"La structure des groupes infinis," Séminaire de Mathématiques, 4th year, 1926-1937; Oeuvres Complètes, v. 2, part II, 1336-1384.

THE STRUCTURE OF INFINITE GROUPS ([1])

By Élie CARTAN

(*Séminaire de Mathématiques*, 4th year, 1936-1937)

Translated by D. H. Delphenich

At this conference and in what follows, I propose to present the general ideas in the theory of finite and infinite continuous groups that I have developed in two memoirs to the *Annales de l'École Normale* (1904-1905, and 1908). The groups in question will be the ones that were considered by S. Lie; they are the groups of analytic transformations that act on a finite number of variables and are characterized by the property that the most general transformation of the group is the general solution of a system of partial differential equations that give the transformed variables as unknown functions of the original variables. The finite and continuous Lie groups belong to that general class because any system of functions in n variables and a certain number of arbitrary constants constitutes the general solution of a completely integrable system. However, the group:

$$x' = f(x), \qquad y' = f(y),$$

which was pointed out by Lie himself, where $f(x)$ is an arbitrary analytic function of its argument, *is not* a Lie group in the preceding sense. Moreover, only Lie groups ever enter into the applications that one makes of the theory of groups to differential systems, in reality.

The generalization to infinite groups of the structure theory of finite groups that was due to Lie and was founded upon the consideration of infinitesimal transformations is shown to be very difficult, which is not to say impossible, despite the works by S. Lie, F. Engel, Medolaghi, etc., that were dedicated to this question. What will explained here begins with a completely different principle: It is in the defining equations of the group – when put into a convenient form – that one may find a point of departure for the theory that uses the theory of equivalence problems that was presented in an earlier conference and the theory of systems in involution.

The notion of *abstract group* will not be presented here with the same purity as in the case of finite group, and this comes down to the difficulty in finding a simple analytical characterization of the notion of isomorphism. It is remarkable that, simultaneously,

---

([1]) Presented by Élie Cartan, Monday, 1 March 1937.

Vessiot, in his beautiful works on automorphic functions, and myself have been led to the same new definition of the isomorphism of two Lie groups, a definition that is, moreover, equivalent to the classical definition in the case of finite groups. This definition rests upon on the notion of the *prolongation* of a group. If one is given a group G that operates on n variables $x^1, x^2, \ldots, x^n$ then a group G' will be called a prolongation of G if it operates on the same variables $x^1, x^2, \ldots, x^n$, but at the same time, on other some variables $y^1, y^2, \ldots, y^p$, in such a way that it transforms the x variables amongst themselves in the same manner as the group G. Thus, a transformation of G corresponds to at least one transformation of G', so the prolongation is called *holomorphic* if it corresponds to only one, and in this case there is a bijective correspondence between the transformations of the two groups. In the contrary case, the prolongation is called *meromorphic*. It is clear that in the former case, there is a holomorphic isomorphism – in the classical sense of the word – between G and G'; in the latter case, G is meromorphically isomorphic to G'.

Having said this, two groups G_1 and G_2 are called *isomorphic* (*holomorphic*) if they admit two similar holomorphic prolongations (i.e., they have the same number of variables and are reducible to each other by a change of variables); if there exists a holomorphic prolongation of G_1 that is similar to a meromorphic prolongation of G_2 then we say that G_2 is *meromorphically isomorphic* to G_1 . One proves without difficulty that two groups that are holomorphically isomorphic to a third one are isomorphic and that if G_1 is holomorphically isomorphic to G_2 and G_2 is meromorphically isomorphic to G_3 then G_1 is meromorphically isomorphic of G_3 .

The fundamental theorem.

The theorem that is at the basis for the theory of Lie groups is the following one:

Any Lie group G admits a holomorphic prolongation that operates on a certain number r of variables x^i and is defined as the set of transformations that leave invariant:

1. *A certain number of functions of the x*;
2. *r Pfaff forms $\omega^j(x, y, z)$ that are linearly independent with respect to the differentials dx_i and whose coefficients may depend upon other auxiliary variables y_r ; finally, the defining equations of the prolongation considered have first order.*

The hypotheses that were made on the prolonged group show that this prolonged group is a Lie group; we confirm that one may always assume that it is defined by a first-order system of partial differential equations.

We pass on to the proof. By definition, the given group, being a Lie group, is comprised of the set of solutions of a system of partial differential equations (viz., the defining equations) that one may always assume are in involution. Let $x^1, x^2, \ldots, x^n$ be the initial variables and let $X^1, X^2, \ldots, X^n$ be the transformed variables. To begin with, the defining equations possibly contain a certain number $n - \nu$ of finite relations between the x and the X, relations that one may assume have been solved with respect to $X^{n-\nu+1}, \ldots, X^n$:

$$X^{n-\nu} = F^k(x^1, \ldots, x^n; X^1, \ldots, X^\nu) \qquad (k = 1, 2, \ldots, n), \tag{1}$$

where the coefficients a_k^i are analytic functions of $x^1, \dots, x^n$; $X^1, \dots, X^\nu$ and of p_1 variables u, upon assuming that the first-order equations may be solved for $n\nu - p_1$ of them as functions of p_1 of the other x and X. If the system contains second-order equations then one may write them in the form:

(3) $$du^h = b_k^h(x, X, u, v)\,dx^k \qquad (h = 1, 2, \dots, p_1),$$

upon introducing p_2 new variables v, and so on. We thus have a series of systems (1), (2), (3), …; if the group G is finite then the latter system of equations will not introduce any new variables.

Before everything else, we remark that equations (1) may be simplified. Indeed, perform a specific transformation of the group on the x and let $\bar{x}$ be the transformed variables. One may naturally pass from the $\bar{x}$ to the X by a transformation of the group, and one will have, as a result:

(4) $$F^k(x^1, \dots, x^n; X^1, \dots, X^\nu) = F^k(\bar{x}^1, \dots, \bar{x}^n; X^1, \dots, X^\nu) \qquad (k = 1, 2, \dots, \nu).$$

These relations become identities in $x^1, \dots, x^n$; $X^1, \dots, X^\nu$ if one replaces the $\bar{x}$ in them with their values, or else one will have at least one non-identical relation:

$$\varphi^k(x^1, \dots, x^n; X^1, \dots, X^\nu) = 0.$$

Now, there always exists a transformation of the group that takes arbitrarily given values of the x to arbitrarily given values of the $X^1, \dots, X^\nu$, which leads to a contradiction. We add that the functions $F^k(x, X)$ are independent when regarded as functions of the x; otherwise, from equations (1), one could deduce at least one non-identical relation in $X^1, \dots, X^\nu, X^{\nu+1}, \dots, X^n$, which is absurd.

It results from these remarks that if one gives the X fixed numerical values X_0 (that are not too specific) then this will show that the ν functions $F^k(x^1, \dots, x^n; X^1, \dots, X^\nu)$ are invariants of the group and number $n - \nu$. One may then assume that the variables are chosen in such a manner that these invariants are $x^{\nu+1}, \dots, x^n$, in such a way that equations (1) take the form:

(1′) $$X^{n+\nu} = x^{n+\nu} \qquad (k = 1, \dots, n - \nu).$$

Having posed these preliminaries, imagine that one has made a change of independent variables in the differential system (1), (2), (3), …, namely, the change that is defined by a specific transformation S of the group G, such as:

(5) $$\bar{x}^i \equiv \varphi^k(x^1, x^2, \dots, x^n) \qquad (\varphi^{\nu+1} \equiv x^{\nu+1}, \dots, \varphi^n \equiv x^n).$$

The system (1), (2), (3), … necessarily preserves the same form with these new independent variables, since if one considers an arbitrary transformation T of the group that takes the x to the X then the transformation TS^{-1} will take the $\bar{x}$ to the X, except that the new equations that involve the partial derivatives (u, v, …) will be changed. We will obviously have the following relations, which are deduced from (2):

$$a_k^i(\bar{x}, X, \bar{u})\frac{\partial \bar{x}^k}{\partial x^h} = a_k^i(x, X, u). \tag{6}$$

One may regard these relations as equations in $\bar{u}^1, \ldots, \bar{u}^{p_1}$. [One must assume that the $\bar{x}$ and the $\partial \bar{x}^k / \partial x^h$ have been replaced with by their values $\varphi^k(x)$ and $\partial \varphi^k / \partial x^h$.] In these equations, we regard the x, the X, and the u as *independent* arguments. It is easy to see that these equations, *which are* νn *in number and have* p_1 *unknowns* u, *are compatible.* Indeed, otherwise there would exist at least one non-identical relation:

$$\psi(x, X, u) = 0.$$

That relation must have meaning no matter what numerical values are given to the arguments $x^1, \ldots, x^n, X^1, \ldots, X^n, u^1, \ldots, u^{p_1}$. This is impossible: Indeed, there always exists at least one solution of the system (1), (2), (3), … in G that corresponds to arbitrarily given initial values for these $n + \nu + p_1$ quantities. Equations (6) are thus soluble for the $\bar{u}$:

$$\bar{u}^k = \psi^k(x, X, u). \tag{7}$$

One may continue the argument by passing from equations (2) to equations (3), which will give formulas:

$$\bar{v}^h = \chi^k(x, X, u, v), \tag{8}$$

and so on. We thus have to add equations (7) and (8) to equations (5). One may likewise ultimately add the relations:

$$\bar{X}^i = X^i \qquad (i = 1, 2, \ldots, \nu).$$

Any transformation S *of the group* G *may thus be prolonged in one and only one way to a transformation* Σ *that acts on the variables* $x, X, u, v, \ldots$, *and from the manner itself by which this prolongation was performed, this prolonged transformation enjoys the following properties:*

1. *It leaves invariant the variables* $x^{\nu+1}, \ldots, x^n, X^1, X^2, \ldots, X^n$.
2. *It leaves invariant the differential system* (2), (3), *etc., of the defining equations for the group* G.

Conversely, take a transformation Σ that enjoys the preceding properties. The invariance of the dX^i, on the one hand, and that of the differential system (2), (3), …, on the other hand, shows that Σ leaves invariant the forms $\omega^i = a_k^i(x, X, u, dx^k)$; as a result, any differential $d\bar{x}^i$ is a linear combination of the dx^k, and the transformed variables $\bar{x}^i$ depend only upon the $x^1, x^2, \ldots, x^n$. Σ thus defines a well-defined transformation S of the x. *This transformation belongs to the group* G. Indeed, since the transformation Σ leaves the defining equations of G invariant, any transformation T ($x \to X$) of the group G

transforms into another transformation $\bar{T}(\bar{x} \to X)$ of *G*; as a result, the transformation *S* is equal to $\bar{T}^{-1}T$, so it belongs to *G*. It is clear that the transformation Σ results from the prolongation of *S* in the way that was constructed above.

The group G may thus be prolonged holomorphically into a group Γ *that transforms the variables x, X, u, v, …, and is defined by the invariance properties that were stated above.*

It now remains for us to show that Γ is a Lie group and that its defining equations are of first order. The first property is deduced from the theory of equivalence problems. It is obvious if the defining equations of Γ are of first order, since Γ is the set of transformations that leave invariant the variables $x^{\nu+1}, \dots, x^n, X^1, \dots, X^n$, as well as the ν forms $\omega^j(x, X, u, dx)$, to which one may add the $n - n$ forms $\omega^{\nu+k} = dx^{\nu+k}$.

We carry out the proof in the case where the defining equations of *G* are of second order, so they are consequently composed of equations (1), (2), (3). Since the group Γ leaves the ν forms ω^j invariant, it will leaves their exterior differentials $d\omega^j$ invariant. Now, each term of $\omega^j = a_k^i(x, X, u, dx^k)$ contains a differential dx^i, so the form $d\omega^j$ may be written:

$$d\omega^j = \omega^k \varpi_k^i ,$$

where the ϖ_k are linear in the dx^i, dX^i, and du^k; these forms ϖ_k^i are determined, moreover, only up to linear combinations of the ω^k. For each value of *i*, the coefficients of the linear combinations of the ω^k that one may add to the ϖ_k^i form a symmetric matrix. Be that as it may, for any transformation Σ of Γ one will have:

$$\omega^i(\bar{x}, \bar{X}, \bar{u}, d\bar{x}^k) = \omega^j(x, X, u, dx^k),$$

so:

$$\omega^k[\varpi_k^i(\bar{x}, \bar{X}, \bar{u}; d\bar{x}^k, d\bar{X}, d\bar{u}) - \varpi_k^i(x, X, u; dx^k, dX, du)] = 0;$$

therefore one has, for each pair of indices *i*, *k*:

$$\varpi_k^i(\bar{x}, \bar{X}, \bar{u}; d\bar{x}^k, d\bar{X}, d\bar{u}) = \varpi_k^i(x, X, u; dx^k, dX, du) \qquad (\text{mod } dx^h).$$

It is clear that p_1 of the forms ϖ_k^i are linearly independent in the du^h. Choose these p_1 forms; each of them may be written:

$$c_h\, du^h + g_k\, dX^i \qquad (\text{mod } \omega^j) \qquad (c_h,\ \gamma_k \text{ are functions of } x, X, u),$$

or further, if one refers to the equations (2) and (3):

$$c_h[du^h - b_l^k(x, X, u, v)dx^l] + \gamma_k(dX^k - \omega^k) \qquad (\text{mod } dx^h).$$

The analogous form in terms of the variables $\bar{x}$, $\bar{X}$, $\bar{u}$, $\bar{v}$ must be equal to the preceding one (mod dx^k), but since the group Γ leaves equations (2) and (3) invariant, and each of the two forms considered is a linear combination of the left-hand sides of these equations, the two forms will be not only *congruent* (mod dx^k), but *equal*. Let ϖ^α ($\alpha = 1, 2, \ldots, p_1$) denote the p_1 forms thus obtained. One sees that the *group* Γ *leaves invariant not only the variables* $x^{\nu+1}, \ldots, x^n, X^1, \ldots, X^\nu$, *but also the* $n + p_1$ *Pfaff expressions* ω^j, ϖ^α, *which are linearly independent in the differentials of the variables x, X, u, and have coefficients that depend upon the auxiliary variables v.* Q. E. D.

If the defining equations of *G* are of third order then one can follow the argument in the same manner in such a way that the fundamental theorem is completely general:

Any Lie group admits a holomorphic prolongation Γ *that is defined as the set of transformations that leave invariant a certain number of variables and a certain number of Pfaff expressions that are equal in number to the variables and are linearly independent in the differentials of these variables.*

We add a very simple, but important, remark: The variables $x^1, \ldots, x^n$ of *G* are transformed amongst themselves by the group Γ. Since the variables $X^1, \ldots, X^\nu$ of Γ are invariant, one will not change the manner by which Γ transforms $x^1, \ldots, x^n$ by giving the X^i fixed numerical values everywhere. *One may thus assume that the forms* ω^j *and* ϖ^α *that are invariant under* Γ *no longer contain either the X or the dX.* One thus sees that the variables *u*, when added to the variables *x* in order to form the transformed variables under Γ, are nothing but the values that are taken at a fixed point (X_0) by the partial derivatives of the functions *X* of *x* in the transformations of the group that take (*x*) to (X_0). The set of variables (*x*, *u*) constitutes the analogue of a *frame* in the theory of finite groups.

The structure equations and the second fundamental theorem.

We again place ourselves in the case where the defining equations of *G*, which are assumed to be in involution, are of second order. We have holomorphically prolonged *G* to a group in the variables *x*, *u* that is characterized by the invariance of $x^{\nu+1}, \ldots, x^n$ and the $n + p_1$ forms ω^j, ϖ^α. The equations:

$$\bar{x}^{\nu+1} = x^{\nu+1}, \quad \ldots, \quad \bar{x}^n = x^n,$$
$$\bar{\omega}^i = \omega^j, \qquad \bar{\varpi}^\alpha = \varpi^\alpha,$$

are, in a certain sense, the defining equations of *G*, but written in a manner that *has the advantage of being completely symmetric in the two series of variables – viz., the initial and transformed ones.* These equations thus comprise a system in involution, but one may also consider them to be the defining equations of the group Γ in the variables *x*, *u*; from this viewpoint, one sees that the group Γ has defining equations of first order. We call a Lie group *normal* when its defining equations are of first order, in such a way that any Lie group admits a normal holomorphic prolongation. If the Lie group is finite then the normal holomorphic prolongation has $r + h$ variables, if r is the number of parameters in the group and h is the number of its invariants; there are thus r parameters when the group is transitive. Similar things are true for the simply transitive group of parameters.

Now take a normal group G in n variables x^i, $n - \nu$ of which $x^{\nu+1}, \ldots, x^n$ are invariant. It is characterized by the property of invariance of these $n - \nu$ variables and n linear forms $\omega^i(x, u; dx)$ (where $\omega^{\nu+k} = dx^{\nu+k}$). The exterior differential $d\omega^i$, as we have already remarked, may be written:

$$d\omega^i = \omega^k \varpi_k^i ,$$

in which the ϖ_k^i are linear in the p differentials $du^1, \ldots, du^p$, but determined only up to linear combinations of the dx^k; i.e., the ω^k. Take p of the forms ϖ_k^i that are linearly independent in the du^k and call them $\varpi^1, \varpi^2, \ldots, \varpi^p$. The form $d\omega^i$ will be an expression that is constructed in a well-defined manner from the ω^k and the ϖ^α. Upon adding linear combinations with *undetermined* coefficients to the ϖ^α, one may profit from these indeterminates to annul the largest possible number of coefficients of the forms $d\varpi^\alpha$. Having done this, one will have:

$$d\omega^i = a_{k\rho}^i \omega^k \varpi^\rho + \frac{1}{2} c_{k\rho}^i \omega^k \omega^\rho \qquad (c_{kh}^i = - c_{hk}^i). \tag{10}$$

The remaining constants are obviously invariants, and consequently, functions of $x^{\nu+1}, \ldots, x^n$. This result is the generalization of the *second fundamental theorem* of Lie. One may remark that nothing will prevent us from making a linear substitution of the ω^i with coefficients that are functions of the group invariants. Equations (10) preserve the same form, but the differentials $dx^{\nu+1}, \ldots, dx^n$, now become linear combinations of the ω^i with coefficients that are functions of the invariants.

One must finally take into account the hypothesis by which the defining equations of G are of first order, which amounts to saying that the system:

$$\bar{x}^{\nu+1} = x^{\nu+1}, \ldots, \bar{x}^n = x^n, \qquad \bar{\varpi}^i = \omega^i \tag{11}$$

is in involution. Now, the exterior differentiation of these equations gives:

$$a_{k\rho}^i \omega^k (\bar{\varpi}^\rho - \varpi^\rho) = 0. \tag{12}$$

The coefficients $a_{k\rho}^i$ *thus form an involutive matrix.* From the theory of systems in involution, one may recognize that this is true by calculating the characters $\sigma_1, \sigma_2, \ldots, \sigma_n$ of the system (11) and by looking for the number of arbitrary parameters that the most general integral element in n dimensions depends upon. The system will be in involution if this number of arbitrary parameters is equal to $\sigma_1 + 2\sigma_2 + \ldots + r\sigma_r$; this number of arbitrary parameters is, moreover, nothing but the number of arbitrary parameters that enter into the solution of the equations:

$$a^i_{k\rho}\omega^k\varpi^\rho = 0 \qquad (i = 1, 2, \ldots, n),$$

where the ϖ^ρ are unknown linear forms in $\omega^1, \omega^2, \ldots, \omega^n$. As for the numbers σ, they are obtained in the following manner: One forms the system of equations in ϖ^ρ:

(13) $$a^i_{1\rho}\varpi^\rho = 0 \qquad (i = 1, 2, \ldots, n),$$

(14) $$a^i_{2\rho}\varpi^\rho = 0,$$

$$\ldots$$

(15) $$a^i_{h\rho}\varpi^\rho = 0.$$

σ_1 is the number of independent equations in (13); $\sigma_1 + \sigma_2$ is the number of independent equations in (13) and (14), and so on; finally $\sigma_1 + \sigma_2 + \ldots + \sigma_r$ is the number of independent equations in (13), (14), …, (15) (which is p, here). It is important to add that one has assumed that a linear substitution has been performed on the ω^j beforehand such that the numbers σ_1, $\sigma_1 + \sigma_2$, … are successively as large as possible.

Equations (10) bear the name of the *structure equations* of the normal group G.

The second fundamental theorem admits a converse that is, moreover, rather obvious:

Let ω^j be n forms that are linear in $dx^1, dx^2, \ldots, dx^n$ and are linearly independent with coefficients that are functions of the variables x and some other variables u. Suppose that these forms satisfy equations of the form (10), *where the coefficients $a^i_{k\rho}$, $c^i_{k\rho}$ are functions of just the variables $x^{\nu+1}, \ldots, x^n$. Suppose, in addition, that the differentials $dx^{\nu+1}, \ldots, dx^n$, which are expressed linearly in terms of $\omega^1, \omega^2, \ldots, \omega^n$, involve only coefficients that are functions of these variables. Finally, suppose that the matrix of the $a^i_{k\rho}$ is involutive. Under these conditions, there exists a normal group G that admits the invariants $x^{\nu+1}, \ldots, x^n$, such that equations* (10) *are the structure equations.*

Indeed, the equations:

$$\bar{x}^{\nu+1} = x^{\nu+1}, \ldots, \bar{x}^n = x^n, \qquad \bar{\varpi}^i = \omega^j$$

obviously form a system in involution, and the $\bar{x}$, when considered as functions of the x, thus constitute the general solution to a system of partial differential equations of first order that are in involution.

If the coefficients of the structure equations are constants then the group G is transitive and these coefficients are *structure constants*. All transitive groups that have the same structure equations are similar.

Before going further, we give some examples:

Example 1. – Start with a finite group $X = ax + b$, whose defining equations are obviously:

$$dX = \omega^1 = u\, dx,$$
$$du = 0$$

(which gives an equation $d^2X / dx^2 = 0$ of second order). Here, one has:

$$d\omega^1 = du\, dx = \frac{du}{u}\omega^1 .$$

The group Γ that is the prolongation of G to the variables x and u is therefore defined by the invariance of the forms:

$$\omega^1 = u\, dx, \qquad \omega^2 = \frac{du}{u} .$$

Example 2. – Let G be the group of two variables x, y and an invariant y whose finite equations are:

$$X = x + ay, \qquad Y = y.$$

The defining equations are:

$$Y = y, \qquad dX = \omega^1 = dx + \frac{X-x}{y}dy ;$$

the group Γ thus coincides with G. In order to get the structure equations, one may thus replace X with a fixed number in the form ω^1 – for example, zero. One then has:

$$\omega^1 = dx - \frac{x}{y}dy \qquad d\omega^1 = -\omega^1 \frac{dy}{y} ,$$

or further:

$$\omega^1 = dx - \frac{x}{y}dy , \qquad \omega^2 = dy,$$

with:

$$d\omega^1 = \frac{1}{y}\,\omega^2\omega^1, \qquad d\omega^2 = 0.$$

Example III.- Let G be the group of homographic transformations of one variable:

$$X = \frac{ax+b}{cx+d} .$$

One knows that the defining equation of the group is:

(1) $$X'X''' - \frac{3}{2}\,X''^2 = 0.$$

Upon setting:

$$X' = u, \qquad X'' = v,$$

we have the system:

(2) $$dX = \omega^1 = u\, dx,$$

(3) $$du = v\, dx,$$

(4) $$dv = \frac{3}{2}\frac{v^3}{u}dx.$$

One has:

$$d\omega^1 = du\,dx = \frac{du - v\,dx}{u}u\,dx = \frac{du - v\,dx}{u}\omega^1;$$

the form $\frac{du - v\,dx}{u}$ is thus invariant; call it ω^2:

$$\omega^2 = \frac{du}{u} - \frac{v}{u}dx.$$

From this, one deduces that:

$$\begin{aligned} d\omega^2 &= -\frac{1}{u}dv\,dx + \frac{v}{u^2}du\,dx = \left(-\frac{1}{u^2}dv + \frac{v}{u^3}du\right)\omega^1 \\ &= \left[-\frac{1}{u^2}\left(dv + \frac{v^2}{u}dx\right) + \frac{v}{u^3}(du - v\,dx)\right]\omega^1, \end{aligned}$$

from which, one obtains the new invariant form:

$$\omega^2 = -\frac{1}{u^2}dv + \frac{v}{u^3}du + \frac{1}{2}\frac{v^2}{u^3}dx.$$

From this, one deduces:

$$d\omega^2 = \frac{1}{u^3}du\,dv + \frac{v}{u^3}dv\,du - \frac{3}{2}\frac{v^2}{u^4}du\,dx = \omega^3\omega^2.$$

The structure equations are, in turn:

$$\begin{aligned} d\omega^1 &= \omega^2\omega^1, \\ d\omega^2 &= \omega^3\omega^1, \\ d\omega^3 &= \omega^3\omega^2. \end{aligned}$$

Example IV. – Let the group G be defined by the equations:

$$X = x + f(x), \qquad Y = y,$$

where $f(y)$ is an arbitrary analytic function of y. The defining equations are:

(1) $$Y = y,$$
(2) $$dX = dx + u\,dy.$$

One thus has:

$$\omega^1 = dx + u\,dy, \qquad \omega^2 = dy,$$

with the structure equations:

$$d\omega^1 = \varpi\, dy = \varpi\, \omega^2, \qquad\qquad d\omega^1 = 0,$$

in which ϖ denotes the form du plus an arbitrary multiple of dy.

Example V. – Let a transitive group be defined by:

$$X = f(x), \qquad Y = \frac{y}{f'(x)},$$

where $f(x)$ is an arbitrary function of x; $f'(x)$ is its derivative. The defining equations are:

$$dX = \frac{y}{Y} dx,$$
$$dY = \frac{Y}{y} dy + u\, dx;$$

they are of first order. Give Y the numerical value 1 in their right-hand sides; we will then have:

$$\omega^1 = y\, dx, \qquad \omega^2 = \frac{dy}{y} + u\, dx,$$

with the structure equations:

$$d\omega^1 = \omega^2 \omega^1, \qquad d\omega^2 = \varpi\, \omega^1 \qquad \left[\varpi = \frac{du}{y} (\mathrm{mod}\, dx)\right].$$

We remark here that the group G is the holomorphic prolongation of the group $X = f(x)$, whose defining equation is $dX = u\, dx$, with:

$$\omega^1 = u\, dx, \qquad\qquad d\omega^1 = \varpi\, \omega^1.$$

The third fundamental theorem.

The question whose answer is the third fundamental theorem is the following one: May the coefficients $c^i_{kh} = -c^i_{hk}$, $a^i_{k\rho}$ that enter into the defining equations of a normal group be chosen arbitrarily as functions of the group invariants? We limit ourselves to the case where the group is transitive; one will have a few modifications to make in the case where the group is intransitive.

In summation, the problem is the following one:

Is it possible to find $n + p$ forms ω^i, ϖ^α that are independent of the $n + p$ variables that satisfy the equations:

(10) $$d\omega^i = a^i_{k\rho}\omega^k\varpi^\rho + \frac{1}{2}c^i_{kh}\omega^k\omega^h \qquad (c^i_{hk} = -c^i_{kh}),$$

where the coefficients are given constants, such that $a^i_{k\rho}$ *form an involutive system?*

Indeed, it suffices that one may find forms that satisfy these conditions, and then the equations $\omega^i = 0$ obviously form, from (10), a completely integrable system, so one may then suppose that $x^1, x^2, \ldots, x^n$ are first integrals of the this system, and from the converse of the second fundamental theorem there will indeed exist a normal group G that operates on these n variables and admits (10) for its structure equations. More generally, one may look for forms ω^i and ϖ^α that are linear in $N \geq n + p$ given variables, with the reservation that the $n + p$ forms are linearly independent. We call these variables ξ^n.

If we set:

(16) $$\omega^i = p^i_\lambda\, d\xi^\lambda, \qquad \omega^\alpha = q^\alpha_\lambda\, d\xi^\lambda$$

then putting the problem into an equation furnishes quadratic exterior equations:

(I) $$dp^i_\lambda \,\mathrm{d}dx^\lambda = a^i_{k\rho}\omega^k\varpi^\rho + \frac{1}{2}c^i_{kh}\omega^k\omega^h,$$

and on the right-hand side of them one supposes that the forms ω^i and ϖ^α have been replaced by their values (16).

From the general theory of Pfaff system, one must add to equations (I) the ones that one deduces by exterior differentiation, taking equations (I) into account. The calculation gives:

(II) $$a^i_{k\rho}\omega^h d\varpi^\rho = a^i_{k\rho}a^k_{h\rho}\omega^h\varpi^\rho\varpi^\alpha + (c^i_{kh}a^k_{l\rho} + \frac{1}{2}c^k_{hl}c^i_{k\rho})\omega^h\omega^l\varpi^\rho + \frac{1}{2}c^i_{kh}c^h_{lm}\omega^h\omega^l\omega^m,$$

in which one has naturally assumed that the forms ω^i and ϖ^α are replaced by their expression (16).

One obtains necessary conditions for compatibility by expressing the possibility of satisfying equations (II) upon replacing the forms $d\varpi^\alpha$ with certain quadratic forms that are constructed with the ω^i and ϖ^ρ in such a way that:

(17) $$d\varpi^\alpha = \frac{1}{2}\gamma^\alpha_{\lambda\mu}\varpi^\lambda\varpi^\mu + \delta^\alpha_{k\kappa}\omega^k\varpi^\lambda + \frac{1}{2}\varepsilon^i_{kh}\omega^k\omega^h.$$

By identification, one finds the relations:

(18) $$a^i_{k\beta}a^k_{h\alpha} - a^i_{k\alpha}a^k_{h\beta} - a^i_{k\rho}\gamma^\rho_{\alpha\beta} = 0 \qquad (i, h = 1, 2, \ldots, n;\ \alpha, \beta = 1, 2, \ldots, p);$$

(19) $$c^i_{k\beta}a^k_{l\alpha} - c^i_{kl}a^k_{h\alpha} + a^i_{k\alpha}c^k_{kl} - a^i_{h\rho}\delta^\rho_{i\beta} + a^i_{l\rho}\delta^\rho_{h\alpha} = 0 \qquad (i, h, l = 1, 2, \ldots, n;\ \alpha = 1, 2, \ldots, p);$$

(20) $$c^i_{kh}a^k_{lm} + c^i_{kl}c^k_{mh} + c^i_{km}c^k_{hl} - a^i_{h\rho}\varepsilon^\rho_{lm} - a^i_{l\rho}\varepsilon^\rho_{hm} - a^i_{m\rho}\varepsilon^\rho_{hl} = 0,$$

$$(i, h, l, m = 1, 2, \ldots, n).$$

It is necessary that these equations be compatible, when considered as linear equations in the unknowns $\gamma^{\alpha}_{\lambda\mu}$, $\delta^{\alpha}_{k\lambda}$, $\varepsilon^{\alpha}_{kh}$; this naturally entails algebraic relations between the constants $a^{i}_{k\rho}$ and c^{i}_{kh}.

Suppose that these relations are verified. The system (I) is then in involution. In order to prove this, one must evaluate the number of arbitrary parameters that the most general n-dimensional integral element depends upon, calculate the characters S_1, S_2, …, S_n of the system, and verify that the former number has the value that the theory of system in involutions gives for the characters S_i .

In order to make the first evaluation (viz., the number of arbitrary parameters upon which the mist general N-dimensional integral element depends), first consider equations (II). By hypothesis, they are verified by the values of the $d\varpi^{\alpha}$ of the form (17), with values that are determined by the coefficients γ, δ, ε. The most general solution of (II) will be obtained upon adding the most general forms Π^{α} that satisfy the equations:

$$a^{i}_{k\rho}\,\omega^{k}\,\Pi^{\rho} = 0 \tag{21}$$

to the values (17). Now, the equations $a^{i}_{k\rho}\,\omega^{k}\,\varpi^{\rho} = 0$, in which the unknowns are the linear forms ϖ^{α}, admit a general solution that depends upon $\sigma_1 + 2\sigma_2 + \ldots + r\sigma_r$ arbitrary parameters, namely:

$$\varpi^{\alpha} = b^{\alpha}_{\lambda k}\; t^{\lambda}\;\omega^{k},$$

where the t^{λ} are the parameters in question. It is then obvious that equations (21) will be verified if one sets:

$$\Pi^{\alpha} = b^{\alpha}_{\lambda k}\;\omega^{k}\;\chi^{\lambda}, \tag{22}$$

in which the χ^{λ} are forms *with arbitrary coefficients* that are linear in $d\xi^1$, …, $d\xi^N$. One thus introduces $N(\sigma_1 + 2\sigma_2 + \ldots + r\sigma_r)$ arbitrary coefficients in the most general expressions for $d\varpi^{\alpha}$ that satisfy equations (II). However, this number must be reduced by the fact that there are two terms in Π^{α}. One of them is in $\omega^1\omega^2$ and the other one is in $\omega^2\omega^1$, and the reduction into just one of these terms and analogous terms diminishes the number of arbitrary coefficients. One may rigorously calculate the number by which one must reduce $N(\sigma_1 + 2\sigma_2 + \ldots + r\sigma_r)$ in order to have the exact number of remaining arbitrary constants. We leave aside this determination in order to point just the result: The number H of the arbitrary coefficients (that are independent of the p^{i}_{λ} and q^{α}_{λ}) that enter into the solution of the equations (II) in the $d\varpi^{\alpha}$, as we have envisioned it, is:

$$(N-1)\,\sigma_1 + (2N-3)\,\sigma_2 + (3N-6)\,\sigma_3 + \ldots = \sum_{i=1}^{r}\left[iN - \frac{i(i+1)}{2}\right]\sigma_i\,.$$

In reality, the number H may be greater than the preceding number, since there may exist solutions of equations (21) that do not take the form (22). One must then write:

$$H \geq \sum_{i=1}^{r} \left[iN - \frac{i(i+1)}{2} \right] \sigma_i .$$

This still does not give us the number of arbitrary parameters that the most general N-dimensional integral element of equations (I) and (II) depends upon. In order to obtain it, we may replace equations (II) with the equations:

$$\text{(II}'\text{)} \qquad dq_\lambda^\alpha d\xi^\lambda = d\varpi^\alpha,$$

where we have replaced $d\varpi^\alpha$ with the value obtained that involves H arbitrary parameters. Any N-dimensional integral element will be defined by:

$$dp_\lambda^i = p_{\lambda\mu}^i d\xi^\mu ,$$
$$dq_\lambda^\alpha = q_{\lambda\mu}^\alpha d\xi^\mu .$$

One sees immediately that, upon taking the terms in $d\xi^\lambda\, d\xi^\mu$ in equations (I) and (II′), that these equations have the form:

$$p_{\lambda\mu}^i - p_{\mu\lambda}^i = \ldots$$
$$q_{\lambda\mu}^i - q_{\mu\lambda}^i = \ldots,$$

in which the right sides are quantities that are well-defined functions of the p_λ^i, the q_λ^α, and the H arbitrary parameters that were in question above. One thus has $\frac{N^2(N-1)}{2}$ linear relations in N^3 unknowns for fixed values of these H arbitrary parameters. As a result, the total number of arbitrary parameters that the most general N-dimensional integral element of the system (I), (II) depends upon is:

$$\frac{N^2(N-1)}{2} + H \geq \frac{N^2(N-1)}{2} + \sum_{i=1}^{n} \left[iN - \frac{i(i+1)}{2} \right] \sigma_i .$$

We pass on to the calculation of the characters $S_1, S_2, \ldots, S_{N-1}$ of the system. We shall determine the integers Σ_1, $\Sigma_1 + \Sigma_2$, … such that one has:

$$S_1 \geq \Sigma_1, \qquad S_1 + S_2 \geq \Sigma_1 + \Sigma_2, \qquad \ldots,$$
$$S_1 + S_2 + \ldots + S_{N-1} \geq \Sigma_1 + \Sigma_2 + \ldots + \Sigma_{N-1} .$$

In order for the system to be in involution, it is necessary and sufficient that the number $\frac{N^2(N-1)}{2} + H$ that we just found, which may not exceed:

$$N^2 - (N-1)\, S_1 - (N-2)\, S_2 - \ldots - S_{N-1} ,$$

must be equal to it. Now, this latter number is itself less than or equal to:

$$N^2 - (N-1)\,\Sigma_1 - (N-2)\,\Sigma_2 - \ldots - \Sigma_{N-1}\,.$$

We shall prove that this latter number is equal to:

$$\frac{N^2(N-1)}{2} + \sum_{i=1}^{n}\left[iN - \frac{i(i+1)}{2}\right]\sigma_i\,.$$

One will then have:

$$\frac{N^2(N-1)}{2} + H \geq \frac{N^2(N-1)}{2} + \sum_{i=1}^{n}\left[iN - \frac{i(i+1)}{2}\right]\sigma_i$$
$$\geq N^2 - (N-1)\,S_1 - (N-2)\,S_2 - \ldots - S_{N-1}\,.$$

As a result, the system is in involution, and the sign in the equality must be replaced everywhere by the inequality sign. In will then result that, on the one hand, $S_i = \Sigma_i$, and on the other hand, one can be sure that equations (22) provide the most general solution to (21).

In order to calculate the numbers Σ, one forms the left-hand sides of equations (I) and (II); they are the only parts of these equations in which the differentials of the unknown functions p_λ^i, q_λ^α enter. One obtains:

(I) $$dp_\lambda^i\, d\xi^\lambda = \ldots$$

(II) $$a_{k\rho}^i\, dp_\lambda^k\, dq_\mu^\rho\, d\xi^\lambda d\xi^\mu = \ldots$$

We will have a number $\Sigma_1 \leq S_1$ by taking the coefficients of dx_1 in (I), which gives the n differentials dp_1^i; one will then take $\Sigma_1 = n$. We will get a number $\Sigma_1 + \Sigma_2 \leq S_1 + S_2$ upon taking the coefficients of $d\xi^1$ and $d\xi^2$ in (I) and the coefficients of $d\xi^1\, d\xi^2$ in (II), which gives:

$$dp_1^i,\ dp_2^i,\ a_{k\rho}^i(p_1^k dq_2^\rho - p_2^k dq_1^\rho)\,,$$

i.e., at least $2n + \sigma_1$ linearly independent solutions. (For example, it will suffice to set $p_1^1 = 1$, while the other p_1^k are zero.) We thus take $\Sigma_2 = n + \sigma_1$. Upon following through on this, one finds:

$$\Sigma_1 = n, \qquad \Sigma_2 = n + \sigma_1, \qquad \Sigma_3 = n + \sigma_1 + \sigma_2,$$
$$\Sigma_4 = n + \sigma_1 + \sigma_2 + \sigma_3\,, \qquad \ldots, \qquad \Sigma_{N-1} = n + \sigma_1 + \sigma_2 + \ldots + \sigma_{N-1}\,,$$

if one sets:

$$\sigma_{n-1} = \ldots = \sigma_{N-1} = 0.$$

One then easily finds the stated equality:

$$N^2 - (N-1)\,\Sigma_1 - (N-2)\,\Sigma_2 - \ldots - \Sigma_{N-1} = \frac{N^2(N-1)}{2} + \sum_{i=1}^{n}\left[iN - \frac{i(i+1)}{2}\right]\sigma_i\,.$$

It is pointless to remark that this third fundamental theorem generalizes that of Lie, since the classical Lie relations between the c_{kh}^i reduce to the relations (20), in which one has suppressed the terms in ε_{kh}^i.

The linear stability group; systatic plane element.

If one is given a Lie group G, which is not necessarily normal, then the set of transformations of the group G that leaves a generic point (x) fixed forms a subgroup g of G that one calls the *stability group* of the point (x). If the group G is finite then g is a Lie group, but this is not true in general when the group G is infinite. Be that as it may, the group g linearly transforms the vectors dx^i that issue from the point (x), and the group that results from transforming them is a linear group that we call the *linear stability group* of the point (x). We shall show that the infinitesimal transformations of this group appear in the structure equations of the group G.

Start with the original form of the defining equations of the group:

(1) $$X^{\nu+1} = x^{\nu+1}, \quad \ldots, \quad X^n = x^n,$$

(2) $$dX^i = a_k^i(x, X, u)\,dx^k \qquad (i = 1, 2, \ldots, n).$$

If one fixes the point (x) and the point (X) then formulas (2) indicate how a transformation T of G takes (x) to (X) transforms the vector (dx) into the vector (dX). This transformation translates into the linear substitution S_u of the coefficients a_k^i that is carried out on the components dx^k of the vector (dx). If we now consider two transformations T and T' of G that both take (x) to (X) then the transformation $T'^{-1}T$ of G belongs to the stability group of (x), and the corresponding linear substitution of the linear stability group is the substitution $S_{u'}^{-1}S_u$. The substitution $S_{u'}^{-1}S_u$ then generates the linear stability group γ when one varies u and u'. One may also regard it as being generated by the substitutions $S_0^{-1}S_u$, where S_0 corresponds to fixed numerical values of u'. The substitutions $\Sigma_u = S_uS_0^{-1}$ also generate a group, since one has:

$$S_uS_0^{-1} = S_0(S_0^{-1}S_u)S_0^{-1};$$

it is the transform of the group γ by S_0. It is again the linear stability group, but only when one takes the components of the vector (dx) to be the quantities that result from the substitution S_0 that is performed on the dx^k – i.e., the quantities $\omega^j(x, X, u_0, dx)$. We again denote this group by γ.

Having said this, consider the infinitesimal substitution $\Sigma_{u+\delta u}\Sigma_u^{-1}$ of γ. It is the one that gives the $\omega^j(x, X, u + \delta u, dx)$ when it is applied to the quantities $\omega^j(x, X, u, dx)$. Now, one has:

$$\omega^i(x, X, u + \delta u, dx) = \omega^i(x, X, u, dx) + \frac{\partial \omega^l}{\partial u^k}\delta u^k .$$

However, from the structure equations:

$$d\omega^i = \frac{1}{2}c^i_{kh}\omega^k\omega^h + a^i_{h\rho}\omega^h\varpi^\rho ,$$

one has:

$$\frac{\partial \omega^l}{\partial u^k} = a^i_{k\rho}\omega^k e^\rho ,$$

upon letting e^ρ denote the result that follows when one replaces dx^k with 0 in ϖ^ρ and du^k with $-\delta u^k$. It then results that *the equations of the infinitesimal transformations of the linear stability group are:*

(23) $$\delta\omega^i = a^i_{k\rho}e^\rho\omega^k ,$$

with p_1 *parameters* $e^1, e^2, \ldots, e^{p_1}$.

The preceding has implicitly assumed that the group G is normal; in this case, if one expresses the idea that the infinitesimal transformations:

(24) $$E^\alpha f = a^i_{k\alpha}\xi^k \frac{\partial f}{\partial \xi^i}$$

generate a structure group $\gamma^\alpha_{k\mu}$ then one finds equations (18) precisely. *The compatibility of these equations, which is necessary for the direct statement of the third fundamental theorem, thus simply expresses the idea that the transformations* (23) *generate a group*.

If the group G is not normal then the preceding results subsist, but the ϖ^α are then the forms ω that are invariant under the normal holomorphic projection of G that presents itself during the consideration of those defining equations for G that are of first order.

In any case, the preceding results lead us to a new notion. Take a vector (dx) whose components annul the np_1 forms $a^i_{k\alpha}\omega^k$ ($i = 1, 2, \ldots, n$; $\alpha = 1, 2, \ldots, p_1$) that are introduced in the right-hand sides of equations (23). Any infinitesimal transformation of γ is left invariant, and conversely, any vector that is invariant under γ annuls these np_1 forms. We the name of *systatic system* to the Pfaff system:

(25) $$a^i_{k\alpha}\omega^k = 0 \qquad (i = 1, \ldots, n;\ \alpha = 1, \ldots, p).$$

The vectors that issue from the point (x) and satisfy the equations of that system generate a plane element E that we call the *systatic plane element that is attached to the point* (x). This terminology is explained by the fact that any transformation of G that leaves the point (x) fixed simultaneously leaves fixed all of the infinitely neighboring points to (x) that are situated in the plane element E that is associated with (x).

From this, it results that *the system* (25) *is completely integrable.* One may verify this by calculation; however, one may also account for this in an intuitive manner ([1]). Consider a curve (*C*) that is tangent at each of its points to the systatic element that is associated with that point. Any transformation of *G* that leaves fixed a point (*x*) of the curve will likewise leave fixed the infinitely neighboring points, and therefore, in succession, all of the points of *C*. The locus of points that are invariant under the transformations of the (nonlinear) stability group of a point (*x*) is therefore a manifold that admits all of the tangents to the systatic element at each of its points. These manifolds are the integral manifolds of the systatic system (25), which is therefore completely integrable.

Naturally, it might happen that the systatic plane element reduces to the point (*x*) itself, in which case, the group *G* is called *asystatic;* this will happen when the number of equations (25) is the number *n* of independent ones.

Essential invariants.

The consideration of the systatic Pfaff system leads us to an important notation: that of *essential invariant.*

We first remark that if one makes an arbitrary change of variables then a systatic system remains invariant; this is a simple consequence of its geometric significance. Having said this, among all of the linear combinations of the equations of this system, consider the ones that depend upon only the differentials $dx^{\nu+1}, \ldots, dx^n$ of the group invariants. One may assume that, with a small change of notations, they are the differentials of a certain number of invariants that we call $y^1, \ldots, y^r$; we denote the other invariants of the group by $z^1, \ldots, z^r$. We finally reserve the letter x to denote the variables $x^1, x^2, \ldots, x^r$. We may likewise assume that the first integrals of the systatic system are $x^1, x^2, \ldots, x^q$ (and the invariants $y^1, \ldots, y^r$).

Having made these conventions, consider the defining equations of the group *G*, which is not necessarily normal, and among them, the equations:

$$\bar{y}^k = y^k, \qquad \bar{z}^h = z^h, \qquad \bar{\varpi}^i = \omega^j,$$

where the independent variables are the x and the unknown functions are the $\bar{x}$. We regard the quantities $x^1, \ldots, x^q, y^1, \ldots, y^r$, which are first integrals of the systatic system, as constant *parameters*, while the quantities $\bar{x}^1, \ldots, \bar{x}^q$ are unknown functions of the other variables $x^{q+1}, \ldots, x^\nu, z^1, \ldots, z^s$. Since the quantities $a^i_{k\rho}\omega^k$ then all become zero, and, in turn, the quantities $a^i_{k\rho}\bar{\varpi}^k$, as well (by virtue of the equations $\bar{\varpi}^i = \omega^j$), the exterior differentiation of the equations $\bar{\varpi}^i = \omega^j$ in n unknown functions $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^\nu$ gives identities, since the forms $d\bar{\varpi}^i - d\omega^j$ are annulled when one takes the equations $\bar{\varpi}^i = \omega^j$ into account. The system considered is therefore completely integrable. Let:

$$(26) \qquad \bar{x}^i = F^i(x, y, z, A^1, A^2, \ldots, A^n) \qquad (i = 1, 2, \ldots, n)$$

([1]) The proof that follows has been ultimately judged to be dubious by E. Cartan.

be the general solution, in which the A are integration constants.

This result proves that any transformation of the group G may be put into the form (26), in which the F^i are well-defined functions of their arguments and the A^i are certain functions of the first integrals $x^1, x^2, \ldots, x^q, y^1, y^2, \ldots, y^r$ of the systatic system. One may add that for a *given* transformation S of the group G the functions A^i are perfectly determined; one necessarily finds functions that depend only upon the $x^1, x^2, \ldots, x^q, y^1, y^2, \ldots, y^r$.

This being the case, now give fixed numerical values x_0^k to the ν variables x^k, and make the change of variables that is provided by the equations:

$$x^i = F^i(x_0, y, z, \xi^1, \xi^2, \ldots, \xi^\nu), \tag{27}$$

equations which give the ξ^i as well-defined functions of the x, y, and z. Then perform a well-defined transformation T of the group on the point (x_0, y, z), and let (ξ, y, z) be the transformed point (expressed with the new coordinates). The transformation T corresponds to well-defined functions $A^i(x^1, \ldots, x^q, y^1, \ldots, y^r)$, which reduce to well-defined functions $A^i(y)$ for $x = x_0$. A comparison of formulas (26) and (27) shows that the coordinates of the transformed point are simply the quantities $A^i(y)$. If one then applies the same transformation T to the point (x_0, y, z') then one will obtain a new point (ξ, y, z') that has the same values $A^i(y)$ of the ξ.

Now, let:

$$\bar{\xi}^i = \varphi^i(\xi, y, z)$$

be the equations of an arbitrary transformation S of the group (expressed in terms of the new variables). Give (arbitrary) fixed numerical values to the ξ, y, z; thus there always exists at least one transformation T of the group that takes the point (x_0, y, z) (the old variables) to the point (ξ, y, z) (the new variables), and the transformation ST takes the same point (x_0, y, z) to the point $(\bar{\xi}, y, z)$. Now, start with the point (x_0, y, z'). The transformation T will give the point (ξ, y, z') and the transformation ST will give the point $(\bar{\xi}, y, z')$, with the same values of the ξ^i and the $\bar{\xi}^i$. As a consequence, the transformation S simultaneously takes the point (ξ, y, z) to the point $(\bar{\xi}, y, z)$ and the point (ξ, y, z') to the point $(\bar{\xi}, y, z')$. One thus has:

$$\bar{\xi}^i = \varphi^i(x, y, z) = \varphi^i(\xi, y, z').$$

The equalities:

$$\varphi^i(\xi, y, z) = \varphi^i(\xi, y, z')$$

are true no matter what numerical value one gives to the arguments ξ, y, z, z', so the functions φ^i depend upon only z.

One may thus perform a change of variables in such a manner that the variables ξ and y are transformed amongst themselves by the group G. The group G thus presents itself as the direct product of a group G in $n - s$ variables with r invariants and a group in n variables that is transformed identically.

Now, this is true no matter how one disposes of the $z^1, z^2, \ldots, z^s$. *However, it is impossible to do more in this sense,* since, as we have said, the systatic system is invariant under any change of variables; it will thus always contain the equations $dy^1 = 0, \ldots, dy^r = 0$. If one may eliminate any of the invariants y as one has eliminated the invariants z then the differentials of these invariants y are thus eliminated, and they obviously may not appear in the left-hand sides of the systatic equations of G, which are the same as for G'.

We say that *the invariants y, which are first integrals of the systatic system, are the essential invariants of the group.*

If the group is finite and of order ν then the systatic plane element fills up all space. There is therefore no systatic system and no essential invariant: *Any finite group is isomorphic to a transitive group*, which is well-known. This theorem is not true for infinite groups [example: $X = x + f(y)$, $Y = y$, where y is an essential invariant, and remains an essential invariant for all isomorphic groups].

Example I. – Recall the example that was already cited of the group:

$$X = x + ay, \qquad Y = y \qquad\qquad (\nu = 1,\ n = 1),$$

with:

$$d\omega^1 = \frac{1}{y}\omega^2\omega^1, \qquad d\omega^2 = 0;$$

there is no systatic system and consequently no essential invariant. Indeed, upon setting $x = \xi y$, one has:

$$\bar{\xi} = \xi + a, \qquad \bar{y} = y.$$

This group is the direct product of the transitive group $\bar{\xi} = \xi + a$ with the group $\bar{y} = y$. One may see directly that the stability group of the point (x_0, y_0) reduces to the identity transformation. The systatic plane element is therefore two-dimensional, and there is no systatic system.

Example II. – Take the group:

$$X = x + ay + b, \qquad Y = y \qquad\qquad (\nu = 1,\ n = 2).$$

One has:

$$\omega^1 = dx + u\,dy, \qquad \omega^2 = dy,$$

with:

$$d\omega^1 = du\,dy = \varpi^1\omega^2, \qquad d\omega^2 = 0 \qquad (\varpi^1 = du),$$

and:

$$d\varpi^1 = 0.$$

The systatic system is formed from the equation $\omega^2 \equiv dy = 0$, so the invariant y is essential.

Example III. – Let the group be:

$$X = x + ay + bz, \qquad Y = y, \qquad Z = z \quad (\nu = 1, n = 3).$$

One has:

$$\omega^1 = dx - x\frac{dz}{z} + u\left(dy - y\frac{dz}{z}\right), \qquad \omega^2 = dy, \qquad \omega^3 = dz,$$

with:

$$d\omega^1 = -\,\omega^1\frac{dz}{z} + \varpi^1\left(\omega^2 - \frac{y}{z}\omega^3\right), \qquad (\varpi^1 = du)$$

and:

$$d\varpi^1 = 0.$$

The systatic system is:

$$\omega^2 - \frac{y}{z}\omega^3 \equiv dy - \frac{y}{z}dz = 0;$$

the invariant y / z *is essential.* Indeed, upon setting $\xi = x / z$, one has:

$$\xi = \xi + a\frac{y}{z} + b,$$

and all that remains is the essential invariant y / z.

Bibliography.

This conference presentation contains the substance of the first part of the memoir on the structure of infinite groups (*Annales École Normale*, 1904) and part of the memoir that followed it (*Annales École Normale*, 1905). The notations have been modified slightly. The proof of the fundamental theorem is precise, as well as everything that concerns the linear stability group (which was called the "adjoint group" in the memoir of 1905) and the theory of essential invariants. The proof that was sketched out at this conference of the converse of the third fundamental theorem is simplified and made more rigorous by the use of the theorems of Kaehler. Finally, the terms "normal group," "stability group," and "systatic plane element" are new.

____