms satisfy the Darboux structure equations and that that the three forms $\omega_1, \omega_2, \omega_3$ must be linearly independent. If we are given the point of space that corresponds to the coordinates (u_1^0, u_2^0, u_3^0) and a tri-rectangular trihedron at that point then we can determine the point in space that corresponds to arbitrary coordinates and the tri-rectangular trihedron that one agrees to attach to it. From another viewpoint, one can say that *being given six differential forms $\omega_1, \omega_2, \omega_3, \omega_{23}, \omega_{31}, \omega_{12}$ in a three-dimensional continuum that satisfy the structure equations of the Euclidian group will permit one to organize the continuum in a Euclidian way* (and even in an infinitude of ways), and to construct a Euclidian space, in some sense, at each point to which one attaches a well-defined tri-rectangular trihedron.

If one given the six forms under consideration then can, if one so desires, also associate each pair of infinitely-close points of the continuum with an infinitesimal

⁽¹²⁾ From another viewpoint, any choice of generating element corresponds to a choice of well-defined subgroup of G that leaves the generating element fixed, as an origin. Conversely, any subgroup will correspond to a family of generators.

Euclidian displacement whose parameters are $\omega_1, \omega_2, \omega_3, \omega_{23}, \omega_{31}, \omega_{12}$ precisely. In that sense, *a continuum that has been organized in a Euclidian way will support infinitesimal Euclidian displacements*. However, one must remark that these displacements that are associated with the various pairs of infinitely-close points are not arbitrary, since their components must satisfy the structure equations.

Being given six components ω_i, ω_j corresponds to choosing the trihedra that are associated with the various points of space; it will then be arbitrary up to a point. Since the infinitesimal displacement that is associated with two infinitely-close points has the form $S_u^{-1} S_{u+du}$, if S_u and S_{u+du} denote the displacements that make the origin frame coincide with the frames that are attached to those two points, one can replace S_u with any other displacement that takes the origin to the point considered. Now, these displacements have the form $S_u R$, if R denotes an arbitrary rotation around the origin. It then results that if one associates each point (u_1, u_2, u_3) of the continuum with a rotation R_u around the origin by some law then the infinitesimal displacement $S_u^{-1} S_{u+du}$ will be replaced by the infinitesimal displacement $R_u^{-1} (S_u^{-1} S_{u+du}) R_{u+du}$.

The preceding result can be stated in the following manner: *If one replaces the infinitely-small displacement $T_{u;du}$ that is associated with two infinitely-close points with the displacement:*

$$T'_{u;du} = R_u^{-1} T_{u;du} R_{u+du}$$

then one will get the same Euclidian organization of the continuum. The change of givens corresponds to a simple change of trihedra that are attached to the various points of space.

More generally, one can take an *arbitrary* rotation that depends upon three parameters v_1, v_2, v_3 , instead of R_u , and one will have then endowed the continuum with a complete system of frames that depend upon six parameters. One will know the linearly-independent components ω_i, ω_j of the infinitesimal displacement of that frame, and one can, in turn, apply the method of moving frames with the given curvilinear coordinates.

37. – We clarify the preceding by indicating explicitly how things appear in the plane. We then have a two-dimensional continuum that is defined by means of the two coordinates u and v , and we give a system of three forms:

$$\omega_1 = \xi du + \xi_1 dv, \quad \omega_2 = \eta du + \eta_1 dv, \quad \omega_{12} = r du + r_1 dv$$

that satisfy the structure equations:

$$(37) \quad \left\{ \begin{array}{l} \frac{\partial \xi}{\partial v} - \frac{\partial \xi_1}{\partial u} = \eta r_i - r \eta_1, \\ \frac{\partial \eta}{\partial v} - \frac{\partial \eta_1}{\partial u} = -\xi r_i + r \xi_1, \\ \frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} = 0. \end{array} \right.$$

The quantities ω_1 , ω_2 , ω_{12} are: The first two are the components of a translation that is parallel to the (moving) axes, and the last one is the component of a rotation around the (moving) origin.

If we change the orientation of the axes – for example, we rotate it through an angle θ – then we will obtain the most general values:

$$(38) \quad \bar{\omega}_1 = \omega_1 \cos \theta + \omega_2 \sin \theta, \quad \bar{\omega}_2 = -\omega_1 \sin \theta + \omega_2 \cos \theta, \quad \bar{\omega}_{12} = \omega_{12} + d\theta$$

that permit one to apply the method of moving frames. For example, the straight lines are characterized by the property that one can attach them to a frame whose first axis is fixed, which gives:

$$\bar{\omega}_2 = 0, \quad \bar{\omega}_{12} = 0;$$

one then deduces that:

$$\tan \theta = \frac{\omega_2}{\omega_1}, \quad \text{so} \quad \omega_{12} + \frac{\omega_1 d\omega_2 - \omega_2 d\omega_1}{\omega_1^2 + \omega_2^2} = 0.$$

One then obtains the differential equation of straight lines by replacing ω_1 , ω_2 , ω_3 with their given values.

A circumference of radius a will likewise be characterized by the equations:

$$\bar{\omega}_2 = 0, \quad \bar{\omega}_{12} = \frac{1}{a} \bar{\omega}_1,$$

etc.

Naturally, all of this applies to no particular Klein space that is defined in arbitrary curvilinear coordinates.

38. – In order to prepare for the introduction of generalized spaces, it remains for us to show what the profound geometric significance of the structure equations is.

Take a Klein space and attach an infinitesimal displacement $T_u ; du$ that satisfies the structure equations to each pair of infinitely-close points (u_i) and $(u_i + du_i)$; that amounts to attaching a frame (R_u) to each point; if that frame is deduced from the frame at the origin by the displacement S_u then one will have:

$$T_u ; du = S_u^{-1} S_{u+du} .$$

Imagine a closed circuit in space – or *cycle* – that we divided into a large number of very small arcs that are subdivided by points M_0, M_1, \dots, M_{n-1} . Let (R_i) denote the frame that is attached to the point M_i , and let S_i be the displacement that takes the initial frame (R_0) to (R_i) . The displacement that takes (R_0) to (R_1) is $S_0^{-1}S_1$, and the one that takes (R_0) to (R_2) is:

$$S_0^{-1}S_2 = (S_0^{-1}S_1)(S_1^{-1}S_2),$$

and so on. The displacement that takes (R_0) to (R_p) will then be:

$$(S_0^{-1}S_p) = (S_0^{-1}S_1)(S_1^{-1}S_2) \dots (S_{p-1}^{-1}S_p);$$

all of these displacements are referred to (R_0) . Finally, when one has described the cycle, one will return to the frame (R_0) by a displacement (which is necessarily zero) that will be the product of the infinitely-small displacements that are attached to the arcs into which the cycle has been composed.

If one then lets $T_{1,2}$ denote the infinitely-small displacement, referred to (R_i) , that takes (R_i) to (R_{i+1}) then one will have:

$$(39) \quad T_{0,1} T_{1,2} \dots T_{n-2,n-1} T_{n-1,0} = 1.$$

39. – *The preceding relation, which is assumed to be true for all cycles in an n -dimensional continuum, will be sufficient for the components of the infinitesimal displacement $T_{u,du}$ that is associated with infinitely-close points of the continuum to satisfy the structure equations.* Indeed, make a particular point (u_i^0) of the continuum correspond to a well-defined point A_0 of the Klein space and a well-defined frame (R_0) at the origin A_0 . Now, let (u_i) be an arbitrary point of the continuum. Join (u_i^0) to (u_i) by a continuous path, and divide that path into a large number of partial arcs, and attach an infinitesimal displacement $T_{u,du}$ to each of these arcs. Then, construct the successive frames that start with (R_0) and are deduced from each other by the corresponding infinitesimal displacement that was given, where that displacement is always assumed to be defined analytically with respect to the frame that one displaces. One will then arrive at a final frame (R_u) , which will be well-defined in the limit when one increases the number of partial arcs indefinitely, while each of them tends to zero. The frame (R_u) that is then attached to the point (u_i) of the continuum will not depend upon the path that is followed in the continuum in order to go from the point (u_i^0) to the point (u_i) , and that is precisely because of the hypothesis that was made about the cycles of the continuum, and that translates into the relation (39). One easily shows, in turn, that the displacement that takes (R_u) to (R_{u+du}) is precisely the given displacement $T_{u,du}$ when it is referred to (R_u) . That will obviously suffice for the components of $T_{u,du}$ to satisfy the structure equations.

40. – From the preceding, it would seem that there is an absolute equivalence between the structure equations and the relations (39). In reality, the structure equations are nothing but the relations (39) *when they are applied to an arbitrary infinitely-small cycle.*

A closer examination will show that if one is given the displacement $T_{u,du}$ arbitrarily, with components ω_i , and if one considers an elementary parallelogram whose summits are:

$$(u_i^0) = (u_i), \quad (u_i^1) = (u_i + du_i), \quad (u_i^2) = (u_i + du_i + \delta u_i + d\delta u_i), \quad (u_i^3) = (u_i + \delta u_i),$$

where d and δ are two mutually-interchangeable differentiation symbols, then the components of the displacement:

$$T_{0,1} T_{1,2} T_{2,3} T_{3,0}$$

will be precisely the quantities:

$$\Omega_i = d\omega_i(\delta) - \delta\omega_i(d) - \sum_{j,h} c_{jhi} \omega_j(d) \omega_h(\delta).$$

It is almost obvious that the relations (39), which are assumed to be true for infinitely-small cycles, will still be true for the finite cycles, but on the condition that those cycles are reducible to a point by a continuous deformations. *Therefore, the structure equations do not necessarily imply the relations (39) for the cycles that do not satisfy that condition.*

X

41. – We are now in a position to understand how one can generalize the notion of Klein space.

The first generalization of this type goes back to Gauss, who naturally could not assert the same viewpoint that we do. Recall the Darboux equations (37) that relate to the plane. Knowing the components $\xi, \xi_1, \eta, \eta_1, r, r_1$ of a displacement in the plane as functions of the two parameters u and v will suffice to recover the Euclidian organization of the plane. However, we can remark that knowing ξ, ξ_1, η, η_1 is sufficient, because the first two equations (37) will permit one to deduce the values of r and r_1 . On the other hand, instead of being given the two forms ω_1 and ω_2 that define ξ, ξ_1, η, η_1 , we can, as we saw in no. 37, just as well give the two forms:

$$\bar{\omega}_1 = \omega_1 \cos \theta + \omega_2 \sin \theta, \quad \bar{\omega}_2 = -\omega_1 \sin \theta + \omega_2 \cos \theta.$$

That amounts to saying that merely knowing the quadratic form $\omega_1^2 + \omega_2^2$ will suffice to recover the Euclidian organization of the plane. That quadratic form is nothing but the ds^2 of the plane, which is the square of the distance between two infinitely-close points. That is a well-known result whose deeper reason is that one can base Euclidian geometry upon only the notion of distance.

ds^2 cannot be given arbitrarily if one desires that the structure equations should be satisfied. Meanwhile, suppose that we are given an arbitrary ds^2 ; i.e., we are given ξ, ξ_1, η, η_1 as arbitrary functions of u and v . We can further infer r and r_1 from the first two structure equations, but the third one will not be verified. As we know, the two-dimensional continuum that is endowed with the given ds^2 can be assimilated to a surface and will enjoy all of the geometric properties of surfaces that are attached to their ds^2 . On such a surface, *the theory of curves will be identically the same as in the plane, and*

nothing will be change in the application of the method of the moving frame. Upon introducing the most general frame with the components $\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3$ for the infinitesimal displacement, one attaches a moving frame to a curve with the condition that $\bar{\omega}_2 = 0$, which will give us the curvature (*geodesic curvature*, in the sense of Gauss):

$$\frac{1}{\rho} = \frac{\bar{\omega}_{12}}{\bar{\omega}_1}.$$

The difference between the surface and the plane is distinguished by the fact that the third structure equation (37) will no longer be verified; one will have:

$$\frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} = K (\xi \eta_1 - \eta \xi_1),$$

or, with our notations:

$$d\omega_{12}(\delta) - \delta\omega_{12}(d) = -K \begin{vmatrix} \omega_1(d) & \omega_2(d) \\ \omega_1(\delta) & \omega_2(\delta) \end{vmatrix};$$

the coefficient K is the *total curvature*. One can interpret this by imagining an infinitely-small cycle and attaching a frame to the various points of the cycle and displacing it step-by-step without rotation. Upon returning to the starting point, it will take a different position from the initial position, and which will be deduced from it by the rotation $K d\sigma$, where $d\sigma$ denote the area that is bounded by the cycle.

One can imagine some other generalizations of the Euclidian plane by associating any pair of infinitely-close points of a two-dimensional continuum with an infinitesimal Euclidian displacement $T_{u,du}$ for which the first two structure equations are not verified. For example, one can give the functions ξ, ξ_1, η, η_1 arbitrarily, while taking $r = r_1 = 0$. The Euclidian frames that are attached to the different points of the continuum are then deduced from each other by a simple translation. One will have a *space with Euclidian connection that is endowed with absolute parallelism*. In that space, the theory of curves will again be the same as it is in the Euclidian plane. Upon introducing the most general frames, one will have:

$$\bar{\omega}_1 = \omega_1 \cos \theta + \omega_2 \sin \theta, \quad \bar{\omega}_2 = -\omega_1 \sin \theta + \omega_2 \cos \theta, \quad \bar{\omega}_{12} = d\theta,$$

and the *straight lines* will be defined by:

$$\bar{\omega}_2 = \bar{\omega}_{12} = 0$$

or

$$\eta + \eta_1 \frac{dv}{du} = C \left(\xi + \xi_1 \frac{dv}{du} \right),$$

in which C is an arbitrary constant that defines the direction of the straight line. One can define a *Euclidian connection* with absolute parallelism on a surface that is given by its

ds^2 by decomposing its ds^2 into a sum of two squares $\omega_1^2 + \omega_2^2$ and taking $\omega_{12} = 0$. For example, for a sphere that is referred to its longitude φ and its colatitude θ , one can take:

$$\omega_1 = d\theta, \quad \omega_2 = \sin \theta d\varphi, \quad \omega_{12} = 0.$$

The *straight lines* will be the loxodromes that admit the two poles of the sphere for their poles and are defined by the equation:

$$\sin \theta \frac{d\varphi}{d\theta} = C.$$

In this generalization, which is distinct from that of Gauss, the displacement $T_{01} T_{12} \dots$ that is associated with a cycle is not a simple rotation, but a translation whose components are:

$$\Omega_1 = d\omega_1(\delta) - \delta\omega_1(d) + \begin{vmatrix} \omega_2(d) & \omega_{12}(d) \\ \omega_2(\delta) & \omega_{12}(\delta) \end{vmatrix},$$

$$\Omega_2 = d\omega_2(\delta) - \delta\omega_2(d) - \begin{vmatrix} \omega_1(d) & \omega_{12}(d) \\ \omega_1(\delta) & \omega_{12}(\delta) \end{vmatrix};$$

this is the *torsion* of the space, as opposed to the Gaussian *curvature* ⁽¹³⁾.

Finally, the most general two-dimensional space with Euclidian connection will be obtained by taking $\xi, \xi_1, \eta, \eta_1, r, r_1$ to be absolutely arbitrary functions of u, v ; one will then have its *curvature* and *torsion*. However, once again, the theory of curves and the application of the method of moving frames will be identical to what it is in the plane.

42. – If one passes from the Euclidian plane to Euclidian space with an arbitrary number of dimensions then one will recover classical Riemannian geometry, in particular. The structure equations of Euclidian space can be divided into two classes:

1. The equations that are written in condensed form as:

$$(40) \quad \omega'_i = \sum_h [\omega_h \omega_{hi}].$$

2. The equations:

⁽¹³⁾ With Darboux's notation, torsion is defined analytically by the two coefficients a and b of the equations:

$$\frac{\partial \xi}{\partial v} - \frac{\partial \xi_1}{\partial u} = a (\xi \eta_1 - \eta \xi_1),$$

$$\frac{\partial \eta}{\partial v} - \frac{\partial \eta_1}{\partial u} = b (\xi \eta_1 - \eta \xi_1).$$

$$(41) \quad \omega'_{ij} = \sum_h [\omega_{ih} \omega_{hj}].$$

If one is then given the n forms ω_i (viz., the components of the translation), which are assumed to be linearly-independent in du_1, \dots, du_n , on an n -dimensional continuum then equations (40) will permit one to unambiguously deduce the forms ω_j (viz., the components of the rotation of the moving frame). On the other hand, the ω_i will be subjected to an arbitrary orthogonal transformation under a change of frame that is attached to a point, in such a way that being given:

$$ds^2 = \omega_1^2 + \omega_2^2 + \dots + \omega_n^2$$

for the space will be sufficient for it to be organized in a Euclidian way, but with the condition that the structure equations (41) must be satisfied, of course.

If one is given the ds^2 arbitrarily then one can obtain a Riemann space. One can further arrange that it should satisfy the structure equations (40), which will permit one to define the infinitesimal Euclidian displacement that will bring two infinitely-close rectangular frames into coincidence according to a law that is intrinsically coupled to the given ds^2 , or what amounts to the same thing, to define the angle between two directions that emanate from two infinitely-close points in space. One will then arrive at Levi-Civita's ⁽¹⁴⁾ notion of *parallelism* step-by-step. The structure equations (41) are no longer verified, in general; one must add complementary terms to the right-hand side that define the *Riemannian curvature* of the space.

In Riemannian space, the application of the method of moving frames to the theory of curves is identically the same as it is in Euclidian space; i.e., the classification of curves, and the notions of curvature and torsion are the same. In other words, *all of the operations of Euclidian geometry that refer to the study of curves preserve the same significance in Riemannian geometry*. The application of the method of moving frames to the theory of surfaces also comes about in the same manner as in Euclidian geometry, but the results are not the same, in the sense that in Euclidian geometry, as in Riemannian geometry, one must take into account the values of the bilinear covariants ω'_i and ω'_{ij} , and those expressions will not be the same when the number of variables exceeds unity. That is why the notions of straight line, circumference, helix, etc, generalize automatically upon passing from Euclidian geometry to Riemannian geometry, but the notion of *plane* does not generally exist in Riemannian geometry, if one wishes to at least define the plane by the same *differential* properties as in Euclidian geometry. Indeed, in a three-dimensional space, those differential properties translate into the equations:

$$\omega_3 = 0, \quad \omega_{13} = 0, \quad \omega_{23} = 0,$$

and those equations will no longer be completely integrable if the structure equations cease to be verified.

⁽¹⁴⁾ T. LEVI-CIVITA, "Nozioni di parallelismo in una varietà qualunque," Rendiconti Circ. mat. Palermo **42** (1917).

Naturally, one can imagine spaces *with Euclidian connection* that are more general than Riemannian spaces, for which the first structure equations (40) will cease to be verified. *The Riemannian spaces with absolute parallelism* that were recovered by Einstein are the ones for which the last structure equations (41) are verified, but not the first ones (40). They have no curvature, but they do have torsion.

43. – One now easily sees how each group G with n variables can be associated with an infinitude of generalized spaces that admit G for their fundamental group. Those spaces can be regarded as n -dimensional continua in which one has attached a well-defined infinitesimal transformation $T_{u,du}$ of the group G to each pair of infinitely-close points (u_i) and $(u_i + du_i)$ in such a manner that the structure equations cease to be verified⁽¹⁵⁾. In these spaces, the operations of differential geometry of the Klein space with group G continue to keep their significance. The theory of curves in them is the same as in Klein space; the method of moving frames also applies in the same manner. However, the classification of surfaces is not the same, nor are some of their properties.

Any cycle of the generalized space for which one is given the initial point A_0 is associated with a transformation of the group G . One can represent it in the following manner: Imagine a sequence of observers that are spaced out along the cycle, all of which belong to a Klein space, and each of which adopts a well-defined frame. The observer that is placed at A_0 can try to represent the sequence of positions of the frames of his colleagues in the Klein space to which he believes that he belongs, provided that each of them transmits the position of the infinitely-close frame with respect to their own frame. When the observer that is placed at A arrives at the end of the cycle, he will confirm that he must attribute a position to his own frame that is different from the one that he really has. The displacement that is required in order to return to his initial position will be the displacement that is associated with the cycle that is endowed with the origin A_0 . It is obvious that the displacement, when considered from the purely-geometric viewpoint by the observer that is placed at A_0 , will not depend upon the sequence of frames that is chosen by the intermediate observers, but its analytic expression will depend upon the choice of the initial frame at the origin A_0 . One can base an important notion upon the considerations of the displacements that are associated with the various cycles *with the given origin* A_0 , namely, that of the *holonomy group* of space. However, we shall not enter into that subject here.

If the cycle is an elementary parallelogram then the components of the infinitesimal displacement that are associated with the cycle are the complementary terms that one must add to the right-hand sides of the structure equations in order for those equations to become exact. They are bilinear expressions that are alternating with respect to the two series of differentials du_i and δu_i , or further, with respect to the two series of components $\omega_i(d)$ and $\omega_i(\delta)$ ($i = 1, 2, \dots, n$) if one supposes that the differential equations of the

⁽¹⁵⁾ More generally, in a continuum of dimension $m \neq n$ for which each point can be defined by an arbitrary coordinate system u_1, \dots, u_m , one can attach an infinitesimal displacement $T_{u,du}$ of the group G to each pair of infinitely-close points. One will have a space that again preserves some of the notions of the differential geometry of a Klein space whose fundamental group is G .

points are obtained by annulling the first n forms ω_i (¹⁶). The space is *torsionless* if the complementary terms that relate to the first n are all zero. That is what happens for the classical Riemannian spaces, so the displacement that is associated with an elementary cycle with origin A_0 will be a rotation of the frame that is attached to A_0 around its origin.

44. – The Kleinian geometry with the fundamental group G can exhibit different aspects according to the chosen *generating element*. In the preceding, we have supposed that it was a point. With a different choice of generating element, one will get generalized geometries that are *essentially different* from the preceding ones. If one regards space in ordinary geometry as a locus of planes then the notion of point will persist, since the point can be regarded as a particular family of planes that depend upon three parameters. However, just like in a Riemann space, which is a *point-like* space with Euclidian connection, the notion of plane will disappear. Similarly, in a *space of planes* with Euclidian connection, the notion of point will disappear, and geometry in such a space will be completely different from Riemannian geometry.

If we are not to remain in these generalities then it will be interesting to see how one can imagine a ruled space with Euclidian connection analytically. If one associates each tri-rectangular trihedron in Euclidian space with the third axis as *origin line* then the components of the infinitesimal displacement of the trihedron that will be annulled when the original line remains fixed are, as we have seen (no. **35**):

$$\omega_1, \omega_2, \omega_{12}, \omega_{23}.$$

One defines a ruled space with Euclidian connection by giving six differential forms in four variables u_1, u_2, u_3, u_4 , while the four forms $\omega_1, \omega_2, \omega_{12}, \omega_{23}$ are linearly independent. One can remark here that the knowledge of these four forms in Euclidian space will imply that of the other two. With Darboux's notations, one gives ξ_i, η_i, p_i, q_i ($i = 1, 2, 3, 4$); the equations:

$$(42) \quad \left\{ \begin{array}{l} \frac{\partial \xi_i}{\partial u_j} - \frac{\partial \xi_j}{\partial u_i} = \eta_i r_j - r_i \eta_j - \zeta_i q_j + q_i \zeta_u, \\ \frac{\partial \eta_i}{\partial u_j} - \frac{\partial \eta_j}{\partial u_i} = \zeta_i p_j - p_i \zeta_j - \xi_i r_j + r_i \xi_u, \\ \frac{\partial p_i}{\partial u_j} - \frac{\partial p_j}{\partial u_i} = q_i r_j - r_i q_j, \\ \frac{\partial q_i}{\partial u_j} - \frac{\partial q_j}{\partial u_i} = r_i p_j - p_i r_j \end{array} \right.$$

determine the ζ_i and the r_i unambiguously.

⁽¹⁶⁾ These bilinear expressions are not arbitrary: They satisfy some identities (viz., the Bianchi identities, in Riemannian geometry) that constitute the *theorem of the conservation of curvature and torsion*.

In order to have a ruled space with a *torsionless* Euclidian connection, one must give the functions $\xi_i, \eta_i, p_i, q_i, \zeta_i, r_i$ that satisfy the preceding 24 equations. However, we know that there is an infinitude of ways to choose the functions for the same space according to the particular choice of frames that one makes. As far as the forms $\omega_1, \omega_2, \omega_{13}, \omega_{23}$ are concerned, one confirms that the things that do not vary with the choice of frame are the two quadratic forms:

$$\omega_{13}^2 + \omega_{23}^2 \quad \text{and} \quad \omega_1 \omega_{23} - \omega_2 \omega_{13},$$

the first of which represents the square of the angle between two infinitely-close lines, and the second of which represents the product of that angle with the shortest distance between them. However, those two quadratic differential forms cannot be chosen arbitrary if one desires that equations (42) should be verified for a convenient choice of ξ_i and r_i . In particular, it is necessary that the first form $\omega_{13}^2 + \omega_{23}^2$ can be expressed as a quadratic differential form in only two variables. A particularly simple manner of choosing the two fundamental quadratic forms consists of taking:

$$(43) \quad \omega_{13}^2 + \omega_{23}^2 = du_1^2 + du_2^2,$$

$$\omega_1 \omega_{23} - \omega_2 \omega_{13} = du_2 du_3 - du_1 du_4.$$

The variables u_1 and u_2 define the *direction* of the straight line. One sees that in the corresponding ruled geometry, the notion of point (when considered to be the center of a sheaf of lines) and the notion of plane (when considered to be a network of lines) still persist⁽¹⁷⁾. The condition for two lines (u_i) and (u'_i) to *belong to the same point* and the condition for two lines (u_i) and (u'_i) to *belong to the same plane* are the same, namely⁽¹⁸⁾:

$$(u'_2 - u_2)(u'_3 - u_3) - (u'_1 - u_1)(u'_4 - u_4) = 0.$$

45. – It is clear that a large number of generalized geometries are only geometric curiosities, up to now. Meanwhile, they have the double advantage of casting a bright light onto the fundamentals of differential geometry themselves, and providing an inventory of geometric schemas into which mathematics and mathematical physics can be drawn some day. Moreover, that is why Riemannian geometry with absolute parallelism, which is at the basis of Einstein's recent research, enters into the general schema that we have presented. Likewise, Weyl spaces are torsionless spaces that admit the group of similitudes for their fundamental group. The theory of curves is given by the Frenet formulas (8) in those spaces. Up to the present, besides the Riemannian spaces and Weyl spaces, it has been the spaces with affine, projective, and conformal

⁽¹⁷⁾ If the space is torsionless then the notion of plane will necessarily persist, but the same thing will not generally be true for the notion of point.

⁽¹⁸⁾ That ruled geometry is linked in a very close way to the classical kinematics of the plane, where the straight line represents a uniform, rectilinear motion. See E. CARTAN, "La cinématique newtonienne et les espaces à connexion euclidienne," Bull. math. Soc. roumaine des Sciences **35** (1933).

connections that have been studied, above all. The last two categories can be attached to some very old problems of analysis that one can therefore cloak in a suggestive geometric form. For example, if one is given a system of second-order ordinary differential equations in n variables, $n - 1$ of which are dependent, and one of which is independent, can one regard the integral curves of that system as playing the role of straight lines in a continuum that is endowed with a projective connection? One sees immediately that this will be possible only if the equations have particular form. For example, in the case of $n = 2$, the given differential equations must have the form:

$$(44) \quad \frac{d^2 y}{dx^2} = A \left(\frac{dy}{dx} \right)^3 + B \left(\frac{dy}{dx} \right)^2 + C \left(\frac{dy}{dx} \right) + D,$$

with coefficients A, B, C, D that are functions of x, y . The problem then involves an infinitude of solutions. However, among all of the projective connections that answer that question, there exists one and only one that is linked in an intrinsic manner to the given equation. That must say that the law by which one associates that projective connection with the differential equation will remain invariant under an arbitrary change of variables ⁽¹⁹⁾. The geometry of the two-dimensional space of (x, y) with projective connection – which is called *normal*, thus-defined – then provides all of the properties of the equation (44) that do not depend upon the choice of variables x, y .

If the differential equation does not have the particular form (44), but is arbitrary, then one can further regard the integral curves as playing the role of straight lines in a space with projective connection, but with the condition that one must not take the point to be the generating element of the geometry, but the *linear element* (viz., the set of a point and a line that passes through that point), while the fundamental group is always the projective group in the plane. Then again, among all of the projective connections that make the integrals of the differential equation take the form of straight lines, one of them will be privileged, and the corresponding geometry will provide all of the properties of the differential equations that do not depend upon the choice of variables x, y .

One can likewise regard the integral curves of an arbitrary third-order differential equation as the circumference of a plane by regarding the continuum of points (x, y) as a generalized space whose fundamental group is the group of contact transformations that changes an oriented circle into an oriented circle. There once more exists a privileged connection, and the corresponding geometry will provide all of the properties of the differential equation that are invariant under an arbitrary contact transformation.

46. – One last example will bring us back to Euclidian geometry. One knows that Riemann envisioned some expressions that were general than the square root of a quadratic differential form in order to define the distance between two infinitely-close points. In the case of two dimensions, one can take an arbitrary homogeneous function of degree one in dx, dy , which one can always write as $F(x, y, y') dx$ by setting $y' = dy / dx$.

⁽¹⁹⁾ The determination of that intrinsic projective connection enters into the general method of moving frames, but only when it is applied to the case of an *infinite* group G , namely, the group of all point transformations in two variables.

On the other hand, the calculus of variations, in the simplest case of an integral $\int F(x, y, y') dx$, leads to some notions that are very analogous to some notions of elementary geometry; for example, transversality is closely analogous to perpendicularity. Certain authors have developed a generalization of Riemannian geometry that is based upon some considerations of that nature. That generalization enters into our general schema. One can make the extremals of the integral $\int F(x, y, y') dx$ into the Euclidian *straight lines* by introducing a Euclidian connection, but one must then not take the point to be the generating element, but the linear element. That amounts to saying that it is in the neighborhood of a linear element, when considered to be an extremal element, that space has the character of a Euclidian plane, but that character will be lost when one considers the neighborhood of a point – i.e., the set of linear elements whose center is close to a given point. One can then define angle between two infinitely-close linear elements in a Euclidian way, the distance between their centers, etc. However, it is important to remark that the distance between those centers can vary if one makes the linear elements turn around their fixed centers ⁽²⁰⁾.

The fact that one found a geometry with Euclidian connection upon the given of the analytical expressions for the distance between two infinitely-close points in an intrinsic manner poses the question of knowing whether one can realize something analogous by being given the analytical expression for the area of a surface element in a three-dimensional continuum ⁽²¹⁾. It is very remarkable that this is possible in general. One can associate an integral $\iint F(x, y, z, p, q) dx dy$ with a Euclidian connection according to an intrinsic law such that the integral will represent the Euclidian area of a surface. However, there are two exceptional cases here; I shall cite only the case of the integral $\iint (p^2 + q^2) dx dy$. Since that integral is invariant under the infinite group of point-like transformations:

$$(45) \quad \begin{cases} x' + iy' = f(x + iy), \\ z' = z + a, \end{cases}$$

in which $f(x + iy)$ denotes an arbitrary analytic function, and a denotes an arbitrary constant, it is certain that it is impossible to associate it with a Euclidian connection according to an intrinsic law (i.e., one that is invariant under an arbitrary change of variables), because any Euclidian connection can remain invariant only under a group with a maximum of six variables, whereas the integral must remain invariant under the infinite group (45) ⁽²²⁾.

The preceding example is very suggestive. It first shows the possibility of basing Euclidian differential geometry in space upon only the notion of area, just as it is possible to found it upon just the notion of length. However, it also shows that if, upon generalizing the Euclidian analytical expression for *length*, one can always base a geometry that preserves the fundamental notions of differential Euclidian geometry, then the same thing will not always be true upon generalizing the Euclidian analytical

⁽²⁰⁾ See ÉLIE CARTAN, *Les espaces de Finsler*, Exposés de Géométrie, II, 1934.

⁽²¹⁾ During the correction of the proofs, L. KOSCHMIEDER pointed out some papers to me [Math. Ann. **94** (1925). Math. Zeit. **24** (1925). Math. Zeit. **25** (1926). Proc. Akad., Amsterdam **31** (1928)], in which that problem was begun, and some interesting results were obtained.

⁽²²⁾ See E CARTAN, *Les espaces métriques fondés sur la notion d'aire*, Exposés de Géométrie, I, 1933.

expression for area, and that in its own right will open up new horizons in regard to the fundamentals of elementary geometry itself.
