ON THE EQUILIBRIUM

OF

ELASTIC SYSTEMS

MEMOIR

BY

CASTIGLIANNO ALBERTO

ENGINEER OF THE RAILWAYS
OF HIGH ITALY

Translated by D. H. Delphenich

ROYAL PRINTSHOP IN TURIN

OF G. B. PARAVIA, AND CO.

1875

INTRODUCTION

In the year 1857, General Luigi Federico MENABREA read a paper to the Accademia delle Scienze di Torino, in which he proposed and sought to prove a new theorem that he called the *principle of elasticity* or *minimum work*, according to which, whenever an elastic system is deformed by the action of external forces, the final tensions that exist in the system will be the ones that render a minimum for the expression for the molecular work that is done during the deformation. The following year, the Paris Academy communicated his research on that subject.

However, the proof that General MENABREA gave did not seem acceptable, and he published another paper in the year 1867 in which, after having exhibited some particular examples in which his theorem led to exact results in those cases, he proposed a new general proof. Nonetheless, it still did not seem to be any more rigorous than the first one, because despite the great beauty and obvious utility of the theorem of minimum work, no one that I know of was able to attend the Societa degle Ingegnieri ed industriali do Torino in the year 1872 when the engineer Giovanni SACHERI read a paper in which he proved that theorem and applied it to the example of the stability of the ribs in the great canopy of the seaport of Arezzo. However, I do not need to say anything about that paper, since it contained only a numerical example, but did not, in fact, go on to prove the theorem.

Early in the year 1873, in which I studied the equilibrium of elastic systems, after having thought of a method that would certainly have to lead to exact results, I proposed to compare it with the theorem of minimum work, thinking that if it were false then I could easily recognize that with some examples, and if it were true then I might have found in that comparison the way to prove it in a general manner.

While pursuing that idea, I seemed to find myself in the latter case, and I published the result of my research in my Laurea dissertation.

I must now add, out of impartiality, that it was not General MENABREA that was the first to propose the theorem of minimum work (or at least, he did not found the program) by mentioning that the theorem was preceded by some analogous theorems. Already in 1827, Captain VÉNE had proposed a *principle* according to which, when *a rigid body* (i.e., an inelastic one) is supported by more than two points on a line or more than three points in a plane, the pressure of the body on the line or plane is distributed over the various support points in such a manner as to render the sum of the squares of the pressures a minimum. If one says "an elastic body," instead of "a rigid one," then VÉNE's principle will be true sometimes and can be regarded as a special case of MENABREA's.

In the year 1828, A. COURNOT published a paper in the Bolletino di matematiche di FÉRUSSAC in which he extended VÉNE's principle and sought to prove it, although, to be sure, his proof was nothing but a vicious cycle. Some have wanted to see the theorem of minimum work in full generality in that paper by COURNOT. However, one should be warned that COURNOT spoke first of all of the way of determining the pressures in a *rigid* body that is supported by another one at several points, and then considered the case of two *absolutely-rigid* bodies that are coupled by *absolutely-rigid* rods; finally, he extended his theorem to the case of a *rigid body* that is supported by *elastic supports.* In the last case, COURNOT's theorem was true and was contained within that of MENABREA. However, COURNOT did not go any further; i.e., he did not attempt to state a theorem of general applicability to all elastic systems.

In fact, almost at the end of his paper, he spoke of the importance of knowing how the pressure of a body is distributed on its supports, which he expressed as follows:

"The knowledge of the manner by which the pressures are effectively and individually distributed is then indispensible, and *although our formulas give it only for the abstract case of absolute rigidity*, it is clear that the solution of that abstract case will shed light upon the solutions of the various cases in nature. It is in that way that all of the theories of pure mathematics are applicable to the needs of practice."

After all, it does not matter who was the first to find the theorem of minimum work. In this, as in all of the other discoveries, one proceeds by degrees, and there is merit to all that have contributed. Therefore, VÉNE and COURNOT, and also PAGANI and MOSSOTTI, but most of all MENABREA, deserve the credit for having intuited the theorem, and although I will be able to give a rigorous proof and to show its utility, I considered myself to have been justly compensated by what little part of the credit that intelligent people believe that I deserve.

Now let me tell you why I wrote this: Since I published by dissertation, I have been meditating upon it (when that was possible), and although that quest has been diverted for whole months by the many occupations of my career and other miscellaneous things, I also think that I have found some new proofs that are simpler or more rigorous than the one that I gave originally. In addition, in order to make the contribution that can be inferred from the theorem of minimum work more obvious to the engineers, I shall apply it to the example of the stability of the ribs of the canopy in the seaport of Bra. I have not reproduced any of the important applications that made in my dissertation, in order to not overextend things.

I do not know if this paper contains anything good, but nevertheless, I hope that I will be excused for having published it, because with my research, though it might be a trifle, I might as well have paved the way for others or to remove all doubt in the truth of the theorem of minimum work and to infer consequences that are yet unknown, or to prove its falsity, which would still be a truth that is acquired by science.

EQUILIBRIUM

OF

ELASTIC SYSTEMS

1. – Consider a system that is composed of elastic rods that are joined at articulated joints and subjected to forces that are applied to the vertices. Refer them to three orthogonal axes whose origin is at one vertex, while the x-axis passes through another vertex and the xy-plane passes through a third vertex that is not on the same line as the first two. Suppose that the three axes move along with the three aforementioned vertices when the system is deformed. In that way, if one considers only the deformation of the system and not its absolute motion in space then it will be as if the axes were immobile, with the vertex that is at their origin fixed, and anything that is found along the x-axis can move only along that axis, while anything that is in the xy-plane cannot leave that plane.

Call an arbitrary vertex of the system V_p, and let x_p, y_p, z_p be its coordinates before the deformation, while X_p, Y_p, Z_p are the components of the applied force parallel to the axes. Let ξ_p, η_p, ζ_p be the increments in its coordinates due to the deformation, or its displacements parallel to the axes. Furthermore let $V_p V_q$ the rod that joins the two vertices V_p, V_q, let Ω_{pq} be the area of its section, let l_{pq} be its length, let E_{pq} be the elastic coefficient of the substance that it is composed of, let λ_{pq} be its elongation due to the deformation, and let T_{pq} be its final tension. Furthermore, let α_{pq}, β_{pq}, γ_{pq}, and α'_{pq}, β'_{pq}, γ'_{pq} be the angles that it makes with the axes before and after the deformation.

Take $p = 0$ for the vertex that is at the origin and is regarded as fixed, take $p = 1$ for the one that is on the x-axis, and take $p = 2$ for the one that is contained in the xy-plane; one will have:

$$\xi_0 = 0,\ \eta_0 = 0,\ \zeta_0 = 0,\ \xi_1 = 0,\ \eta_1 = 0,\ \zeta_1 = 0. \tag{1}$$

In addition, set:

$$\frac{E_{pq}\,\Omega_{pq}}{l_{pq}} = \varepsilon_{pq}\,, \tag{2}$$

in general, so one will have:

$$T_{pq} = \varepsilon_{pq}\,\lambda_{pq}\,,$$

$$l_{pq} = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2 + (z_q - z_p)^2}\,,$$

and

$$l_{pq} + \lambda_{pq} = \sqrt{(x_q - x_p + \xi_q - \xi_p)^2 + (y_q - y_p + \eta_q - \eta_p)^2 + (z_q - z_p + \zeta_q - \zeta_p)^2}\,.$$

If the differences $\xi_q - \xi_p$, $\eta_q - \eta_p$, $\zeta_q - \zeta_p$ are very small in comparison to the other ones $x_q - x_p$, $y_q - y_p$, $z_q - z_p$ then one can develop λ_{pq} in a convergent series in ascending powers of those small differences, which will give us:

$$\lambda_{pq} = \frac{x_q - x_p}{l_{pq}}(\xi_q - \xi_p) + \frac{y_q - y_p}{l_{pq}}(\eta_q - \eta_p) + \frac{z_q - z_p}{l_{pq}}(\zeta_q - \zeta_p) + \theta_{pq},$$

in which θ_{pq} comprises all of the terms in the development that contain powers of $\xi_q - \xi_p$, … that are higher than the first, such that its ratio with λ_{pq} tends to zero when the differences $\xi_q - \xi_p$, … tends to zero.

Now, one has:

$$\frac{x_q - x_p}{l_{pq}} = \cos \alpha_{pq}, \qquad \frac{y_q - y_p}{l_{pq}} = \cos \beta_{pq}, \qquad \frac{z_q - z_p}{l_{pq}} = \cos \gamma_{pq},$$

so:

$$\lambda_{pq} = (\xi_q - \xi_p) \cos \alpha_{pq} + (\eta_q - \eta_p) \cos \beta_{pq} + (\zeta_q - \zeta_p) \cos \gamma_{pq} . \tag{3}$$

The angles α'_{pq}, β'_{pq}, γ'_{pq} that the rod $V_p V_q$ make with the axes after the deformation are given by the equations:

$$\cos \alpha'_{pq} = \frac{x_q - x_p + \xi_q - \xi_p}{l_{pq} + \lambda_{pq}}, \quad \text{etc.,}$$

or, if one develops that in a convergent series that is ordered in positive, increasing powers of $\xi_q - \xi_p$, $\eta_q - \eta_p$, $\zeta_q - \zeta_p$:

$$\cos \alpha'_{pq} = \frac{x_q - x_p}{l_{pq}} + \omega_{pq}^{(x)} = \cos \alpha_{pq} + \omega_{pq}^{(x)},$$

$$\cos \beta'_{pq} = \frac{y_q - y_p}{l_{pq}} + \omega_{pq}^{(y)} = \cos \beta_{pq} + \omega_{pq}^{(y)},$$

$$\cos \gamma'_{pq} = \frac{z_q - z_p}{l_{pq}} + \omega_{pq}^{(z)} = \cos \gamma_{pq} + \omega_{pq}^{(z)},$$

in which $\omega_{pq}^{(x)}$, $\omega_{pq}^{(y)}$, $\omega_{pq}^{(z)}$ are functions that contain no constant terms and tend to zero when the differences $\xi_q - \xi_p$, $\eta_q - \eta_p$, $\zeta_q - \zeta_p$ tend to zero.

2. – After the deformation, the system will be in equilibrium, and it is clear that the tensions in all of the rods that are concurrent to the vertex V_p must equilibrate the external force X_p, Y_p, Z_p ; one will then have the equations:

$$\left.\begin{aligned} X_p + \sum T_{pq} \cos \alpha'_{pq} = 0, \\ Y_p + \sum T_{pq} \cos \beta'_{pq} = 0, \\ Z_p + \sum T_{pq} \cos \gamma'_{pq} = 0, \end{aligned}\right\} \tag{4}$$

in which the sum that is indicated by the symbol $\sum$ relates to all values of q that correspond to the vertices that are joined to the vertex V_p by rods.

One will have three equations that are analogous to the preceding ones for each vertex of the system except for the three vertices V_0, V_1, V_2. There is no equation for V_0, which can be regarded as fixed. One has only one of them for V_1, which cannot leave the x-axis, and one will have two of them for V_2, which can move only in the xy-plane. It will then follow that if one has as many equations as there are displacements ξ_p, η_p, ζ_p, ξ_q, …, then the latter can be determined, and therefore, the tensions in all of the rods after the deformation, as well.

However, equations (4) and their analogues are very complicated, and solving them rigorously is practically impossible. On the contrary, the solution will become very simple if one is content with approximate results, but approximate in such a way that one can generally regard them as exact.

In fact, one has:

$$
\begin{aligned}
T_{pq} \cos \alpha'_{pq} &= \varepsilon_{pq}[(\xi_q - \xi_p)\cos\alpha_{pq} + (\eta_q - \eta_p)\cos\beta_{pq} + (\zeta_q - \zeta_p)\cos\gamma_{pq} + \theta_{pq}](\cos\alpha_{pq} + \omega_{pq}^{(x)}) \\
&= \varepsilon_{pq}\left\{\begin{array}{l} \quad [(\xi_q - \xi_p)\cos\alpha_{pq} + (\eta_q - \eta_p)\cos\beta_{pq} + (\zeta_q - \zeta_p)\cos\gamma_{pq}]\cos\alpha_{pq} \\ + \, [(\xi_q - \xi_p)\cos\alpha_{pq} + (\eta_q - \eta_p)\cos\beta_{pq} + (\zeta_q - \zeta_p)\cos\gamma_{pq}]\,\omega_{pq}^{(x)} \\ + \, \theta_{pq}\cos\alpha_{pq} + \omega_{pq}^{(x)}\,\theta_{pq}, \end{array}\right\}
\end{aligned}
$$

in whose right-hand side, one sees that of the four terms that are contained in the outer brackets, the first one has degree one with respect to the differences $\xi_q - \xi_p$, $\eta_q - \eta_p$, $\zeta_q - \zeta_p$, while the other three contain only powers of those differences that have degree greater than one. Therefore, the ratio of the sum of the last three terms to the first one will tend to zero when those differences tend to zero.

Hence, if they are very small, as they always are in practice, then the last three terms can be neglected in comparison to the first, which will give:

$$T_{pq} \cos \alpha'_{pq} = \varepsilon_{pq}\,[(\xi_q - \xi_p) \cos \alpha_{pq} + (\eta_q - \eta_p) \cos \beta_{pq} + (\zeta_q - \zeta_p) \cos \gamma_{pq}] \cos \alpha_{pq} .$$

However, one will see in that way that one can suppose that:

$$T_{pq} = \varepsilon_{pq}\,[(\xi_q - \xi_p) \cos \alpha_{pq} + (\eta_q - \eta_p) \cos \beta_{pq} + (\zeta_q - \zeta_p) \cos \gamma_{pq}], \quad \cos \alpha'_{pq} = \cos \alpha_{pq}\,; \quad (5)$$

i.e., in the expression for the tensions, one takes only the terms of first degree in the displacements and the directions of the rods are considered to be unvarying under the deformation.

One will also have that if one wishes to express the tensions by formulas (5) then one will necessarily need to suppose that $\alpha'_{pq} = \alpha_{pq}$, $\beta'_{pq} = \beta_{pq}$, etc.; i.e., one must suppose that the directions of the rods are unvarying, because if one accepts that formula, and one would still like to take into account the change in direction of the rods, which are meaningless in it, then in the expression for T_{pq} cos α'_{pq}, one must take into account the terms:

$$\varepsilon_{pq}\,[(\xi_q - \xi_p) \cos \alpha_{pq} + (\eta_q - \eta_p) \cos \beta_{pq} + (\zeta_q - \zeta_p) \cos \gamma_{pq}]\,\omega_{pq}^{(x)} ,$$

which translate into:

$$\varepsilon_{pq}\ \theta_{pq} \cos \alpha_{pq} ,$$

which has the same order of magnitude.

Therefore, in order to determine, first of all, all the displacements in the rods of the system and then the final tensions in them, one will have the equations:

$$\left.\begin{array}{ll} X_1 + \sum T_{1q} \cos\alpha_{1q} = 0, & \\ X_2 + \sum T_{2q} \cos\alpha_{2q} = 0, & Y_2 + \sum T_{2q} \cos\beta_{2q} = 0, \\ \dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots \\ X_p + \sum T_{pq} \cos\alpha_{pq} = 0, & Y_{pq} + \sum T_{pq} \cos\beta_{pq} = 0, \\ Z_{pq} + \sum T_{pq} \cos\gamma_{pq} = 0, & \\ \dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots \end{array}\right\} \tag{6}$$

in place of equations (4) and their analogues.

3. Theorem of minimum work. – The molecular work that is produced by the deformation of the rod $V_p\, V_q$ can be expressed by the formula:

$$\frac{1}{2}\frac{T_{pq}^2}{\varepsilon_{pq}},$$

so the molecular work that is done by the deformation of all of the system can be expressed by the formula:

$$\frac{1}{2}\sum \frac{T_{pq}^2}{\varepsilon_{pq}} . \tag{7}$$

I shall now say that:

The tensions in the rods of the system after the deformation are the ones that satisfy equation (6) *and render the expression for the molecular work of the system a minimum.*

In order to comprehend the significance of that theorem, observe that equations (6) are $3n - 6$ in number, if one calls the number of vertices n. Therefore, if the number of rods is $3n - 6$ (one can never have less if the system, which is assumed to be rigid, is to have an unvarying form) then equations (6) will serve to determine the tensions of all of the rods *independently of the deformations*. However, if the number of rods is greater than $3n - 6$, as is generally the case, then the number of unknown tensions will exceed the number of equations (6), which will not suffice to determine the tensions without first expression them as functions of the displacements of the vertices. Lacking that, equations (6) can be satisfied by an infinitude of systems of values of the tensions, and each of them will correspond to a different value of the expression:

$$\frac{1}{2}\sum \frac{T_{pq}^2}{\varepsilon_{pq}}.$$

Now, the stated theorem consists of saying that of all the systems of tensions, the one that will exist effectively after the deformation of the rods will be the one that gives a minimum to the expression (7).

In fact, in order to find the values of the tensions T_{pq} that satisfy that condition, one has the equation:

$$\sum \frac{T_{pq}\,dT_{pq}}{\varepsilon_{pq}} = 0, \tag{8}$$

in which the differentials dT_{pq} are coupled with each other by the equations:

$$\left.\begin{array}{l}
\sum dT_{1q}\cos\alpha_{1q} = 0, \\
\sum dT_{2q}\cos\alpha_{2q} = 0, \quad \sum dT_{2q}\cos\beta_{2q} = 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\sum dT_{pq}\cos\alpha_{pq} = 0, \quad \sum dT_{pq}\cos\beta_{pq} = 0, \quad \sum dT_{pq}\cos\gamma_{pq} = 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{array}\right\} \tag{9}$$

which are obtained by differentiating (6).

Multiply each of equations (9) by a constant coefficient and generally let A_p, B_p, C_p denote the coefficients by which the three equations that relate to the vertex V_p are multiplied, and then sum the left-hand sides of equations (9), multiplied by the constant coefficients in the left-hand side of equation (8), and equate the coefficients of all of the differentials of the tensions to zero. One will then obtain as many equations as tensions. For example, equate the coefficients of T_{pq} to zero and obtain the equation:

$$\frac{T_{pq}}{\varepsilon_{pq}} = (A_q - A_p)\cos\alpha_{pq} + (B_q - B_p)\cos\beta_{pq} + (C_q - C_p)\cos\gamma_{pq},$$

which will be no different from (5) *if one changes the symbols* ξ, η, ζ *into the symbols* A, B, C.

If one now combines the equations thus-obtained with equations (6) then one will first obtain the values of the constants A_p, B_p, C_p, …, and then those of the tensions T_{pq}. However, since the preceding equation and its analogues will be no different from (5) and its analogues when one changes the symbols ξ, η, ζ into the symbols A, B, C, it is obvious that one will find the same values for the constants A_p, B_p, C_p, etc., that were obtained for the displacements ξ_p, η_p, ζ_p, …, and therefore the values of the tensions that one obtains will be effectively the ones that exist after the deformation.

One then proves the theorem of minimum work for articulated systems, which echoes the one that I stated in the text, i.e., that:

The constants by which one multiplies equations (9) *are nothing but the displacements of the vertices parallel to the axes.*

4. Expression for the molecular work done on an articulated system. – Recall formula (5), which can be put into the form:

$$\frac{T_{pq}}{\varepsilon_{pq}} + \xi_p \cos\alpha_{pq} + \eta_p \cos\beta_{pq} + \zeta_p \cos\gamma_{pq} + \xi_q \cos\alpha_{pq} + \eta_q \cos\beta_{pq} + \zeta_q \cos\gamma_{pq} = 0.$$

If one multiplies this by T_{pq} then one will get:

$$\frac{T_{pq}^2}{\varepsilon_{pq}} + \xi_p\, T_{pq} \cos\alpha_{pq} + \eta_p\, T_{pq} \cos\beta_{pq} + \zeta_p\, T_{pq} \cos\gamma_{pq}$$
$$+ \xi_q\, T_{pq} \cos\alpha_{pq} + \eta_q\, T_{pq} \cos\beta_{pq} + \zeta_q\, T_{pq} \cos\gamma_{pq} = 0.$$

If one applies that equation to all of the rods of the system, sums the corresponding sides of the equations thus-obtained, and collects all of the terms that contain the same displacement then one will get:

$$\sum \frac{T_{pq}^2}{\varepsilon_{pq}} + \xi_1 \sum T_{1q} \cos\alpha_{1q} + \xi_2 \sum T_{2q} \cos\alpha_{2q} + \eta_2 \sum T_{2q} \cos\beta_{2q} + \ldots$$
$$\ldots + \xi_p \sum T_{pq} \cos\alpha_{pq} + \eta_p \sum T_{pq} \cos\alpha_{pq} + \zeta_p \sum T_{pq} \cos\gamma_{pq}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0.$$

Now, from equations (6), one has:

$$\sum T_{1q} \cos\alpha_{1q} = -X_1\,, \qquad \sum T_{2q} \cos\alpha_{2q} = -X_2\,, \qquad \sum T_{3q} \cos\alpha_{3q} = -X_3\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\sum T_{pq} \cos\alpha_{pq} = -X_p\,, \qquad \sum T_{pq} \cos\alpha_{pq} = -Y_p\,, \qquad \sum T_{pq} \cos\alpha_{pq} = -Z_p\,,$$

so the preceding equation will become:

$$\sum \frac{T_{pq}^2}{\varepsilon_{pq}} = X_1\, \xi_1 + X_2\, \xi_2 + X_3\, \xi_3 + \ldots + X_p\, \xi_p + Y_p\, \eta_p + Z_p\, \zeta_p + \ldots,$$

or, more briefly:

$$\sum \frac{T_{pq}^2}{\varepsilon_{pq}} = \sum (X_p\ \xi_p + Y_p\ \eta_p + Z_p\ \zeta_p).$$

If R_p is the resultant of the three forces X_p , Y_p , Z_p , and r_p is the projection of the displacement of the vertex V_p onto the direction of the force R_p then one will have:

$$X_p\ \xi_p + Y_p\ \eta_p + Z_p\ \zeta_p = R_p\ r_p\ ,$$

so

$$\sum \frac{T_{pq}^2}{\varepsilon_{pq}} = \sum R_p\ r_p\ .$$

The left-hand side of that equation expresses twice the molecular work that is provided by the deformation of the system, so that work can also be expressed as a function of the external force and the displacements of the vertices by the formula (*):

$$\tfrac{1}{2} \sum R_p\ r_p\ .$$

5. – I shall now pass on to an examination of the case in which a system is found to already be in equilibrium under the action of external forces, and one applies a new force whose effect one wishes to study. That is precisely the most common case in nature.

One can also have that the various parts of a system are already found to be tensed or compressed before the application of the external force. Such will be the case, e.g., in an articulated quadrilateral that is composed of six rods that are arranged along the sides and diagonals when one of the rods does not have precisely the length in its natural state that the natural length of the other five require. Now, it is just that case that is also included in the one that shall I treat.

Perhaps the present section will seem pointless to everybody. However, I do not know how to suppress it, since it seems that it serves to make my proof more complete and rigorous.

Let X_p^0, Y_p^0, Z_p^0 be the components parallel to the axes of the force that is initially applied to the vertex V_p . Let X_p, Y_p, Z_p be the components of the force that is applied to that same vertex afterwards, while ξ_p , η_p , ζ_p are its displacements that are produced by the application of the new force to the system, and α_p , β_p, γ_p are the angles that the rod V_p V_q makes with the axes before the new deformation. T_p^0 is the tension in the rod V_p V_q before the application of the force X_p , Y_p , Z_p, etc., and T_{pq} is the increment in that tension that is produced by that force.

Since the system is equilibrium before the application of the force X_p , Y_p , Z_p , etc., and returns to it afterwards, one will have two groups of equations:

(*) It seems to me that the argument by which I have obtained that formula leaves nothing to be desired in terms of simplicity or rigor.

$$\left.\begin{array}{ll} X_1^0 + \sum T_{1q}^0 \cos\alpha_{1q} = 0, & \\ X_2^0 + \sum T_{2q}^0 \cos\alpha_{2q} = 0, & Y_0^0 + \sum T_{2q}^0 \cos\beta_{2q} = 0, \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ X_p^0 + \sum T_{pq}^0 \cos\alpha_{pq} = 0, & Y_{pq}^0 + \sum T_{pq}^0 \cos\beta_{pq} = 0, \\ Z_p^0 + \sum T_{pq}^0 \cos\gamma_{pq} = 0, & \end{array}\right\} \tag{10}$$

$$\left.\begin{array}{r} X_1^0 + X_1 + \sum (T_{1q}^0 + T_{1q}) \cos\alpha_{1q} = 0, \\ X_2^0 + X_2 + \sum (T_{2q}^0 + T_{2q}) \cos\beta_{2q} = 0, \\ Y_2^0 + Y_2 \ + \sum (T_{2q}^0 + T_{2q}) \cos\gamma_{2q} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ X_p^0 + X_{pq} + \sum (T_{pq}^0 + T_{pq}) \cos\alpha_{pq} = 0, \\ Y_p^0 + Y_{pq} \ + \sum (T_{pq}^0 + T_{pq}) \cos\gamma_{pq} = 0, \\ Z_p^0 + Z_{pq} \ + \sum (T_{pq}^0 + T_{pq}) \cos\gamma_{pq} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \tag{11}$$

and if the first of them is subtracted from the second one then one will get another one:

$$\left.\begin{array}{ll} X_1 + \sum T_{1q} \cos\alpha_{1q} = 0, & \\ X_2 + \sum T_{2q} \cos\alpha_{2q} = 0, & Y_2 + \sum T_{2q} \cos\beta_{2q} = 0, \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ X_p + \sum T_{pq} \cos\alpha_{pq} = 0, & Y_{pq} + \sum T_{pq} \cos\beta_{pq} = 0, \\ Z_p + \sum T_{pq} \cos\gamma_{pq} = 0, & \dots\dots\dots\dots \end{array}\right\} \tag{12}$$

Now, the molecular work that is done on the rod $V_p\ V_q$ while its tension passes from T_{pq}^0 to $T_{pq}^0 + T_{pq}$ is expressed by:

$$\frac{(T_{pq}^0 + T_{pq})^2 - (T_{pq}^0)^2}{2\varepsilon_{pq}} = \left(T_{pq}^0 + \tfrac{1}{2} T_{pq}\right) \frac{T_{pq}}{\varepsilon_{pq}},$$

so the molecular work done on the entire system will be:

$$\sum \left(T_{pq}^0 + \tfrac{1}{2} T_{pq}\right) \frac{T_{pq}}{\varepsilon_{pq}}.$$

One will then have:

$$\frac{T_{pq}}{\varepsilon_{pq}} = (\xi_p - \xi_q) \cos\alpha_{pq} + (\eta_p - \eta_q) \cos\beta_{pq} + (\zeta_p - \zeta_q) \cos\gamma_{pq},$$

or

$$\frac{T_{pq}}{\varepsilon_{pq}} = -\xi_p \cos\alpha_{pq} - \eta_p \cos\beta_{pq} - \zeta_p \cos\gamma_{pq} - \xi_q \cos\alpha_{pq} - \eta_q \cos\beta_{pq} - \zeta_q \cos\gamma_{pq} .$$

Multiply that equation by $T_{pq}^0 + \frac{1}{2} T_{pq}$ and take the sum of all the equations that one obtains from all of the rods of the system and get:

$$\sum \left(T_{pq}^0 + \tfrac{1}{2} T_{pq}\right) \frac{T_{pq}}{\varepsilon_{pq}} =$$

$$\sum \left[\begin{array}{c} -\xi_p \sum \left(T_{pq}^0 + \frac{1}{2} T_{pq}\right)\cos\alpha_{pq} - \eta_p \sum \left(T_{pq}^0 + \frac{1}{2} T_{pq}\right)\cos\beta_{pq} \\ -\xi_p \sum \left(T_{pq}^0 + \frac{1}{2} T_{pq}\right)\cos\alpha_{pq}, \end{array} \right]$$

or, if one eliminates the sum that is contained inside the parentheses by means of equations (10) and (12):

$$\sum \left(T_{pq}^0 + \tfrac{1}{2} T_{pq}\right) \frac{T_{pq}}{\varepsilon_{pq}} = \sum \left[\left(X_p^0 + \tfrac{1}{2} X_p\right)\xi_p + \left(Y_p^0 + \tfrac{1}{2} Y_p\right)\eta_p + \left(Z_p^0 + \tfrac{1}{2} Z_p\right)\zeta_p \right]. \tag{13}$$

That is therefore the expression for the molecular work done as a function of the external force.

If the external forces X_p^0, Y_p^0, Z_p^0 are zero then the expression for the molecular force will reduce to:

$$\sum \left(T_{pq}^0 + \tfrac{1}{2} T_{pq}\right) \frac{T_{pq}}{\varepsilon_{pq}} = \sum (X_p \xi_p + Y_p \eta_p + Z_p \zeta_p) , \tag{14}$$

no matter what the initial tensions T_p^0 in the rods are.

6. – If one supposes that the forces X_p, Y_p, Z_p in formula (13) are infinitely small, so the increments T_{pq} in the tensions in the rods will be, as well, and the displacements ξ_p, η_p, ζ_p, etc., of the vertices will be infinitely small, and if one changes X_p, Y_p, Z_p, …, ξ_p, η_p, ζ_p, …, T_{pq} , … by dX_p^0, dY_p^0, dZ_p^0, …, $d\xi_p^0$, $d\eta_p^0$, $d\zeta_p^0$, …, dT_{pq}^0, … then suppresses the index 0 everywhere, for simplicity, and neglects second-order infinitesimals then one will get the formula:

$$\sum \frac{T_{pq}\, dT_{pq}}{\varepsilon_{pq}} = \sum (X_p d\xi_p + Y_p d\eta_p + Z_p d\zeta_p)$$

for the increment of the molecular work that is produced by the given increments in the external forces.

Now, if one differentiates formula (14) then the left-hand side will give:

$$\sum \frac{(T_{pq}^0 + T_{pq})\, dT_{pq}}{\varepsilon_{pq}},$$

and since $T_{pq}^0 + T_{pq}$ expresses the tension in the rod V_p V_q after the external force X_p, Y_p, Z_p, … is applied to the system when the tension in the preceding formula is represented in it by T_{pq}, in order to keep the same symbols in the formulas, one can write T_{pq}, instead of $T_{pq}^0 + T_{pq}$. Hence, after differentiating equation (14), one will get:

$$\sum \frac{T_{pq}\, dT_{pq}}{\varepsilon_{pq}} = \tfrac{1}{2} \sum (X_p d\xi_p + Y_p d\eta_p + Z_p d\zeta_p) + \tfrac{1}{2} \sum (\xi_p dX_p + \eta_p dY_p + \zeta_p dZ_p).$$

If one equates the two expressions that are obtained for $\sum \frac{T_{pq}\, dT_{pq}}{\varepsilon_{pq}}$ then it will result that:

$$\sum (X_p d\xi_p + Y_p d\eta_p + Z_p d\zeta_p) = \sum (\xi_p dX_p + \eta_p dY_p + \zeta_p dZ_p), \qquad (15)$$

so the increment in the molecular work that is produced by the given increments dX_p, dY_p, dZ_p, … in the external forces can be expressed by either the left-hand side of equation (15) or the right-hand side.

If the external forces have constant directions (which is the only case that I find important enough to consider) then let R_p denote the resultant of the forces X_p, Y_p, Z_p, and let λ_p, μ_p, ν_p be the angles that they make with the axes, and one will have:

$$X_p d\xi_p + Y_p d\eta_p + Z_p d\zeta_p = R_p\, (d\xi_p \cdot \cos \lambda_p + d\eta_p \cdot \cos \mu_p + d\zeta_p \cdot \cos \nu_p),$$

or, if one lets dr_p denote the projection of the elementary displacement of the vertex V_p onto the direction of the force R_p :

$$X_p d\xi_p + Y_p d\eta_p + Z_p d\zeta_p = R_p\, dr_p\,.$$

However, if one further has:

$$dX_p = dR_p \cdot \cos \lambda_p\,, \qquad dY_p = dR_p \cdot \cos \mu_p\,, \qquad dZ_p = dR_p \cdot \cos \nu_p$$

then:

$$\xi_p dX_p + \eta_p dY_p + \zeta_p dZ_p = dR_p \cdot (\xi_p \cos \lambda_p + \eta_p \cos \mu_p + \zeta_p \cos \nu_p) = r_p\, dR_p\,,$$

and since equation (15) can also be written:

$$\sum R_p\, dr_p = \sum r_p\, dR_p\,. \qquad (16)$$

However, one should be cautioned that one that equation (15) will always be true, while (16) is true only when the direction of the external forces is constant.

Observation. – When the formula:

$$\frac{T_{pq}}{\varepsilon_{pq}} = (\xi_p - \xi_q) \cos \alpha_{pq} + (\eta_p - \eta_q) \cos \beta_{pq} + (\zeta_p - \zeta_q) \cos \gamma_{pq}$$

is applied to all of the rods of the system and combined with equations (12) in the same way as in the no. **4**, that will lead to the formula:

$$\tfrac{1}{2}\sum \frac{T_{pq}^2}{\varepsilon_{pq}} = \tfrac{1}{2}\sum (X_p \xi_p + Y_p \eta_p + Z_p \zeta_p)$$

so from (13), one will have:

$$\sum \frac{T_{pq}^0\, T_{pq}}{\varepsilon_{pq}} = \sum (X_p^0\, \xi_p + Y_p^0\, \eta_p + Z_p^0\, \zeta_p)\,.$$

Now, $\tfrac{1}{2}\sum \frac{T_{pq}^2}{\varepsilon_{pq}}$ expresses the molecular work that is done on the system by only the increments T_{pq} in the tensions of the rods, as if the initial tensions were zero, and $\sum \frac{T_{pq}^0\, T_{pq}}{\varepsilon_{pq}}$ expresses the molecular work that is done on the system by the original tensions T_{pq}^0 due to the elongations of the rods by the increments T_{pq} in their tensions. Hence, the right-hand sides of the two preceding equations will give those two works as functions of the external forces and the displacements of the vertices.

One will see easily from the preceding that the theorem of minimum work will also be true for an articulated system in which the initial tensions are not zero, or one that is already found in equilibrium under the action of arbitrary forces when one applies the forces whose effect one wishes to study and then takes the expression for the molecular work to be the one that is produced by the latter forces as if the former ones did not exist, and the initial tensions in the rods were zero.

7. Principal property of the theorem of minimum work. – In an arbitrary articulated system, imagine a surface S that encloses a certain number of vertices within it. Some rods will be cut by the surface S; i.e., connect the vertices V_r, V_r', … that are inside that surface to some vertices V_s, V_2', … that are external to it and represent their tensions by T_{rs}, T_{rs}', etc.

The molecular work that is produced by the deformation of the system is expressed by the formula $\frac{1}{2}\sum\frac{T_{pq}^2}{\varepsilon_{pq}}$, which can be put into the form $L + L'$, if one lets L denote the sum of all the terms that relate to the rods that are inside the surface S, and L' is the sum of all the ones that relate to the rods outside the surface S or cut it.

If one equates the molecular work that is done on the system to zero then one will get:

$$dL + dL' = 0.$$

Now, differentiate equations (6) as in no. **3**, multiply each of them by an undetermined constant, and sum the products with the preceding equation. One can divide those equations into two groups, one of which is composed of the equations of equilibrium that relate to the vertices that are placed inside the surface S, and the other one relates to the vertices that are placed outside. The sum of the terms that are provided by the first group is represented by:

$$dM + \sum (A_r \cos \alpha_{rs} + B_r \cos \beta_{rs} + C_r \cos \gamma_{rs})\, dT_{rs}\,,$$

which thus ignores all of the terms that contain the tensions in the rods that cut the surface S. The sum of the terms that are provided by the second group is represented by dM'. One then obtains the equation:

$$dL + dL' + dM + dM' + \sum (A_r \cos \alpha_{rs} + B_r \cos \beta_{rs} + C_r \cos \gamma_{rs})\, dT_{rs} = 0.$$

In order to find the tensions in all of the rods of the system, one needs to equate the coefficients of the differentials of all the tensions that are contained in the preceding equation to zero and combine the equation thus-obtained with that of equilibrium at all vertices. Now, it is easy to see that the terms dL, dM contain only the tensions in the rods that are enclosed by the surface S and can contain no other ones, and that the other terms cannot contain any of those tensions. Therefore, the equation that one finds immediately splits into two:

$$dL + dM = 0,$$

$$dL' + dM' + \sum (A_r \cos \alpha_{rs} + B_r \cos \beta_{rs} + C_r \cos \gamma_{rs})\, dT_{rs} = 0.$$

The first one is precisely the one that would be obtained if one considered the system that is contained inside the surface S to be a free system and regarded the tensions in the rods that cut the surface S as external forces. Hence, if one equates the coefficients of all the differentials that are contained in the equations $dL + dM = 0$ to zero and combines the equations thus-obtained with those of equilibrium at all of the vertices that are contained inside the surface S then it will be clear that one has obtained the values of the constants A_r, B_r, C_r, and the tensions in all of the rods that are contained inside the surface S as functions of the tensions in the rods that do not cut it.

However, it will result from no. **3** that if the three vertices V_0, V_1, V_2 – the first of which is regarded as fixed and placed at the origin, the second of which is constrained to remain on the x-axis, and the third of which must remain in the xy-plane – are three vertices that are contained inside the surface S then the values of the constants A_r, B_r, C_r will be nothing but the displacements of the vertex V_r parallel to the axes. Therefore, if one lets t_r denote the projection of the displacement of the vertex V_r onto the direction of the rod $V_r V_s$ then one will have:

$$t_r = A_r \cos \alpha_{rs} + B_r \cos \beta_{rs} + C_r \cos \gamma_{rs} ,$$

so the second of the equations above will become:

$$dL' + dM' + \sum t_r \, dT_{rs} = 0, \tag{17}$$

and since it results from no. **2** that the direction of the rod must be regarded as constant during the deformation, as was proved in no. **6**, it will follow that the sum of the $t_r \, dT_{rs}$ will be nothing but the differential of the work that is done on the system that is contained inside the surface S by the tensions in the rods that cut it. Hence, the sum:

$$dL' + \sum t_r \, dT_{rs}$$

will express the differential of the molecular work that is done on the whole system as a function of the tensions in the rods that are external to the surface S or cut it.

Therefore, equation (17) is the same as the one that would be obtained if one had expressed the idea that the molecular work that is done on the whole system, when expressed as a function of only the rods that are external to the surface S or cut it, is a minimum and take into account the equations of equilibrium at the vertices that are external to the surface S.

Since, on the other hand, we know that when we equate to zero the coefficients of all the differentials that are contained in equation (17) and combine the equations that are then obtained with those of equilibrium at the vertices that are external to the surface S, we will obtain the tensions in all of the rods that are external to that surface and cut it, we conclude that:

If an articulated system is deformed by given forces and one expresses the molecular work that is done by one part that is contained inside a certain surface S as a function of the tensions in the rods that connect that part with the remaining ones then one will obtain the tensions in those rods and the ones that are external to the surface S by expressing that the molecular work that is done on the whole systems is a minimum, taking into account the equations of equilibrium that relate to all of the vertices that are external to the surface S.

8. Displacements of the vertices as functions of the external forces. – We have seen that the molecular work that is done by the deformation of a system can be expressed by:

$$\tfrac{1}{2} \sum (X_p \, \xi_p + Y_p \, \eta_p + Z_p \, \zeta_p),$$

and its differential by:

$$\sum (\xi_p\, dX_p + \eta_p\, dY_p + \zeta_p\, dZ_p).$$

Now, when we have found the tensions in all of the rods as functions of the external forces, we can easily obtain the molecular work done on all of the system as a function of those forces, as well. If we call that work L then its differential with respect to the variations of the external forces will be:

$$\sum \left(\frac{dL}{dX_p} dX_p + \frac{dL}{dY_p} dY_p + \frac{dL}{dZ_p} dZ_p \right),$$

so we will have:

$$\sum (\xi_p\, dX_p + \eta_p\, dY_p + \zeta_p\, dZ_p) = \sum \left(\frac{dL}{dX_p} dX_p + \frac{dL}{dY_p} dY_p + \frac{dL}{dZ_p} dZ_p \right),$$

and since that equation must be true for any increments dX_p, dY_p, dZ_p, …, it will generally follow that:

$$\frac{dL}{dX_p} = \xi_p\,, \qquad \frac{dL}{dY_p} = \eta_p\,, \qquad \frac{dL}{dZ_p} = \zeta_p\,.$$

Let R_p be the resultant of the forces X_p, Y_p, Z_p, and let α, β, γ be the angles that an arbitrary line makes with those axes. If one lets P denote the projection of the force R_p onto the line (α, β, γ) then one will have:

$$P = X_p \cos \alpha + Y_p \cos \beta + Z_p \cos \gamma.$$

Now, since the force X_p, Y_p, Z_p is equal to R_p, multiplied by the cosines of the angles that it makes with the axes, we see that the work L can be expressed as a function of the single external force R_p, and therefore also as a function of only its projections P. Suppose that it is expressed in such a way that one obtains:

$$\frac{dL}{dX_p} = \frac{dL}{dP}\frac{dP}{dX_p} = \frac{dL}{dP} \cos \alpha, \quad \frac{dL}{dY_p} = \frac{dL}{dP} \cos \beta, \qquad \frac{dL}{dZ_p} = \frac{dL}{dP} \cos \gamma,$$

and if one sums those equations, after having multiplied them by $\cos \alpha$, $\cos \beta$, $\cos \gamma$, in succession, it will result that:

$$\frac{dL}{dX_p} \cos \alpha + \frac{dL}{dY_p} \cos \beta + \frac{dL}{dZ_p} \cos \gamma = \frac{dL}{dP},$$

or

$$\frac{dL}{dP} = \xi_p \cos \alpha + \eta_p \cos \beta + \zeta_p \cos \gamma.$$

Let σ_p denote the projection of the displacement of the vertex V_p onto the direction P, so one will have:

$$\sigma_p = \xi_p \cos \alpha + \eta_p \cos \beta + \zeta_p \cos \gamma,$$

so we will further have:

$$\frac{dL}{dP} = \sigma_p .$$

That is:

If one differentiates the molecular work done on an articulated system when it is expressed as a function of the forces that are applied to its vertices with respect to the force that is applied at one vertex, projected onto a given direction, then the derivative that one will obtain will express the projection of the displacement of the vertex considered onto the given direction.

It will then follow that the derivative of the expression for the molecular work with respect to the resultant R_p is the projection of the displacement of the vertex onto the direction R_p .

Do not forget that all of this will be true only if the directions of the forces are constants, since otherwise the angles α_p, β_p, γ_p that the force R_p makes with those axes will be functions of that force, and when one takes the derivatives, one will get other terms from the ones that were written down.

9. Articulated systems that contain fixed points. – Suppose that some of the vertices in an articulated system are fixed. Let V_r be one of them: If one lets $-X_r$, $-Y_r$, $-Z_r$ be the components parallel to the pressure that the vertex exerts upon a point of the constraint then it will be clear that one can consider the vertex V_r to be free and subjected to forces X_r, Y_r, Z_r that are parallel to the axes. Now, suppose that one has obtained the expression for the molecular work that is done on the system as a function of the external forces, the reactions X_r, Y_r, Z_r , etc., of the fixed points, and the tensions in any rod, but none of the ones that are concurrent at the fixed points. Represent the expression for the molecular work done on all of the system by $F(X_r, Y_r, Z_r, \ldots, T_{pq}, \ldots)$.

I say that the values of the reactions X_r, Y_r, Z_r, …, and of the unknown tensions T_{pq} are the ones that will give a minimum to the molecular work, if one takes into account the condition equations between the tensions T_{pq} .

In fact, if one equates the differential of the molecular force to zero then one will get:

$$\frac{dL}{dX_r} dX_r + \frac{dL}{dY_r} dY_r + \frac{dL}{dZ_r} dZ_r + \ldots + \frac{dL}{dT_{pq}} dT_{pq} = 0.$$

Now, since none of the tensions T_{pq} belong to the rods that are concurrent at the fixed points, it will be clear that the reactions X_r, etc., will not enter into any of the condition equations, so the preceding equation will split into these equations:

$$\frac{dF}{dX_r} = 0, \qquad \frac{dF}{dY_r} = 0, \qquad \frac{dF}{dZ_r} = 0, \text{ etc.,} \qquad \sum \frac{dF}{dT_{pq}} dT_{pq} = 0.$$

If one combines the latter in the usual way with the conditions equations then one will obviously get the same equations as if the forces X_r , Y_r, Z_r , … were known; i.e., one can determine all of the unknown tensions T_{pq} as functions of the unknown reactions at the fixed points. It is then necessary to augment those equations with ones that express the idea that the displacements of the fixed vertices are zero. Now, according to the theorem that was stated in no. **10**, the functions dF / dX_r , dF / dY_r , dF / dZ_r , etc., express the displacements of the vertex V_r parallel to the axes. Therefore, if one equates them to zero then one will have expressed precisely the notion that the vertex V_r is fixed.

One can arrive at this result more directly by imagining that each fixed vertex is detained by three perfectly-rigid rods that are parallel to the axes. In fact, if one imagines a surface S that cuts all of those rigid rods and in such a way that it contains all of that part of the system in which one expresses the work as functions of the external forces and tensions in the other rods (whether compressed or rigid), and if one represents, as we just did, the molecular work in the entire system by:

$$F(X_r, Y_r, Z_r, \ldots, T_{pq}, \ldots)$$

then it will be clear that the values of the unknowns (including the ones in the added rigid rods) can be obtained by rendering a minimum to the function F, while taking into account the condition equations. Now, since one has no condition equations at the vertices V_r besides the one that was proved in no. **7**, one will see that one first has the equations:

$$\frac{dF}{dX_r} = 0, \qquad \frac{dF}{dY_r} = 0, \qquad \frac{dF}{dZ_r} = 0, \qquad \text{etc.,}$$

and then, the equation:

$$\sum \frac{dF}{dT_{pq}} dT_{pq} = 0,$$

which must be combined with the condition equations.

One should note that the three orthogonal rods that are substituted for each fixed vertex are supposed to be rigid, since the molecular work will be zero then, and therefore the molecular work that is done on the system will not be altered. Nevertheless, one will obtain the same result by supposing that one substitutes three orthogonal elastic rods at each fixed vertex and then reduces their elasticity indefinitely.

10. Utility of the theorem of minimum work. – In practice, it almost never happens that one employs simply-articulated elastic systems; i.e., systems that are composed of nothing but elastic rods that are coupled at nodes. Rather, one is continually employing systems that one calls *mixed*, which are composed of beams that are reinforced by tierods or crossbars; i.e., of elastic rods that are coupled to junction points on the beams at various points along their length and between them.

Therefore, in order for a theorem that concerns elastic systems to have any practical utility, it must be applicable to *mixed* systems. The theorem of minimum work has precisely that character, and it is for just that reason that I have adopted it, when I could, to show its accuracy and utility.

However, since the property of that theorem that relates to simply-articulated systems is also maintained for mixed ones, as will be shown shortly, I shall now speak of some advantages that it presents in other methods for calculating articulated systems.

It is initially clear that it is permissible to determine the tensions in all of the rods of the system with any method that serves to find the minimum of a function of several variables when one is given some condition equations between those variables.

In addition, from what was proved in no. **7**, it will result that if one obtains, in any manner, the expression for the molecular work that is done on an articulated system as a function of the tensions in only some of the rods that comprise it then one will obtain the values of those tensions by expressing the notion that the molecular work that is done on the system is a minimum, if one takes into account the condition equations between the unknowns.

Finally, if one has expressed the molecular work in an articulated system by means of the tensions in some of the rods, and if those tensions can be expressed as functions of the other quantities m_1, m_2, … then it will be clear that the molecular work that is done on the system can also be expressed as a function of the m_1, m_2, …, and the condition equations between the unknown tensions can be converted into other ones in the quantities m_1, m_2, etc., or better yet, the values of m_1, m_2, … that are obtained from the condition that the molecular work that is done on the system will be a minimum when it is expressed in terms of them, if one takes into account the condition equations that constrain them. That last observation has great importance.

11. Observations in regard to the theorem of minimum work. – There are some cases in which one can doubt that the theorem of minimum work is applicable. I will choose one of them, and the argument that one makes for it can also serve as norm for the other ones.

Let a body be perfectly rigid, and apply elastic rods to it in such a way they define a system that is arbitrary, except that it has an unvarying form, apart from small deformations that are due to the elasticity of the axes.

It is clear that it would not change the conditions on the system if one replaces the rigid rods that connected the points of the rigid body with elastic rods in all possible ways. Now, imagine a surface S that encloses all of the rigid rods, and cuts the elastic rods that connect with them.

Next, consider another system that does not differ from the one that was just considered, except for the fact that one substitutes elastic rods for the rigid ones, and supposes that the sum of the molecular works that are done on those rods that are contained inside the surface S is expressed as function of the tensions in the rods that are cut by that surface and the external forces, and then have the molecular work that is done on the system as a function of the tensions in the rods that are external to the surface S or cut it. It will result from no. **7** that the values of those tensions are the ones that will render a minimum to the molecular work that is done on the system if one takes into account the condition equation between those tensions. That proposition will be true no matter degree of elasticity exists in the rods that are contained inside the surface *S*, provided that the deformations of the system are always very small. Therefore, it will also be true when that rod is rigid, in which case, its molecular work will be zero, and therefore the molecular work that is done on all of the system will reduce to just the molecular work that is done on the elastic rod.

If one is therefore cautioned that the tensions in the rods that are enclosed by the surface *S* cannot enter into the condition equations then one can conclude that in the case of a rigid body that is constrained by elastic rods, one will obtain the tensions in those rods by expressing the idea that the sum of their molecular works is a minimum, if one takes into account the condition equation between the unknown tensions.

The case that I considered obviously includes that of a rigid, planar panel that leans on an arbitrary number of elastic supports.

12. Considerations pertaining to perfectly-rigid systems. – Imagine an articulated system that is composed of perfectly-rigid beams. It is clear that it can be regarded as the limit of another one that is composed of elastic rods whose degree of elasticity diminishes indefinitely, or the coefficient of elasticity *E* increases indefinitely. Therefore, suppose, first of all, that the system whose vertices are *n* in number is elastic and then determine the tensions in all of the rods.

We saw in no. **2** that we have $3n - 6$ equations (6) between the unknown tensions and that they can be expressed as functions of the $3n - 6$ displacements of the vertices by means of formula (5), so we can take those displacements to be the unknowns, and we will have just as many first-degree equations as unknowns. Therefore, they can all be expressed in the same way; i.e., by the ratio of two determinants whose denominator will be the same for all of them and have order $3n - 6$, and therefore it will be a homogeneous function of degree $3n - 6$ in the coefficients ε_{pq} . The numerator can be deduced from the determinant of the denominator when one replaces a column with the constant terms in equations (6) or the components X_1, X_2, Y_2, …, X_p, Y_p, Z_p, … of the external forces. One will again have determinants of order $3n - 6$, but they will be homogeneous functions of degree $3n - 7$ in the coefficients ε_{pq} .

Therefore, if the expressions that were found for the displacements of the vertices are substituted in equation (5) in order to obtain the tensions in the rod then we will see that each of the tensions will be expressed by the ratio of two homogeneous functions of degree $3n - 6$ with respect to the coefficients ε_{pq} , and therefore, it will depend upon only the ratios of those coefficients, and not at all upon their absolute values.

If one varies those ratios then one will vary the values of the tensions. Now, if one supposes that the elasticity coefficients of all the rods increase indefinitely then the system will become ever more rigid, but in such a way that the ratios of the elasticity coefficients will remain complete arbitrary, and therefore the values of the tensions will not tend to any finite limit, but will remain indeterminate.

Therefore, in a perfectly-rigid system, it is impossible to determine the tensions in the rods unless their number is only $3n - 6$, and their arrangement is such that it gives an unvarying form to the system, in which case, equations (6) will be satisfied independently of formula (5).

As I said in the introduction, VÉNE and COURNOT believed that they had discovered a principle for determining the pressures and tensions in perfectly-rigid systems, and the illustrious Prof. MOSSOTTI found the ideas of those two authors confusing, so he believed that it would be difficult to judge whether that principle could exist. VÉNE and COURNOT started with the idea that if one is given a system of arbitrary form that is perfectly rigid then the tensions and pressures in the various parts of it will necessarily have well-determined values that can be found, and their opinion is confirmed by regarding rigid systems as the limit of elastic systems.

Now, there exist no perfectly-rigid systems in nature, at least, not on the Earth's surface. However, in their place, as one can derive from the preceding proof, it can happen that it is impossible to determine the tensions in their parts, although it is obvious that those tensions can have perfectly-determined values in each particular case.

13. Non-simply articulated systems. – So far, I have spoken only of articulated systems – i.e., ones that are composed of elastic rods that are connected to each other at nodes. They have the peculiarity that the rods can only be found to be subject to tensions or compressions – i.e., forces that are directed along their axes – so they can rotate freely around their extremities during the deformation of each rod. However, such systems are never used in constructions, so the research that is concerned with them can arrive at doctrines that will be rich and elegant, but useless in practice, unless their results can also be extended to the systems that one effectively uses.

Now, I propose to show that the theorem of minimum work is applicable to all systems.

Assume, above all, that the bodies are composed of molecules of dimensions that are very small in comparison to their separation distances, which are also extremely small, and that in a body in equilibrium, those molecules are kept at a well-defined distance from each other as a result of attractions and repulsions that they exert upon each other. Call the masses of two molecules m, m', and the distance between them r, so their mutual attraction can be expressed by $m\,m'\,f(r)$, in which $f(r)$ is an unknown function of the distance r. If external forces are applied to the body then it will be deformed, and it will assume a new equilibrium condition in which the distance between the two molecules m, m' will become $r + \Delta r$, and therefore their mutual attraction will become:

$$m\,m'\,f(r) + m\,m'\,f'(r)\,\Delta r,$$

if one supposes that Δr is very small in comparison to r.

Therefore the attraction of the two molecules will be augmented by:

$$m\,m'\,f'(r)\,\Delta r\;;$$

i.e., it will be proportional to the increment Δr in the distance, which is precisely what one will have for the increment of the tension in the rod.

It will then follow that an arbitrary body can be regarded as a system of very small rods that are connected at nodes that are, perhaps, subject to certain pressures or tensions, even while the system is also not subjected to any external forces.

One should note that it is not necessary for the function $f(r)$ to have the same form for all pairs of molecules, but it is enough that it is continuous for each pair, at least, for very small variations of r when one starts from the value that corresponds to the natural equilibrium of the body. That observation will be necessary, because so far little is known about the molecular constitution of bodies, as can be seen in the best treatments (*).

We conclude that the theorem of minimum work is applicable to a body or a system of bodies that exhibits very small deformations under the action of external forces, so the separation of any two molecules is very small with respect to their original distance. Therefore, if the state of the system after the deformation can depend upon a small number of quantities that are linked with each other by some condition equations, and if the molecular work done on the system during the deformation is expressed by means of only those quantities then one will get their values by considering them to be variables that are linked by condition equations and seeking the system of values for them that gives a minimum to the expression for the molecular work.

Suppose, e.g., that an elastic body is subject to forces of any sort that are applied to the nodes, which are also connected with each other at various points of the elastic rods or with other elastic rods. The mathematical theory of elasticity for solid bodies teaches us how to find the equilibrium condition for the body under the action of the forces that are applied, including the tensions in the rods that are linked to it directly. Hence, by means of CLAPEYRON's formula, one can get the molecular work that is done during the deformation of the body as a function of the external forces and the tensions in any rod, and therefore the molecular work that is done upon the entire system as a function of the tensions in all of the rods.

Imagine a surface S that envelops the given body and cuts the rods that are applied directly, so the case that was just considered is precisely the one that was studied in number **7**, and therefore one can find the unknown tensions in the rods of the system by expressing the idea that the molecular work that is done upon it is a minimum, if one takes into account the equations of equilibrium at all of its vertices where only the elastic forces are concurrent.

14. Caveat. – I gave the proof in number **7** precisely in order to be able to pass to the consequences that I treated there by a rigorous argument.

It might seem to some that I could have also proved the theorem in number **7** in fewer words. The molecular work that is done on the entire articulated system, when expressed as a function of the tensions in the rods that are cut by the surface S or external to it, is such that one will obtain the molecular work done on the system, expressed as a function of all of the tensions, after eliminating the tensions in the rods that are internal to the surface S. One will then find the values of the tensions that this expression contains,

(*) LAMÉ, *Théorie mathématique de l'élasticité des corps solides*, Lecture 24, no. 134.
See also the note and Appendices IV and V of SAINT-VENANT in *Trattato della resistenza dei solidi di Navier*.

moreover, by seeking its minimum when one takes into account the equations of equilibrium at the vertices that are external to the surface *S*.

The argument will be sound if the equations of equilibrium at the vertices that are internal to the surface *S* are sufficient to express the tensions in the rods that are internal to that surface as functions of the tensions in the ones that are cut by it. However, since they are not generally sufficient, the elimination that was just described cannot be done.

To some, it might see that the theorem in no. **7** can be deduced from the one in no. **3** intuitively. However, I believe that is it is not enough in mathematics that something must present the appearance of truth to the mind, but that it is necessary that it should be proved rigorously, and that is all the more true when one treats theorems that have been shown to be exact and will be used continually in practice. Hence, I would prefer to be lengthy and precise, rather than brief and confused.

15. Applications. – In my Laurea dissertation, which was published in November 1873, I proposed to show the utility of the theorem of minimum work by showing how simply one could derive the conditions of elastic equilibrium from it for those systems that are used most frequently in practice, and that had been less-perfectly studied until then. In that dissertation, I saw that by applying the theorem of minimum work, one will easily obtain either the CLAPEYRON equation that relates to beams that are supported at various points or the known formula for the calculation of POLONCEAU roof timbers, English roof timbers, and reinforced beams. I gave limits between which those formulas would be exact, and what terms would need to be added to them in order to make them rigorous.

I shall not repeat what they were, but rather, I will give an example of how to apply the theorem of minimum work to the study of the stability of ribs of arbitrary form, and perhaps that study will not prove to be pointless, because so far I have always been seen to start from arbitrary hypotheses that sometimes conform to the facts of reality very little, and by which one can hope for neither progress in science nor results that one can trust.

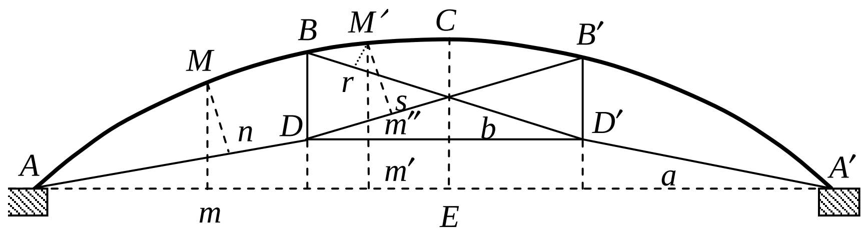
I know well that many believe that it is sufficient in practice to have a sound criterion for construction that is aided all the more by some empirical formulas, and nonetheless believe that for many projects (such as, e.g., the ribs of large canopies), none of which can generally be taken to be the norm for all other analogous constructions, it is indispensible to determine by an exact calculation the strains that one finds the various parts to have been subjected to in order to be able to assign dimensions to each of them that will ensure an indefinite lifespan for the work.

16. Ribs of the canopy in Bra. – Those ribs are composed of wooden arcs, to which are connected five rods and two crossbars, as is represented in the figure. Suppose that the rib is loaded uniformly along its horizontal projection, which is regarded as permanent weight, as well as overload, and that the rib is plated with zinc, which gives 70 kg distributed along each meter of the horizontal projection, which is composed of the proper weight of the rib, the weight of snow, and wind pressure. Therefore, if one calls the distance between two ribs *D* then one will have:

$$p = 70\,D$$

for the weight in Kg that is distributed over a meter of the horizontal projection of the rib.

Suppose further that the rib is supported at its extremities over a horizontal plane without friction, so each support will exert only a vertical pressure that is equal in Kg to $6.15 \times 70D = 430.5\,D$, which we shall call Q.



$AA' = 12.30$ m $\quad CE = 2.10 \quad Dd = 0.60 \quad AD = 1.20$

$BD' = DB' = 1.19 \quad BD = 1.25 \quad DD' = 4.00$

If we would like to see whether the rib is in a good condition of stability then we must first determine the unknown tensions in the rods and crossbars. We must therefore express the molecular work that is done on the whole system as a function of those unknown tensions, and then look for the values of the latter that will give a minimum to that expression, taking into account the equilibrium equations at the vertices D and D', or rather, only the vertex D, since it is enough to consider only the half of the system, by symmetry.

Let t, t_1, t_2, t_3 denote the tensions in the rods DD', AD, DB', DB, respectively, which will be equal to the tensions in their symmetric images. Let l, l_1, l_2, l_3 denote the lengths of those rods, while ω, ω_1, ω_2, ω_3 denote the areas of their sections, e is the elastic coefficient of the three rods DD', AD, DB', which are made of iron, and e_1 is that of the cast-iron rod DB; a and b are the angles DAE, $B'DD'$.

As we just said, consider only half the system, so the molecular work that is done on the four rods will be expressed by:

$$\frac{1}{2e}\left(\frac{t^2}{\omega}\frac{l}{2}+\frac{t_1^2}{\omega_1}l_1+\frac{t_2^2}{\omega_2}l_2\right)+\frac{1}{2e_1}\frac{t_3^2}{\omega_3}l_3\,. \tag{18}$$

One must add the molecular work that is done by the deformation of the arc AMC to this. Then, if one lets μ denote the flexural moment with respect to the point M, while N denotes the sum of the components parallel to the tangent at M of all the forces that are applied to the arc to the left of that point, including the reaction of the support, T denotes the sum of the components perpendicular to the aforementioned tangent (being careful that all of the forces are contained in a vertical plane), s, S denote the lengths of the arcs AB, AC, resp., and ds is the infinitesimal element of arc length for AC then the molecular work that is done by the deformation of that arc will be expressed by the formula:

$$\frac{1}{2EI}\left[\int_0^s \mu^2 d\sigma + \int_s^S \mu^2 d\sigma\right] + \frac{1}{2E\Omega}\left[\int_0^s N^2 d\sigma + \int_s^S N^2 d\sigma\right] + \frac{1}{2E_1\Omega}\left[\int_0^s T^2 d\sigma + \int_s^S T^2 d\sigma\right], \quad (19)$$

in which Ω and I denote the area of the section of the arc and its moment of inertia with respect to the horizontal that passes through the center of gravity, resp., while E and E_1 are the longitudinal and transversal elastic coefficients, resp., of the substance that the arc is made of (*).

Now, if one bases the perpendiculars Mm, Nn to AE, AD, resp., at the point M and the perpendiculars $M'm'$, $M'r$, $M's$ to AE, BD', DB', resp., at the point M', and if one lets φ, φ' denote the angles between the tangents at M, M' and the horizontal, resp., then one will have:

$$\mu = \left(Q - \tfrac{1}{2}\, p \cdot \overline{Am}\right)\overline{Am} - t_1 \cdot \overline{Mm},$$
$$N = \left(Q - p \cdot \overline{Am}\right)\sin\varphi + t_1 \cdot \cos(\varphi - \alpha),$$
$$T = \left(Q - p \cdot \overline{Am}\right)\cos\varphi - t_1 \cdot \sin(\varphi - \alpha),$$

at the point M and:

$$\mu = \left(Q - \tfrac{1}{2}\, p \cdot \overline{Am'}\right)\overline{Am'} - t \cdot \overline{M'm''} - t_2 \cdot (\overline{M'r} + \overline{M's}),$$
$$N = \left(Q - p \cdot \overline{Am'}\right)\sin\varphi' + (t + 2t_2\cos\beta)\cos\varphi',$$
$$T = \left(Q - p \cdot \overline{Am'}\right)\cos\varphi' - (t + 2t_2\cos\beta)\sin\varphi'$$

at the point M'.

One can eliminate t_2 from the last three expressions, and one can eliminate the tensions t_2, t_3 in formula (18), since the tensions in the four rods that are concurrent at the point D must be in equilibrium, and will have two equations:

$$t + t_2\cos\beta - t_1\cos\alpha = 0,$$
$$l_1\sin\alpha - l_3 - l_2\sin\beta = 0,$$

(*) The third term in this formula expresses the work that is done by the transversal sliding, but in a non-rigorous form, and the coefficient f is measured in any special case precisely by assigning a convenient value to it that would allow one to obtain exact results. That coefficient will depend upon either the form of the section of the solid or the distribution law for the forces. However, so far, the value has not been found exactly, except in some cases that are very simple, but nonetheless quite important, which were solved for the first time by SAINT-VENANT.

Therefore, Professors BRESSE and CURIONI started from the hypothesis that the sections of the solid are kept planar during the deformation and obtained $f = 1$. However, from the work of SAINT-VENANT, it results in that way that one can commit an error in the calculation of the transverse sliding that will have the same order of magnitude as the quantity that one would like to calculate, and that, on the contrary, one can already obtain much better approximations by taking into account the flexure of the sections, but while assuming that the flexion happens along cylindrical surfaces. Starting from that idea, I have obtained the third term in formula (19) in order to express the molecular work that is due to transversal sliding, in which the coefficient f must therefore be regarded as a function of only the form of the section. I found:

$$f = \frac{6}{5}$$

for the arc, which was considered to be rectangular in section, with a horizontal side.

That research, along with some other analogous ones, will be treated at another time.

or, if one substitutes the values of the angles:

$$t_2 = 1.035\, t_1 - 1.05\, t,$$
$$l_2 = 0.313\, l - 0.153\, l_1 .$$

If one replaces the letters with the numbers in the formulas just obtained and performs the integrations then one will get:

$$\int_0^s \mu^2 d\sigma = 2650000\, D^2 + 3.44 t_1^2 - 6070\, D\, t_1 ,$$

$$\int_s^S \mu^2 d\sigma = 3240000\, D^2 + 1.56 t_1^2 + 0.052\, t_2 - 7700\, D\, t_1 + 755\, Dt - 0.92\, t\, t_1 ,$$

$$\int_0^s N^2 d\sigma = 102000\, D^2 + 4.20 t_1^2 + 1120\, D\, t_1 ,$$

$$\int_s^S N^2 d\sigma = 370\, D^2 + 6.90 t_1^2 + 1.68\, t_2 + 59\, D\, t_1 - 20.3\, Dt - 6.62\, t\, t_1 ,$$

$$\int_0^s T^2 d\sigma = 330000\, D^2 - 710\, D\, t_1 + 0.408 t_1^2 ,$$

$$\int_s^S T^2 d\sigma = 13800\, D^2 + 0.062 t_1^2 + 0.0142\, t_2 - 46.50\, D\, t_1 + 21.0\, Dt - 0.058\, t\, t_1 ,$$

One can now substitute those expressions in formula (19), then add that formula to (18) and equate the derivatives of the sum with respect to t, t_1 to zero. However, one should observe that the section of the arc is a rectangle with horizontal sides of 0.12 m, while the other sides are 0.20 m, so one will have:

$$\Omega = 0.024, \qquad I = 0.00008,$$

so if one supposes that the arc is made of larch, and one can then take:

$$E = 1{,}500{,}000{,}000, \qquad E_t = 500{,}000{,}000,$$

then one will get:

$$E\,\Omega = 36{,}000{,}000, \qquad E_t\,\Omega = 12{,}000{,}000, \qquad EI = 120{,}000.$$

One will therefore see that the quantities $1 / \Omega E$, $1 / E_t \Omega$ will both be equal to only 1/300, and that $1 / EI$ will be equal to 1 / 100. It will then follows that, with a degree of approximation that is much greater than the one that occurs in practice, one can neglect the two terms in the molecular work that is done on the arc by compression and transversal sliding in comparison with the work that is done by flexion. Therefore, I have neglected them, but I have not seen that any difficulty would arise from that.

In regard to formula (18), one has that the rods *AD*, *DB′*, *DD′* are all made of iron and have circular sections with a diameter of 0.035 m, while the crossbar *BD* is made of cast iron and has a cruciform section whose two branches have lengths 0.08 m and area of 0.00215 m^2. One then has:

$$e = 15{,}000{,}000{,}000, \qquad e_1 = 12{,}000{,}000{,}000$$

for iron and cast iron, resp., so one will have:

$$e\,\omega = e\,\omega_1 = e\,\omega_1 = 14{,}400{,}000, \qquad e_1\,\omega_1 = 25{,}800{,}000,$$

resp., and if one replaces the letters with numbers then one will get:

$$\tfrac{1}{2}t^2 l + t_1^2 l_1 + t_2^2 l_2 = 8.67 t_1^2 + 6.60\, t_2 - 8.85\, t\, t_1\,,$$
$$t_3^2 l_3 = 0.0292 t_1^2 + 0.122\, t_2 - 0.120\, t\, t_1\,,$$

in addition.

If one substitutes of all these numerical results in formula (18) and is cautioned that the quantities $1 / e\omega$, $1 / e_1\,\omega_1$ are much less than 1 / 100 in the quantity $1 / EI$ then one will see that all of the terms that yield the molecular works that are done on the rods can be neglected in comparison to the work that is done by the flexion of the arc.

If one then takes only the last term into account and equates the partial derivatives with respect to t, t_1 to zero then one will get two equations:

$$16\, t_1 - 0.92\, t = 13770\, D,$$
$$0.92\, t_1 - 0.104\, t = 755\, D,$$

from which, one infers that:

$$t_1 = 900\, D, \qquad t = 615\, D,$$

and therefore:

$$t_2 = 285\, D, \qquad t_3 = 50.5\, D.$$

If the distance between the ribs is 5 meters then one will have:

$$t_1 = 4500, \qquad t = 3075,$$
$$t_2 = 1435, \qquad t_3 = 252.5,$$

and if one then supposes, as one usually does, that the resistance of the iron to the tension strains is 6 kg per mm^2 then one will find that the diameter of the rod *AD* is 30.7 mm, that of *DD′* is 25.6 mm, and that of *DB′* is 17.4 mm.

The resistance of cast iron to the tension strains can be taken to be 1.5 kg per mm^2. Therefore, it is sufficient that the area of the section of the crossbar should be 252.5 / 1.5 = 168 mm^2.

The builders of the canopy at Bra made the crossbar *BD* from cast iron and gave it a cruciform section, since they probably believed that it would be found to be compressed, while the result of the preceding calculation was that it would be tensed. However, a closer examination of things will show that it is not difficult to show the reason why that crossbar if found to be tensed. In fact, the more that the rod *DD′* approaches the line *BB′* – i.e., the shorter that the two crossbars *BD*, *B′D′* become – the more the tensions in them must increase, as would result from the disposition of the rods that are concurrent to the point *D*. However, if the rod *DD′* approaches the chord *AA′* then the tension in the crossbar *BD* will diminish, and will become zero when the tension in the rod *AD* becomes equal to the resultant of the tensions in the two rods *B′D*, *DD′*. If one starts from the point and the rod *DD′* continues to approach the chord *AA′* then the tension in the crossbar *BD* will become negative – i.e., the pressure will change.

Having determined the tensions in all of the rods of the system, one will have no difficulty in evaluating the maximum tension and the maximum pressure that is generated in the arc, and thus, its degree of stability.

I have not performed that calculation, although it is quite brief, because it would not add to what I have wished to show by way of example, namely, the great utility of the theorem of minimum work. Rather, I will add that the simplifications that occurred for the ribs of the canopy at Bra that originated in the smallness of some of the terms with respect to others will occur in almost all cases. It is very useful to know how and why once can greatly abbreviate calculations without committing errors that might have a pernicious influence in practice.

Turin, 27 December 1874

CASTIGLIANO ALBERTO