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ELECTRICITY

DEDUCED FROM EXPERIMENT

AND REDUCED TO THE PRINCIPLE OF VIRTUAL WORK

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PREFACE

The idea behind this book is expressed in its title: My goal is the study of fixed or variable electrical states, and my method is the mechanical interpretation of experimental laws. My main conclusion is the statement of two fundamental laws that generalize the two laws of Kirchhoff and which will base an electrical field on a system of constraints that obey the principle of virtual work.

All physicists are in agreement on the necessity of improving the theory of electricity, because although Maxwell's work has earned their deserving admiration, they nonetheless recognize that is it obscure and that his commentators have not sufficiently clarified it.

Now, in the work of the master, one theory that is taken from it has my unreserved admiration, namely, the theory of induced currents that are treated by the Lagrange equations. It is one of the much-too-rare theories in which one can say that it does not contain any other hypothesis than the mechanical interpretation of experimental laws. A system of electrical circuits is associated with a system with constraints whose coordinates are of two types: On the one hand, there are geometric coordinates that fix the form and position of the circuits, and on the other, the quantities of electricity that they lose. In all cases, the Lagrange equations that apply to that system give equations that determine a system of functions of time, and their number is equal to that of the coordinates. That is the theory. It is satisfying, but limited to the particular case of filamentary conductors that are embedded in a homogeneous dielectric. Can it be extended to the electrical manifestations of an arbitrary system?

Here are the ideas that I was led to in my study of that problem:

Do electrical phenomena satisfy a general law of constraint? That is the first question that one poses, and Maxwell has answered it in the affirmative, although in an indirect and somewhat confused fashion in which one does not clearly distinguish the parts that correspond to observation, calculation, and hypothesis. The constraint resides in the law of total current flux, which is analogous to the incompressibility constraint in hydrodynamics. Can one simply borrow it from experiments? I will show that the answer is yes. That is the generalized first law of Kirchhoff.

Does the principle of virtual work apply to the system of constraints that will define the electric field from now on? Experiments show that the answer to that second question is also yes. That is the generalized second law of Kirchhoff.

The foundations of the theory have been laid. What did I say? The theory is done, because all that remains is to present its mathematical development. One sees that it is borrowed exclusively from observation with no other hypothesis than the mechanical interpretation of the observed results. That is the greatest degree of certainty that one can demand in the physical sciences.

The plan of the book resulted from what one just read. It contains two parts: In the first, I shall present the known theory of the phenomena of induction that filamentary conductors present. In the second part, I shall extend my study to all of the electrical manifestations of an arbitrary system. The method of presentation is the same in the two parts: I observe and state the experimental laws and then I give the interpretation in the language of rational mechanics. If the observations are missing from some difficult cases then the mechanical interpretation will show the generalization that it imposes upon the law that is observed in the simple cases.

Let me say a few more words about the details of the presentation, due to their importance. What obscures the work of Maxwell is the multiplicity of theories and viewpoints, the absence of unity in the definitions, and finally the lack of a distinction between the purely-experimental truths and then ones that are deduced from the theory. My main preoccupation has been to avoid that obscurity. To that end, I have isolated each part of the subject in a special paragraph. Each paragraph is preceded by an introduction and followed with some conclusions in which the results that were obtained are recorded. On the other hand, each notion has been defined precisely and uniquely. That has sometimes obliged me to reproduce some things that the reader knows well. He can pass over them, but he can refer back to them if the train of thought seems to demand that. In principle, I assume that one knows about magnetism, electrostatics, and the elementary laws of currents (Ohm's law). Meanwhile, I will not hesitate to recall certain points from those theories when clarity seems to demand it, and notably as it concerns flux.

I have adopted Maxwell's notations and conventions in regard to senses and signs. Nonetheless, in order to simplify the writing and to shorten the treatise, I shall employ the same letter to denote each vector that also serves to denote its first component. That is why α will denote the magnetic force, as well as the first of its components α, β, γ. No confusion should result from that. I have also replaced the quaternion notation with that of Grassmann, which is more advantageous. I think that all that it will imply is much greater ease for the reader.

March 1902

Since 1903, the ideas that one just read about have served as the basis for some lectures that I taught at Charliat's l'École pratique d'Électricité industrielle and that I set down in a book that was published in 1904 ([1]). They have also passed from the realm of Science to that of elementary teaching and practice.

March 1907.

([1]) E. CARVALLO, *Leçons d'Électricité*, Paris, Béranger, 1904.

PART ONE

Theory of induction currents according to Helmholtz and Maxwell.

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INTRODUCTION

1. – The theory of induction currents that is generally adopted is that of Helmholtz. It is very simple in the case of the relative displacement of a permanent magnet and a circuit with no battery, but its development will usually become more obscure, sometimes erroneous, and always less satisfying for the more complex cases in which the circuits contain batteries. Another theory exists that has my unreserved admiration, namely, that of Maxwell, although it is less known, much less appreciated, and even wrongly accused of inexactitude ([2]). The two theories are so different that they seem opposite to each other, on first glance, since the authors have followed Helmholtz by generally regarding the intrinsic energy of currents as a potential energy, while Maxwell considered it to be a kinetic energy. The opposition is only apparent. The proof is in the fact that I take Helmholtz's theory to be an introduction to Maxwell's theory. In reality, Helmholtz's theory, when presented correctly, includes no hypothesis on the nature of the energy in question. Might one say that is an advantage? Most certainly as a starting point, and that is the reason why I took the method of Helmholtz to be an introduction to that of Maxwell. However, as a stopping point, I think not. Is it not the goal of the physicist to penetrate the nature of things by any means necessary, such as experiments and inductive and deductive reasoning? The Helmholtz method consists of writing down only the equation of energy, which is a method that is sacrosanct, but insufficient, because it gives only one equation for the problem, whereas one really needs several. One succeeds in developing the theory by a sequence of hypotheses that are often disguised by faulty arguments. Furthermore, they do not give a complete, neatly-formulated, general solution; or rather, that solution will be Maxwell's solution. That is why I came to present Maxwell's theory as the complement and the coronation of Helmholtz's theory. I think that one of the interesting aspects of my presentation is to show how one can start along the Helmholtz path with no hypothesis and be led by the facts and the aspects of some formulas to specify the nature of the forces and energies that come into play in conclusions such as this one:

The intrinsic energy of currents is kinetic energy. The electromotive forces of induction, the electrodynamical, electromagnetic, and magnetic forces are inertial forces.

([2]) Vaschy (*Théorie de l'Électricité*, Baudry, 1896, pp. XII of the Introduction) declared that Maxwell's explanations were insufficient (t. II, § 573). I have pointed out that error to some authors who had followed Helmholtz in my theory of the unicycle and the bicycle [J. Ec. Poly. (2) Cahiers VI and VII]. It resulted clearly from two articles by Sarrau [C. R. Acad. Sci. **133**, (August-September 1901), pp. 402 and 421]

Once those facts have been established, Maxwell's theory can be developed with no hypothesis by a simple application of Lagrange's equations. Everything will become clear, and all formulas will appear to be necessary, while everything in Helmholtz's theory will become inexplicable when one regards the energy of currents as potential energy.

Take the simplest case, namely, induction by the approach of a permanent magnet to a circuit with no battery. The energy equation indeed gives the total quantity of induced electricity, but nothing else. One needs a hypothesis in order to find the induced electromotive force. One must ask a very serious question: Why does current arise in the circuit? It is a phenomenon that the theory does not explain and which will remain incomprehensible when one appeals to the idea of potential energy. In Maxwell's theory, it becomes as natural as the effects of centrifugal force in rational mechanics.

As I said, the Lagrange equations are the basis for that theory. To be clear, it seems to me necessary to recall it before embarking upon Maxwell's theory. I have reduced the presentation to only that which is most essential and most appropriate to my goal. What will result is a very personal and suggestive form that I hope will not be devoid of interest for the reader.

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CHAPTER I

HELMHOLTZ'S THEORY

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§ 1. – *Electromagnetic force function. Magnetic induction.*

2. Introduction. – Experiments show that a magnetic needle is subject to a director couple. If it is very small then the couple will depend upon only two vectors: the *magnetic moment* that is attached to the needle and the *magnetic force* that is determined at each point in space by the surrounding currents or magnets.

Experiments with a broken magnet show that magnetism is a property of the particles like gravity, in such a way that the magnetic moment of the needle is the resultant of the magnetic moments of its elements. If we divide the magnetic moment of an element by its volume then we will have the *magnetization A* at the point of the matter where that element is taken.

The vector *a* suffices to define the action of the field, not only on the magnetic needle, but also on a conducting element that is the site of a current. The law of the fields that are due to magnets was discovered by Laplace and the law for the fields that are due to currents was discovered by Ampère. Maxwell stated it in full generality in the following form:

An element dx of a circuit where the intensity of the current is i and the magnetic force is α is subject to a force that is called the ***electromagnetic force****. It can be represented by the same vector as the oriented area of the parallelogram that is constructed from the two vectors i dx and α.*

Conforming to Grassmann's notation, I shall denote it by | [*i dx* α].

One deduces the expression for the elementary work done by electromagnetic forces for a displacement of the circuit (with or without deformation) from that law. It is the exact differential of a function that is called the *force function,* and which we shall establish. However, before we do that, we must recall some indispensible notions.

3. The flux law of the magnetic force. – The distribution of the vector α depends upon the magnets and currents that are present. One can vary that distribution by one's choice of those elements, but not in an entirely arbitrary fashion. It must obey this fundamental law:

The flux of the vector α will be zero when it crosses any closed surface that does not contain a magnet.

I would like to make the meaning of that more precise: Consider a surface element *dS* and the vector α at the point where one finds the element *dS*, and then construct the cylinder that has dS for its base and α for its generator. That cylinder is the *flux* of α that crosses dS. It is interesting to learn not only the size of the cylinder, but also the edge of the surface where it is found. In order to do that, one gives a sense to the surface and a sign to the flux, as I would like to explain: The *sense* of the element dS is the one in which one agrees to traverse its contour. That contour, with its sense, is what I shall call the *circuit* of the element dS. With Maxwell, I agree to define the normal ν to that element to have a sense such that an observer whose feet are on dS and whose head is towards the point ν will see his circuit as being traversed from right to left. *The flux of the vector* α *upon crossing* dS *will be the volume of the cylinder* $[dS, \alpha]$, *preceded with the + sign if the cylinder is on the same side as the normal* ν *to the element* dS *and preceded with the – sign in the contrary case.* In other words, the flux of the vector α is the product of dS with the projection of α onto ν; I shall denote it by $[dS \cdot \alpha]$. The flux of α upon crossing a finite surface is the integral of the elementary flux $\int [dS \cdot \alpha]$.

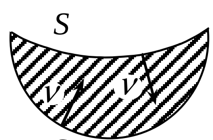

Figure 1.

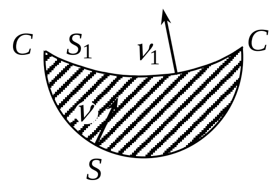

Figure 2.

When one considers a closed surface S (Fig. 1), continuity will lead one to always carry the normal on the same side of the surface. If the normal points to the interior of the volume that is enclosed by the surface then the integral will represent the *inward flux*. On the contrary, it will be the *outward flux* when the normal points to the exterior.

When one considers two surfaces S and S_1 that are bounded by the same circuit C (Fig. 2), the sense of the surface will be determined by that of the circuit, in such a way that if the normal ν to the first one is interior to the volume that is enclosed by S and S_1 then the normal ν_1 to the second one will be exterior to that same volume. The flux that leaves the volume that is bounded by SS_1 is equal to the excess $\varphi_1 - \varphi$ of the flux that crosses S_1 over the flux that crosses S. If the vector considered is the magnetic force α then Φ will be zero, which is the law. In other words, $\varphi_1 = \varphi$: *The flux of the magnetic force is the same upon crossing all surfaces that are bounded by the same circuit.* We can call that invariant the *flux of the magnetic force upon crossing the circuit.* That is rightfully the electromagnetic force function that I would like to explain.

4. Electromagnetic force function. – Since electromagnetic forces are proportional to the intensity (no. **2**), it will suffice to treat the case of a circuit that is traversed by the current $i = 1$. That is what I mean by an electrical circuit in the following statements, which are consequences of the elementary law of electromagnetic action ([3]) (no. **2**).

([3]) The proofs are classical. They have a purely-geometric character, in such a way that the reader can, if desired, establish them or assume the results with no loss of clarity to the subject for him.

1. If the element dx of an electric circuit submits to an infinitely-small displacement δ then the work done by the electromagnetic force will be equal to the volume of the parallelepiped $(dx \cdot \alpha\,\delta) = (\delta\,dx \cdot \alpha)$. From our definition in no. **3**, it is the flux of the vector α that crosses the surface $(\delta\,dx)$ that is swept out by the element dx.

2. For an infinitely-small displacement of the circuit (with or without deformation), the work done by electromagnetic forces is equal to the flux of the vector α upon crossing the area that is swept out by the contour of the circuit.

3. As a consequence of the properties of the flux of the magnetic force that was studied in no. **3**, it is the differential of the flux of α when it crosses the circuit.

Hence, the flux of the magnetic force upon crossing the circuit is indeed the electromagnetic force function that the neighboring magnets and currents exert upon the electric circuit.

5. Magnetic induction. – The surfaces that one envisions in the study of magnetic flux upon crossing a circuit must not meet the magnet, since it is necessary for the vector α to be defined there. It is important to lift that restriction, which can be awkward, especially when the magnet has the form of a ring and when the circuit and magnet are linked like two consecutive rings of a chain. Therefore, suppose that a surface cuts the magnet. Along the geometric section, imagine that an infinitely-thin cut has been made in the material, but which does not change the magnetization of the two remaining parts of the magnet. The electromagnetic forces are not changed by that infinitely-small suppression of magnetized matter. Henceforth, α will be defined in the cut. An inconvenience in the vector α thus-defined is that it varies with the obliquity of the cut with respect to the magnetization. On the contrary, the vector a that corresponds to a cut that is normal to the magnetization will be an invariant. Now, it will imply the same result for the flux, as one can show. There is a great advantage to introducing the vector a in preference to the other.

Therefore, imagine a vector a that is equal to the existing magnetic force α outside the magnet, but equal to the magnetic force that one observes in an infinitely-thin cut that is normal to the magnetization inside the magnet. That is Maxwell's *magnetic induction* vector. From its definition, it will satisfy the same law as α, namely: *The flux of the magnetic induction is zero across any closed surface.* However, although one must consider only surfaces that do not cut the magnetic force α, the law will apply to all surfaces, without restriction, for a . The new vector will permit one to state Laplace's law in this form:

The electromagnetic force function that is exerted on an electric circuit is equal to the flux of the magnetic induction that crosses the circuit.

6. Conclusions. – I recalled:

1. The definitions of the magnetic moment, magnetization, magnetic force, the elementary law of electromagnetic action in the form that was given by Maxwell, the definitions that relate to flux, and the fundamental property of magnetic flux.

2. How I concluded the electromagnetic force function that acts upon an electric circuit from the elementary law.

3. The generalization of the expression that was found that led me to Maxwell's notion of magnetic induction. The fundamental law of that vector is that its flux upon crossing any closed surface is zero. It permits one to state the second of our conclusions thus:

4. The electromagnetic force function that is exerted upon a circuit that is traversed by a current is equal to the intensity of the current, multiplied by the flux of the magnetic induction that crosses the circuit.

§ 2. – *Energy equation. Induced electromotive force. Self-induction.*

7. Introduction. – When an observer displaces a circuit C that is a filamentary conductor with no battery in the field of a magnet A, a current i will manifest itself in the circuit during the duration of the motion. It will be zero before and after it. That is the *induction current*, which is the topic of the present chapter. Like any current, it is subject to the two laws of energy:

First law. – The electromagnetic force function that is exerted on the circuit is the product $i\,\Phi$ of the current intensity with the flux of the magnetic induction upon crossing the circuit. That is the law that was stated in the preceding paragraph.

Second law. – In a circuit of resistance r that is traversed by a current i, the heat that is released is called the *Joule heat*. It is equal to $r\,i^2$ per unit time. That is *Joule's law.*

Moreover, the induced current satisfies a qualitative law of its own: The sense of the current is always such that the electromagnetic forces oppose the motion that the observer imposes upon the circuit. That is *Lenz's law.*

There are three energies in effect: the work produced by the observer, that of the electromagnetic forces that resist him, and that of the Joule heat. In addition, there is the *vis viva* of the conductor, and we see that the current possesses a proper energy that is analogous to *vis viva*, but those two energies will be zero when the circuit is at rest and when it is no longer traversed by any current.

Those are the energies that experiments reveal. If one assumes that they are the only ones then one can apply to the system: on the one hand, the *vis viva* theorem of rational mechanics, and on the other, the more general principle of energy. Upon eliminating the work done by the observer from the two equations obtained, one will get a relation between the Joule heat and the work done by electromagnetic forces. That is the *Helmholtz equation.*

8. Energy equation according to Helmholtz. – I consider the system that is composed of the magnet and the conductor. It starts out at rest and ends at rest. The *vis viva* does not change. Therefore, the sum of the works done by the forces that are applied to the system will be zero. That is the work done by the electromagnetic forces $\mathcal{T}$ and the work done by the observer $\mathcal{T}_1$:

(1) $$\mathcal{T} + \mathcal{T}_1 = 0.$$

I let the Joule heat dissipate. The system returns to the initial temperature. Its energy has not changed, moreover. Therefore, the sum of the energies provided to the system is zero. They are the work done by the observer $\mathcal{T}_1$ and the Joule heat, with the opposite sign – J. As for the electromagnetic forces, they must not be taken into account, since those forces are internal forces that are exerted between the magnet and the circuit:

(2) $$\mathcal{T}_1 - J = 0.$$

When one subtracts corresponding sides of equation (2) from equation (1), one will get:

(3) $$\mathcal{T} + J = 0.$$

The values of $\mathcal{T}$ and J are defined by the two laws that were stated in no. **7**; they are:

$$\mathcal{T} = \int i\,d\Phi, \qquad J = \int r\,i^2\,dt .$$

Equation (3) is then written:

(4) $$\int i\,d\Phi + \int r\,i^2\,dt = 0 .$$

That is the Helmholtz equation; it shows that $\int i\,d\Phi$ is negative. In other words, the electromagnetic forces oppose the motion; that is Lenz's law.

Nothing allows us to conclude from the fact that the two integrals in equation (4) are equal, up to sign, that the corresponding elements of the integrals will be equal. In fact, they are not, since the current i is endowed with a type of energy. At the beginning, the work done by the observer is partially expended in overcoming that inertia, so for that reason, it will be greater than the Joule heat. At the end, the energy that was stored is expended as Joule heat, which comes, in turn, from the work done by the operator.

9. Proper inertia of a current. Self-induction. – Experiments have shown the following ([4]):

A double circuit *ANBCRDA* (Fig. 3) has a branch *ANB* that is composed of a key (i.e., a switch) that moves around the point *N*, which permits one to establish the current in either *A* or *B*. The three parallel branches contain:

BC, a battery *P*,
NR, a coil of wire *b*,
AD, a galvanometer *G*,

respectively.

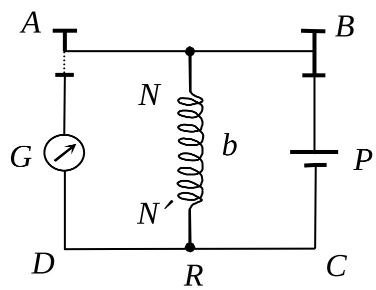

Figure 3.

One establishes the current in the circuit on the right and then presses the key *A* in such a fashion as to cut the circuit on the right and establish one on the left. One confirms with the galvanometer *G* that there is an impulse that has the opposite sense to the deviation that the galvanometer would get from the battery if the contacts were simultaneously established at *A* and *B*. That impulse shows that the current in the coil *b* will continue for some time after the battery *P* has ceased to act. The magnitude of the impulse depends upon the constitution of the coil *b*. If it is wound in the natural fashion from its extremity *N* to its extremity *N′* then the deviation will be larger when the coil has more turns. Moreover, the effect increases if one gives the coil a core of soft iron. However, if one winds the wire in pairs by starting from its middle *M* (Fig. 4) and in such a fashion that the two extremities *N* and *N′* are at the same end of the coil then the deviation of *G* will be zero. One then perceives that the inertia of the current depends upon the induction flux that crosses its circuit by reason of the current itself. That is *self induction*.

10. Induced electromotive force. – The type of inertia that we just observed opposes the equality of the corresponding elements in the integrals of the Helmholtz equation. However, if the regime is established at a given moment then the elements will be equal at that moment, and one will have:

([4]) That form of the *Faraday experiment* is due to Cornu. It provoked a criticism from Potier. It is easy to substitute the form that Potier preferred for the Faraday form; that is what I did in my *Leçons d'Électricité*. However, due to the schematic character of that pamphlet, it seemed preferable to me to preserve Cornu's more intuitive form here.

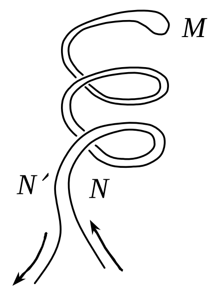

Figure 4.

(1) $$- i\, d\Phi = r\, i^2\, dt,$$

from which, one infers that:

(2) $$- \frac{d\Phi}{dt} = ri\,.$$

If one recalls Ohm's formula $e = ri$ then one will see that the quantity (2) is an electromotive force. It is the *induced electromotive force.* It was calculated by supposing that the current i has been established; however, it exists independently of it. One can show that by means of a dynamo in an open circuit when one connects the two poles to an electrometer, or more practically, by opposing the dynamo in a closed circuit with a battery of accumulators. The latter experiment will show, moreover, that the induced electromotive force is independent of the other electromotive forces that can be found in the same circuit; experimental verifications of that fact abound. The law that is expressed by the formula:

(3) $$e = - \frac{d\Phi}{dt}$$

will then have a degree of generality that is greater than that of the proof. One can observe its generality along a different path as follows: Instead of approaching the induced circuit with a permanent magnet from a great distance, one creates an equivalent electromagnet in its place by passing a current through it. The total impulse $\int e\, dt$ = $\int r\, i\, dt$ will be the same. One can verify that with the aid of a ballistic galvanometer. It will give the same deviation in both cases. Therefore, the total impulse of induction will depend upon only the variation of the induction flux Φ, as one sees from formula (3).

11. – Electromotive force of self-induction. – If one assumes the complete generality of formula (3) then one can deduce the electromotive force that is due to the induction of a current itself thus: Let L be the induction flux that is caused by the current itself and which crosses its circuit when the intensity is equal to + 1; that is the coefficient of self-induction. Its value depends upon only the form of the circuit. Moreover, it is essentially positive, which is a consequence of Ampère's rule about the sense of the magnetic field that is due to a current, as well as our conventions in regard to the sense and sign of the magnetic induction flux (no. **3**). For an intensity of i, the flux will be $\Phi = Li$. If one assumes the generality of formula (3) then the electromotive force that is due to the variation of i will then be:

$$e = -\frac{d\Phi}{dt} = -L\frac{di}{dt}.$$

That electromotive force has the opposite sign to di / dt. It opposes the establishment and stopping of the current. In summary, it behaves like an inertial force.

That is the electromotive force of self-induction that formula (3) will lead to when one assumes that it is complete general.

12. Conclusions:

1. I have defined the integral equation of the energy that relates to the displacement of a circuit that starts out at rest and returns to rest in the presence of a magnet.

2. I have exhibited a property of currents that is analogous to inertia that one calls *self-induction.* It opposes the differentiation of the integral equation in the variable regime.

3. Differentiation is permissible when the derivative of the intensity is zero. It will lead to the notion and the expression for the induced electromotive force. Although the expression was proved only in that particular case, it is nonetheless general. Experiments bear witness to that fact.

§ 3. – *Currents in the variable regime. Mechanical interpretation.*

13. Introduction. – The general law of induction that was recalled in the preceding paragraph is stated thus: *The induced electromotive force in a filamentary circuit is equal to the derivative of the magnetic induction flux that crosses the circuit with the sign changed.* As an application, I would like to establish the equations of the currents in the variable regime and give them a dynamical interpretation.

14. Equation for the current induced in a circuit with no battery. Experiments that are independent of the self-induction. – The magnetic induction flux that crosses the circuit is the sum of two terms: Li, which is due to the current itself, and Φ, which is due to other effects of the magnetic field. The electromotive force of induction is the derivative of that sum with the sign changed: $-L\frac{di}{dt}-\frac{d\Phi}{dt}$. From Ohm's law, that electromotive force is equal to ri. The equation of the current is then:

$$-L\frac{di}{dt}-\frac{d\Phi}{dt} = ri.$$

I suppose that the circuit is displaced without deformation. I then consider an instant before the displacement and an instant after a sufficiently-long time when the current has ceased; i is zero at those two limits. The first term will then disappear in the integration. What will remain is:

$$-\Delta\Phi = r\int i\,dt\,.$$

Now, $\int i\,dt$ is the quantity of electricity q that is lost by the circuit; $-\Delta\Phi$ will then be the decrease in the induction flux. Therefore:

The total quantity of induced electricity is equal to the quotient of the decrease in induction flux that crosses that circuit over the resistance of the circuit.

Since the ballistic galvanometer records q precisely, that formula will give rise to some easy ways to verify it, and notably by experiments on zero. For example, if one creates a flux by exciting an electromagnet and then annuls it by displacing the circuit or electromagnet then the galvanometer will not budge.

15. Currents from batteries in the variable regime. – Keeping the preceding notations, I consider a circuit that is equipped with batteries and electrolytes whose resultant electromotive force is E. One must add to that the electromotive force that is due to the induction of the current on itself $-L\dfrac{di}{dt}$ in order to get the total electromotive force $E - L\dfrac{di}{dt}$. It is equal to ri, which is Ohm's law. The equation of the current is then:

$$E - L\frac{di}{dt} - ri = 0. \tag{I}$$

I have supposed that the current is isolated from any magnetic variation other than the one that comes from the current itself.

Now let two currents be present. I shall distinguish the givens for the two circuits by the indices 1 and 2. For each current, one must add the electromotive force that is induced by the variations of the other current to the terms that correspond to the ones in equation (I). Let M_{12} be the induction flux that crosses the first circuit when the second one is traversed by the current $i = +1$. That coefficient depends solely upon the form of the figure that is presented by the totality of the two circuits. It will be the same when one inverts the roles of the two circuits ($M_{12} = M_{21}$); that is a consequence of Laplace's law that relates to the magnetic field of a current. From it, the electromotive force that is induced by the second current on the first one will be $-M_{12}\dfrac{di_2}{dt}$, and the electromotive force that is induced by the first current on the second one will be $-M_{12}\dfrac{di_1}{dt}$. The simultaneous equations of the two currents will then be:

$$
\text{(II)} \qquad \begin{cases} (1) \quad E_1 - L_1 \dfrac{di_1}{dt} - M_{12} \dfrac{di_2}{dt} - r_1\, i_1 = 0, \\[2ex] (2) \quad E_2 - L_2 \dfrac{di_2}{dt} - M_{12} \dfrac{di_1}{dt} - r_2\, i_2 = 0. \end{cases}
$$

Upon writing the equations that relate to when an arbitrary number of fixed currents are present similarly, one will have as many equations as unknowns. If the currents are moving and their form is invariable then the coefficients of mutual induction M will be functions of the position coordinates. One must add terms that come from the variation of M to the current equation. On the other hand, one must write the equations of motion of the circuits that move under the action of inertial forces, electromagnetic forces, and forces of a purely-mechanical origin. One will then have a system of equations whose number is equal to the number of unknowns.

16. Mechanical interpretation. Principles of energy and virtual work. – In order to interpret equation (I), I must first choose the most appropriate definition of the intensity i. I shall adopt $i = dq / dt$, which is the rate of dissolution of zinc in the battery or the rate of decomposition of an electrolyte that is placed in the circuit. The quantity q will then be regarded as the *coordinate of the circuit*, and it indeed has the character of a coordinate, because the state of the circuit will be known when one gives q and dq / dt. Finding q as a function of time is the problem that corresponds to the problem of finding the coordinate x of a machine with complete constraints. With the introduction of the coordinate q, equation (I) will be written:

$$E - L \frac{d^2 q}{dt^2} - r \frac{dq}{dt} = 0.$$

The analogy between that equation and the equation of a simple machine is obvious: E corresponds to the applied forces, power, and resistance, $-L\dfrac{d^2 q}{dt^2}$ corresponds to the inertial force, and $-r\dfrac{dq}{dt}$ corresponds to the force of friction. As in mechanics, *the power, resistance, inertial force, and passive resistances are in equilibrium.* As in mechanics, one forms the energy equation by the multiplying the sides of the equation of dynamical equilibrium by the differential of the *electric displacement* $dq = i\, dt$. I will then get:

$$E\, dq - d\left[\tfrac{1}{2} L \left(\frac{dq}{dt}\right)^2\right] - r \left(\frac{dq}{dt}\right)^2 dt = 0.$$

$E\, dq$ is the excess of the energy that is provided by the generators over the energy that is absorbed by the receivers. $\tfrac{1}{2} L \left(\dfrac{dq}{dt}\right)^2$ is Maxwell's *electro-kinetic energy*, which is analogous to the *vis viva*. Like the *vis viva*, it is proportional to the square of the velocity dq / dt. Like the *vis viva* of a flywheel, it opposes the starting and stopping of it, and it is stored during the first period to be released during the second one. Finally, $r\, i^2\, dt$ is the energy released to Joule heat by the passive resistance ri that is analogous to friction:

The work done by power is equal to the work done by the resistance plus the increase in kinetic energy and the energy that is lost to heat by the effect of passive resistance.

Hence, a single circuit that is endowed with a single coordinate q is completely analogous to a machine with complete constraints.

Similarly, a system of two or more fixed or moving circuits of variable or invariable form that presents a number of coordinates that is greater than 1 will be analogous to a system with incomplete constraints. In order to not complicate the explanations, I shall confine myself to examining the case that was developed in no. **15** of a system of two fixed circuits. Equations (II) of the system must be associated with the two *Lagrange equations* that correspond to the coordinates q_1 and q_2 . Indeed, they express the idea that the total work done by electromagnetic forces will be zero for the two virtual displacements δq_1 and δq_2 . As in mechanics, the energy equation will be obtained by multiplying the first equation by dq_1 and the second one by dq_2 and adding them, which will give:

$$(E_1 - r_1\, i_1)\, dq_1 + (E_2 - r_2\, i_2)\, dq_2 = dT, \tag{3}$$

with

$$\left\{\begin{aligned} dT &= \left(L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt}\right) dq_1 + \left(M_{12} \frac{di_1}{dt} + L_2 \frac{di_2}{dt}\right) dq_2, \\ T &= \tfrac{1}{2}[L_1\, i_2^2 + 2M_{12}\, i_1\, i_2 + L_2\, i_1^2]. \end{aligned}\right. \tag{4}$$

The left-hand side of equation (3) represents the work done by applied forces: namely, powers, resistances, and passive resistances. Therefore, the expression T must be analogous to the kinetic energy; it is Maxwell's *electro-kinetic energy*. The expression for that energy is found to be precisely equal to the electromagnetic force function that is exerted on the particles of the two circuits. It then results that the electromagnetic forces must be analogous to inertial forces, as well as the electromotive induction forces themselves.

17. Conclusions:

1. I showed how the general law of induced electromotive force permits one to form the equations of a system of fixed currents. I indicated that it will permit one to also solve the problem of moving currents.

2. I gave the dynamical interpretation of such a system: It is analogous to a system of constraints in rational mechanics that obeys the law of virtual work. One can apply the Lagrange equations to it by means of the following interpretations:

a. The Lagrangian coordinates q are the quantities of electricity that are lost by the circuits.

b. The electro-kinetic energy is the electromagnetic force function that is exerted on the particles of the circuits.

c. Lagrange's resultant applied forces Q are the electromotive forces $E - ri$ that result from the generators, the receivers, and the Joule resistance. The electromagnetic forces of induction and the electromagnetic forces are inertial forces.

Taking those principles as the point of departure constitutes Maxwell's theory. That is the theory that we must present in full generality. However, we must meanwhile recall the principle of virtual work and the Lagrange equations.

CHAPTER II

GENERAL EQUATION OF DYNAMICS

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§ 1. – *Theorem of virtual work.*

18. Introduction. – The components of a machine interact with each other. Hence, a solid that is pushed by another one will be deformed by the pressure until the elastic force that is due to the deformation equilibrates the pressure that acts upon it. When the deformation is negligible, the work done by the elastic reaction will also be negligible. I call:

The elastic forces that limit the deformations to very small values so that the work they do will be negligible the *forces of constraint.*

A system whose elastic forces are forces of constraint a *system with constraints.*

Displacements for which the deformations that produce the forces of constraint are zero *displacements that are compatible with the constraints.*

As a result of those definitions, the work done by forces of constraint will be zero for all displacements that are compatible with the constraints ([5]).

In a natural system, the material points are uncountable. One must then give up on writing out all of their equilibrium equations under the action of applied forces, inertial forces, and elastic forces. The problem will be simplified if one neglects the deformations of the components. The position of the system will then depend upon a small number of parameters. For example, the position of a solid that can slide along one axis and turn around that axis depends upon two parameters q_1 and q_2, namely, the sliding and the rotation; the *degree of freedom* of the system is 2. Two equations will suffice to exhibit the two parameters as functions of time, provided that they do not contain unknown constraint forces, but only the forces that are applied to the system, which are supposed to be given. Those equations are produced by the following method:

19. Theorem of virtual work. General equation of dynamics. – From d'Alembert's principle, each particle in the system is in equilibrium under the action of its inertial force, the given forces that are applied to it, and unknown elastic forces. The total work done by those forces will then be zero for arbitrary displacement of the particles of the system. Such a displacement is called *virtual*. The work done by forces of constraint is non-zero for all of those displacements, and notably, for the ones that deform the solid pieces. However, it will be zero for the displacements that are compatible with the constraints. Now, give the system the virtual displacement that corresponds to the set of values δq and δq_1 of the two parameters that were chosen to define its position. The total work done by forces will then have the form $P\ \delta q + P_1\ \delta q_1$. I write that it is zero:

([5]) I shall intentionally exclude the cases in which the constraints are unilateral or depend upon time from my study.

(1) $$P\,\delta q + P_1\,\delta q_1 = 0.$$

That equation must be verified for any arbitrary displacement (δq, δq_1). It will then imply the two necessary equations:

(2) $$P = 0, \qquad P_1 = 0.$$

They do not include the constraint forces, since the work done by the constraints is zero because the displacement (δq, δq_1) is compatible with the constraints. Moreover, equations (2) determine the variable coordinates q and q_1 as functions of time. They are then the equations of motion. Thus:

The equations of motion are obtained by writing down that the total work done by applied forces and inertial forces is zero for any virtual displacement that is compatible with the constraints.

That is the *theorem of virtual work*. The equation (1) that provides it is the *general equation of dynamics*. Equations (2), when written in a less specialized form, bear the name of Lagrange.

The equilibrium equations are deduced by setting:

$$q' = q'' = q_1' = q_1'' = 0.$$

In the particular case of equilibrium, P and P_1 depend upon only the applied forces.

We have assumed that there are two degrees of freedom in the system. If that number of n then the method will give n equations that determine the n parameters q that the position of the system depends upon. In order to simplify the writing and the language, we shall continue to suppose that the number of degrees of freedom is 2. That will not impair the generality of the arguments and formulas.

20. Extending the idea of force that is deduced from the notion of energy. Electromotive force. – The work done by the forces for the displacement δq is proportional to δq, namely, $P\,\delta q$. The dimensions of the coefficient P with respect to the fundamental quantities (length, time, and mass) depend upon the dimensions of the coordinate q in such a fashion that $P\,\delta q$ will have the dimensions of energy. When q is a length, P will be force, properly speaking, which is equal to the product of a mass with an acceleration. When q is an angle, P will be a moment, which has the dimensions of energy. Although those two quantities – viz., force and moment – do not have the same dimensions with respect to fundamental quantities, one cannot deny that they have a commonality of origin and nature: Whether P is a force or a moment, it is the coefficient of a displacement in the expression for energy; it is the quotient of the energy over the variation δq of the coordinate. In the two cases, P measures the weight of the forces in the system (momentum), which is tendency to displace by δq. That is why the moments play the same role in the study of rotations that the forces do in the study of translations.

If q is now the quantity of electricity that is lost by a circuit then it will be the Faraday-Maxwell *electro-kinetic displacement*, as one will see later. The factor P measures the tendency of that coordinate to displace by δq. It then deserves the name of *electromotive force*. From that, the electromotive force of a generator or a receiver of electrical energy is indeed defined by the coefficient of δq in the expression for the energy that it produces or absorbs. It is the quotient of the energy in play by δq. That definition of the electromotive force seems to us to be the only one that does not lead to contradictions and inextricable difficulties, notably as far as the electromotive forces of contact and the Peltier effect ([6]) are concerned.

21. Conclusions:

1. I considered a system with constraints to be the limiting case of a natural system in which the elastic deformations do negligible work.

2. I deduced the theorem of virtual work and the general equation of dynamics.

3. I showed how the notion of energy leads to an extension of the idea of force that includes the electromotive force.

§ 2. – *Work done by inertial forces. Lagrange equations.*

22. Introduction. – In § 1, we saw that the left-hand side of the general equation of dynamics is the work done by forces for the displacement (δq, δq_1), namely, $P\ \delta q + P_1\,\delta q_1$. That work will include the work done by inertia, which deserves a special study because it often presents great difficulties. When we are dealing with only a machine that is composed of solid pieces whose particular constitution we indeed know, the direct calculation of the inertial forces from the integral of its elements is already very difficult and can become inextricable. However, when we are dealing with systems in which electricity enters into play, the direct calculation of the inertial forces becomes impossible, due to our ignorance of the nature of electricity. Lagrange made an important discovery on that subject by giving some formulas that permitted one to easily calculate the work done by inertial forces when one knows the expression for the *vis viva* as a function of the coordinates of the system and their velocities, without it being necessary to know anything else about the mechanism under study. Now, that is precisely the case for electrical currents in filamentary circuits. Furthermore, we understand the great interest that the Lagrange equations attract in the study of electricity. Unfortunately, their application involves a restriction that is really quite important (no. **25**): It is necessary that the mobility parameters must be true coordinates that permit one to fix the state of the system when one knows those parameters at the moment t. That is not the case for the parameters that are natural to the study of rolling motions, in general, and notably for the hoop. That also does not seem to be the case for the experiments with the Barlow

([6]) Congrés de Physique in 1900. Papers by Arrhénius, Christiansen, and L. Poincaré.

wheel (no. **83**). That remark is essential to this book. It explains why the Lagrange equations lead to two results that are contrary to the facts when they are applied to the Barlow wheel, and it reestablishes the agreement between the theory and the experiments.

23. Lagrange's expression for the work done by inertial forces. – Let M be a particle of the system, and let m be its mass. *I suppose that its coordinates x, y, z depend upon only two parameters q and q_1.* In other words, I suppose that the given of q and q_1 imply the positions of all the points of the system.

The variation δq of the parameter q implies a displacement of the point M whose components are $\frac{dx}{dq}\delta q$, $\frac{dy}{dq}\delta q$, $\frac{dz}{dq}\delta q$. Since the components of the inertial force are $-m\,x''$, $-m\,y''$, $-m\,z''$, the work that is due to the inertial of the particle M under the displacement δq will be:

$$-\,m\left(x''\frac{dx}{dq}+y''\frac{dy}{dq}+z''\frac{dz}{dq}\right)\delta q\,.$$

The total work done by inertial forces is the sum of the works over all particles of the system, namely:

$$I\,\delta q=-\sum m\left(x''\frac{dx}{dq}+y''\frac{dy}{dq}+z''\frac{dz}{dq}\right)\delta q\,. \tag{1}$$

It is the presence of accelerations x'', y'', z'' that makes the calculation difficult and Lagrange avoided them by means of the following transformation: It consists of an integration by parts:

$$x''\frac{dx}{dq}=\left(x'\frac{dx}{dq}\right)'-x'\left(\frac{dx}{dq}\right)'. \tag{2}$$

The hypothesis is that x is a function of only the parameters q and q_1, which are functions of time t. Moreover, one can invert the order of derivations with respect to q and t in the right-hand side of formula (2). The second term can then be transformed thus:

$$x'\left(\frac{dx}{dq}\right)'=x'\frac{dx'}{dq}=\frac{d}{dq}\left(\tfrac{1}{2}x'^2\right). \tag{3}$$

I would now like to show that one can replace dx / dq with dx' / dq in the first term. In order to do that, I remark that x depends upon t only by the intermediary of q and q_1. One will then have:

$$x'=\frac{dx}{dq}q'+\frac{dx}{dq_1}q_1'\,. \tag{4}$$

That formula shows that x is a linear function of q' and q_1', and that from a purely-formal standpoint, the derivative of x' with respect to q' is dx / dq. One will then have:

$$\frac{dx}{dq} = \frac{dx'}{dq'}.$$

Furthermore, the first term on the right-hand side of formula (2) is the derivative of:

$$x' \frac{dx'}{dq'} = \frac{d}{dq'}\left(\tfrac{1}{2} x'^2\right),$$

and formula (2) will then be written:

$$x'' \frac{dx}{dq} = \left[\frac{d}{dq'}\left(\tfrac{1}{2} x'^2\right)\right]' - \frac{d}{dq}\left(\tfrac{1}{2} x'^2\right). \tag{5}$$

If one performs the same transformation on the other two terms of the expression (1) then one will get:

$$x'' \frac{dx}{dq} + y'' \frac{dy}{dq} + z'' \frac{dz}{dq} = \left[\frac{d\,\frac{1}{2}(x'^2 + y'^2 + z'^2)}{dq'}\right]' - \frac{d\,\frac{1}{2}(x'^2 + y'^2 + z'^2)}{dq}.$$

It will suffice to multiply this by m in order to exhibit the *vis viva* of the particle M in the right-hand side. The sum over all particles then exhibits the kinetic energy T of the system; hence:

$$\sum m\left(x'' \frac{dx}{dq} + y'' \frac{dy}{dq} + z'' \frac{dz}{dq}\right) = \left(\frac{dT}{dq'}\right)' - \frac{dT}{dq}. \tag{6}$$

That is (up to sign) the particularly advantageous form that Lagrange gave to the part of P that comes from the inertia, and which I have denoted by I [eq. (1)]. If I, with Lagrange, call the part of P that comes from the applied forces Q then I will get the equations of motion [eq. (2), no. **19**] in the form of:

$$\left\{\begin{aligned} Q &= \left(\frac{dT}{dq'}\right)' - \frac{dT}{dq}, \\ Q_1 &= \left(\frac{dT}{dq_1'}\right)' - \frac{dT}{dq_1}. \end{aligned}\right. \tag{7}$$

Those are the *Lagrange equations.*

24. *Vis viva* theorem. – If one replaces the arbitrary virtual displacement (δq, δq_1) with the real displacement $dq = q\,dt$, $dq_1 = q_1'\,dt$ in the equation of virtual work:

$$(1) \qquad P\,\delta q + P_1\,\delta q_1 = 0$$

then the left-hand side will represent the total work done by the applied and inertial forces. Now, one knows that the work done by inertial forces is equal to the decrease in *vis viva*. The equation:

$$(2) \qquad P\,dq + P_1\,dq_1 = 0$$

then means that the work done by applied forces is equal to the increase in the *vis viva*.

25. Modifying the Lagrange equations when the mobility parameters are not coordinates, properly speaking ([7]). – I take the motion of the hoop as an example.

The natural mobility parameters are the tilt angle θ, the conversion angle γ, and the rolling angle σ ([8]). Their velocities θ, γ, σ suffice to determined the velocity x of a Cartesian coordinate of any arbitrary particle of the system. Meanwhile, x cannot be calculated as a function of θ, γ, σ. Indeed, if one says that the hoop starts from Paris with the angles θ_0, γ_0 and rolls through σ until the former angles have become θ, γ then one cannot say whether the hoop has arrived in Bordeaux, Lyon, or any other place. It is necessary that one must know, in addition, what the sequence of correlated values of γ and σ were if one is to be able to trace the trajectory of the contact point with the ground and determine the present position of the hoop. To conclude, *the parameters γ and σ are not coordinates, properly speaking*. x is not a function of just θ, γ, σ; it is a function of θ and the sequence of correlated values of γ and σ. In other words, the expression for dx as a function of $d\theta$, $d\gamma$, and $d\sigma$ is not integrable. (Refer to no. **23** and the hypothesis that is emphasized there.) The transformation of the last term in formula (2) from $x'\left(\dfrac{dx}{dq}\right)'$ into $x'\dfrac{dx'}{dq}$ is once more permissible for the variable θ, but not the variables γ and σ. The corresponding Lagrange equation is still valid for θ, but for γ and σ, the equation of Lagrange type must be replaced with this one:

([7]) E. CARVALLO, "Théorie du monocycle et de la bicyclette," nos. **71** and **72**, J. Ec. Poly. (2) Cahiers VI and VII.

([8]) Namely:

θ	is the angle between the plane of the hoop and the ground
γ	is the angle between the tangent to the lowest point and the initial position of that line
σ	is the angle through which the hoop has rolled from the origin.

$$Q = \left(\frac{dT}{dq'}\right)' - \sum m\left[x'\left(\frac{dx}{dq}\right)' + y'\left(\frac{dy}{dq}\right)' + z'\left(\frac{dz}{dq}\right)'\right].$$

Those are the modifications that must be made to the Lagrange equations when the mobility parameters are not coordinates, properly speaking.

26. Conclusions:

1. I gave the expression that Lagrange discovered for the work done by inertial forces.

2. I showed how the Lagrange equations must be corrected when the parameters q are not coordinates, properly speaking.

We are now in a position to begin the study of induced currents according to Maxwell.

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CHAPTER III

MAXWELL'S THEORY.

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§ 1. – *Induced currents according to Maxwell.*

27. Introduction. – In Chapter II, I established the Lagrange equations. The number of them is equal to the number of parameters q that define the system and have the type:

$$Q = \left(\frac{dT}{dq'}\right)' - \frac{dT}{dq}.$$

In order to be able to write them, it is necessary to:

1. Define the coordinates q that fix the configuration of the system.

2. Know the expression for the kinetic energy T as a function of q and q'.

3. Know the coefficients Q.

One can form the Lagrange equations with those givens. They will determine the q as functions of time.

Recall that Q is the coefficient of δq in the expression for the work that is done on the system under the displacement δq. With the extension that I gave to the idea of force, it is the resultant of the applied forces and will tend to displace the system by δq (no. **20**).

As for T, it is the kinetic energy $\frac{1}{2}\sum mv^2$. It is a homogeneous function of degree two – viz., a *quadratic form* – in the velocities q'. For two variables, it will be an expression of the form $\frac{1}{2}(A q'^2 + 2B q' q_1' + A_1 q_1'^2)$.

The coefficients A and B are functions of the q that are well-defined for each system and which can be reduced to constants, moreover.

Those are the results that pertain to the Lagrange equations. On the other hand, in Chapter I, I deduced the simultaneous equations of the currents when two fixed circuits are present from the experimental laws of induced currents, and I saw that they are precisely the two Lagrange equations that relate to the system by means of some analogies that I shall recall (no. **17**).

a. The coordinates q are the two quantities of electricity that are lost by the circuits.

b. The kinetic energy is the electromagnetic force function:

$$T = \tfrac{1}{2}\left(L_1\, i_1^2 + 2M_{12}\, i_1\, i_2 + L_2\, i_2^2\right).$$

c. The applied forces Q are the electromotive forces $E - ir$ that result from the generators, the receivers, and the resistance due to the Joule effect.

So much for fixed circuits. We shall now begin with moving circuits.

28. Maxwell's theory. – In order to study currents in moving circuits (whether deformable or not), one can follow the same analytical route as in Chapter I: Write out the expression for the experimental laws and note that the equations obtained are the Lagrange equations that relate to the system. With Maxwell, I would like to follow the opposite route: I assume, *a priori*, that the Lagrange equations are applicable. I apply them and then compare the formulas to the facts. If there is any disagreement then the theory must be rejected; if there is agreement then it must be accepted.

As in Chapter I, I will treat only the simplest cases. The argument will be no less general. The language and notation will be lightened, and the spirit will rest upon a concrete example that is easy to imagine.

29. Lagrange's equations for moving filamentary circuits of invariable form. – As in no. **15**, I will consider two circuits C_1 and C_2 and preserve the same notations. Moreover, I shall suppose that C_1 is moving with a rectilinear translatory motion. What are the quantities that must be considered in the Lagrange equation?

a. Coordinates q. – These are, first of all, the coordinates that take the form of the currents q_1 and q_2 (no. **27**) and then the mobility coordinate x of the circuit C_1 .

b. Kinetic energy T. – When the circuit C_1 is immobile, we saw (no. **16**) that one must take the kinetic energy to be $T_e = \tfrac{1}{2}\left(L_1\, i_1^2 + 2M_{12}\, i_1\, i_2 + L_2\, i_2^2\right)$. That is the electromagnetic force function. Since that energy is due exclusively to the electric motions, Maxwell called it *electro-kinetic*. In regard to the expression for T_e , the intensities i_1 and i_2 are equal to the rates q_1' and q_2' of the electric displacements q_1 and q_2 , respectively. L_1 and L_2 depend upon only the form of the circuits, so they are constants. The coefficient M_{12} depends upon the relative position of the circuits, in addition, so it will be a function of x.

We must add the *vis viva* T_m of the matter in the circuit C_1 to the term T_e ; if m is the mass of that circuit is m then the *vis viva* will be $\tfrac{1}{2} m\, x'^2$. In total, one has:

$$T = T_e + T_m = \tfrac{1}{2}\left[m\, x'^2 + L_1\, i_1^2 + 2M_{12}\, i_1\, i_2 + L_2\, i_2^2\right].$$

Maxwell was struck by a peculiarity of this: The expression for T indeed contains terms in the squares of the velocities x'^2, i_1^2 , i_2^2 , but also the product $i_1\, i_2$ of the electrical velocities.

Why does it not contain any terms in $x' \, i_1$ and $x' \, i_2$? We shall return to that point (§ **2**).

c. Applied forces Q. – For the electric displacements δq_1 and δq_2, those forces are $Q_1 = E_1 - r_1 \, i_1$, $Q_2 = E_2 - r_2 \, i_2$ (no. **17**). For the material displacement δx, it will be the force F that the observer's hand, for example, applies to the circuit C_1 in order to displace it. As for the electromotive forces of induction, on the one hand, and the electromagnetic forces, on the other, there is no reason to introduce them into the expressions for Q, because they can be regarded as inertial forces (no. **17**).

With those givens, one will have the left-hand sides of the Lagrange equations:

$$Q = F, \qquad Q_1 = E_1 - r_1 \, i_1, \qquad Q_2 = E_2 - r_2 \, i_2 .$$

The right-hand sides will be:

$$\left(\frac{dT}{dx'}\right)' = m\,x'', \qquad \frac{dT}{dx} = \frac{dM_{12}}{dx} i_1 \, i_2 ,$$

$$\left(\frac{dT}{dq_1'}\right)' = \frac{d}{dt}[L_1 \, i_1 + M_{12} \, i_2], \qquad \frac{dT}{dq_1} = 0,$$

$$\left(\frac{dT}{dq_2'}\right)' = \frac{d}{dt}[M_{12} \, i_1 + L_1 \, i_2], \qquad \frac{dT}{dq_2} = 0 .$$

In summary, the Lagrange equations are:

$$\text{(I)} \qquad \begin{cases} (1) & F = mx'' - \dfrac{dM_{12}}{dx} i_1 \, i_2, \\ (2) & E_1 - r_1 \, i_1 = \dfrac{d}{dt}[L_1 \, i_1 + M_{12} \, i_2], \\ (3) & E_2 - r_2 \, i_2 = \dfrac{d}{dt}[M_{12} \, i_1 + L_2 \, i_2]. \end{cases}$$

What if we want to deduce the energy equations? It will suffice to add the latter together after multiplying them by dx, dq_1, dq_2, respectively. After simplifying, we will get:

$$\text{(II)} \qquad F\,dx + (E_1 - r_1 \, i_1)\,dq_1 + (E_2 - r_2 \, i_2)\,dq_2 = dT.$$

30. Comparison of Lagrange's equations with experiments. – Equations (1) will then indeed satisfy the principle of the conservation of energy. They determine the motion of the system from the purely mechanical standpoint, as well as the standpoint of electricity, by telling one the coordinates x, q_1 , q_2 as functions of time. Do they agree with experiments? Yes, because they give the electromagnetic force and the induced electromotive force that are revealed by experiments, as I will now explain.

1. *Electromagnetic force.* – Equation (1) means that the applied force F is added to two forces that will be in equilibrium with it: namely, the inertial force $-m\,x''$ and the force $\frac{dM_{12}}{dx} i_1\, i_2$, which is the derivative with respect to x of T_e = $\frac{1}{2}\left[L_1\, i_1^2 + L_2\, i_2^2 + 2M_{12}\, i_1\, i_2\right]$. Now T_e is the electromagnetic force function, so $\frac{dM_{12}}{dx} i_1\, i_2$ is indeed the electromagnetic force that is given by experiment and was studied by Ampère.

2. *Induced electromotive force.* – Equation (2), for example, means that the applied electromotive force $E_1 - r_1\, i_1$ must be combined with an electromotive force $-\frac{d}{dt}[L_1\, i_1 + M_{12}\, i_2]$ that brings about equilibrium with it. Now, $L_1\, i_1 + M_{12}\, i_2$ is (nos. **14** and **15**) the flux of magnetic induction that crosses the circuit C_1 , so $-\frac{d}{dt}[L_1\, i_1 + M_{12}\, i_2]$ will be the derivative of that flux with the sign changed. That is, in fact, the induced electromotive force that is verified by experiments (nos. **9** and **10**).

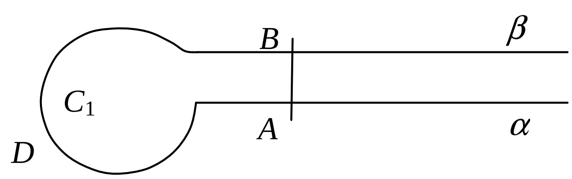

Figure 5.

31. Lagrange's equations for deformable filamentary circuits. – Once more, consider two circuits C_1 and C_2 . However, instead of supposing that the circuit C_1 is moving and has an invariable form, suppose that is composed as follows: One part D is fixed, open, and extends along two rails α and β. An element AB forms the bridge between those two rails and can be displaced by a translation parallel to the rails (Fig. 5). The notations are the same as in no. **29**, except that m and x represent the mass and the coordinate of the moving element AB. F will be the force that the observer, for example, applies to the element AB. The kinetic energy of the system will again be:

$$T = T_e + T_m = \frac{1}{2}\left[m\, x'^2 + L_1\, i_1^2 + 2M_{12}\, i_1\, i_2 + L_2\, i_2^2\right].$$

The only difference is that now L_1 is a function of x, as is M_{12}. The three Lagrange equations are:

$$\text{(I)}\qquad \begin{cases} (1) \quad F = mx'' - \dfrac{1}{2}\left[\dfrac{dL_1}{dx} i_1^2 + 2\dfrac{dM_{12}}{dx} i_1 i_2\right], \\ (2) \quad E_1 - r_1 i_1 = \dfrac{d}{dt}[L_1 i_1 + M_{12} i_2], \\ (3) \quad E_2 - r_2 i_2 = \dfrac{d}{dt}[M_{12} i_1 + L_2 i_2]. \end{cases}$$

As in nos. **29** and **30**, those equations determine the unknowns x, q_1, q_2. They lead to the energy equation and to the experimental laws of electromagnetic forces and induced electromotive forces.

32. Conclusions. – I have specified the analogies that permit one to regard a system of filamentary conductors that are traversed by currents as a system with constraints, as in rational mechanics, namely:

1. The *coordinates q* are of two types: the geometric coordinates that fix the form and position of the circuits and the quantities of electricity that are lost by each of them.

2. The *kinetic energy T* contains two parts: the *vis viva* of the moving parts of the circuit and the electromagnetic force function.

3. The *applied forces Q* are of two types: the purely-mechanical forces, like the effort exerted by the observer's hand and the electromotive force $E - r\,i$ that is due to generators, receivers, and Joule resistance.

4. With those analogies, I applied the Lagrange equations to moving filamentary circuits, whether deformable or not.

They determine the system as a function of time, satisfy the law of energy, and agree with experiments by giving two types of forces, namely, the electromagnetic forces and the induced electromotive forces.

Those forces must be regarded as inertial forces.

§ 2. – *Maxwell's research on the kinetic energy of currents with moving conductors.*

33. Introduction. – In no. **29**, I treated the case of two circuits C_1 and C_2 that are traversed by currents $i_1 = dq_1 / dt$, $i_2 = dq_2 / dt$. One of them was fixed, while the other one was moving and had a coordinate x. I assumed that the kinetic energy T was the sum of two terms, namely, the *vis viva* T_m of the matter and the moving circuit and the electromagnetic force function T_e, so in total:

$$T = T_e + T_m = \tfrac{1}{2}\left[m\,x'^2 + L_1\,i_1^2 + L_2\,i_2^2 + 2M_{12}\,i_1\,i_2\right].$$

That quadratic form in x', i_1, i_2 indeed contains the squares of the three variables and the product $i_1\ i_2$ of the intensities. Why does not contain terms like:

$$A_1\,x'\,i_1 + A_2\,x'\,i_2 = T_{me}\,,$$

which are products of a velocity with an intensity, as well? That is the question that was posed in no. **29** and which we shall now examine. Of course, I shall assume that the coefficients A_1 and A_2 can depend upon x, but not on q_1 and q_2 , as I was led to assume for M_{12} (no. **15**). Following Maxwell's method, I shall suppose that the terms $A_1\,x'\,i_1 + A_2\,x'\,i_2$ exist; inertial forces will result that can be calculated by Lagrange's formulas. Are they verified by experiments? If that were true then the existence of T_{me} would be established. If, on the contrary, the inertial forces that are calculated do not exist then the coefficients A_1 and A_2 of T_{me} would have to be zero, as we assumed in § 1.

34. Calculating inertial forces by the Lagrange equations. – There are three inertial forces that come from the part of the kinetic energy:

$$T_{me} = A_1\,x'\,i_1 + A_2\,x'\,i_2\,,$$

corresponding to the three displacements δx, δq_1 , δq_2 . They are, up to sign:

$$(\delta x) \qquad \left(\frac{dT_{me}}{dx'}\right)' - \frac{dT_{me}}{dx} = (A_1\,x'\,i_1 + A_2\,x'\,i_2)' - \frac{d}{dx}(A_1\,x'\,i_1 + A_2\,x'\,i_2),$$

$$(\delta q_1) \qquad \left(\frac{dT_{me}}{dq_1'}\right)' = (A_1\,x')',$$

$$(\delta q_2) \qquad \left(\frac{dT_{me}}{dq_2'}\right)' = (A_2\,x')'.$$

The first one is a force in the usual sense of the word, namely, the *ponderomotive force.* The other two are electromotive forces. We shall examine those two types of forces.

35. Electromotive force. – From its very expression:

$$(\delta q_1) \qquad \left(\frac{dT_{me}}{dq_1'}\right)' = (A_1\, x')',$$

the electromotive force is independent of the intensities, and as a result, of the magnetic field. It depends upon only the motion of the conductor. Now, no induction phenomenon has ever been observed in moving conductors that are outside of any magnetic field. One must then conclude the non-existence of the terms T_{me} in the expression for energy.

That experimental proof has great value due to the extreme sensitivity of the galvanometers that permit one to observe extraordinarily small electromotive forces.

Although that is sufficient and indeed better than the result of considering the ponderomotive force, we shall nonetheless examine that force.

36. Ponderomotive force. – From its expression:

$$(\delta x) \qquad \left(\frac{dT_{me}}{dx'}\right)' - \frac{dT_{me}}{dx} = (A_1\, x'\, i_1 + A_2\, x'\, i_2)' - \frac{d}{dx}(A_1\, x'\, i_1 + A_2\, x'\, i_2),$$

the ponderomotive force consists of two parts:

$$F_1 = A_1\, i_1' + A_2\, i_2',$$

$$F_2 = \left(\frac{dA_1}{dx}x'\, i_1 + \frac{dA_2}{dx}x'\, i_2\right) - \left(\frac{dA_1}{dx}x'\, i_1 + \frac{dA_2}{dx}x'\, i_2\right).$$

The second part F_2 is found to be zero here when one reduces the parentheses, but that particular situation is due to the fact that one has taken a system that is endowed with only one moving coordinate x. If there are more than one then the same reduction will not take place; some terms will remain that are proportional to both the velocities x' and the intensities i. In order to make them noticeable, one must employ both large velocities and large intensities. The character of those terms is such that they must change sign when one changes the signs of the velocities or intensities.

The first part F_1 gives forces that are proportional to the rates of variation of the intensities. One must seek them in the opening and closing of circuits. They must change sign when one passes from one of those operations to the other. Maxwell tried two experiments that were intended to exhibit F_1 and F_2 , respectively. They gave negative results. Since they have a less compelling character than the experiments on electromotive forces (no. **35**) due to their difficulty, I shall dispense with a discussion of them here.

37. Conclusion. – The kinetic energy of a system of moving circuits (whether deformable or not) is composed of the *vis viva* of the matter in the circuits and the electro-kinetic energy, which is the function of the electromagnetic forces that are applied to the circuits. It does not contain terms in the products of the velocities of the circuits with the intensities of the currents that traverse them.

§ 3. – *On the role of magnets in Maxwell's theory according to Sarrau.*

38. Introduction. – In his theory of induced currents, Maxwell excluded magnets from the Lagrange equations, which is a curious, if not almost paradoxical, fact, because the experiment in which one brings a magnet close to a current had served as the basis for Helmholtz's theory, which led to Maxwell's theory. How would he not understand that fundamental experiment?

Despite all appearances, there is nothing abnormal in that. Indeed, when the energy equation is applied to the experiment in question, it does not establish the general law of induction. It only permits one to guess the law by a bold inductive argument. The law is verified by experiments, and for that reason, we shall assume it. It is very good, but then it is borrowed from direct experiments, and not from theoretical consequences of the fundamental experiment. That is not to be found in the theory, and it would by unjust to reproach Maxwell for such an oversight.

Meanwhile, the question is most important and deserves to be addressed, if not solved. I shall examine the case of permanent magnets and that of temporary magnets.

39. Ampère's hypothesis. – The fields that are produced by magnets are in no way distinguished from the fields that are produced by the currents (no. **2**). On the other hand, magnetization is special property (no. **2**). Those two laws lead to *Ampére's hypothesis:* Each magnetic particle is analogous to a small closed current whose plane is normal to the magnetization vector. Moreover, the electro-kinetic energy of a system of currents and magnets is composed of three parts: The first one comes from only the currents, so one knows its expression. The second one comes from the combination of currents with the magnets. One must then assume that it is the sum of the products of the intensities with the induction flux of magnetic origin that crosses it. The third part comes from only the magnets. To say that a magnet is permanent is to say that its particular currents have constant intensities.

40. Moving a permanent magnet towards a circuit. – I shall preserve the previous notations (no. **29**). In addition, I shall let Φ denote the induction flux that is due to the magnet that crosses the circuit and let $\frac{1}{2}A$ denote the electro-kinetic energy of the magnet; by hypothesis, $\frac{1}{2}A$ is constant and Φ depends solely upon x, so the kinetic energy of the system is:

$$T = \tfrac{1}{2}[m\,x'^2 + L\,i^2 + 2i\,\Phi + A].$$

If E is the electromotive force of the battery in the circuit and F is the force that is applied by the observer's hand, for example, then the Lagrange equations that correspond to δx and δq will be:

(δx) $$F = m\,x'' - i\,\frac{d\Phi}{dx},$$

(δq) $$E - r\,i = L\,i' + \Phi'.$$

They determine the two unknowns x and i as functions of time. That is the theory that Sarrau proposed ([9]).

Equations (δx) and (δq) an indeed verified by experiments: The first one gives the electromagnetic force, while the second one gives the electromotive force of induction. In that regard, the theory is satisfying. However, a difficulty remains in regard to energy, and it should be studied.

41. Remark concerning energy. – There is an erroneous argument that I believe is worth pointing out because it will serve as an introduction to the exact argument in the following number, which will exhibit certain characteristics of permanents magnets, notably their coercive force:

Equations (δx) and (δq) of no. **40** define the system of two necessary and sufficient equations for determining the two unknowns x and q of the problem. Can one not conclude, from no. **24**, that they satisfy the law of energy?

No. Indeed, recall the calculation that served to pass from Lagrange's equations to that of energy. One must multiply the first one by $dx = x'\,dt$ and the second one by $dq = i\,dt$. In that way, I will get:

$$F\,dx + (E - r\,i)\,dq = d\left[\tfrac{1}{2}m\,x'^2 + \tfrac{1}{2}L\,i^2\right].$$

The work done by applied forces is not equal to the increase in kinetic energy.

The error? It is this: Equations (δx) and (δq) indeed form the system of two Lagrange equations that determine the *two unknowns of the physical problem*, namely, x and q. However, although they are the equations and unknowns of the physical problem, they are not *all of the variables and equations of the mechanical system*. Those are uncountable, since each particle has at least one *electrical coordinate* that corresponds to the current that traverses it, from Ampère. Hence, the system $[(\delta x), (\delta q)]$ is incomplete from the mechanical standpoint. It does not have to satisfy the law of energy then. In order to recover the energy equations, it will be necessary to add a line to it that would account for the equations of magnetization. That is what we shall do.

42. Coercive force of the magnet. – In order to avoid the complication of particle currents, I shall compare the magnet to a single current $i_1 = dq_1 / dt$. What must the peculiarities of our fictitious current be?

1. The intensity i_1 is constant, since the magnet is assumed to be permanent.

([9]) SARRAU, C. R. Acad. Sci. **133** (2 September 1901), pp. 421.

2. The applied electromotive force E_1 varies in such a fashion that it will keep the intensity i_1 reasonably constant.

3. The resistance r_1 is zero, because magnets do not give rise to anything that resembles Joule heat.

4. The energy of self-induction $\frac{1}{2}L_1 i_1^2$ is constant. The energy of mutual induction is $i\,\Phi = M\,i\,i_1$. M is a function of only x.

In summary, the fictitious current exhibits the character of a perfect machine without friction this is endowed with an ideal regulator.

Under those conditions, the system under study of the magnet and the current will have three equations in the form of:

$$(\delta x) \qquad F = m\,x'' - i\,i_1\frac{dM}{dx},$$

$$(\delta q) \qquad E - r\,i = L\,i' + (M\,i_1)',$$

$$(\delta q_1) \qquad E_1 = (M\,i)'.$$

If I multiply the three equations by $dx = x'\,dt$, $dq = i\,dt$, $dq_1 = i_1\,dt$, respectively, then I will get the equation:

$$F\,dx + (E - r\,i)\,dq + E_1\,dq_1 = d\left[\tfrac{1}{2}m\,x'^2 + \tfrac{1}{2}L\,i^2 + M\,i\,i_1\right].$$

It means that the work done by the forces is indeed equal to the increase in the kinetic energy. The law of energy is now verified. However, that is only true thanks to the electromotive force E_1 ; that is the *coercive force* that the physicists imagined. Its value will be provided by the formula (δq_1) as soon as one knows the value of $M = \Phi / i_1$, which is a given that characterizes the magnet.

If the magnet is not permanent, but varies with the currents that are present, then the fictitious current i_1 can only be regarded as a constant. What is more, the magnetization of each particle varies in its own way, and it will be impossible to replace the particle currents, even fictitiously, with a single current. This topic presents great complexity, moreover; it has not been addressed up to now. Be that as it may, the two equations (δx) and (δq) will remain true, but they will no longer suffice to determine the system, which depends upon an uncountable number of unknowns. It remains for one to establish the equations that relate to the electric coordinates that correspond to the magnetization of the particles. That remark is quite important, so it will dominate all of Part Two of this book.

43. Conclusions:

1. Maxwell excluded magnets from his theory of induced currents, and one cannot logically grieve for him because of that fact.

2. Sarrau assumed that magnets are composed of particle currents; that was Ampère's hypothesis. With that hypothesis, when the Lagrange equations are applied to a system of permanent magnets and currents, they will give results that conform to experiments. They determine all of the unknowns, which are the position coordinates and electric coordinates.

3. Those equations will lead to the equation of energy only if one adds equations for the magnets. They will determine the coercive force of the permanent magnets.

4. The equations of soft iron have not been addressed up to now, but the Lagrange equations that relate to position coordinates and electric coordinates can always be written out: They are necessary, but not sufficient, equations for the problem.

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CONCLUSIONS FROM PART ONE

44. The electromagnetic and magnetic forces are inertial forces, like the electromotive force of induction. – We have seen how Helmholtz's theory, aided by experiments, led us to two truths: that the electromotive forces of induction are inertial forces and that the proper energy of a system of currents:

$$\tfrac{1}{2}[L_1\, i_1^2 + 2M_{12}\, i_1\, i_2 + L_2\, i_2^2]$$

is a kinetic energy. Those results led me to Maxwell's theory, in which the electromagnetic forces also appear to be inertial forces. That was an important discovery by Maxwell. Unfortunately, it is still unknown, and for that reason I would like to stop there in my conclusions. Why is it unknown? Perhaps that is due to the name of *electrodynamical potential* that was given to the energy with the sign changed, which might lead one to think that it is a form of potential energy. Meanwhile, whether or not one accepts Maxwell's remarkable theory, it is hard to deny the kinetic character of the proper energy of currents: Like that of flywheels, it is stored in the beginning to be released as work during the period when things stop. Now, that energy is precisely the electromagnetic force function. A necessary consequence is that the electromagnetic forces are inertial forces. Having misunderstood that, physicists frequently made sign errors with the induced electromotive forces in their development of Helmholtz's theory. From the ideas of Ampère and Sarrau, one must also regard magnetic forces as inertial forces.

Since that is quite remarkable, it will suffice to examine the magnetic and electromagnetic forces in order to recognize the character of inertial forces. Indeed, what is Laplace's law, which is, as we have seen, the key to all of the theory?

A South magnetic pole exerts a force on a current element that is perpendicular to the plane of the element and the pole, and proportional to the mass of the pole, the length and intensity of the current element, the sine of the angle between the element and the line that connects it to the pole, and inversely proportional to the square of the distance. The sense of the force is such that it carries the element to the left of an observer that turns his back to the pole and is traversed by the current from his feet to his head.

This strange fictitious law is almost as unacceptable as the elementary law: It is fictitious since it introduces the notion of isolated magnetic pole, which is an unrealizable object, and strange since it gives a force that is not directed along the line that connects the elements. By virtue of the principle of the equality of action and reaction, one must suppose that the action of the current element on the pole is applied, not to the location whether the pole is, but to the location where the current element is. Forces of that type are unacceptable as applied forces that act from element to element.

On the contrary, recall the expression that we gave to the transformed Laplace law.

The force that is exerted on an element of the circuit *ds* that is traversed by a current *i* at a point where the magnetic field is α is represented by the same vector that represents the oriented area of the parallelogram $[i\ ds \cdot \alpha] = -[\alpha \cdot i\ ds]$. It is, if one prefers, the velocity (with the sign changed) of the extremity of the vector *i ds* under a rotation that is represented by the vector α. In that expression, one recognizes (up to a factor of 2) the composite centrifugal force – 2 [Ω V_r], in which the vector Ω represents the guiding velocity and V_r represents the relative velocity. Is there harmony in the signs of that double analogy? Certainly, and here is why: *i ds* is analogous to a velocity, so we must choose a sense for that velocity. We choose the same sense as the current. What is the sense of the rotation Ω that one must associate with the vector α then? In order to see it, we examine what the magnetic force of that current is at the center of a small circular current *C*. It is perpendicular to the plane of *C* and has a sense such that we will see the current *C* turn in the opposite sense to the hands of a watch. It is the axis of rotation *C* that is described in astronomy, and also with Maxwell's conventions. Ω will then have the same sense as α. Hence, a current element *i ds* that is placed in a magnetic field *a* is analogous to a material point that is animated with the relative velocity $V_r = i\,ds$ in a medium whose guiding rotation is $\Omega = \alpha$. The composite centrifugal force has the opposite sense to the composite acceleration; i.e., to the velocity of the extremity of the vector $V_r = i\,ds$ that is due to the rotation $\Omega = \alpha$. That is precisely the sense of the electromagnetic force that acts on the element *i ds*. In summary, the electromagnetic force indeed presents all of character, magnitude, direction, and sense of a composite centrifugal force; i.e., an inertial force. The same thing will be true for the electrodynamical force, which is not distinguished from the electromagnetic force.

The coherence of the analogies that we just verified continues into the magnetic forces. The gyroscope always tends to make its axis parallel to that of the Earth. Similarly, the magnet will tend to make its magnetization axis parallel to the magnetic field. There is more: The director couple has the same expression in both cases. It is represented by the oriented parallelogram that is constructed from the two corresponding vectors, namely, the rotation of the Earth and the rotation of the gyroscope, on the one hand, and the magnetic field and magnetization vector, on the other.

Hence, the magnetic forces, like the electromagnetic forces and the electrodynamical forces, as well as the electromotive forces of induction, indeed present all of the character of inertial forces.

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PART TWO

Electricity reduced to the principle of virtual work

INTRODUCTION

45. – In Part One of this book, we saw how the interpretation of the experimental laws led us to regard a system of currents that circulate in filamentary conductors as a system of constraints that satisfy the principle of virtual work in rational mechanics.

1. There are two types of Lagrangian coordinates q : the geometric coordinates that fix the forms and position of the circuit and the quantities of electricity that are lost by each of them.

2. The kinetic energy T contains two parts: the semi-*vis viva* of the moving parts of the circuits and the electromagnetic force function.

3. There are two types of applied forces Q : the purely-mechanical forces, such as the effort exerted by the observer's hand, and the electromotive forces $E - ri$, which come from the generators and receivers and the Joule effect.

4. As for the electromagnetic forces and the electromotive induction forces, they must be regarded as inertial forces and must be calculated by means of T using the Lagrange formula, like the inertial forces of rational mechanics.

Those are the principles of Maxwell's theory. It is important to point out that they are based upon Faraday's law: *In a monofilament circuit, the quantity of electricity that is lost q (viz., the quantity of electrolyte that has decomposed) is the same in all parts of the circuit.* That is the fundamental law of constraint that characterizes electrical phenomena, which is a constraint that is analogous to the constraint that a fluid should be incompressible, and for that reason, one can call it the *incompressibility constraint.*

The law of currents that is derived from Faraday's law is a first extension of that law, which was further generalized by Kirchhoff into this statement:

In a network of filamentary conductors that are traversed by currents, the algebraic sum of the current intensities that terminate at an arbitrary point of the network is zero.

Maxwell understood that this law is even more general and can first be extended to three-dimensional conductors and then to dielectrics, thanks to his brilliant notion of the *displacement current.* I will show how the experimental laws that one observes in the charging of a condensor by a current from a battery will lead to Maxwell's results in the simplest and most natural way.

The law of incompressibility is completely generality, moreover, so the principle of virtual work must be applicable to all imaginable electric displacements, not only the ones that traverse conductors, but also the ones that traverse dielectrics, provided that they are compatible with the incompressibility constraint. Now, displacements that are compatible with incompressibility can take place only along closed contours. The following law is then imposed:

The total work done by electric forces (applied and inertial) *under an electric displacement* δq *along an arbitrary closed contour is zero.*

In particular, perform the displacement dq along one of the closed circuits that one borrows from a filamentary network, and that will give Kirchhoff's second law:

The sum of the electromotive forces (due to generators and receivers and the Joule effect) *that one meets along the contour will be zero.*

Hence, it is the extension of Kirchhoff's laws that we shall pursue, and which will reduce electricity to the principle of virtual work. I shall carry out that generalization in only one experiment. Once the two general laws have been established – viz., the law of incompressibility and the law of virtual work – I will point out the analytical developments that result from them for bodies at rest and bodies in motion.

An unexpected difficulty arises from bodies in motion:

Experiments with the Barlow wheel prove that the Lagrange equations are not applicable to three-dimensional conductors. Nonetheless, the fundamental idea of this book remains intact: I will show that the Barlow wheel presents a remarkable analogy with the rolling of the hoop. The Lagrange formulas cease to be applicable, but the principle of virtual work will remain true. There is a new difficulty: Whereas rational mechanics further permits one to calculate the inertial forces of the hoop (for example, by the formulas that I have given), that calculation is forbidden with the Barlow wheel by our ignorance of the elementary constitution of electrical currents. One must borrow the inertial forces of electricity from experiments. The study of the experimental laws of electrical inertia must precede the equations of electrodynamics, just as the study of moments of inertia must precede the dynamics of solid bodies.

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CHAPTER I

THEORY OF ELECTRICITY IN BODIES AT REST

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§ 1. – *Extending Kirchhoff's laws to three-dimensional conductors.*

46. Introduction. – The problem of the distribution of currents in the permanent regime in a network of filamentary conductors that are endowed with given generators and receivers can be solved entirely by using Kirchhoff's two laws:

First law. – The algebraic sum of the intensities of the currents that terminate at a point of the network is zero.

Second law. – The algebraic sum of the electromagnetic forces for the generators, receivers, and Joule heat that one meets along a closed contour that belongs to the network is zero.

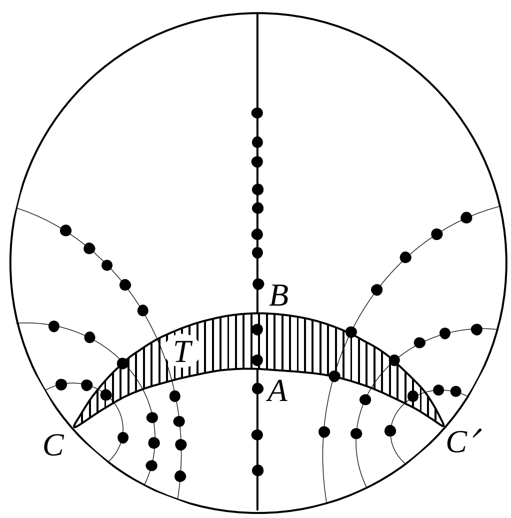

Figure 6.

47. Kirchhoff's experiment with a plate. – With Kirchhoff, consider (Fig. 6) a thin metallic plate *CC´* of circular form. We connect the two poles of the battery to two points *C* and *C´* on the circumference. Current flows throughout all of the plate. (One can see that by means of an electrometer or galvanometer.) In that experiment, the points that gave the same voltage are sketched on the plate as *equipotential lines, or level lines.*

Trace out an orthogonal trajectory *CAC´* to the level lines and cut the plate with a knife along that line; that operation will change nothing in regard to the output of the battery or the distribution of the potentials. The opposite situation will take place for a section that is oblique to the level lines. For that reason, the line *CAC´* is called a *current* *line.* The plate is cut into filamentary conductors along the current lines. One calls them *current tubes,* because in the plate that has not been cut, the currents will distribute themselves in the same way as in the filamentary network that is obtained by cutting the plate. One understands, moreover, that Kirchhoff's laws apply to the plate. They therefore extend to three-dimensional conductors, because of one considers a fluid mass, instead of a plate, then exploring the points with the same voltage will produce equipotential surfaces whose orthogonal trajectories again cut the medium into current tubes.

Let us therefore develop the consequences of Kirchhoff's plate experiment.

48. The current vector and its flux. – At each point of the plate, the current possesses two qualities: its *direction*, which is the same as the current line, and its *density*, which is the current intensity per unit cross-section of the current tube at that point.

Maxwell was then led to represent the current by a vector p that he called the *current vector*. Its direction is that of the current and its length is its density. When one gives the vector p at a point, one will know the intensity in a current tube whose cross-section is an arbitrary surface element $d\sigma$ that passes through that point. That will be the flux $[p\ d\sigma]$ of the vector p that crosses the element $d\sigma$.

49. Extending Kirchhoff's first law. – I consider a point O of a network. I surround it with a surface S. To say that the algebraic sum of the intensities i of the currents that terminate at the point O is zero is to say that the total flux of the currents p that cross the surface is zero.

In that form, it is clear that the law is general, because if one traces out a surface on Kirchhoff's plate then the current tubes that enter it will leave with the same content, in such a fashion that the total flux that enters it will be zero. Hence:

The total flux of the current that crosses an arbitrary closed surface that is traced in the system will be zero.

50. Counter-electromotive force of the Joule effect. – It is always resistive; i.e., contrary to the current like the forces of friction. As for its magnitude, we shall calculate it by Ohm's law. Consider a cylindrical element MN that is defined thus: Its generators are parallel to the current p and have length ds. Its cross-section is $d\sigma$. By the definition of the vector p, the element $d\sigma\, ds$ is traversed in the sense of the generators by a current of intensity:

$$i = p\ d\sigma .$$

From Ohm's law, it possesses a resistance r that is proportional to its length ds, inversely proportional to its cross-section $d\sigma$, and has a conductance C such that:

$$r = \frac{ds}{C\,d\sigma}.$$

The counter-electromotive force of the Joule effect has an absolute value of:

$$r\,i = \frac{ds}{C\,d\sigma} \times p\,d\sigma = \frac{p}{C}\,ds\,.$$

That expression appears to be the product of the vector p / C with the length ds. However, the element ds was taken to have the same direction as the vector p. What expression must one take if the element $ds = MP$ is oblique to p ? The answer is found by exploring the Kirchhoff plate with an electrometer (no. **47**): Indeed, displace the contact with the electrometer along the equipotential line from the point P to the point N where the current line p that issues from M meets the equipotential line. The electrometer will not be influenced by that displacement from P to M. Therefore, the counter-electromotive Joule force will be $(p / C) \times MN$. It is the product of the vector p / C with the projection of ds onto it, which is a product that we, with Grassmann, will denote by $\left.\frac{p}{C}\right|ds$. The magnitude and sign of the counter-electromotive Joule force from a point A to a point B of a conductor are then represented by the integral $-\int_A^B \left.\frac{p}{C}\right|ds$.

51. Electromotive force of the Peltier effect. – It has an invariable sense that is contrary to that of the Joule effect. On the other hand, if the electromotive force that results when a current that traverses the separation surface between two metals is to have a constant magnitude then it will suffice that the current should be normal or oblique to the separation surface. The measure of that constant will suffice in our formulas, but it would be appropriate to explain the paradoxical appearance of that force and to make it comparable to the Joule force that is spread throughout the volume. Imagine a vector e with the same physical dimensions as q / C, is normal to the interface surface AB, and has a well-defined magnitude at each distance from AB, but has measurable values only in a transition layer of very small thickness ε. The integral of the vector e along an arbitrary line CD that is oblique to AB is:

$$E = \int_C^D e \,|\, ds\,.$$

The same thing will be true for any path that one follows upon crossing the transition layer, provided that one crosses it completely. That integral will be an electromotive force. It enjoys all of the properties of the electromotive force that corresponds to the Peltier effect.

What was just said in the context of the Joule and Peltier effects can be repeated for the Thomson effect and for electromotive forces of chemical origin.

52. Extending Kirchhoff's second law. – As a result of nos. **50** and **51**, one can define two vectors at each point that represent electromotive forces: the vector e for the generators and receivers and the vector $- p / C$ for the Joule effect. With those notations, the electromotive force between two points will be the integral of the sum $e - p / C$, and Kirchhoff's second law can be state thus:

The integral of the electromotive force along any closed contour that is traced out in a conductor is zero.

It is easy to confirm that the law applies to Kirchhoff's plate and three-dimensional conductors, just as it does to networks of filamentary conductors.

53. Conclusions:

1. I defined a *current vector* and an *electromotive force* vector at each point in a conductor.

2. *The current flux across an arbitrary surface is zero.* That is Kirchhoff's first law.

3. *The integral of the electromotive force along an arbitrary contour that is traced in the conductors is zero.* That is Kirchhoff's second law.

§ 2. – *Extending Kirchhoff's laws to the variable regime and to dielectrics.*

54. Introduction. – The two generalized Kirchhoff laws are:

1. The current flux across an arbitrary surface is zero.

2. The integral of the electromotive force along an arbitrary contour that is traced in the conductors is zero.

I would like to extend those laws to currents in the variable regime and to dielectrics by means of two new notions that relate to dielectrics: The displacement current for the first law and the electrostatic electromagnetic force for the second law. It will suffice for me to analyze a well-known experiment.

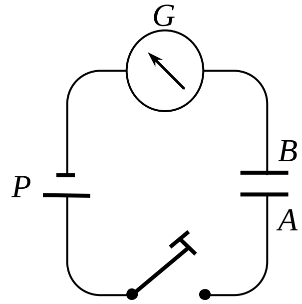

Figure 7.

55. Charge on a condensor from the current from a battery. – Place a key *C*, a condensor *AB*, and a ballistic galvanometer *G* in the circuit *PCABGP* of a battery *P*. (Fig. 7)

Let i be its intensity, when measured positively in the sense of *PAB*.

During its total duration, the current output will be a quantity of electricity:

$$q = \int i\,dt$$

that is measured by the galvanometer.

A first experimental law is this one:

The faces *A* and *B* of the condensor will take on charges that are positive and negative, respectively, and which will prove to be proportional to the output q that is recorded by the galvanometer when they are measured by electrostatic procedures (for example, the attraction of two armatures).

Those charges will be exactly $+ q$ and $- q$ when one adopts the *electromagnetic unit* for the quantity of electricity.

They are distributed over the surface S of the condensor in such a way that the surface density is q / S. In that system of units (viz, the *electromagnetic system*) the coefficient k of Coulomb's law $\left(\text{viz.}, F = k\dfrac{qq'}{r^2}\right)$ will have the dimensions of the square of a velocity, and experiments show that this velocity will be that of light. I shall replace the constant k with $1 / K$ in order to conform to Maxwell's notations. Moreover, the electric force X of the field between the two faces of the condensor is coupled to its charge q, as measured by the galvanometer, by the electrostatic formula:

$$X = \frac{4\pi}{K}\frac{q}{S}. \tag{1}$$

A second experimental law is that the current will stop when the difference in values of the potential φ between the armatures *A* and *B* is equal to the electromotive force E of the battery with a charge that is *measured by dynamical processes:*

$$E = \varphi_A - \varphi_B . \tag{2}$$

Those two laws contain the generalization of Kirchhoff's laws, as I will explain.

56. Extending Kirchhoff's first law. – I consider a closed surface Σ that surrounds the positive armature *A* of the condensor, but leaves the armature *B* exterior to it. I consider two fluxes across that surface at the moment t during the charging period: the flux of the current p that *penetrates* the charge conductor, namely:

$$\text{flux } p = i = \frac{dq}{dt}, \tag{1}$$

and then the flux of the electric force X that *leaves* it through the part of the surface that is located between the two armatures of the condensor. From formula (1) of no. **55**, that will be:

$$X \cdot S = \frac{4\pi}{K} q .$$

The flux that *enters* from the boss (*chef*) into Σ will then be:

$$\text{flux } X = -\frac{4\pi}{K} q , \tag{2}$$

so when one differentiates that with respect to t, one can infer that:

$$\text{flux } \frac{dX}{dt} = -\frac{4\pi}{K} \frac{dq}{dt} . \tag{2′}$$

One then deduces this from the relations (1) and (2′):

$$\text{flux } \left(p + \frac{K}{4\pi} \frac{dX}{dt} \right) = 0. \tag{3}$$

Hence, the vector $\frac{K}{4\pi} \frac{dX}{dt} = p_1$ completes the vector p in such a fashion that the total flux of those two vectors will be zero. p_1 is Maxwell's *displacement current*, while $p + p_1$ is the *total current*, which reduces to p in a perfect conductor and to p_1 in a perfect dielectric. Kirchhoff's first law then generalizes to this:

The total current flux that crosses an arbitrary closed surface is zero.

57. Extending Kirchhoff's second law. – Formula (2), which represents the second law of no. **55**, is written:

$$0 = E + \varphi_B - \varphi_A .$$

Therefore, $\varphi_B - \varphi_A$ is an electromotive force that equilibrates E. By the definition of the potential, it is the integral (with the sign changed) of the electric force X along the path AB. From that, *if a closed contour contains a piece AB that is contained inside a dielectric in which the electric force X prevails then an electromotive force will result along that contour that is equal to the integral of* $- X = P_1$ *along AB.* It will equilibrate the other electromotive forces E on the contour that are measured by dynamical processes.

Therefore, one finds an electromotive force at each point: viz., P in the conductor and P_1 in the dielectric. If the magnetic field varies then the study of induction currents will teach us that one must add the derivative (with the sign changed) of the magnetic induction a that crosses a contour to the integral of the electromotive forces along the contour. Those are the electromotive forces that the experiment will reveal. Provided that one takes them into account, the statement of the second law will remain unchanged:

The total electromotive force that prevails in any closed contour is zero.

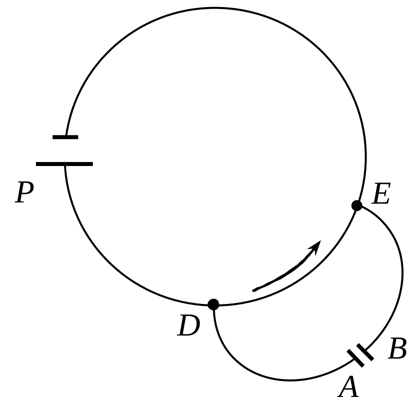

Figure 8.

58. Experimental verification of the generalized Kirchhoff laws. – This will result from the well-known exploration of the circuit with an electrometer. One establishes a derived circuit *DABE* that contains an electrometer *AB* (Fig. 8) at the two points *D*, *E* of the circuit *PDEP* of the battery *P*. At the moment when one establishes the derived circuit, an instantaneous current will traverse it; equilibrium will soon be established. At that moment, the laws of no. **54** will give the following results:

First law:

1. The current in the conductors *DA* and *BE* is zero.

2. The intensity is the same in the branches *PD*, *DE*, *EP*.

Second law: The equilibrium equations of the circuits *PDEP* and *DEBAD* are:

$$E - R\,i = 0. \tag{1}$$

$$\varphi_A - \varphi_B - r\,i = 0. \tag{2}$$

The notations in those formulas are the same as in the preceding ones, with the one modification that R is the resistance of the circuit *PDEP* and r is that of the portion *DE*.

It needs to be pointed out that those results are verified by experiments.

59. Electrostatic field created by currents. – In the experiment of no. **58**, nothing prevented one from progressively lengthening the two armatures of the condensor *AB* that forms the electrometer and diminishing their areas as much as one desires, until the metallic parts *DA* and *BE* disappear. The entire path *DABE* will then be traced inside the dielectric, and equation (2) of no. **58** will become:

$$\varphi_D - \varphi_E - r\,i = 0 .$$

Hence, $\varphi_D - \varphi_E$ is not zero. That will show that, even in the absence of any condensor, in any open derived circuit, the existence of a current will imply an electric field in the surrounding medium. That fact is quite obvious in experiments, because in high-tension currents, it is not rare to see sparks emitted from one part of a conductor and another.

60. Conclusions:

1. When a condensor is charged by the current from a battery, I drew upon two notions that relate to dielectrics: the displacement current and the electromotive force that is due to the potential difference.

I found these two laws:

2. The total current flux across any closed surface is zero.

3. The total electromotive force that prevails in any closed contour is zero.

The notion of electric displacement is due to Maxwell (*Traité d'Electricité*, nos. **60** and **61**). However, since it was introduced in the early days of electrostatics, it does not exhibit the same clarity that it does here, and it seems quite arbitrary in the work of that great physicist. The law of total current flux is likewise found there. In no. **61**, it appeared with its true significance. In equation (E) of no. **607**, it appears as a mathematical consequence of a hypothesis on the magnetic field that is due to an open current; nowhere does it appear as directly-observable experimental fact. As for the law of electromotive forces, Maxwell's formulas do not lead to it. I believe that it is new. It is interesting to confirm that the two laws are only extensions of Kirchhoff's laws.

§ 3. – *General equations of electricity for bodies at rest.*

61. Introduction. – I have arrived at the two fundamental laws of electrodynamics:

First law: *The total current flux across any closed surface is zero.*

Second law: *The total electromotive force in any closed circuit is zero.*

I would like to explain their mechanical interpretations and formulate their analytical expressions. I will then have the two fundamental laws of electromagnetism. I shall apply them to the particular case of perfect conductors and perfect dielectrics. Finally, I shall compare my results to those of Maxwell.

62. Mechanical interpretation of the two fundamental laws. – The elements that enter into the two laws are interpreted thus:

The total current u of the first law is the velocity of the electric coordinate q. It is equal to the conduction current p in perfect conductors and to the displacement current p_1 in perfect dielectrics, and $p + p_1 = u$ in all cases.

We pass on to the second law.

The applied electromotive force vector U consists of several parts: The Joule part, which is equal to the quotient of the current by the conductance $P = - p / C$; it is found in conductors and correspond to friction. The electric force, with the sign changed, $P_1 = - X = -\frac{4\pi}{K}\int p_1\,dt$ is found in dielectrics and corresponds to the reaction of a spring. The electromotive force of the generators and receivers P_2 corresponds to the applied force that is due to dynamical generators and receivers such as waterfalls and machine tools. The integral of those forces along a closed circuit represents the total work done by applied forces for the virtual electric displacement $\delta q = 1$ along the circuit.

One must add the work done by inertial forces. The electrical inertial forces are the electromotive forces of induction (no. **45**, 4.). Their virtual work is $- d\,\Phi / dt$ (no. **10**); it is the derivative (with the sign changed) of the magnetic induction flux a (no. **5**) that crosses the circuit. In other words, it is the vector flux $- da / dt$ or $- a'$, which is the derivative (with the sign changed) of the magnetic induction vector a.

The total virtual work thus-calculated for the displacement $\delta q = 1$ along a closed contour must be equal to zero, and that equality must be satisfied for all imaginable closed contours. It is the generalization of Kirchhoff's second law.

The interpretation of the two laws is obvious: The first law corresponds to the incompressibility constraint in hydrodynamics. The second law expresses that the total work done by forces is zero for any displacement δq that is compatible with the incompressibility constraint.

They must then contain the general equations of electricity for bodies at rest. There are two types of equations according to whether one considers the mass of a continuous medium or the separation surface between two different media. I would like to establish those equations.

63. Indefinite equations in a continuous medium. – Apply the first law to the surface of the parallelepiped $dx\,dy\,dz$, whose first summit O has the coordinates x, y, z (Fig. 9), and calculate the flux of the vector (u, v, w) that *leaves* that surface. Upon crossing the surface $OBC = dy\,dx$, the flux is $- u\,dy\,dz$. Upon crossing the parallel face that is drawn through A, the flux will be $+\left(u+\frac{du}{dx}dx\right)dy\,dz$, so in total, $(du / dx)\,dy\,dy\,dz$. One will likewise find the flux across the pairs of faces $dz\,dx$ and $dx\,dy$. When one equates the sum to zero and suppresses the common $dx\,dy\,dz$, one will find:

(1) $$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} = 0.$$

That expresses the law of incompressibility. Upon applying the second law to the contours that bound each of the three faces *OAB*, *OBC*, *OCA* of the same parallelepiped, one will find by a similar argument that:

$$(2)\qquad \begin{cases} \dfrac{dW}{dy} - \dfrac{dV}{dz} - a' = 0, \\ \dfrac{dU}{dz} - \dfrac{dW}{dx} - b' = 0, \\ \dfrac{dV}{dx} - \dfrac{dU}{dy} - c' = 0. \end{cases}$$

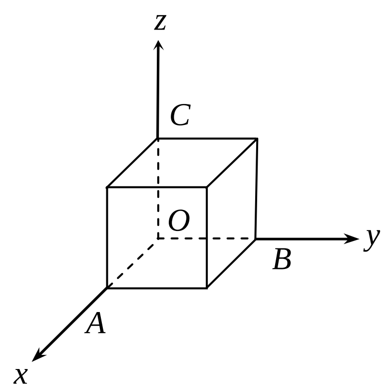

Figure 9.

In Grassmann's notation, equations (1) and (2) are written:

$$(1)\qquad \frac{d}{dx}\bigg|\, u = 0,$$

(cf., MAXWELL, no. **607**),

$$(2)\qquad \bigg|\left[\frac{d}{dx}U\right] - a' = 0,$$

resp. (cf., MAXWELL, eq. (A), (B), (I); nos. **591**, **598**, **611**).

According to whether the body that is taken at the point *O* is a conductor or a dielectric, those equations will become:

$$\frac{d}{dx}\bigg|\, u = 0, \qquad \bigg|\frac{d}{dx}\left(-\frac{p}{C} + P_2\right) = a' \quad \text{(conductor)},$$

$$\frac{d}{dx}\bigg|\,\frac{K}{4\pi}\frac{dX}{dt} = 0, \qquad \bigg|\frac{d}{dx}\left(-X + P_2\right) = a' \quad \text{(dielectric)},$$

respectively.

If the mass is homogeneous, one must suppose, in addition, that P_2 is zero and C (or K) are constant.

64. Equations for the separation surface between two media. – Take the axes Ox, Oy to be parallel to the surface and the axis Oz to be perpendicular to it, and then apply the two laws to the element $dx\,dy\,dz$, as in the previous number, and upon putting a prime on the letters that relate to the second medium, we will get the equations:

$$\text{(III)} \qquad U = U', \qquad V = V', \qquad w = w'.$$

They are stated thus: *The tangential components of the electromotive force are continuous. The normal components of the total current are continuous.* Three cases present themselves according to whether the media are both conductors, or both dielectrics, or one is a conductor and the other is a dielectric. The equations are written:

$$\text{(1)} \qquad \frac{p}{C} - P_2 = \frac{p'}{C'} - P_2', \qquad \frac{q}{C} - Q_2 = \frac{q'}{C'} - Q_2', \qquad r = r',$$

$$\text{(2)} \qquad X - P_2 = X' - P_2', \qquad Y - Q_2 = Y' - Q_2', \qquad K\frac{dZ}{dt} = K'\frac{dZ'}{dt},$$

$$\text{(3)} \qquad \frac{p}{C} - P_2 = X' - P_2', \qquad \frac{q}{C} - Q_2 = Y' - Q_2', \qquad r = \frac{K'}{4\pi}\frac{dZ'}{dt},$$

respectively.

In particular, if each of the media is homogeneous then P_1, P_2', Q_2, Q_2' will be zero.

If the initial field (X, Y, Z) is zero then the integration of last two equations in the last column will give:

$$K\,Z = K'Z', \qquad \int r\,dt = \frac{K'}{4\pi}Z',$$

respectively.

65. Equivalence of the two fundamental laws with the system of equations (I), (II), (III). – As we have seen, the two laws imply equations (I), (II), and (III). Conversely, when those equations are extended to all of space, they will suffice to insure that the laws are verified. One sees that by a classical argument that I shall not reproduce here. If they are extended over only a volume such as an ellipsoid then they will once more insure that the two laws are verified in the entire interior of the ellipsoid, but if the volume has the form of a torus, for example, then one must further express the idea that the second law is verified for a contour that is taken in the volume and goes completely around the torus. For example, one can choose the circumference that is described by the center of the meridian circle of the torus. The necessity of verifying the Kirchhoff's second equation on that contour will amount to the fact that such a contour cannot be replaced with a set of infinitely-small contours that are taken in the torus (cf., MAXWELL, v. I, nos. **18**, *et seq.*).

66. Comparison of our theory with Maxwell's. – Equation (I) is in Maxwell. Although formula (II) deviates from Maxwell, it will lead to the same equations as the ones that one can deduce from formula (A), (B), (I) in Maxwell (nos. **591**, **598**, **611**) for conductors, on the one hand, and dielectrics, on the other. Nonetheless, there is a fundamental difference in the interpretation of the various electromotive forces that Maxwell seems to have confused with each other.

The differences are accentuated in the formulas of no. **64**: Equations (1) agree with the ones that one finds in no. **310** of Maxwell, but one does not find formulas (2) and (3), which I believe to be new.

Two characteristics distinguish our theory from Maxwell's: On the one hand, there is the exclusive consideration of closed contours in order to write the equations of dynamical equilibrium for electricity, and on the other hand, the distinction between and localization of the various electromotive forces. I only associate the Joule electromotive force with the conductors, and that comes from the potential differences in dielectrics. Both types of forces are found in the *imperfect* dielectrics.

Our method seems to suggest itself by its simplicity, the absence of hypotheses, and finally by the mechanical interpretation that I presented in no. **62**, and which introduces the laws of electromagnetism into the general principle of virtual work.

67. Conclusions:

1. I gave the mechanical interpretation of the two fundamental laws of electricity:

The first one is the condition of incompressibility that relates to electrical displacements.

The second one is the expression for the principle of virtual work for the electric displacements that are compatible with the incompressibility condition.

Here, as in Part One, one is no longer dealing with a system of constraints with a finite number of degrees of freedom. One is dealing with an indefinite medium that is analogous to an incompressible fluid with the following analogies:

a. The unknowns, which are infinite in number, are the values of the electric displacement $q = \int u\,dt$ at each point of the medium, which are displacements that must satisfy the laws of incompressibility.

b. The kinetic energy T is a volume integral that we shall determine later (§ **5**). Here, it suffices to know the work done by inertial forces for the displacement $\delta q = 1$ and along a closed contour. Experiments teach us that it is the flux of the vector $-a'$ that crosses the contour.

c. The applied forces Q are the electromotive forces of the generators, receivers, and the Joule effect of the dielectric.

2. I gave an analytical expression for the two laws in continuous media. They consist of two partial differential equations:

The numerical equation (I) for the law of incompressibility.
The vectorial equation (II) for the law of virtual work.

3. For the discontinuities, they are the equations (III), namely:

The continuity of the normal component of the total current, which relates to incompressibility.

The continuity of the tangential components of the applied electromotive force, which relates to virtual work.

4. I compared both the results and methodology of the present theory to Maxwell's.

§ 4. – *The problem of electromagnetism and electro-optics.*

68. Introduction. – I was led to two fundamental laws that give the general equations of electricity, whether one defines them indefinitely or on the separation surface between two media. They contain the current p, the electric force X, and the magnetic induction a. One of the first two vectors will be zero in perfectly-conducting or isolated bodies, and two unknown vectors will remain at each point. Another group of relations is necessary to determine the phenomena. One can obtain them in the case where the bodies are not magnetic: The induction a then agrees with the magnetic field α, and it will be given by the generalized Biot-Savart law. I would like to present that topic and show its links to optics, and infer a consequence of it that relates to the nature of dielectrics.

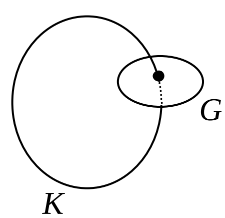

Figure 10.

69. The Biot-Savart law. – One way of formulating that law is this:

If one follows a line K that links a conducting circuit G in which a closed current of intensity i circulates (Fig. 10) *then the integral of the magnetic force that is due to the current will be* $4\pi i$.

Now, i is the current flux p upon crossing an arbitrary surface that is bounded by the contour K. Assume that the law is also true for open conduction currents. I can deform the arbitrary surface until is found entirely within the dielectric. In order for the expression to keep the same value, one must then replace the conduction current p with the displacement current p_1 . In a word:

The integral of the magnetic flux along any closed contour is equal to the total current flux, multiplied by 4π.

The analytical expression for that law was obtained in paragraph **3**; for continuous media, it is:

$$\left|\frac{d}{dx}\alpha\right. = 4\pi u, \qquad \begin{cases} u = p & \text{in conductors,} \\ u = p_1 = \dfrac{K}{4\pi}\dfrac{dX}{dt} & \text{in dielectrics,} \end{cases}$$

and for discontinuity surfaces, it is:

$$\alpha = \alpha', \qquad \beta = \beta'.$$

It is suitable to add to these equations the equations of the incompressibility of the magnetic induction vector, which is an incompressibility that was recalled in no. **5** ; as in nos. **63** and **64**:

$$\frac{d}{dx}\left| a\right. = 0, \qquad \text{for continuous media,}$$

$$c = c' \qquad \text{for a discontinuity surface.}$$

With our hypothesis of a non-magnetic medium, a, b, c are equal to α, β, γ, respectively.

70. Equations of the conduction current and the electric force. – We have obtained two vectorial equations:

$$\left|\frac{d}{dx}U\right. = \alpha', \qquad \left|\frac{d}{dx}\alpha\right. = 4\pi u.$$

In order to eliminate α, I perform an operation on the first one that I shall represent by $\left|\frac{d}{dx}\right.$. I will then replace $\left|\frac{d}{dx}\alpha\right.$ in the right-hand side by its value $4\pi u$ that is derived from the second equation. I will then get:

$$\left|\frac{d}{dx}\right|\frac{d}{dx}U = 4\pi u'.$$

According to whether the medium is a conductor or a dielectric, that equation can be written:

$$\left|\frac{d}{dx}\right|\frac{d}{dx}\left(-\frac{p}{C}+P_2\right)=4\pi\frac{dp}{dt}, \tag{1}$$

$$\left|\frac{d}{dx}\right|\frac{d}{dx}(-X+P_2)=K\frac{d^2X}{dt^2}, \tag{2}$$

respectively.

If the medium is homogeneous then P_2 will be zero, while C (or K) will be constant. If the medium is crystalline then that constant must be replaced with a system of constant, namely, a *linear system* that defines a linear vectorial function.

71. Determining the problem of electromagnetism. – For a surface, I shall isolate a system of perfect, non-magnetic conductors and dielectrics in a field. In that medium, one knows the indefinite vector equations for the conduction current p or the electric force X (eq. 1 or 2 in no. **70**). It has order two in the coordinates. In addition, one knows the vector equation of each discontinuity surface (no. **64**). That will yield the following givens:

1. p or X at each point at the initial time.

2. Those vectors and their derivatives along the normal to the boundary surface at each moment in time,

and the problem of calculating p and X in all of space and every moment will have been well-defined.

72. Comparison with light. Constitution of dielectrics. – The problem is posed in the same way in optics. Now, my research on the wave surface and the dispersion of colors in crystals has led me to precisely the same equation that is obtained for dielectrics that are either isotropic or crystalline. One then identifies the two theories. The equation of dielectrics gives the theory of the propagation of light and diffraction. The equations of two dielectrics, combined with the equation at the separation surface, gives reflection and refraction. The equations of a dielectric and a conductor, combined with the equation on the surface, give metallic reflection.

Meanwhile, there is this remarkable anomaly: Whereas for electricity, K is a characteristic constant of the medium itself, in optics, it is a function of the wave length, except in the ether of the vacuum. Moreover, certain types of radiation are absorbed, and the body will heat up. That phenomenon corresponds to the liberation of Joule heat. Meanwhile, a dielectric does not allow any measurable permanent current to pass through it. One must then assume that the dielectric is composed of conducting corpuscles that are separated by an isolating medium. That system will act like a complex condensor. In the context of electricity, that configuration explains the greater capacity of a condensor with an arbitrary dielectric in it when compared to one *in vacuo*. In optics, it explains:

1. The heating of the body by alternating currents in the particular conductors.

2. The necessity of considering two vectors at each point, namely, the mean value of the conduction current p and the mean value of the electrostatic force X, which is a necessity that led me to the direct study of the dispersion of colors ([10]).

73. Conclusions:

1. I gave the partial differential equations of the conduction current in conductors and the electric force in perfect dielectrics.

2. I explained that those equations, when combined with the equations on each discontinuity surface, will permit one to calculate those vectors when one knows the initial conditions and the boundary conditions on the field.

3. I showed the link between those equations and those of optics.

4. From a comparison of the results of the two theories, I deduced a consequence in regard to the constitution of dielectrics.

The last remark is new, I believe. Since the other notions are known, it is good to add them here when they have specialized by using our formulas.

§ 5. – *Electric energy.*

74. Introduction. – The equations of motion for a system with constraints are expressed by writing down that the total work done by applied and inertial forces is zero for any virtual displacement that is compatible with the constraints. In order to get the differential equation of energy, one writes the same condition, not only for an arbitrary virtual displacement, but for one that is effectively produced during the time interval dt. We applied the first of those rules in paragraph 2; we shall now apply the second one.

The displacement that is produced can be decomposed into an infinitude of others: They correspond to all of the effective displacements dq that are produced during the time interval dt along the various current tubes that one finds at the moment t. It will suffice for me to consider one of those tubes and combine the result with the ones that are found by considering the other tubes.

([10]) In most imperfect dielectrics, that conduction, when limited to particles and which forms the polarization of the dielectric, is accompanied by a conduction that is very weak, properly speaking.

75. Energy equation. – Consider a current line *C* whose element is *dx* and a surface *S* that is bounded by *C*. Let *n* be the normal to the element *dS*. From paragraph 1, the equation of equilibrium of the forces on the contour *C* is:

$$\int_C U \,|\, dx - \int_S \nu \,|\, a' dS = 0. \tag{1}$$

The second integral is taken over an arbitrary surface. It requires a prior study in order to arrive at the choice of surfaces that correspond to the various current tubes in such a fashion that one will avoid ones that intersect each other, which would be necessary in order to perform the summation that relates to those tubes. One circumvents the difficulty by transforming the surface integral into a line integral. In order to do that, I observe that, by definition, the magnetic induction *a* satisfies the incompressibility condition $\frac{d}{dx}\Big| a = 0$. In that case, one knows that one can find a distribution (and even an infinitude of distributions) of vectors *F* that satisfies the relation:

$$a = \Big| \frac{d}{dx} F ,$$

from which, one will deduce that:

$$a' = \Big| \frac{d}{dx} F' \tag{3}$$

upon differentiating with respect to time.

I substitute that value of a' in the surface integral and get:

$$i \int_S \nu \,|\, a' dS = i \int_S \left(\nu \frac{d}{dx} F' \right) dS = i \int_C F' \,|\, dx .$$

Equation (2) can then be written:

$$dt \int_C (U - F') \,|\, i\, dx = 0.$$

Now, *i dx* is the product of the current vector *a* and the volume element $d\tau$ of the part of the current tube that has length *dx*. Hence, the part of the energy equation that corresponds to a current tube is expressed by an integral over only the elements $d\tau$ of that tube. One will get the complete equation by extending the integrals over all of space. In summary, the differential equation of energy is then:

$$0 = dt \int_C (U - F') \,|\, u\, d\tau = dt \left(\int U \,|\, u\, d\tau - \int F' \,|\, u\, d\tau \right). \tag{4}$$

76. Various types of electric energy. – From the ideas that were presented in paragraph **3**, the first term in equation (4) represents the work done by applied forces.

The second term represents the work done by inertial forces. The product of the second integral by dt then represents the variation dT of the kinetic energy. The first integral is the power that results from the various applied forces. That power itself contain two parts: The first one $\mathcal{P}$ comes from the exchange of energy between the system and its exterior. It contains the energy of the batteries, and the Peltier, Thomson, and Joule effects. It is analogous to the power that results from generators, receivers, and passive resistances in ordinary mechanics. The second part of the power comes from the potential energy (W) that is due to the electric force, and which we compared to that of a spring strip in no. **52**. We shall calculate those three parts to the energy:

$$\mathcal{P}\,dt - dW - dT. \tag{4'}$$

77. Work done by applied forces. Power of generators. Potential energy of a dielectric. – The power of the applied forces is:

$$\frac{d\mathcal{T}}{dt} = \int U \,|\, u\,d\tau .$$

If the bodies that are present are perfect conductors and dielectrics then the integral will split into two others, namely:

(5)
$$\mathcal{P} = \int\left(-\frac{p}{C} + P_2\right) |\, p\cdot d\tau, \qquad \text{which is extended over the conductors,}$$
$$-\frac{dW}{dt} = \int [-X + P_2] \,|\, \frac{K}{4\pi}\frac{dX}{dt}\cdot d\tau, \qquad \text{which is extended over the dielectrics.}$$

The first integral represents the total power $\mathcal{P}$ that is provided to the system by the generators, receivers, and the Joule effect. The second one is the derivative of the work that is produced by the reactions of the dielectric. It is derived from the potential:

$$W = \frac{K}{8\pi}\int X \,|\, X\,d\tau - \frac{K}{4\pi}\int P_2 \,|\, X\,d\tau . \tag{6}$$

In general, the electromotive forces P_2 that reside in dielectrics are negligible, and the potential energy reduces to the first term:

$$W = \frac{K}{8\pi}\int X \,|\, X\,d\tau = \frac{K}{8\pi}\int (X^2 + Y^2 + Z^2)\,d\tau . \tag{6'}$$

78. Work done by inertial forces. Electro-kinetic energy. – We have found (no. **75**) that the differential of the kinetic energy is:

$$dT = dt \int F' \,|\, u \, d\tau .$$

One must eliminate the auxiliary vector F and return to the vectors a and α, which are the ones that are detected by experiments. In order to achieve that goal, I shall first replace u with its value:

$$u = \frac{1}{4\pi} \left| \frac{d}{dx} \alpha \right. \qquad \text{(no. **69**).}$$

I will then get:

$$\frac{dT}{dt} = \frac{1}{4\pi} \int F' \frac{d}{dx} \alpha \, d\tau .$$

In that formula, the derivations with the symbol d / dx apply to α. Upon integrating by parts, one will apply them to F' and obtain:

$$\frac{dT}{dt} = \frac{1}{4\pi} \int \alpha \frac{d}{dx} F' \, d\tau ,$$

or, upon replacing $\left| \frac{d}{dx} F' \right.$ with a' [equation (3) in no. **74**] :

$$\frac{dT}{dt} = \frac{1}{4\pi} \int \alpha \,|\, a' \, d\tau . \tag{7}$$

If there are no magnets in the field then a' will be equal to α', and formula (7) will give the kinetic energy in the form:

$$T = \frac{1}{8\pi} \int \alpha^2 \, d\tau . \tag{7′}$$

79. Conclusions:

1. I deduced the equation of electric energy from the equations of electromagnetism, just one deduces the energy equation in rational mechanics from the equations of dynamics.

2. I gave an expression for each of the electrical energies. It is composed of, first of all, the power that is provided to system from its external environment or conversely, then the potential energy that is called electrostatic, and finally the electro-kinetic energy. Those expressions are represented by formulas (5), (6) or (6′), (7) or (7′), resp.

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CHAPTER II

THEORY OF ELECTRICITY IN MOVING BODIES

§ 1. – *Maxwell's theory and the Barlow wheel.*

80. Introduction. – In Part One of this book (Chap. III, § 1), I have specified the analogies that permit one to regard a system of filamentary conductors that are traversed by currents as a system of constraints in rational mechanics, namely:

1. There are two types of *coordinates:* the ones that fix the displacements of the moving parts in the circuits and the ones that fix the electric displacements.

2. The *kinetic energy T* contains two parts: the semi-*vis viva* of the moving parts of the circuits and the electromagnetic force function or electro-kinetic energy.

3. There are two types of *applied forces:* the purely-mechanical forces, like the effort of the observer's hand, and the electromotive force $E - ri$ that comes from the generators, electric receivers, and the Joule resistance.

With those analogies, I applied the Lagrange equations to moving filamentary circuits, whether deformable or not. They determine the system as a function of time and agree with experiments.

The Lagrange equations then give entirely satisfactory results for filamentary circuits. That will not be true when the conductors are not filamentary. I would like to show that.

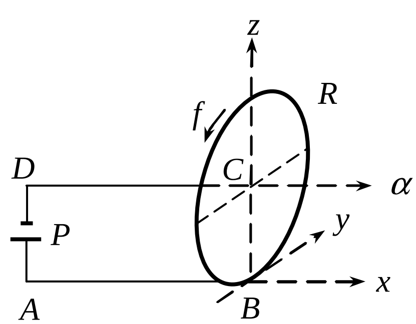

Figure 11.

81. The Lagrange equations break down in experiments with the Barlow wheel. – Consider two electric circuits that are composed in the following manner (Fig. 11): The first one *ABCD* is in the plane of diagram. *AB* and *CD* are two horizontal wires. *DA* is a vertical wire on which there is a battery *P*. Finally, the second vertical edge *BC* is composed of the vertical radius of a wheel *R* that is perpendicular to *CD*, is planar and metallic and moves around its axis *DC*. The second circuit is composed of a solenoid that creates a magnetic field α in the direction of *DC*. From that fact, the magnetic flux that crosses the circuit *ABCD* is zero. As a result, the mutual induction of the two circuits is zero, and the electro-kinetic energy of the system reduces to the sum of their self-induction energies, namely, while keeping the previous notations:

$$\tfrac{1}{2}(L_1\, i_1^2 + L_2\, i_2^2)\,.$$

I further suppose that the wheel *R* turns. Let θ be its angle of rotation, counted positively in the trigonometric sense for an observer that is placed along $C\alpha$, which is an extension of *DC*. That is the sense of the arrow *f*. Let *I* be the moment of inertia of the wheel; its semi-*vis viva* is $\tfrac{1}{2} I\, \theta'^2$. The total kinetic energy of the system is then:

$$T = \tfrac{1}{2}[I\,\theta'^2 + L_1\, i_1^2 + L_2\, i_2^2\,].$$

The mobility parameters of the system, which are three in number, are θ, q_1, q_2, whose velocities are $\theta' = d\theta / dt$, $i_1 = d\theta_1 / dt$, $i_2 = d\theta_2 / dt$. The forces that correspond to q_1 and q_1 are $E_1 - r_1\, i_1$ and $E_2 - r_2\, i_2$. I let *Q* denote the moment of the external forces (the observer's hand, for example) with respect to the wheel. The three Lagrange equations are:

$$\text{(I)} \qquad \left\{ \begin{array}{ll} (1) & Q = I\,\theta'', \\[4pt] (2) & E_1 - r_1\, i_1 = L_1 \dfrac{di_1}{dy}, \\[4pt] (3) & E_2 - r_2\, i_2 = L_2 \dfrac{di_2}{dy}. \end{array} \right.$$

That includes two results that are contrary to experiments:

1. No electromagnetic force will tend to make the wheel turn [eq. (1)].

2. The motion of the wheel will not produce any induced electromotive force [eq. (2)].

These are the facts:

1. The wheel turns spontaneously under the action of electromagnetic forces.

2. The rotation of the wheel produces an induced electromotive force in its own circuit.

As for the magnitude of each of those forces, it is easy to guess by the following fiction:

The element of the circuit *BC* is borrowed from the plane of the wheel, in such a way that the trajectory of the current is *fixed in space* and moves with respect to the wheel ([11]). Instead of that, suppose that the element *BC* is borrowed from a wire that has the same position and *the same velocity* as the radius *BC* at that same instant. The wire *BC* will then be the site of an electromagnetic force and an induced electromotive force that the laws of filamentary conductors permit us to calculate.

Those forces are precisely the ones that affect the radius *BC* of the wheel. That is the experimental fact that must be expressed in order to get the exact equations of the Barlow wheel.

82. True equations for the Barlow wheel. – We must first examine the electromagnetic forces that are exerted on the wheel and their moments with respect to the rotational axis. Those forces are given by Laplace's law. Let *r* be the distance *CM* from a point *M* of the radius *CB* to the center, and let *dr* be the length of an element that is situated at that point. The force that is exerted on the element *dr* has a magnitude of $\alpha i_1 \, dr$. Its direction is that of the axis *Oy* that is normal to the plane of the diagram *ABCD*, to the left of an observer that is standing along *BC* and looking at α. It has the same direction as the displacement of the point *M* under the rotation $\delta\theta$. That displacement has a value of $r \; \delta\theta$. The work done by electromagnetic forces is then $\alpha i_1 \, r \, dr \, \delta\theta$. The sum of the works done by the forces that are applied to the various points of the wheel will then be:

$$\alpha i_1 \, \delta\theta \int_0^R r \, dr = \alpha i_1 \frac{R^2}{2} \delta\theta .$$

The coefficient of $\delta\theta$ is the moment of the electromagnetic forces with respect to the axis $C\alpha$, and that moment must be added to the moment of the applied forces (observer's hand, for example).

The first equation of virtual work, which is the one that relates to the parameter *q*, is then:

$$Q + \alpha i_1 \frac{R^2}{2} = I \, \theta''. \tag{1}$$

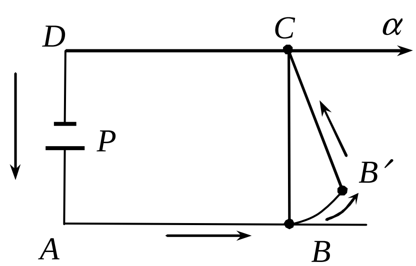

Figure 12.

([11]) *See* my response to Liénard on that subject in a footnote in no. **81**.

Now examine the induced electromotive force in the fictitious part *BC* of the circuit, which is a filamentary part that moves with the wheel (Fig. 12). Under the real displacement $d\theta$, the radius *BC* goes to *B´C*, in such a way that the circuit *ABCDA* is replaced with *ABB´CDA*. The magnetic flux of the vector α that crosses the circuit is zero in the initial position; in the final position, it will have the value:

$$\alpha \times CBB'C = +\alpha \frac{R^2}{2} d\theta .$$

The increase $d\,\Phi$ in the flux results in an induced electromotive force:

$$-\frac{d\Phi}{dt} = -\frac{\alpha R^2}{2}\theta' .$$

That must be added to the applied electromotive force $E_1 - r_1\, i_1$, in such a way that the second equation of virtual work, which is the one that relates to the electric coordinate q_1 in the first circuit, will be:

(2) $$E_1 - r_1\, i_1 - \frac{\alpha R^2}{2}\theta' = L_1 \frac{di_1}{dt} .$$

As for the third equation:

(3) $$E_2 - r_2\, i_2 = L_2 \frac{di_2}{dt} ,$$

it is not true with no modifications.

All that remains is to replace the magnetic field α with its value in equations (1) and (2). Now, the magnetic field α is proportional to the current i_2 and to a coefficient that depends upon only the geometric dimensions of the apparatus. I can then denote the field α by $\frac{2K}{R^2} i_2$, in such a fashion that one will have:

$$\frac{\alpha R^2}{2} = K\, i_2 .$$

One sees that without a doubt I have neglected the current i_1 itself in the calculation of the two forces that act upon *BC*, but that part of the field is normal to the plane *ABCD* and gives zero as a result for the two forces that are evaluated here.

To conclude, *by virtue of the experiments*, the three dynamical equations for the system are:

(II) $$\left\{ \begin{aligned} &(1) \quad & Q &= -K\, i_1\, i_2 + I\,\theta'', \\ &(2) \quad & E_1 - r_1\, i_1 &= +K_1\, i_2 \theta' + L_1 \frac{di_1}{dy}, \\ &(3) \quad & E_2 - r_2\, i_2 &= \qquad\qquad + L_2 \frac{di_2}{dy}. \end{aligned} \right.$$

The energy equations are deduced from these equations by adding them, after multiplying by $d\theta = \theta' \, dt$, $dq_1 = i_1 \, dt$, and $dq_2 = i_2 \, dt$, respectively. That addition will eliminate the two terms in K that we have to add to equations (1) and (2), in such a way that one will indeed recover the expression for kinetic energy in no. **81**, as if those additional terms did not exist, namely:

$$T = \tfrac{1}{2}[I\,\theta'^2 + L_1\,i_1^2 + L_2\,i_2^2].$$

Equations (II) are incompatible with the system (I) (no. **81**) of Lagrange equations.

Must Maxwell's theory be abandoned then? In the form that consists of applying the Lagrange equations, the answer must necessarily be yes. However, at its basis, I do not think that this must be the case, since the Lagrange equations are not always applicable to systems with constraints. On the subject of the hoop, I showed that they break down when the mobility parameters are not true coordinates ([12]). Let us examine things from that viewpoint.

83. Adapting Maxwell's theory to the Barlow wheel. – Is the state of the system well-known when one gives the three parameters θ, q_1, q_2 ? Yes, if one is dealing with filamentary conductors in which the current always traverses the same material particles, but no in the case of the Barlow wheel, in which current displaces in the wheel.

Pass a quantity of electricity q_1 through the wheel and then turn it through θ, so the Joule heat will be released along the first vertical radius. On the contrary, turn the wheel through θ before passing the current through it. The heat will be released along another radius that makes an angle of θ with the first one. In order to know the state of the system at the moment t, it will not suffice to give θ, q_1, q_2 , since one must further give the law of the simultaneous values of the two parameters θ and q_1 .

Similarly, in order to know the absolute position of the hoop (e.g., in order to know if one will arrive at Bordeaux, Lyon, or any other place when one starts from Paris), one must give not only the present values of the angles of advance, conversion, and tilt, but also the law of the simultaneous values of the first two parameters when one starts from the origin.

The analogy is obvious, and there is good reason to apply my remarks in regard to the hoop to the Barlow wheel: *The third Lagrange equation persists, but one must complete the first two with certain terms in the products of the velocities.*

Those are precisely the peculiarities that the system (II) presents that are deduced from experiments. Unfortunately, while the theory permits one to calculate the complementary terms that modify the Lagrange equations for the hoop, here, one must borrow from the experimental laws of electromagnetism, due to our ignorance of the elementary constitution of the mechanical system that we are dealing with. We know only its mobility parameters q, its kinetic energy T, and the applied forces Q. That will suffice when the Lagrange equations are applicable, but not in the contrary case.

([12]) E. CARVALLO, "Théorie du monocycle et de la bicyclette," J. Ec. Poly. (2) Cahiers VI and VII.

84. Conclusions ([13]):

1. The experiments with the Barlow wheel show that the Lagrange equations are not always applicable to electromagnetic phenomena, and notably in the case of two or three-dimensional conductors.

2. It presents a close analogy with the rolling of the hoop, for which the Lagrange equations break down.

3. Consequently, the fact that the Lagrange equations do not apply to the Barlow wheel, far from invalidating the principle of virtual work, is actually a confirmation of the that principle.

4. The Barlow experiments then seem to confirm the two fundamental principles, as well as their Maxwell correlates: The energy of a system of currents is a kinetic energy. The electromagnetic forces and the electromotive inductions forces are inertial forces.

§ 2. – *Laws of electric inertia.*

85. Introduction. – In the preceding paragraph, I showed that one will obtain a false result when one calculates the electromagnetic phenomena that are provided by the Barlow wheel by means of Lagrange's equations, following Maxwell's idea. I gave the correction terms that one must add to the equations in order to make them agree with experiments. Finally, I showed the dynamical analogy between the case of the Barlow wheel and that of the rolling hoop and the modification that one must consequently make to Maxwell's idea in regard to the application of the general equation to the dynamics of electrical currents.

Now, Maxwell deduced the electromagnetic forces and the electromotive induction forces, when considered as *inertial forces*, from Lagrange's equations. Moreover, the author did not at all give a sufficiently correct and general statement of that. It is then necessary to return to the subject in order to specify the experimental laws of electric inertia.

([13]) LIÉNARD [C. R. Acad. Sci. **134** (20 January 1903), pp. 163] declared that I committed an error by applying the virtual electric displacement to a current tube that is fixed in space. It is necessary that the current tube must move with the wheel. I do not, in turn, accuse him of committing an error, because I do not pretend that nature has unveiled its secrets to me alone, and his way of looking at things might very well be justifiable, but I do claim to have correctly applied the theorem of virtual work by imagining everything to be at rest, except for the virtual displacement that is envisioned (see no. **91**).

LUIGI TRAFELLI (*L'Elettricista*, t. VI, no. 1; 1907) also claimed that I was wrong. Contrary to Liénard, he adopted the analysis of my no. **82**. Contrary to what I said, he concluded from that the analysis that the Lagrange equations are applicable, but he did not apply them. He said that I was wrong to adopt the expression $\frac{1}{2}(L_1 i_1^2 + L_2 i_2^2)$ for electro-kinetic energy in no. **81**, but he did not say why, nor did he say what the right expression for that energy would be.

86. Critique of Maxwell's statements. Two forces of electromotive induction. – Here are Maxwell's statements, condensed and modified in form, but basically respected:

FIRST LAW. – *An element of a three-dimensional conductor in which the current is p and the magnetic induction is a is subject to a ponderomotive force that, when expressed per unit volume, is expressed by the same vector* | [*p a*] ([14]) *as the parallelogram that is constructed from the two vectors p and a.*

SECOND LAW. – *The induced electromotive force in a moving filamentary conductor circuit is the derivative of the magnetic induction flux that crosses the moving circuit, with the sign changed* ([15]).

In the first statement, Maxwell indeed understood the Barlow wheel: The electromagnetic force is attached, not to the current, but to the conductor, in such a fashion that it displaces the wheel, not the current. As for the second statement, it is exact for filamentary conductors, but defective, because it confuses two essentially-distinct parts of the electromotive force, in such a way that it will become false if one applies it to the Barlow wheel. The first part of the induced electromotive force is due to the motion of the conductor: *It is the integral along the circuit of the vector* | [x' *a*], which represents the area of the parallelogram that is constructed from the velocity x' of the conductor and the magnetic induction *a*. The second part of the electromotive force is due to the magnetic variations of the field: *It is the rate of variation, with the sign changed, of the magnetic induction flux a that is enclosed by the circuit, which is considered to be fixed.* The sum of those two parts represents the induced electromotive force in the circuit, in any case. It is not always equal to the derivative that enters into the second statement above when the conductors are not filamentary. The study of the Barlow wheel then leads us to the following three statements, in which we agree to first confine ourselves to the case of non-magnetic bodies. For that reason, instead of the magnetic induction vector *a*, we write the magnetic force α.

87. Three laws of electric inertia:

FIRST LAW. – *A non-magnetic conducting element in which the current is p and the magnetic force is a is subject to an electromagnetic force that is represented by the vector* $|[p\,\alpha]$.

SECOND LAW. – *A moving, non-magnetic conducting element whose velocity is x and the magnetic field is a is the site of an electromotive induction force that is represented by the vector* | [x' *a*].

THIRD LAW. – *The electromotive induction force in a closed contour is the sum of two terms, one of which is the integral of the vector* | [x' α] *along the contour, and the*

([14]) MAXWELL, *Traité d'Électricité*, nos. **490** and **501** (French edition). Grassmann's notation.
([15]) MAXWELL, *ibid.*, no. **531**.

other of which is the derivative (with the sign changed) of the magnetic induction flux a that crosses the contour, which is supposed to be fixed.

Experiments with the Barlow wheel authorize us to state those laws only for non-magnetic conductors. What will they become when one considers bodies that enjoy (together or separately) properties that make then conducting, magnetic, dielectric, or electrolytic? It is due to Maxwell's divinations that one knows that in those bodies, the magnetic induction vector α must replace the magnetic field a, and the total current u must replace the conducting current p. From Maxwell, that is the change that one must make to the statements in order to generalize them. It is important for experiments to control the legitimacy of that extension of the laws that were established for the Barlow wheel.

88. Experimental verifications:

1. Replace the Barlow wheel with a wheel of soft iron, so the electromagnetic force that is applied to the wheel will be increased due to the magnetic induction being increased by the presence of iron. The electromotive induction force that the wheel is the site of will also be increased. That increase will correlate with the first one, by virtue of the principle of energy, and conforming to Helmholtz's calculation.

2. When the wheel is made of steel and magnetized along its axis, we can suppress the current or induction magnetic: The wheel will still turn, and its rotation will once more produce electromotive induction forces along the radii of the wheel.

3. Establish a magnetic field in the N-S direction through a vertical jet of electrolytic liquid and a socket for the circuit (*prise de circuit*) in the E-O direction. An electrical current will traverse the circuit.

4. When the fluid jet is air, charges will distribute themselves on the two electrodes, which are supposed to be isolated.

The second experiment is classical, namely, it is due to Ampère, in which a vertical magnet is embedded in a test tube containing mercury and one turns it around its axis such that one will arrive at a current at the upper extremity of the axis. It dispenses with the rigor of the first experiment, which is hardly in doubt. The third one was successfully realized by Bouty. The fourth one was realized in air by Blondlot; he gave a negative result. What I said about dielectrics leads one to think that the forces $|\,[u\ a]$ and $|\,[x'\ a]$ exist only in the conducting parts for those bodies that are found to be truly distributed throughout them. Be that as it may, we can assume that the expressions $f\cdot|\,[u\ a]$ and $f\cdot|\,[x'\,a]$ are valid, in which the coefficient f is equal to 1 for conductors and electrolytes, roughly zero for air, and has a value that must be determined by experiments for the other dielectrics.

89. Conclusions. – If the coefficient f has the value that I just said then the laws of electric inertia are these:

1. An element in which the total current is u and the magnetic induction is a is subject to an electromagnetic force per unit volume that is represented by the vector $f \cdot |[u\,a]$,

2. An element whose velocity is x and in which the induction is a is the site of an electromotive induction force that is represented by the vector $f \cdot |[x'\,a]$.

3. The electromotive induction force in a closed contour is the sum of two terms, one of which is the integral of the preceding vector along the contour, and the other of which is the derivative (with the sign changed) of the magnetic induction flux that crosses the contour, which is supposed to be fixed.

§ 3. – *Electromagnetism in moving bodies.*

90. Introduction. – My analysis of the charging of a condensor by a voltaic current (no. **53**) has led me to these two laws for bodies at rest:

FIRST LAW: *The total current flux that crosses any closed surface is zero.*

SECOND LAW: *The total electromotive force in any closed contour is zero.*

I have deduced the equations of electromagnetism in bodies at rest. I would like to extend those results to bodies in motion.

91. Extending the two fundamental laws to moving bodies. – The first law is a constraint that is analogous to the incompressibility constraint for liquids. Like the law for liquids, it extends to moving surfaces. The surface across which one measures the flux can be either fixed or moving with the body.

As for the second law, it is the law of virtual work for displacements that are compatible with the incompressibility constraint. How must one apply it to moving bodies? As one knows, one must take displacements that are compatible with the constraints on the system, *which is fictitiously considered to be at rest* in its configuration at the moment t ([16]). They are the electric displacements that one can imagine along all of the closed contours. Hence, the second law still applies to all *fixed closed contours*. Only the expression for the work done by inertial forces will be changed: For bodies at rest, it will be the flux (across the fixed contour) of a vector – a', which is the rate of variation of the magnetic induction, with the sign changed. For moving bodies, it results from paragraph **2** that it will be the same flux, but increased by the integral of the vector $f \cdot |[x'\,a]$ along the contour.

([16]) Cf., footnote in no. **84**, in response to Liénard.

92. Equations of electromagnetism for moving bodies. – One obtains those equations as one did for bodies at rest (no. **63**). They are, first of all for continuous media:

(1) $$\frac{d}{dx}\Big|\, u = 0,$$

(2) $$\frac{d}{dx}\Big|\, U = a' - \Big|\frac{d}{dx} f \cdot |\, x' a\,.$$

For the discontinuity surfaces, the normal component of the current u is continuous. The tangential components of the electromotive force $U + f \cdot |\, x' a$ are continuous.

Equation (1) is that of incompressibility. Equation (2) expresses the principle of virtual work for electric displacements that are compatible with the incompressibility constraint. In addition, one must write the equation of virtual work for displacements that are compatible with the mechanical mobility of bodies that comprise the system. In order to do that, I shall call the *vis viva*, properly speaking, T_1 and one of its purely-mechanical mobility parameters q_1. The virtual work δq_1 will then be composed of two terms: The work done by electric forces with a potential W, which is $- dW / dq_1$, and the work done by other applied forces (the observer's hand, for example), which I denote by Q_1. The work done by inertial forces, which has a purely-mechanical origin, is $-\left(\frac{dT_1}{dq_1'}\right)' + \frac{dT_1}{dq_1}$, when Lagrange's formula is applicable. Finally, the electromagnetic forces that are due to electric inertia per unit volume, namely, $f \cdot |\, [u\, a]$, have the integral:

$$\int f \cdot \left[\frac{dx}{dq_1} u\, a\right] d\tau,$$

which extends over the entire electromagnetic field, for their virtual work. The function under the $\int$ sign is the product that one obtains by multiplying the constant f in no. **88** times the volume of the parallelepiped that is constructed from the three vectors dx / dq_1, u, a. The equation of virtual work that corresponds to the coordinate q_1 will then be:

(3) $$Q_1 - \frac{dW}{dq_1} = \left(\frac{dT_1}{dq_1'}\right)' - \frac{dT_1}{dq_1} - \int f \cdot \left[\frac{dx}{dq_1} u\, a\right] d\tau\,.$$

One must append the equations that are concerned with the magnetic field (no. **69**) to equations (1), (2), (3). For continuous media, they are:

(4) $$\Big|\frac{d}{dx}\alpha = 4\pi u,$$

(5) $$\left| \frac{d}{dx} a \right. = 0.$$

For the discontinuity surfaces, the tangential components of α are continuous, and the normal components of a are continuous.

If the bodies are not magnetic then a will be equal to α. Moreover, equations (1), (2), (3), (4), and the conditions at the discontinuity surfaces will determine the electric motion and the mechanical motion of the system, since they define the complete expression for the general principle of virtual work.

93. Energy equation. – It is the equation of virtual work, in which one has replaced the virtual displacement with the actual displacement. Now, the elementary displacement is composed of the electric displacements dq that are produced in the current tubes and the displacements dq_1 of the body. The first ones lead to the work that I evaluated in no. **76**. It is:

$$\mathcal{P}\, dt - \partial W - dT \qquad \text{[formula (4′)],}$$

in which, from now on, ∂ will denote the partial differential that is due to only dq_1 and ∂_1 will denote the partial differential that is due to dq_1, so $d = \partial + \partial_1$.

One can add to that work the work that is done by the new inertial force | [f [x' a] ; as in no. **75**, one will find that:

$$\sum i\, dt \int f\, [x'\, a\, dx]\, d\tau = dt \int f\, [x'\, a\, u]\, d\tau .$$

Upon adding that result to the preceding one, I will obtain the work done by both applied and inertial forces in the form of:

(1) $$\mathcal{P}\, dt - \partial W - dT + dt \int f\, [x'\, a\, u]\, d\tau .$$

That is the work done by real electric displacements dq . One must add the work done by the displacements dq_1 of matter. The applied forces of mechanical origin give a work $d\,\mathcal{T}_1$ that one will have to calculate in every case by using the principles of rational mechanics. In $d\,\mathcal{T}_1$, there is good reason to distinguish the work done by forces that are derived from a potential – dW_1 and the work done by other forces $\mathcal{P}_1\, dt$. The electrostatic forces give the work $-\sum \frac{dW}{dq_1} dq_1$, which I shall denote by $-\partial_1 W$.

I pass on to the inertial forces: They are the ones of purely-mechanical origin, and are given by – dT_1, as one knows. Finally, the electromagnetic forces do work equal to:

$$\sum_{q_1} dt \int f \cdot \left[\frac{dx}{dq_1} u\, a \right] q_1'\, d\tau = dt \int f \cdot [x'\, u\, a]\, d\tau .$$

The total resultant of the displacements dq_1 of matter is, in summary:

$$\mathcal{P}\,dt - dW_1 - \partial_1 W - dT_1 + dt\int f\,[x'\,a\,u]\,d\tau\,. \tag{2}$$

The energy equation is obtained by equating the sum of the works (1) and (2) to zero. In addition, the last term in the expression (1) cancels the last term in the expression (2), because the volumes $[x'\ u\ a]$ and $[x'\ u\ a]$ are equal and opposite. That is the same reduction that we observed in the case of the Barlow wheel by means of equations (II) of no. **82**. The energy equation is then:

$$\mathcal{P}\,dt + \mathcal{P}_1\,dt = dW + dW_1 + dT + dT_1\,.$$

It signifies that the energy that is provided to the system is equal to the increase in its total energy, which is composed of two potential energies and two kinetic energies, namely, the electric energy and the energy of purely-mechanical origin.

94. Conclusions:

1. I extended the fundamental laws of electromagnetism that were established in Chapter I for bodies at rest to moving bodies. The first one is the incompressibility constraint, while the second one is the manifestation of the principle of virtual work in the particular form that is appropriate to electricity.

2. I gave the general equations of electromagnetism for bodies in motion. They are the equations of virtual work.

3. The energy equation is deduced from them as in rational mechanics.

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GENERAL CONCLUSION

95. My conclusion is the idea of this book itself. – A system that exhibits electrical manifestations can be associated with a system with constraints in rational mechanics. It is not a system with a finite number of degrees of freedom. It is an indefinite medium that is analogous to an incompressible fluid in which ponderable bodies with the usual constraints of rational mechanics are embedded.

There are two types of equations of constraint and equations of virtual work that relate to the motions of the matter, on the one hand, and to the electricity, on the other. The equations that relate to electricity result from two fundamental laws, and Kirchhoff stated a particular case of them, namely, the law of incompressibility and the equilibrium of the electromotive forces in any closed contour. The other equations are the usual equations of rational mechanics.

One knows that the theory of virtual work provides exactly the necessary and sufficient equations for determining an arbitrary system as a function of time. One might be tempted to conclude that the problem of electricity is solved completely by the two generalized Kirchhoff laws. Meanwhile, it is not, for two reasons that I have pointed out and which result from our ignorance of the constitution of magnets, on the one hand (no. **42**), and of dielectrics, on the other. Our ignorance of magnets prevents us from writing the equations of virtual work that relate to the particular currents that constitute magnetization, from Ampère's ideas, and were adopted by Sarrau (no. **39**). We can replace them with the largely defective equations of magnetization ([1]).

As for dielectrics other than the vacuum, the phenomenon of the dispersion of colors reveals to us that they have a complex constitution (no. **72**) whose electric displacement involves not just one unknown vector – viz., *the displacement current* p_1 – at each point, but also a second unknown, namely, the *conduction current* p. What is the law that is necessary for us to determine the second unknown? We do not have it. We only have to replace it with the very inadequate theories and very complex results of the dispersion of colors ([2]).

A vast landscape remains open to experimental research and theories along the two paths that I just indicated. In that research, it would seem that it would be advantageous to take the mechanical ideas that were presented in the book as a guide. They might possibly help one to find a more complete truth, like the ones that led Maxwell to the partial truth for filamentary circuits and the ones that have led me, I think, to a higher truth for the totality of all electric phenomena.

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([1]) MAXWELL, *Traité d'Électricité*, Part 3, Chap. IV.

([2]) E. CARVALLO, "Rapport sur les théories et formules de dispersion," presented to the Congrès de Physique in 1900, t. II, pp. 175.