"Sur les variétes a connexion projective," Bull. Soc. Math. France, t. 52 (1924), 205-241.

On manifolds with projective connections

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For more than two years, in several Notes to the *Comptes rendus de l'Academie des Sciences* ([1]), I have indicated a very general viewpoint, from which one may envision the theory of metric manifolds and its various generalizations. The fundamental idea is attached to the notion of *parallelism* that T. Levi-Civita has introduced in a manner that has proved fruitful ([2]). For example, if we consider a surface in ordinary (Euclidian) space then one may say that a small piece of that surface that surrounds one of its points presents all of the character of a two-dimensional (planar) Euclidian space, properly speaking, but it is only thanks to the notion of parallelism that one may *relate* two small pieces of that surface that surround two infinitely close points to points in that same Euclidian plane. It is the notion of parallelism that endows the surface with a Euclidian *connection*, to employ a term of H. Weyl.

The numerous authors that have generalized the theory of metric spaces have all started with the fundamental idea of Levi-Civita, however, it seems that they have not freed it from the idea of a *vector*. This is not an inconvenience when one is concerned with manifolds with an *affine* connection, in which the theory plays the role in these metric manifolds of affine Geometry with respect to Euclidian Geometry. However, it seems to dash any hope of founding an *autonomous* theory of manifolds with conformal or projective connections. In fact, what is essential in the idea of Levi-Civita is that one is given a means of associating between two small infinitely close pieces of a manifold, it is this idea of *agreement* that proves fruitful. In developing this idea one thus concedes the possibility of arriving at a general theory of manifolds with *affine, conformal, projective,* etc., connections.

In a memoir to the *Annals de l'École Normale supérieure* ([3]) I have developed the general theory of manifolds with *affine* connections, and in a memoir to the *Annales de la Société polonaise de mathématique* ([4]), the theory of manifolds with *conformal* connections ([5]). In that article, I proposed to rapidly indicate the fundamental points of the theory of manifolds with *projective* connections. Perhaps the most interesting point of that theory is the following one:

The *geodesics* of a manifold with projective connection are defined by second order differential equations of a particular form. Now, the class of equations of this form is *identical* to that of the equations that give the geodesics of manifolds with *affine*

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† Translated by D. H. Delphenich.

([1]) *C. R. Acad. Sc.*, t. 174, 1922, pp. 437, 593, 734, 857, 1104.

([2]) *Rendiconti del Circ. Matem. di Palermo*, t. 42, 1917, pp. 173-205.

([3]) *Ann. Éc. Norm.*, 3rd series, t. XL, 1923, pp. 325-412; t. XLI, 192, pp. 1-25.

([4]) *Ann. Soc. pol. Math.*, t. II, 1923, pp. 171-221.

([5]) This memoir is a development of the Note that was cited above (*C. R. Acad. Sc.*, t. 174, 1922, pp. 857-860).

connections. However, whereas it is impossible to distinguish, by simple intrinsic properties, one connection among all of the *affine* connections that endow a manifold with given geodesics, this is possible if one considers *projective* connections. I give that privileged projective connection the name of *normal*. There thus exists a unique correlation between a differential system of type considered and a manifold with a normal projective connection, in such a way that it is the notion of *projective* connection that permits us to give a geometrically satisfying form to the theory of differential systems in question, in particular, to the theory of the geodesic representation. One may say, from another viewpoint, that manifolds with normal projective connections play the same role with respect to these differential systems that Riemannian manifolds (with the Levi-Civita definition of parallelism) with respect to quadratic differential forms.

In the case of $n = 2$ the class of differential equations that are capable of defining the geodesics of a manifold with an affine connection has the form of equations for which $\frac{d^2y}{dx^2}$ is an integer polynomial of at most third degree in $\frac{dy}{dx}$. One may demand this if one does not generalize the theory in such a manner that the integral curves of *no particular* second order differential equation might also be regarded as geodesics. In the second part (§ VII and VIII) of that article, I showed that one may achieve this objective if one introduces the notion of a *manifold of elements* with a projective connection, in which the preceding manifolds are regarded as *point-like*. A manifold of elements with a normal projective connection is associated in an intrinsic manner (i.e., independently of any point-like transformation) with any second order differential equation, and the point-like manifolds with normal projective connections reappear as a particular case in these new manifolds. The notion of projective connection thus confers upon the theory of differential invariants of a second order differential equation a very unintended geometric aspect vis-à-vis the point-like group ([1]). There is no doubt that this geometrization may be effected for many other analogous questions. For example, I cite the theory of characteristics of an involutive system of second order partial differential equations in an unknown function of two independent variables. In that theory, the general projective group of the plane is replaced by a certain simple group with 14 parameters ([2]).

I. – THE NOTION OF MANIFOLD WITH PROJECTIVE CONNECTION.

1. A manifold (or space) with projective connection is a numerical manifold that presents all of the character of a projective space ([3]) in the immediate neighborhood of any point and is endowed, moreover, with a law that permits us to associate the two small

([1]) This theory has been the object of an important memoir by A. Tresse: *Determination des invariants ponctuels de l'équations de l'équations différenetielle ordinaire du second ordre* $y'' = \omega(x, y, z)$, a memoir that was run by l'Academie Jablowski: S. Hirkel, Leipzig (1896).

([2]) The method and its calculations are virtually identical to the ones in my memoir: *Les systèmes du Pfaff à cinq variables et les equations aux derives partielles du second ordre (Ann. Ec. Norm.*, 3rd series, t. XXVII, 1910, pp. 109-192).

([3]) When I say "projective space," I refer to a space in which the only properties of the figures that one regards as essential are the ones that are preserved under the most general projective (or homographic) transformations.

pieces of such a projective space that surround two infinitely close points. In order to give a precise sense to this definition, it suffices to imagine that one has attached a projective space to which that point belongs to each point of the manifold, and that one has a law that permits us to associate the projective spaces that attached to two infinitely close points of the manifold; this is the law that defines the projective connection on the manifold. Analytically, one chooses, in a manner that is arbitrary, moreover, a *frame* in the projective space that is attached to each point of the manifold, which defines a system of projective coordinates (trilinear in the two-dimensional case, tetrahedral in the three-dimensional case, etc.). The agreement between the projective spaces that are attached to two infinitely close points **a** and **a**′ analytically translates into a homographic transformation that permits us to pass from the coordinates of a point **m**′ in the projective space that is attached to the point **a** that coincides with **m**′ to the coordinates of the point **m** in the projective space that is attached to the point **a** when one has defined a way of relating these two spaces. Naturally, one supposes that this homographic transformation is infinitely close to the identity transformation. The coefficients of the formulas for that transformation will define the projective connection on the manifold.

One may take a slightly different viewpoint by interpreting the homographic transformation in question as the analytical translation of the *projective displacement* that permits one to move a frame attached to the point **a** to one attached to the point **a**′ in the unique projective space that defines the relationship between the projective spaces attached to **a** and **a**′. The knowledge of this projective displacement (once one has chosen the frames attached to the various points of the manifold) determines the projective connection on this manifold.

It naturally results from the preceding that the projective connection of the manifold may be analytically defined in an infinitude of different ways according to the choice of frames that are attached to the various points of the manifold. Likewise, one may choose – and there is often an advantage to proceeding this way – a frame at each point that depends upon arbitrary parameters; the analytical components of the projective connection then depend upon these parameters. If the frame is chosen in the most general manner possible then the component functions of the projective connection that are independent of these parameters will give an *intrinsic* analytical definition of the projective connection ([1]).

2. Before going further, it is useful to indicate what we mean by a projective *frame* in a precise manner. We adopt a system of homogeneous coordinates (Cartesian, for example) in an n-dimensional space. We agree to denote the set of $n+1$ coordinates $(x_1, x_2, \ldots, x_{n+1})$ by the letter **m**, in such a way that a geometric point will be just as well denoted by the symbol t**m**, where t is an arbitrary numerical coefficient, as by the symbol **m**. We nevertheless agree to speak of the point **m**, which we regard as distinct form the point 2**m**, the point 3**m**, etc. ([2]).

Having said this, take $n+1$ points:

([1]) This is similar to the way that the Euclidian connection of a Riemannian manifold is defined by starting with the notion of parallelism of Levi-Civita, which implies that this connection is completely defined by the ds^2 of the manifold.

([2]) In a sense, the symbol **m** denotes a point that is given a mass.

$$\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n+1}$$

such that the determinant of their coordinates is non-null. Any point **m** may, in one and only one way, and upon adopting a notation that is self-explanatory, be put into the form:

$$\mathbf{m} = y^1 \mathbf{a}_1 + y^2 \mathbf{a}_2 + \ldots + y^n \mathbf{a}_n + y^{n+1} \mathbf{a}_{n+1}.$$

The numerical coefficients $y^1, y^2, \ldots, y^{n+1}$ constitute the coordinates of the point **m** with respect the *frame* formed from the n+1 points $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n+1}$. A projective frame is therefore a set of n+1 points that are not situated in the same n–1-dimensional hyperplane; we call them the *vertices* of the reference (n+1)-hedron.

If one chooses two different frames:

$$\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n+1},$$
$$\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n+1}$$

then one obviously has formulas of the form:

$$\mathbf{b}_i = \alpha_i^1 \mathbf{a}_1 + \alpha_i^2 \mathbf{a}_2 + \ldots + \alpha_i^{n+1} \mathbf{a}_{n+1} \qquad (i = 1, 2, \ldots, n+1) \tag{1}$$

that analytically define the position of the second frame with respect to the first one, or furthermore, the projective displacement that makes the first one coincide with the second one. The formulas:

$$x^i = \alpha_1^i y^1 + \alpha_i^2 y^1 + \ldots + \alpha_i^{n+1} y^{n+1} \qquad (i = 1, 2, \ldots, n+1), \tag{2}$$

which analytically translate into the geometric equality:

$$x^1 \mathbf{a}_1 + x^2 \mathbf{a}_2 + \ldots + x^{n+1} \mathbf{a}_{n+1} = y^1 \mathbf{b}_1 + y^2 \mathbf{b}_2 + \ldots + y^{n+1} \mathbf{b}_{n+1},$$

defines the passage from the coordinates *yi* of a point with resoect to the second frame to the coordinates *xi* of the same point with respect to first frame. The $(n+1)^2$ coefficients α_i^j are the same in formulas (1) and (2).

However, it is important to remark that the homographic transformation (2) *does not change* if one multiplies all of the coefficients α_i^j by the same factor. In reality, all that is geometrically essential is the mutual ratios of the α_i^j, or, furthermore, the ratios of the α_i^j by their determinant ([1]).

3. We now return to manifolds with projective connection. It is natural to take each point **a** if the manifold to be one of the vertices of the frame attached to this point; we shall do so from now on. We denote the other n vertices by $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Now, let **a** and **a′** be two infinitely close points of the manifold, and let $\mathbf{a}'_1, \ldots, \mathbf{a}'_{n+1}$ denote the last n vertices of the frame attached to **a′**. *When one has defined a relationship* between the projective

([1]) One may further say that one has essentially the same frame when one multiplies all of the vertices by the same numerical factor.

spaces attached to **a** and **a′**, one will have formulas analogous to (1), but in which the coefficients α_i^j will be very close to unity, and the coefficients α_i^j $(i \neq j)$ will be very small. We write:

$$(3) \qquad \begin{cases} \mathbf{a}' = (1+\omega_0^0)\mathbf{a} + \omega^1\mathbf{a}_1 + \cdots + \omega^n\mathbf{a}_n, \\ \mathbf{a}_1' = \omega_1^0\mathbf{a} + (1+\omega_1^1)\mathbf{a}_1 + \cdots + \omega_1^n\mathbf{a}_n, \\ \qquad \cdots \\ \mathbf{a}_n' = \omega_n^0\mathbf{a} + \omega^1\mathbf{a}_1 + \cdots + (1+\omega_n^n)\mathbf{a}_n, \end{cases}$$

or, symbolically:

$$(3') \qquad \begin{cases} d\mathbf{a}' = \omega_0^0\mathbf{a} + \omega^1\mathbf{a}_1 + \cdots + \omega^n\mathbf{a}_n, \\ d\mathbf{a}_1' = \omega_1^0\mathbf{a} + \omega_1^1\mathbf{a}_1 + \cdots + \omega_1^n\mathbf{a}_n, \\ \qquad \cdots \\ d\mathbf{a}_n' = \omega_n^0\mathbf{a} + \omega^1\mathbf{a}_1 + \cdots + \omega_n^n\mathbf{a}_n. \end{cases}$$

The infinitesimal quantities $\omega_0^0, \omega_i^0, \omega^j, \omega_i^j$ define the infinitesimal projective displacement that brings the frame attached to the point **a** into coincidence with the frame attached to the point **a′**, once one has defined a relationship between the projective spaces attached to these two points. However, in reality, the agreement between these two spaces is analytically defined by the mutual ratios of the coefficients (3), i.e., by the quantities:

$$\omega^j,\ \omega_i^0,\ \omega_i^i - \omega_0^0,\ \omega_i^j \qquad (i \neq j;\ i, j = 1, 2, \ldots, n);$$

these are the components of the projective connection of the manifold.

If one an arbitrary system of coordinates $u^1, u^2, \ldots, u^n$ on the given manifold, and if the frame attached to each point of the manifold depends upon the parameters $v^1, v^2, \ldots, v^r$, then the components of the projective connection will naturally be assumed to be linear with respect to du^i and dv^i, since the coefficients are functions (which we assume to be differentiable) of the u^i and v^i.

It is important to remark that the n components $\omega^1, \omega^2, \ldots, \omega^n$ depend linearly only upon the $du^1, du^2, \ldots du^n$, because if one fixes the coordinates u^i then the geometric point **a** does not vary, **a′** is of the form $(1+\varepsilon)$ **a**, and the n expressions ω^j are annulled. Conversely, $du^1, du^2, \ldots du^n$ may be linearly expressed by means of the $\omega^1, \omega^2, \ldots, \omega^n$, because if these last n expressions are annulled then one reaches the same geometric point of the manifold and the du^i are annulled.

4. One may always choose the frames attached to the various points of the manifold in such a way that the sum of the n components $\omega_i^i - \omega_0^0$ is identically null. Indeed, chose the frame attached to one point of the manifold in an arbitrary, but well-defined, way (thus, there are no parameters v^i). Let:

$$\omega^j,\ \omega_i^0,\ \omega_i^i - \omega_0^0,\ \omega_i^j$$

be the corresponding components of the projective connection. Now replace the vertices:

$$\mathbf{a},\ \mathbf{a}_1,\ \ldots,\ \mathbf{a}_n\,,$$

by:

$$\mathbf{a},\ \mathbf{a}_1 + m_1\mathbf{a},\ \ldots,\ \mathbf{a}_n + m_1\mathbf{a},$$

in which we let $m_1, \ldots, m_n$ denote functions of the u^i that are temporarily arbitrary. As one easily sees, the new components $\bar{\varpi}_i^i - \bar{\varpi}_0^0$ of the projective connection will be:

$$\omega_i^i - \omega_0^0 + m_i\,\omega^i + \sum_{k=1}^{k=n} m_k \omega^k\,.$$

We only have to determine the functions m_i by the identity:

$$\sum_{i=1}^{i=n}(\omega_i^i - \omega_0^0) + (n_i + 1)\sum_{k=1}^{k=n} m_k \omega^k = 0$$

if we are to satisfy the demanded condition: this is possible since the Pfaff expression $\sum \omega_i^i - \omega_0^0$, which is linear in $du^1, \ldots, du^n$, may be linearly expressed $\omega^1, \ldots, \omega^n$.

On the other hand, we remark that one may always choose the frame I such a way that one has:

$$\omega^1 = du^1,\ \omega^2 = du^2,\ \ldots,\ \omega^n = du^n.$$

for this, it suffices, and without changing the vertex $\mathbf{a}$ (which depends, as one knows, on an arbitrary factor), to put the expression:

$$\omega^1\mathbf{a}_1 + \omega^1\mathbf{a}_1 + \ldots + \omega^n\,\mathbf{a}_n$$

into the form:

$$du^1\bar{\mathbf{a}}_1 + du^2\bar{\mathbf{a}}_2 + \ldots + du^n\,\bar{\mathbf{a}}_n\,.$$

Finally, the two preceding conditions may be realized simultaneously. Indeed, we may first realize the second one, i.e., suppose, to begin with, that:

$$\omega^i = du^i\,.$$

We may then realize the first condition by replacing $\mathbf{a}_i$ with $\mathbf{a}_i + m_i\,\mathbf{a}_i$, with the coefficients m_i conveniently chosen, which does not change the value of the ω^i.

If we now suppose that the two are conditions are realized:

$$\omega^i = du^i\,, \qquad \sum_{i=1}^{i=n}\omega_i^i = n\,\omega_0^0,$$

then the components ω^j, ω_i^0, $\omega_i^i - \omega_0^0$, ω_i^j *of the affine connection are well defined.* Indeed, suppose that one has made a particular choice of frame that satisfies the indicated conditions. For any other choice, one will have:

$$d\bar{\mathbf{a}} = (m\,\omega_0^0 + dm)\,\mathbf{a} + m\,du^1\mathbf{a}_1 + \ldots + m\,du^n\,\mathbf{a}_n\,.$$

From this, it results that one must take:

$$\bar{\mathbf{a}}_i = m\,\mathbf{a}_1 + m_i\,\mathbf{a} \qquad (i = 1, 2, \ldots, n)\,.$$

One then finds, by a simple calculation:

$$\sum_{i=1}^{i=n} (\bar{\omega}_i^i - \bar{\omega}_0^0) = (n + 1) \sum_{i=1}^{i=n} m_i du^i\,.$$

One must therefore have that the coefficients $m_1, \ldots, m_n$ are null, and the new components of the projective connection are identical with the old ones.

It results form this that when one chooses a system of coordinates $(u^1, \ldots, u^n)$ *on the manifold then the projective connection is uniquely determined by the Pfaff expressions:*

$$\omega_i^0,\ \omega_i^i - \omega_0^0,\ \omega_i^j,$$

once one has arranged to reduce the ω^j *to the* du^i, *and to annul the sum of the n expressions* $\omega_i^i - \omega_0^0$. The projective connection thus depends on $n^2 + n - 1$ arbitrary Pfaff expressions; in other words, on $n(n^2 + n - 1)$ arbitrary functions of the $u^1, \ldots, u^n$.

In the sequel, we will suppose, unless stated to the contrary, that the frames are chosen arbitrarily, in such a way that the components ω^j, ω_i^0, $\omega_i^i - \omega_0^0$, ω_i^j are arbitrary Pfaff expressions.

II. – THE STRUCTURE OF A MANIFOLD WITH A PROJECTIVE CONNECTION.

5. One arrives at the notion of the structure of a manifold with a projective connection by considering what happens when one successively relates each projective space attached to a point of a curve traced out in the manifold with the infinitely close ones. If one first considers two infinitely close points **a** and **a′** of the manifold, each of which has a frame attached to it, then the point (x^i) of the projective space attached to the point **a** will coincide with the point $x^i + dx^i$ of the projective space attached to the point **a′** when one has defined their relationship if one has symbolically, upon accounting for equations (3′):

$$d(x\,\mathbf{a} + x^1\mathbf{a}_1 + \ldots + x^n\,\mathbf{a}_n) = 0\,.$$

This equation decomposes into $n + 1$ other ones:

$$(4)\qquad \begin{cases} dx + \omega_0^0 x + \omega_1^0 x^1 + \cdots + \omega_n^0 x^n = 0, \\ dx^1 + \omega^1 x + \omega_1^1 x^1 + \cdots + \omega_n^1 x^n = 0, \\ \cdots \\ dx^n + \omega^n x + \omega_n^1 x^1 + \cdots + \omega_n^n x^n = 0. \end{cases}$$

If the point **a** describes a closed curve on the manifold then the coordinates $u^1, \ldots, u^n$ may be expressed as functions of the same parameter t, the expressions ω_i^j take the form $p_i^j dt$, and the equations (4) become linear differential equations in the $n+1$ unknown functions $x, x^1, \ldots, x^n$ of the independent variable t. Their integration permits us to successively analytically define the relationship between the projective space attached to the starting point and the projective space attached to the final point along the curve considered.

In particular, suppose that the curve is closed. Under these successive relationships, the projective space that is attached to the starting point does not coincide, in general, with the projective space that is attached to the final point, which will obviously be the case only if the manifold constitutes a projective space in its own right. Under these successive relationships the point $(x, x^1, \ldots, x^n)$ of the projective space attached to the starting point will coincide with the point $(y, y^1, \ldots, y^n)$ of the same space, and the expressions for the y^i will be once more obtained by the integration of equations (4) with the initial values (x^i) of the unknowns.

In addition, if the closed contour is infinitely small then the y^i will be infinitely close to the x^i, and if one sets:

$$y^i = x^i + \Delta x^i,$$

one will have formulas such as:

$$(5)\qquad \begin{cases} \Delta x + \Omega_0^0 x + \Omega_1^0 x^1 + \cdots + \Omega_n^0 x^n = 0, \\ \Delta x^1 + \Omega^1 x + \Omega_1^1 x^1 + \cdots + \Omega_n^1 x^n = 0, \\ \cdots \\ \Delta x^n + \Omega^n x + \Omega_n^1 x^1 + \cdots + \Omega_n^n x^n = 0. \end{cases}$$

in which the Ω_i^j are doubly-extended integral elements on the areas that is bounded by the contour. One may prove ([1]) that one has:

([1]) The proof is analogous to the one that we indicated in the case of manifolds with affine connection (*Ann. Éc. Norm.*, 3rd series, t. XL, 1923, pp. 369-373).

$$(6)\quad \begin{cases} \Omega^i = (\omega^i)' - [\omega_0^0 \omega^i] - \sum_{k=1}^{k=n} [\omega^k \omega_k^i], \\ \Omega_0^0 = (\omega_0^0)' - \sum_{k=1}^{k=n} [\omega^k \omega_k^0], \\ \Omega_i^0 = (\omega_i^0)' - [\omega_i^0 \omega_0^0] - \sum_{k=1}^{k=n} [\omega_i^k \omega_k^0], \\ \Omega_i^j = (\omega_i^j)' - [\omega_i^0 \omega^j] - \sum_{k=1}^{k=n} [\omega_i^k \omega_k^j]. \end{cases}$$

6. Formulas (5) define an infinitesimal projective displacement whose components are:

$$\Omega^i,\ \Omega_i^0,\ \Omega_i^i - \Omega_0^0,\ \Omega_i^j,$$

in such a manner that formulas (4) define the infinitesimal projective displacement of the components:

$$\omega^j,\ \omega_i^0,\ \omega_i^i - \omega_0^0,\ \omega_i^j.$$

One may remark, moreover, that the former components are constructed uniquely from means of the latter ones and their bilinear covariants.

To any infinitely small closed contour that is traced in the manifold there is associated an infinitesimal projective displacement whose components are the double integral elements of the form $\sum g_{ik}\, du^i\, du^k$. These components analytically define the *structure* of the manifold; they generalize the Riemann-Christoffel tensor. One may also say that they define the *curvature* of the manifold, in the sense that they manifest the divergence that exists between the given manifold and a (flat) projective space, properly speaking, when one traverses an infinitesimal closed contour in that manifold.

As in the case of manifolds with affine connections, there exists a *theorem of the conservation of curvature*, which one obtains analytically upon exterior differentiating both sides of formulas (6) and taking into account these formulas themselves:

$$(7)\quad \begin{cases} (\Omega^i)' = -[\Omega_0^0 \omega^i] + [\omega_0^0 \Omega^i] - \sum_{k=1}^{k=n} [\Omega^k \omega_k^i] + \sum_{k=1}^{k=n} [\omega^k \Omega_k^i], \\ (\Omega_0^0)' = [\Omega^k \omega_k^0] + \sum_{k=1}^{k=n} [\omega^k \Omega_k^0], \\ (\Omega_i^0)' = -[\Omega_i^0 \omega_i^0] + [\omega_i^0 \Omega_0^0] - \sum_{k=1}^{k=n} [\Omega_i^k \omega_k^0] + \sum_{k=1}^{k=n} [\omega_i^k \Omega_k^0], \\ (\Omega_i^j)' = -[\Omega_i^0 \omega^j] + [\omega_i^0 \Omega^j] - \sum_{k=1}^{k=n} [\Omega_i^k \omega_k^j] + \sum_{k=1}^{k=n} [\omega_i^k \Omega_k^j]. \end{cases}$$

This theorem admits a geometric formulation that involves an infinitesimal three-dimensional domain in the manifold, but it would prove pointless to discuss it in detail here ([1])

III. – MANIFOLDS WITH PROJECTIVE CONNECTIONS WITH VANISHING TORSION.

7. A manifold with projective connection is said to be without torsion if the infinitesimal projective displacement associated with an arbitrary infinitesimal closed contour that starts from an arbitrary point of that manifold and returns to it leaves the (geometric) point **a** invariant. In order for this to be true, it is necessary and sufficient that formulas (5) give null values for $\Delta x^1, \Delta x^2, \ldots, \Delta x^n$ when the coordinates $x^1, x^2, \ldots, x^n$ are null. In other words, *manifolds without torsion are characterized by the equalities:*

$$\Omega^1 = 0, \quad \Omega^2 = 0, \ldots, \quad \Omega^n = 0 . \tag{8}$$

The components of the projective connection thus satisfy the identities:

$$(\omega_i)' = [\,\omega^j\,(\omega_i^j - \omega_0^0)] + \sum_{k\neq i}[\omega^k \omega_k^i] . \tag{9}$$

In addition, formulas (7) show that the non-null components Ω^i, Ω_i^0, $\Omega_i^i - \Omega_0^0$, Ω_i^j, of the curvature are not arbitrary, because they are coupled by the n relations:

$$[\,\omega^j\,(\Omega_i^j - \Omega_0^0)] + \sum_{k\neq i}[\omega^k \Omega_k^i] = 0 \qquad (i = 1, 2, \ldots, n) . \tag{10}$$

In particular, one sees that if one has taken $\omega^j = du^i$, and if one sets:

$$\omega_i^i - \omega_0^0 = \sum_{r=1}^{r=n} \Gamma_{ir}^i du^r , \qquad \omega_i^j = \sum_{r=1}^{r=n} \Gamma_{ir}^j du^r ,$$

then the coefficients Γ_{ij}^k will satisfy the symmetry law:

$$\Gamma_{ij}^k = \Gamma_{ji}^k .$$

One may further say that there exist n quadratic forms $\Phi^1, \Phi^2, \ldots, \Phi^n$ in $du^1, \ldots, du^r$, such that one has:

$$\omega_i^i - \omega_0^0 = \frac{1}{2}\frac{\partial \Phi^i}{\partial (du^i)}, \qquad \omega_i^j = \frac{1}{2}\frac{\partial \Phi^j}{\partial (du^i)} .$$

As for the identities (10), they show that if one sets:

([1]) See *Ann. Éc. Norm., loc. cit.*, pp. 373-375.

$$\Omega_i^i - \Omega_0^0 = \sum_{h,k}^{1,\ldots,n} \Lambda_{ihk}^i [\omega^h \omega^k], \qquad \Omega_i^j = \sum_{h,k}^{1,\ldots,n} \Lambda_{ihk}^j [\omega^h \omega^k],$$

then one has the identities:

(11) $$\Lambda_{\alpha\beta\gamma}^i + \Lambda_{\beta\gamma\alpha}^i + \Lambda_{\gamma\alpha\beta}^i = 0.$$

8. One obtains an important category of manifolds without torsion by remarking that the projective group leaves a point fixed admits an invariant subgroup. In deed, this group is the dual of the group that leaves a hyperplane fixed, i.e., it is a sum of *affine* groups that admit invariants subgroup of affine transformations that preserve the volumes (Möbius group). Now, the dual of the projective transformation (5) is:

$$\begin{aligned}
\Delta\xi &= \Omega_0^0 \xi, \\
\Delta\xi_1 &= \Omega_1^0 \xi + \Omega_1^1 \xi_1 + \cdots + \Omega_1^n \xi_n, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
\Delta\xi_n &= \Omega_n^0 \xi + \Omega_n^1 \xi_1 + \cdots + \Omega_n^n \xi_n.
\end{aligned}$$

If we use *inhomogeneous* coordinates then it becomes the affine transformation:

$$\Delta\xi_i = \Omega_i^i + (\Omega_i^i - \Omega_0^0)\xi_i + \sum_{k \neq i} \Omega_i^k \xi_k, \qquad (i = 1, 2, \ldots, n).$$

It belongs to the Möbius group if one has:

(12) $$\sum_{i=1}^{i=n} (\Omega_i^i - \Omega_0^0) = 0\,.$$

This is the identity that, along with (8), defines the category of manifolds in question. From (6), it may also be described by:

$$\sum_{i=1}^{i=n} (\omega_i^i - \omega_0^0)' = -(n+1) \sum_{i=1}^{i=n} [\omega^i \omega_i^0].$$

In particular, and one may always realize this (sec. 4), one has:

$$\omega^i = du^i, \qquad \sum_{i=1}^{i=n} (\omega_i^i - \omega_0^0) = 0,$$

then one will have:

$$\omega_i^0 = \frac{1}{2} \frac{\partial \Phi}{\partial (du^i)},$$

in which Φ is a quadratic form in $du^1, du^2, \ldots, du^n$.

From (7), one will also have:

$$\sum_{i=1}^{i=n} [\omega^i \omega_i^0] = 0,$$

or furthermore, if we employ a self-explanatory notation:

$$\Lambda^0_{\alpha\beta\gamma} + \Lambda^0_{\beta\gamma\alpha} + \Lambda^0_{\gamma\alpha\beta} = 0 \qquad (\alpha, \beta, \gamma = 1, 2, \ldots, n).$$

9. A last category of manifolds without torsion, which is more restricted, is composed of ones for which the infinitesimal projective displacement that is associated with an infinitesimal closed contour that starts at **a** and returns to it leaves fixed, not only the point **a**, but also all of the lines through **a**. In order for this to be true, it is necessary and sufficient that $\Delta x^1, \Delta x^2, \ldots, \Delta x^n$ be proportional to $x^1, x^2, \ldots, x^n$ in (5); in other words, that one have the identities:

$$\Omega_1^1 = \Omega_2^2 = \ldots = \Omega_n^n, \qquad \Omega_i^j = 0 \qquad (i \neq j).$$

If this is true then formulas (10) show, upon supposing that $n > 2$, that one has:

$$\Omega_i^i - \Omega_0^0 = 0.$$

Formulas (7) give:

$$[\Omega_i^0 \omega^j] = 0 \qquad (i, j = 1, 2, \ldots, n),$$

from which we have:

$$\Omega_i^0 = 0.$$

All of the components of the curvature are therefore null, and the manifold reduces to the self-same projective space.

Thus, if $n > 2$ *then there is only the projective space, properly speaking, for which the infinitesimal projective displacement that is associated with a closed contour that starts at* **a** *leaves the point* **a** *invariant, along with all of the lines that pass through* **a**.

This conclusion is naturally invalid when $n = 2$ ([1]).

IV. – THE GEODESICS OF MANIFOLDS WITH PROJECTIVE CONNECTIONS.

10. A curve (*C*) that is traced out in a manifold with a projective connection is called a *geodesic* of that manifold if, when one successively relates the projective spaces attached to each of the points of the curve, all of these points define a straight line. One expresses the fact a curve is a geodesic by saying that, by virtue of formulas (3′), which define the projective connection on the manifold, the point $d^2\mathbf{a}$ is situated on the line that joins the point **a** to the point $d\mathbf{a}$. One is thus led to the second order differential equations:

([1]) There exists an analogous theorem in the theory of manifold with conformal connection, but it is true only when *n* is greater than three (*Ann. Soc. Pol. Mat., loc. cit,* pp. 185).

$$\frac{d\omega^1 - \omega^1\omega_0^0 + \sum_{i=1}^{i=n}\omega^i\omega_i^1}{\omega^1} = \frac{d\omega^2 - \omega^2\omega_0^0 + \sum_{i=1}^{i=n}\omega^i\omega_i^2}{\omega^2} = \ldots = \frac{d\omega^n - \omega^n\omega_0^0 + \sum_{i=1}^{i=n}\omega^i\omega_i^n}{\omega^n}.$$

In particular, suppose that one has chosen the frame in such a manner that ω^i reduces to du^i. The equations are of the form:

$$\frac{d^2u^1 + P^1(du)}{du^1} = \frac{d^2u^2 + P^2(du)}{du^2} = \ldots = \frac{d^nu^1 + P^n(du)}{du^n},$$

in which the P^i (du) are quadratic forms in du^1, du^2, …, du^n. One sees that if, for example, one expresses u^1, u^2, …, u^{n-1} as functions of u^n then one obtains the equations:

(13)
$$\begin{cases} \dfrac{d^2u^1}{(du^n)^2} = -P^1\left(\dfrac{du^1}{du^n}\right) + \dfrac{du^1}{du^n}P^n\left(\dfrac{du^1}{du^n}\right) \\ \cdots \\ \dfrac{d^2u^{n-1}}{(du^n)^2} = -P^{n-1}\left(\dfrac{du^i}{du^n}\right) + \dfrac{du^{n-1}}{du^n}P^n\left(\dfrac{du^i}{du^n}\right), \end{cases}$$

in which the P^i are now n (not necessarily homogeneous) integer polynomials of second degree in the $\dfrac{du^1}{du^n}$, …, $\dfrac{du^{n-1}}{du^n}$, with coefficients that are given functions of the u^1, u^2, …, u^{n-1} ([1]).

One sees from this that if one is given a family of curves in an n-dimensional numerical manifold by a system of n–1 second order differential equations then *it is generally impossible to associate a projective connection with the manifold in such a way that these curves become its geodesics*. This is true only if the differential equations can be put into the form (13).

Conversely, if one is given an arbitrary system of differential equations of the form (13) then it will be possible to regard them as the geodesic equations of the manifold when one endows it with a conveniently chosen projective connection. Indeed, one may always suppose that one has $\omega^i = du^i$. It then suffices to take the components $\omega_i^i - \omega_0^0, \omega_i^j$ in such a way that one has:

$$du^i\,(\omega_i^i - \omega_0^0) + \sum_{k\neq i} du^k\omega_k^i = P^i\,(du),$$

and, more generally:

(14)
$$du^i\,(\omega_i^i - \omega_0^0) + \sum_{k\neq i} du^k\omega_k^i = P^i\,(du) + du^i\sum_{k=1}^{k=n} c_k du^k,$$

([1]) In the case of $n = 2$ these equations reduce to just one, which, upon changing the notations, has the form:

$$\frac{d^2y}{dx^2} = A + 3B\frac{dy}{dx} + 3C\left(\frac{dy}{dx}\right)^2 + D\left(\frac{dy}{dx}\right)^3,$$

in which the coefficients A, B, C, D are functions of x, y.

in which the c_k are arbitrary functions.

11. In particular, we seek all of the projective connections *without torsion* that are compatible with the identities. As one has seen one may, with no loss of generality, suppose that:

$$\sum_{i=1}^{i=n} (\omega_i^i - \omega_0^0) = 0. \tag{15}$$

Now, since the manifold is without torsion the expressions $\omega_i^i - \omega_0^0, \omega_i^j$ are the partial semi-derivatives of the right-hand side of the identity (14), which one regards as a quadratic form in $du^1, du^2, \ldots, du^n$. One thus has:

$$\omega_i^i - \omega_0^0 = \frac{1}{2}\frac{\partial P^i}{\partial(du^i)} + \sum_{k=1}^{k=n} c_k du^k + \frac{1}{2} c_i du^i,$$
$$\omega_k^i = \frac{1}{2}\frac{\partial P^i}{\partial(du^k)} + \frac{1}{2} c_k du^i.$$

Condition (15) gives:

$$\sum_{i=1}^{i=n} c_i du^i = -\frac{1}{n+1}\sum_{i=1}^{i=n}\frac{\partial P^i}{\partial(du^i)}.$$

It unambiguously determines the coefficients c_i , and, as a result, the components $\omega_i^i - \omega_0^0, \omega_i^j$ of the desired projective connection. One sees that *the integral curves* (C) *of the system* (13) *may always be regarded as the geodesics of the manifold, which is assumed to have no torsion, and the projective connection that one must necessarily attribute to this manifold depends upon n arbitrary Pfaff expressions* $\omega_1^0, \omega_2^0, \ldots, \omega_n^0$.

V. – MANIFOLDS WITH NORMAL PROJECTIVE CONNECTIONS.

12. It is interesting to inquire whether, among all of the projective connections that endow a given numerical manifold with the same geodesics, there exists one of them that is endowed with particularly simple intrinsic properties. It is natural to suppose that the projective connection is without torsion. We just saw that under that hypothesis there remain n arbitrary Pfaff expressions ω_i^0. We may now arrange that one have:

$$\sum_{i=1}^{i=n} (\Omega_i^i - \Omega_0^0) = \sum_{i=1}^{i=n} (\omega_i^i - \omega_0^0)' + (n+1)\sum_{i=1}^{i=n} [\omega^i \omega_i^0] = 0.$$

When one is given the preceding choice, satisfying this condition amounts to taking the ω_i^0 equal to the partial semi-derivatives of a quadratic form Φ in $du^1, du^2, \ldots, du^n$:

$$\omega_i^0 = \sum_{i=1}^{i=n} \Gamma_{ik} du^k \qquad (\Gamma_{ij} = \Gamma_{ji}).$$

What remains are $\frac{n(n+1)}{2}$ arbitrary coefficients Γ_{ij}. One has:

$$\Omega_i^i - \Omega_0^0 = (\omega_i^i - \omega_0^0)' - \sum_{i=1}^{i=n} [\omega_i^k \omega_k^i] + [\omega^i \omega_i^0] + \sum_{k=1}^{k=n} [\omega^k \omega_k^0] ,$$
$$\Omega_j^i = (\omega_j^i)' - \sum_{k=1}^{k=n} [\omega_j^k \omega_k^i] + [\omega^i \omega_i^0].$$

We let a_{ikl}^i, a_{jkl}^i the coefficients of the forms $\Omega_i^i - \Omega_0^0, \Omega_j^i$ when one annuls the ω_i^0. In the general case, these coefficients become:

$$\begin{aligned} A_{iil}^l &= a_{ill}^i + \Gamma_{il} , \\ A_{ikl}^i &= a_{ikl}^i \qquad (k, l \neq i), \\ A_{jil}^i &= a_{jil}^i + \Gamma_{jl} , \\ A_{jkl}^i &= a_{jkl}^i \qquad (k, l \neq i). \end{aligned}$$

One may assign the coefficients Γ_{ij} in one and only one manner that makes:

$$\sum_{k=1}^{k=n} A_{jik}^k = 0 \qquad (i, j = 1, 2, \ldots, n);$$

it will suffice to take:

$$\Gamma_{ij} = \frac{1}{n-1} \sum_{k=1}^{k=n} a_{ijk}^k .$$

By virtue of formulas (11) the two values obtained for Γ_{ij} are equal (sec. 7) when one replaces i with γ and sums over γ.

By definition, *with the (unique) choice of frame that reduces* ω^i *to* du^i *and* $\sum \omega_i^i - \omega_0^0)$ *to zero. There exists one and only one projective connection that makes the integral curves of* (13) *into geodesics and satisfies the conditions:*

$$(16) \qquad \Omega^i = 0, \qquad \sum_{i=1}^{i=n} (\Omega_i^i - \Omega_0^0) = 0, \qquad \sum_{k=1}^{k=n} A_{ijk}^k = 0 \qquad (i, j = 1, 2, \ldots, n).$$

13. We shall now show that the conditions (16) are *independent of any choice of frame.* This is obvious from the relations:

$$\Omega^i = 0, \qquad \sum_{i=1}^{i=n} (\Omega_i^i - \Omega_0^0) = 0,$$

which have, as we saw in (sec. 8), an invariant significance. If we suppose that these conditions are already then the quantities:

$$B_{ij} = \sum_{k=1}^{k=n} A_{ijk}^{k} \qquad (i, j = 1, 2, \ldots, n)$$

satisfy the symmetry law $B_{ij} = B_{ji}$. We shall now show that under an infinitesimal change of frame they transform among themselves by a linear homogeneous substitution.

Indeed, imagine that the frames depend upon a variable parameter v at each point of the manifold and use the symbol δ for a variation of that parameter while the u^i remain fixed. The symbol d will be reserved for an arbitrary variation of the u and v. Formulas (7):

$$(\Omega_i^i - \Omega_0^0)' = \sum_{k=1}^{k=n}[\Omega_l^k \omega_k^i] + \sum_{k=1}^{k=n}[\omega_l^k \Omega_k^i] - [\omega^i \Omega_i^0] - \sum_{k=1}^{k=n}[\omega^k \Omega_k^0],$$

$$(\Omega_j^i)' = -\sum_{k=1}^{k=n}[\Omega_j^k \omega_k^i] + \sum_{k=1}^{k=n}[\omega_j^k \Omega_k^i] - [\omega^i \Omega_j^0]$$

show that for an infinitesimal variation of v one has:

$$\delta(\Omega_i^i - \Omega_0^0) = \sum_{k \neq i}(e_i^k \Omega_k^i - e_k^i \Omega_i^k),$$

$$\delta(\Omega_j^i) = \sum_{k=1}^{k=n}(e_j^k \Omega_k^i - e_k^i \Omega_j^k),$$

in which e_i^i, e_j^i denote the expressions that $\omega_i^i - \omega_0^0, \omega_i^j$ become for the differentiation symbol δ.

On the one hand, the formulas:

$$(\omega^j)' = [\omega^i(\omega_i^i - \omega_0^0)] + \sum_{k \neq i}[\omega^k \omega_k^i] \qquad (i = 1, 2, \ldots, n)$$

give:

$$\delta\omega^j = -\sum_{k=1}^{k=n} e_k^i \omega^k \qquad (i = 1, 2, \ldots, n),$$

One easily deduces from this that:

$$\delta A_{ikl}^{j} = \sum_{\rho=1}^{\rho=n}(e_j^\rho A_{\rho kl}^{i} - e_\rho^j A_{ikl}^{\rho} + e_k^\rho A_{i\rho l}^{i} + e_i^\rho A_{ik\rho}^{j}), \qquad (i, j, k, l = 1, 2, \ldots, n).$$

and, as a result:

$$\delta B_{ij} = \sum_{\rho=1}^{\rho=n}(e_i^\rho B_{\rho j} + e_j^\rho B_{i\rho});$$

this what we had to prove ([1]).

From this, it results that if the coefficients B_{ij} are null for a particular choice of frame then they are null for any other choice.

We say that a projective connection that satisfies conditions (16) is *normal*. We then see that *if we are given a family of curves in a numerical manifold that are defined by a system of differential equations of the form* (13) *then one may endow the manifold with a normal projective connection such that the curves in question become the geodesics of that manifold, and in one and only one way*.

An important consequence of the preceding theorem is the following: *The analytical study of the invariants of the system* (13) *vis-à-vis an arbitrary change of variables identical with the study of the geometrical properties of manifolds with normal projective connections* ([2]). One sees the importance of the notion of normal projective connection from either the viewpoint of Analysis or that of Geometry.

One may remark that manifolds with normal projective connections are characterized by the property that the infinitesimal projective displacement that is associated with an infinitesimal closed contour that starts from a point **a** leaves the point **a** and *all* of the lines through **a** invariant.

Remark. – Given a system of differential equations of the form (13), it is easy to see that one may, and in an infinitude of way, endow the manifold with an *affine* connection such that the integral curves (13) become the geodesics of the manifold. One may likewise do this in such a way that there exists an *absolute unit of volume* in the manifold. However, also with this supplementary condition, the involves an infinitude of solutions (which depend upon an arbitrary function of n variables u^i), and none of them is distinguished from the others by simple intrinsic properties ([3]). It is therefore the notion of normal projective connection alone that permits us to construct a *geometric* theory that satisfies equations of the form (13).

([1]) One may show in a more elegant manner that the quadratic form:

$$\sum_{k=1}^{k=n}\sum_{i=1}^{i=n} \omega^k \frac{\partial \Omega_k^i}{\partial \omega^i} = \sum B_{ij}\, \omega^i\, \omega^j$$

(in which we have written Ω_i^i in place of $\Omega_i^i - \Omega_0^0$) is *invariant* under an arbitrary frame change.

([2]) Manifolds with normal projective connections play the same role, vis-à-vis systems of differential equations (13), as Riemannian manifolds, vis-à-vis quadratic differential forms, and manifolds with normal conformal connections (*see* the Note cited in *Comptes rendus*, t. 174, pp. 857), vis-à-vis Monge's quadratic equations.

([3]) The indicated supplementary condition makes the tensor $B_{ij} = \sum_k A_{ijk}^k$ symmetric. *See* L.-P. EISENHART, *Spaces with correspondent Paths* (*Proc. Nat. Acad. of Sciences*, t. 8, 1922, pp. 336). In the "Geometry of Paths" by L.-P. EISENHART and O. VEBLEN (*Proceed. Nat. Acad. of Sciences*, t. 8, 1922, pp. 19), geodesics are taken as the point of departure for affine connections.

VI. – THE NORMAL PROJECTIVE CONNECTION ATTACHED TO THE GEODESICS OF A GIVEN ds^2.

14. The problem of the *geodesic representation* of two ds^2 is cast in a new light if one takes the preceding viewpoint. To each ds^2 there corresponds a well-defined manifold with normal projective connection. Two ds^2 admit a geodesic representation in terms of each other if the two manifolds with normal projective connection are *isomorphic*, i.e., if it is possible to establish a point-like correspondence between the two manifolds that preserves the components of the projective connection. We shall not occupy ourselves with the general problem, which may be treated by exactly the same methods as one uses for the problem of the isomorphism (or mappability) of two Riemannian manifold ([1]). We shall content ourselves with indicating how one determines the normal projective connection corresponds to a given ds^2, and to see in which cases this ds^2 is geodesically representable on a Euclidian space ds^2.

Consider a ds^2 that one may always express as a sum of n squares:

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 + \dots + (\omega^n)^2 ;$$

the Riemannian manifold that is defined by this ds^2 admits a Euclidian connection (without torsion) whose components are the ω^j and certain Pfaff expression $\omega_i^j = -\omega_j^i$. This connection may be regarded as a projective one by attributing the value zero to the ω_i^0.

We use the letter ϖ to denote the components of the *normal* projective connection that gives the same geodesics. We have the right to take:

$$\varpi^j = \omega^j, \quad \varpi_i^i - \varpi_0^0 = \omega_i^i = 0, \qquad \varpi_i^j = \omega_i^j .$$

Since $\sum (\varpi_i^i - \varpi_0^0) = 0$, the argument that we made above shows that the ϖ_i^0 have the form:

$$\varpi_i^0 = \sum_{k=1}^{k=n} \Gamma_{ik}\omega^k \qquad (\Gamma_{ij} = \Gamma_{ji}).$$

Set:

$$\Omega_i^j = \sum_{k,l} a_{ikl}^j [\omega^k \omega^l].$$

We have:

$$\Gamma_{ij} = \frac{1}{n-1}\sum_{k=1}^{k=n} a_{ijk}^k = \frac{b_{ij}}{n-1}.$$

The first problem is thus solved, and one has:

$$\Pi_i^i - \Pi_0^0 = [\omega^i \varpi_i^0] = \frac{1}{n-1}\sum_{k=1}^{k=n} b_{ik}[\omega^i \omega^k] \qquad (i = 1, 2, \dots, n),$$

([1]) See my memoir: *Sur les equations de la gravitation d'Einstein* (*J. de Math.*, 1922, fasc. 1).

$$\Pi_i^j = \Omega_i^j + [\omega^j \varpi_i^0] = \sum_{k,l} a_{ikl}^j [\omega^k \omega^l] + \sum_{k=1}^{k=n} b_{ik} [\omega^j \omega^k] \qquad (i, j = 1, 2, \ldots, n).$$

15. If one wishes that a given ds^2 be geodesically representable by a Euclidian ds^2 then it is necessary and sufficient that the components $\Pi_i^i - \Pi_0^0$, Π_i^j, Π_i^0 of the curvature of the manifold with normal projective connection that we just determined are identically null. If we first suppose that $n > 2$ then we know (sec. 9) that for this to be true it suffices that the $\Pi_i^i - \Pi_0^0$ and the Π_i^j be null. One first sees that the b_{ij} are null for $i \neq j$. One then sees that one has:

$$\Omega_i^j = - b_{ii} [\omega^i \omega^j].$$

However, the relation $\Omega_i^j + \Omega_i^j = 0$ shows that the n coefficients b_{ii} are all equal, in such a way that one has:

$$\Omega_i^j = - c[\omega^i \omega^j].$$

From a classical theorem, which is easy to prove moreover, c is a constant, in such a way that the *Riemannian manifold in question has constant curvature*.

This conclusion is also true if $n = 2$, but to show this, it is necessary to take into account the expressions for Π_1^0 and Π_2^0:

$$\Pi_1^0 = (\omega_1^0)' - [\omega_1^2 \omega_2^0] = c\{(\omega^1)' - [\omega_1^2 \omega^2]\} + [dc\ \omega^1] = [dc\ \omega^1],$$
$$\Pi_2^0 = (\omega_2^0)' - [\omega_2^1 \omega_1^0] = c\{(\omega^2)' - [\omega_2^1 \omega^1]\} + [dc\ \omega^2] = [dc\ \omega^2].$$

These two expressions are null only when $dc = 0$.

We thus recover the classical theorem that says *the only Riemannian manifolds that are geodesically representable in Euclidian space are manifolds with constant curvature.*

16. The preceding considerations lead us quite naturally to the projective definition, which is due to Cayley, of manifolds with constant curvature. The simplest *analytical* manner of presenting this definition consists of taking an (*absolutely*) non-degenerate quadratic form in a projective space. Let:

$$\Phi(\mathrm{X}, \mathrm{X}^1, \ldots, \mathrm{X}^n) = 0$$

be such an equation. If one agrees to endow each point of the space with coordinates that satisfy the relation $\Phi = k$, where k is a give constant, then the desired ds^2 is, quite simply:

$$ds^2 = \Phi(d\mathrm{X}, d\mathrm{X}^1, \ldots, d\mathrm{X}^n).$$

Now consider, in a Riemannian manifold of constant curvature, the normal projective connection that makes this manifold into a projective space, properly speaking, and whose components are:

$$\varpi^i = \omega^i, \quad \varpi_i^i - \varpi_0^0 = 0, \quad \varpi_i^j = \omega_i^j, \quad \varpi_i^0 = c\ \omega^i.$$

We likewise suppose – as is permissible – that $\varpi_0^0 = 0$. Having said this, one easily verifies that if one considers the coordinates $(x, x^1, \ldots, x^n)$, in different frames, of a fixed point of projective space then the quantity:

$$(x^1)^2 + (x^2)^2 + \ldots + (x^n)^2 - \frac{1}{c}(x)^2 .$$

This results from formulas (4), which give the variation of the x^i when one passes from one frame to a neighboring one, formulas in which one naturally replaces the letter ω with the letter ϖ. Thus, if one chooses one of the frames to be a *fixed* frame (with coordinates X^i) then one will have the following coordinates (x^i) of a point when referred to the frame attached to a given point **a**:

$$(x^1)^2 + (x^2)^2 + \ldots + (x^n)^2 - \frac{1}{c}(x)^2 = (X^1)^2 + (X^2)^2 + \ldots - \frac{1}{c}(X)^2. \tag{17}$$

In particular, the coordinates (X^i) of the point **a** itself satisfy the relation:

$$(X^1)^2 + \ldots + (X^n)^2 - \frac{1}{c}(X)^2 = -\frac{1}{c}.$$

On the other hand, the relation (17) entails the following relation for two infinitely close points in the space:

$$(dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2 - \frac{1}{c}(dx)^2 = (dX^1)^2 + (dX^2)^2 + \ldots - \frac{1}{c}(dX)^2.$$

We apply this to the point **a** and the infinitely close point **a′**. We have:

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 + \ldots + (\omega^n)^2 = (dX^1)^2 + (dX^2)^2 + \ldots - \frac{1}{c}(dX)^2.$$

We recover Cayley's projective definition, with the equation:

$$\Phi(X, X^1, \ldots, X^n) = (X^1)^2 + (X^2)^2 + \ldots - \frac{1}{c}(X)^2 = 0$$

for the absolute, and the value $\frac{1}{c}$ for the constant k that corresponds to the given constant curvature.

VII. – THE NOTION OF A MANIFOLD OF ELEMENTS WITH A PROJECTIVE CONNECTION.

17. The system of differential equations (13) is not the most general system of second order ordinary differential equations in $n - 1$ unknown functions. The curves of an n-dimensional numerical manifold that are defined by a system that is not of the form (13) may not be regarded as the geodesics of a manifold with projective connection. One may demand that one exists by a generalization that permits us to regard them, among others, as geodesics. We shall not occupy ourselves with the general problem, but only content ourselves with the simplest case $n = 2$, i.e., the case of integral curves of a second order differential equation:

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

which we may always suppose to be written in the form:

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right). \tag{18}$$

The theory of differential invariants of that equation, vis-à-vis the group of point-like transformations at (x, y), have been the object of an important memoir of Tresse ([1]). We shall see that the notion of projective connection permits us to give that theory a simple geometric form.

We start with the notion of *element*, which we regard as the set of a point and a direction through that point, and analytically let (x, y) define the point and the value y that $\frac{dy}{dx}$ takes on when it is displaced in the given direction. The set of all such elements (x, y, y') constitutes a three-dimensional *manifold of elements*.

Having said this, imagine that each element (x, y, y') is attached to a projective plane that contains this element. We will have endowed the manifold of elements in question with a projective connection if we give it a law that permits us to associate the projective planes attached to two infinitely close elements. This law will be arbitrary, but it must nevertheless satisfy the following conditions:

a. If one considers a *multiplicity* (in the sense of S. Lie) in the manifold of elements, i.e., a continuous one-parameter family of elements that satisfy the equation:

$$dy - y'\,dx = 0,$$

([1]) *See* the introduction.

then this multiplicity remains a multiplicity when one successively associates the projective planes attached to the various elements of the given multiplicity ([1]).

b. We choose the frame attached to an element **e** of the manifold in the following manner: Since the element **e** belongs to the projective plane that is attached to it, we take the vertex **a** of the frame to be the *point* of the element **e**, and we restrict ourselves to taking the vertex $\mathbf{a}_1$ on the line that, along with the point **a**, constitutes the element in question of the projective plane. Then let:

$$d\mathbf{a} = \omega_0^0\mathbf{a} + \omega^1\mathbf{a}_1 + \omega^2\mathbf{a}_2,$$
$$d\mathbf{a}_1 = \omega_1^0\mathbf{a} + \omega_1^1\mathbf{a}_1 + \omega_1^2\mathbf{a}_2,$$
$$d\mathbf{a}_2 = \omega_2^0\mathbf{a} + \omega_2^1\mathbf{a}_1 + \omega_2^2\mathbf{a}_2,$$

be the equations that define the projective connection on the manifold. The components ω_k^j are linear in dx, dy, dy' (and also contain the differentials of the parameters that the frame attached to each element of the manifold might possibly depend upon).

We now express the fact that the conditions *a* and *b* are verified. If one displaces along an arbitrary multiplicity of the manifold then it is necessary that the point $d\mathbf{a}$ be on the line $\mathbf{aa}_1$, in other words, that the expression ω^2 be null. As a result, *ω^2 is annulled along with $dy - y'dx$.*

Now, if the point **a** of the manifold remains fixed then it is necessary that the point $d\mathbf{a}$ agrees with **a** geometrically. In other words, ω^1 and ω^2 are linear combinations of the dx and dy. Therefore, *the form ω^1 depends linearly only on dx and dy.*

We add the following remark: If the element **e** remains fixed then the point **a** and the line $\mathbf{aa}_1$ must remain fixed. In other words ω^1, ω^2, and ω_1^2 are annulled with dx, dy, dy', or finally, *the form ω_1^2 depends linearly upon only the dx, dy, dy'*. Moreover, this remark is interesting only if the frame attached to an element of the manifold depends on variable parameters. If this is not true then the forms ω^1 and ω^2 satisfy the conditions that were stated above, and the three forms ω^1, ω^2, and ω_1^2 are linearly independent.

VIII. – MANIFOLDS OF ELEMENTS WITH NORMAL PROJECTIVE CONNECTIONS.

19. We use the term *geodesics* of a manifold of elements with a projective connection to refer to the curves in the manifold that *develop* onto a projective plane along straight lines. These curves are naturally regarded as the point-like supports of multiplicities. If **a** is a point of the curve and $\mathbf{aa}_1$, its tangent, then the curve will be a geodesic if $d\mathbf{a}$ and $d\mathbf{a}_1$ are both one the line $\mathbf{aa}_1$, which gives:

([1]) One may express this in a more intuitive manner by saying that any multiplicity in the manifold is *developed* onto a projective plane along a multiplicity of that plane. Naturally, one may not speak of developing a figure in the manifold that has two parameters.

$$\omega^2 = 0, \qquad \omega_1^2 = 0.$$

These equations are reducible to:

$$dy - y'\,dx = 0, \qquad dy' - f(x, y, y') = 0,$$

and the geodesics are the integral curves of the differential equations:

(18) $$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right).$$

Conversely, the integral curves of an arbitrary second order ordinary differential equation may be regarded as the geodesics of the manifold endowed with a convenient projective connection; indeed, it suffices to take:

$$\omega^1 = \alpha\,dx + \beta\,dy, \qquad \omega^2 = \alpha(dy - y'\,dx),$$
$$\omega_1^2 = \lambda(dy - y'\,dx) + \mu(dy' - f\,dx),$$

with the other ω being arbitrary.

Among all of the projective connections that are defined by a two-parameter family of curves (C), might one distinguish one of them by intrinsic properties? WE shall see that this is effectively true, which will lead us to the notion of a *manifold of elements with a normal projective connection.*

First, we make a remark. No matter what projective connection is on the manifold, one may always suppose that one has:

(19) $$\omega^1 = dx, \qquad \omega^2 = dy - y'\,dx, \qquad \omega_1^2 = k(dy' - f\,dx).$$

Indeed, the multiplication of **a**, $\mathbf{a}_1$, $\mathbf{a}_2$ by the same factor does not change the components of the projective connection so any change in the frame may be realized by preserving **a**. If we order $d\mathbf{a}$ according to dy', dx, $dy - y'dx$ then it suffices to take **a** to be the coefficient of dx and $\mathbf{a}_1$ to be the coefficient of $dy - y'dx$, and one thus reduces ω^1 to dx and ω^2 to $dy - y'dx$. One likewise see that one may further add arbitrary multiples of **a** to $\mathbf{a}_1$ and $\mathbf{a}_2$.

One may now obviously dispose of λ in such a manner that the coefficient of $\mathbf{a}_2$ in $d(\mathbf{a}_1 + \lambda\,\mathbf{a})$, a coefficient that is:

$$\omega_1^2 + \lambda\omega^2,$$

must be proportional to $dy - y'dx$. The proposition is thus proved.

One sees that the frame is no longer completely determined, since one may replace $\mathbf{a}_2$ with $\mathbf{a}_2 + h\,\mathbf{a}$, in which h is an arbitrary coefficient. This coefficient h has no influence on ω_1^1, but it modifies ω_0^0 by the quantity $h\omega^2$. *One may thus arrange things so that one modifies the components* $\omega_1^1 - \omega_0^0$ *by a quantity of the form* $h\omega^2$.

20. We shall now proceed in the following manner to *normalize* the projective connection of the manifold. First, we calculate the component Ω^2 of the curvature:

$$\begin{aligned}\Omega^2 &= (\omega^2)' - [\omega^1\omega_1^2] - [\omega^2(\omega_2^2 - \omega_0^0)] \\ &= (1 - k)\,[dx\,dy'] - [(dy - y'\,dx)\,(\omega_2^2 - \omega_0^0)].\end{aligned}$$

It is always possible to specialize the projective connection in such a manner that one annuls Ω^2. In order to do this, it is necessary to take $k = 1$ and:

$$\omega_2^2 - \omega_0^0 = u\,(dy - y'\,dx).$$

We may likewise annul Ω^1:

$$\Omega^1 = (\omega^2)' - [\omega^2\omega_2^1] - [\omega^1(\omega_1^1 - \omega_0^0)] = -[dx(\omega_1^1 - \omega_0^0)] - [(dy - y'dx)\omega_2^1].$$

Since one may, from the remark made above, modify $\omega_1^1 - \omega_0^0$ by an arbitrary multiple of $dy - y'\,dx$ we may suppose that:

$$\begin{aligned}\omega_1^1 - \omega_0^0 &= v\,dx, \\ \omega_2^1 &= w\,(dy - y'\,dx).\end{aligned}$$

From now one, the frame is chosen unambiguously (except that one must be given a projective connection that annuls Ω^2 and Ω^1).

One now has:

$$\begin{aligned}\Omega_1^2 &= (\omega_1^2)' + [\omega^2\omega_1^0] - [\omega_1^2(\omega_2^2 - \omega_1^1)] \\ &= [dx\,df] - u[(dy' - f\,dx)(dy - y'\,dx)] - v\,[dx\,dy] + [(dy - y'\,dx)\,\omega_1^0].\end{aligned}$$

One may further annul the component Ω_1^2 of the curvature with the condition that one take:

$$v = \frac{\partial f}{\partial y'},$$

$$\omega_1^0 = \frac{\partial f}{\partial y}\,dx - u\,(dy' - f\,dx) + \lambda\,(dy - y'\,dx).$$

One likewise has:

$$\begin{aligned}\Omega_1^1 + \Omega_1^2 - 2\Omega_0^0 &= (\omega_1^1 + \omega_2^2 - 2\omega_0^0)' + 3[\omega^1\omega_1^0] + 3[\omega^2\omega_2^0] \\ &= \left[d\frac{\partial f}{\partial y'}\,dx\right] + [du(dy - y'\,dx)] - 2u\,[dx\,dy'] \\ &\quad + 3\lambda[dx\,(dy - y'\,dx)] + 3[(dy - y'\,dx)\,\omega_2^0],\end{aligned}$$

and one may annul the right-hand side by taking:

$$u = -\frac{1}{2}\frac{\partial^2 f}{\partial y'^2}$$

$$\omega_2^0 = \lambda\, dx - \frac{1}{3}\frac{\partial^2 f}{\partial y \partial y'}\, dx - \frac{1}{6}\, d\frac{\partial^2 f}{\partial y'^2} + u\,(dy - y'\, dx).$$

Finally, one has:

$$\begin{aligned}\Omega_1^1 - \Omega_0^0 &= (\omega_1^1 - \omega_0^0)' + [\omega_1^2 \omega_2^1] + 2[\omega^1 \omega_1^0] + [\omega^2 \omega_2^0] \\ &= \left[d\frac{\partial f}{\partial y'}\, dx\right] - w[(dy' - f\, dx)\,(dy - y'\, dx)] \\ &\qquad + 3\lambda[dx\,(dy - y'\, dx)] + 3[(dy - y'\, dx)\,\omega_2^0], \\ &\qquad + \frac{1}{3}\frac{\partial^2 f}{\partial y \partial y'}[dx\,(dy - y'\, dx)] + \frac{1}{6}\left[d\frac{\partial^2 f}{\partial y'^2}(dy - y' dx)\right].\end{aligned}$$

One may further annul this expression by taking:

$$w = \frac{1}{6}\, d\frac{\partial^2 f}{\partial y'^2},$$

$$\lambda = \frac{2}{3}\frac{\partial^2 f}{\partial y \partial y'} - \frac{d}{dx}\left(\frac{\partial^2 f}{\partial y'^2}\right).$$

In summation, we have obtained the following expressions for the components of the projective connection:

$$(20)\qquad \begin{cases} \omega^1 = dx, \qquad \omega^2 = dy - y' dx, \\ \omega_1^1 - \omega_0^0 = \dfrac{\partial f}{\partial y'}\, dx, \quad \omega_2^2 - \omega_0^0 = -\dfrac{1}{2}\dfrac{\partial^2 f}{\partial y'^2}(dy - y' dx), \\ \omega_1^2 = dy - y' dx, \quad \omega_2^1 = \dfrac{1}{6}\dfrac{\partial^2 f}{\partial y'^2}(dy - y' dx) \\ \omega_1^0 = \dfrac{\partial f}{\partial y}\, dx + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial y'^2}(dy' - f dx) + \left(\dfrac{2}{3}\dfrac{\partial^2 f}{\partial y \partial y'} - \dfrac{1}{6}\dfrac{d}{dx}\dfrac{\partial^2 f}{\partial y'^2}\right)(dy - y' dx), \\ \omega_2^0 = \left(\dfrac{1}{3}\dfrac{\partial^2 f}{\partial y \partial y'} - \dfrac{1}{6}\dfrac{d}{dx}\dfrac{\partial^2 f}{\partial y'^2}\right) dx - \dfrac{1}{6}\, d\dfrac{\partial^2 f}{\partial y'^2} + \mu(dy - y' dx), \end{cases}$$

with an arbitrary coefficient μ, when one realizes the conditions:

$$(21)\qquad \Omega^2 = 0, \qquad \Omega^1 = 0, \qquad \Omega_1^2 = 0, \qquad \Omega_1^1 - \Omega_0^0 = 0, \quad \Omega_2^2 - \Omega_0^0 \neq 0.$$

21. Before going further, note that relations (21) permit us to foresee the sort of expressions we should expect for the remaining components $\Omega_2^1, \Omega_1^0, \Omega_2^0$ of the curvature of the manifold. The identities (7) immediately give us:

$$[\omega^2\Omega_2^1] = 0, \qquad [\omega^2\Omega_1^0] = 0,$$
$$[\omega^1\Omega_1^0]+[\omega^2\Omega_2^0] = 0, \quad [\omega_1^2\Omega_2^1] - [\omega^1\Omega_1^0] = 0,$$

from which, one deduces the general expressions:

$$\begin{aligned}\Omega_2^1 &= a[\omega^2\omega_1^2] + r[\omega^1\omega^2]\\ \Omega_1^0 &= b[\omega^1\omega^2] + r[\omega^2\omega_1^2]\\ \Omega_2^0 &= h[\omega^1\omega^2] + k[\omega^2\omega_1^2] + r[\omega^1\omega_1^2].\end{aligned}$$

We shall se that one may dispose of the arbitrary coefficient μ by annulling r. Indeed, one has:

$$\begin{aligned}\Omega_2^1 &= (\omega_2^1)' + [\omega^1\omega_1^2] + [\omega_2^1(\omega_2^2 - \omega_0^0)]\\ &= \frac{1}{6}\frac{\partial^3 f}{\partial y'^3}[dx\,dy'] + \frac{1}{6}\left[d\frac{\partial^3 f}{\partial y'^3}(dy - y'dx)\right] - \frac{1}{6}\left[dx\,d\frac{\partial^2 f}{\partial y'^2}\right]\\ &\quad + \mu[dx\,(dy - y'\,dx)] + \frac{1}{6}\frac{\partial f}{\partial y'}\frac{\partial^3 f}{\partial y'^3}[dx\,(dy - y'dx)].\end{aligned}$$

First of all, one indeed verifies that the right-hand side is annulled with $dy - y'dx$, i.e., with ω^2. As for the coefficient denoted by r, it is the coefficient of $[dx\ dy]$ in the right-hand side when one replaces dy' with $f\,dx$:

$$r = \mu - \frac{1}{6}\frac{\partial^3 f}{\partial y\partial y'^2} + \frac{1}{6}\frac{\partial f}{\partial y'}\frac{\partial^3 f}{\partial y'^3} + \frac{1}{6}\frac{d}{dx}\frac{\partial^3 f}{\partial y'^3}.$$

We can take the following value for μ:

(22) $$\mu = \frac{1}{6}\frac{\partial^3 f}{\partial y\partial y'^2} - \frac{1}{6}\frac{\partial f}{\partial y'}\frac{\partial^3 f}{\partial y'^3} - \frac{1}{6}\frac{d}{dx}\frac{\partial^3 f}{\partial y'^3}.$$

The projective connection of the manifold is thus unambiguously determined, with expressions of the form:

$$\Omega_2^1 = a[\omega^2\omega_1^2], \qquad \Omega_1^0 = b[\omega^1\omega^2], \qquad \Omega_2^0 = h[\omega^1\omega^2] + k[\omega^2\omega_1^2]$$

for $\Omega_2^1, \Omega_1^0, \Omega_2^0$.

Moreover, one immediately finds:

(23) $$a = -\frac{1}{6}\frac{\partial^4 f}{\partial y'^4},$$

and it would serve no useful purpose to calculate b, h, k explicitly.

22. An important question remains to be answered. We have specialized the projective connection of the manifold by starting with a particular choice of frame. Is the result obtained independent of the choice of frame? In other words, *do the properties of the projective connection expressed by the relations* (21), *combined with the condition* $r = 0$, *have an invariant character?* (under any change of the coordinates x, y).

First, as one easily verifies, the property $\Omega^2 = 0$ signifies that under the infinitesimal projective displacement that is associated with an infinitesimal closed contour of elements starting with an element **e**, the element **e′**, which is the transform of **e**, is *identical* with **e**; this is obviously an *invariant* property.

In the second place, the conditions:

$$\Omega^1 = 0, \qquad \Omega^2 = 0, \qquad \Omega_1^2 = 0$$

express the fact the element **e** remains invariant under the infinitesimal displacement that we just considered (i.e., there is no *torsion*); this is an invariant property. The condition $\Omega_1^1 - \Omega_0^0 = 0$ then expresses the fact that there exists no invariant *isolated* point outside of the point **a** on the line $\mathbf{aa}_1$. The condition $\Omega_2^2 - \Omega_0^0 = 0$ then expresses the fact that there exists no invariant *isolated* point outside of the line $\mathbf{aa}_1$.

The relations (21) thus have essentially an invariant character that is independent of any choice of frame.

The same is true for the relation $r = 0$. One confirms the following geometric significance that one may attribute to it: If one considers an infinitesimal (open) linear family of elements and one closes that family by demanding that at the point (x, y) the *same* path is traversed in the opposite sense so that the direction y' of the element goes from its final value to its initial value *without repeating the same intermediary values*, then this closed contour of elements is associated with an infinitesimal projective displacement (in the projective space attached to the initial element). The relation $r = 0$ signifies that all of the points of the line $\mathbf{aa}_1$ are invariant under this displacement. Thus has an invariant significance indeed.

In conclusion, we thus see that *to any second ordinary differential equation* $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$ *there is associated, in an invariant manner, a manifold of elements with projective connection such that the integral curves of the differential equations are geodesics.* We say that the unique projective connection thus obtained is *normal*. We have previously indicated the geometric properties that characterize normal projective connections.

The problem that was treated by Tresse may thus be stated geometrically as follows: *To study the geometric properties of the manifold of elements with normal projective connection.*

23. In projective geometry, the notion of an *element* is its own proper dual, as well as the notion of *multiplicity*. As a result, any manifold of elements with a projective connection is transformed by duality into another manifold of elements with a projective connection such that the *points* of the former coincide with the *geodesics* of the latter, and conversely. If one denotes the components of the latter projective connection by the letter ϖ then one has, as is easily seen to be true:

$$\begin{array}{lll} \varpi_0^0 = \omega_2^2, & \varpi^1 = \omega_1^2, & \varpi^2 = \omega^2, \\ \varpi_1^0 = \omega_2^1, & \varpi_1^1 = \omega_1^1, & \varpi_1^2 = \omega^1, \\ \varpi_2^0 = \omega_2^0, & \varpi_2^1 = \omega_1^0, & \varpi_2^2 = \omega_0^0. \end{array}$$

Relations (21), which relate to a normal projective connection, are thus transformed under duality into:

$$\Pi^2 = 0, \quad \Pi_1^2 = 0, \qquad \Pi^1 = 0, \qquad \Pi_1^1 - \Pi_2^2 = 0, \quad \Pi_0^0 - \Pi_2^2 = 0;$$

they preserve the same form. As for the condition that the coefficient r of $[\omega^1 \omega^2]$ in Ω_2^1 is null, it becomes the condition that the coefficient of $[\varpi^2 \varpi_1^2]$ in Π_1^0 be null. In other words, *the dual of a manifold of elements with a normal projective connection is again a manifold of elements with a normal projective connection*.

The relation that exists between the two families of geodesics of two dual normal manifolds is obvious. If:

$$F(x, y, a, b) = 0$$

is the general equation of the geodesics on the former manifold when on regards x and y as point-like variables and a, b as arbitrary constants, then it is also the equation of the latter manifold, with the condition that one regard a and b as the point-like variables and x and y as the arbitrary constants. Thus, the relation between dual normal manifolds analytically translates into a certain correspondence between two second order ordinary differential equations (or rather, between two classes of differential equations that one obtains by transforms each of them by an arbitrary point-like transformation). This correspondence has already been studied by A. Koppisch ([1]) in its purely analytical aspect.

24. A particularly remarkable case is the one in which the coefficient a in the form Ω_2^1 is identically null. The identities (7) then give:

$$[\omega^1 \Omega_2^0] = k[\omega^1 \omega^2 \omega_1^2] = 0.$$

In other words, the only non-null components Ω_1^0, Ω_2^0 of the curvature of the manifold are of the form:

$$\Omega_1^0 = b[\omega^1 \omega^2], \qquad \Omega_2^0 = h[\omega^1 \omega^2].$$

They do not involve ω_1^2; they are simply proportional to $[dx\ dy]$. This result may be geometrically interpreted by saying that the relationship between two projective planes

([1]) A. KOPPISCH, *Zur Invariantentheorie der gewöhnlichen Differentialgleichungen zweiter Ordnung. Inaugural Dissertation.* Leipzig, B.-G. Teubner, 1905. *See* also the *Inaugural Dissertation* of A. KAISER; Leipzig, B.-G. Teubner, 1913.

that are attached to two given elements **e** and **e**′ of the manifold depends only on the initial element, the final element, and the intermediary path that is followed by the *point* of the moving element, but they do not depend on law that determines the change in the *direction* of that element. In other words, the manifold of elements is based on a *point-like* manifold with a projective connection, in the sense of the first part of this memoir [with the accessory condition that one has taken a frame at each point (x, y) that depends on a parameter y']. The geodesics of the given manifold are then the geodesics of a point-like manifold with a projective connection. Indeed, this also results immediately from the preceding expression that we found for the coefficient a. If the coefficient is null then the geodesic equation has the form:

$$\frac{d^2y}{dx^2} = \mathrm{A} + 3B\frac{dy}{dx} + 3C\left(\frac{dy}{dx}\right)^2 + D\left(\frac{dy}{dx}\right)^3,$$

which characterizes the curves that are susceptible to being geodesics of a point-like manifold with a projective connection. One may add that the projective connection that was determined is, moreover, *normal*, in the old sense of the word, since Ω^i, $\Omega_i^i - \Omega_0^0$, and Ω_i^j are all null expressions. In other words, *in the case of* $\Omega_2^1 = 0$, *a manifold with a normal projective connection reduces to a point-like manifold with a normal projective connection.*

If one has both $a = 0$, $b = 0$, i.e.:

$$\Omega_2^1 = 0, \qquad \Omega_1^0 = 0,$$

then the identities (7) show that the two coefficients h and k are null, and the manifold reduces to the projective plane. In other words, the differentiation of the geodesics (as well as the dual equation) will be reducible to:

$$\frac{d^2y}{dx^2} = 0.$$

Formulas (20) and (22), which give the components of the projective connection, permit us to calculate b, which gives, in the particular case where a is null:

$$\begin{aligned} b = &\left(2\frac{\partial^2 C}{\partial x\partial y} - \frac{\partial^2 B}{\partial y^2} - \frac{\partial^2 D}{\partial x^2} + 2D\frac{\partial A}{\partial y} + A\frac{\partial D}{\partial y}\right. \\ &\qquad \left. -3D\frac{\partial B}{\partial x} - 3B\frac{\partial D}{\partial x} - 3C\frac{\partial B}{\partial y} + 6C\frac{\partial C}{\partial x}\right) \\ &+\left(2\frac{\partial^2 B}{\partial x\partial y} - \frac{\partial^2 C}{\partial x^2} - \frac{\partial^2 A}{\partial y^2} - 2A\frac{\partial D}{\partial x} - D\frac{\partial A}{\partial x}\right. \\ &\qquad \left. +3A\frac{\partial C}{\partial y} + 3C\frac{\partial A}{\partial y} + 3B\frac{\partial C}{\partial x} - 6B\frac{\partial B}{\partial y}\right) y'. \end{aligned}$$

One immediately derives the *two* conditions that the coefficients A, B, C, D in the given differential equation must satisfy in order for it to be reducible to the equation $\frac{d^2y}{dx^2} = 0$ ([1]).

25. We return to the general case. We leave to the reader the task of verifying the existence of the *integral invariants:*

$$\int \sqrt[4]{ab}\,\omega^2, \quad \iiint \sqrt{ab}\,\omega^1\omega^2\omega_1^2, \quad \iint a^{-\frac{1}{8}}b^{-\frac{5}{8}}\omega^1\omega^2, \quad \iint a^{-\frac{5}{8}}b^{-\frac{1}{8}}\omega^2\omega_1^2 .$$

We also content ourselves with pointing out that in a manifold of elements with a normal projective connection, one may, as one does in the projective plane, construct a theory of (projective) differential invariants of curves, determine the fifth order differential equation of the curves that are *developed* onto the projective plane along a cone; i.e., in summary, it plays the role, vis-à-vis the given two-parameter family of curves, that is played by cones vis-à-vis lines, etc. We also content ourselves by indicating that there is a possible generalization of the preceding theory for an arbitrary number of dimensions.

([1]) The conditions are due to A. Tresse, on page 56 of his memoir cited above. *See* also, A. KOPPISCH, *loc. cit.*, pp. 17.