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Natural projection and transverse derivation in a Riemannian manifold with normal hyperbolic metric.

Memoir of CARLO CATTANEO (Pisa)

To Giovanni Sansone on his 70th birthday.

Abstract. – *On a manifold* V_{n+1} *with a normal hyperbolic metric* (+ + … + –) *that is endowed with a "timelike" congruence* C_0 *of reference, we define a natural decomposition operation for a typical vector or tensor. The operation is successively applied to the definition of a transverse covariant derivative (with respect to* C_0*) that operates on "spacelike" tensor fields and has an invariant character under any internal coordinate transformations of* C_0*. This natural decomposition and transverse derivative are then systematically applied to the differential of a timelike congruence.*

The purpose of this article is to illustrate in detail the properties and geometric significance of a differential operation – *transverse derivation*, ordinary or covariant – that was recently introduced in general relativity ([2], [3], [4], [5]), which subsumes all the geometric operations on which it is based and organically connects the disparate notions that pertain to all timelike congruences (KILLING [10], BORN [1], HERGLOTZ [9], EISENHART [6], LEVI-CIVITA [11], GÖDEL [8], SALZMANN [16], TAUB [16], SYNGE [17] [18], LICHNEROWICZ [13]).

In the sequel, we will refer to a Riemannian manifold V_{n+1} with a normal hyperbolic metric and an arbitrary number of dimensions. Such generality, which is adopted in view of its possible application to penta-dimensional relativity theory, does not impede the use of the expressive spatio-temporal terminology of general relativity, and which is systematically adopted.

I. – NATURAL DECOMPOSITION OF VECTORS AND TENSORS. Σ-PROJECTION AND Θ-PROJECTION.

1. *"Timelike" congruence. Adapted coordinates. Internal coordinate changes.* Let V_{n+1} be an arbitrary Riemannian manifold with a normal hyperbolic metric, x, any of its points, T_x , the space of tangent vectors at x, and let x^i (i = 1, 2…, n, 0) be a typical local coordinate system. In addition, let:

$$ds^2 = g_{ik}\, dx^i\, dx^k \tag{1.1}$$

be the fundamental quadratic form, which is assumed to have the signature + + … + –.

Conforming to the conventional terminology in general relativity, we say that any vector $\boldsymbol{V}$ in T_x that has negative norm ($g_{ik}\, V^i\, V^k < 0$) is *"timelike,"* whereas any vector $\boldsymbol{V}$ that has positive norm ($g_{ik}\, V^i\, V^k > 0$) is *"spacelike."* This manner of speaking – timelike or spacelike – also applies to the *directions* of the corresponding vectors, whereas vectors of null norm ($g_{ik}\, V^i\, V^k = 0$) and their respective directions will be given the name of vectors and directions *of null length*, that is, to also borrow from general relativity, *lightlike* vectors and directions.

At any point x, we refer to the set of all vectors $dP \equiv (dx^i)$ in T_x that have null length:

$$g_{ik}\, dx^i\, dx^k = 0 \tag{1.2}$$

as the *light cone*, or *elementary cone*, at x. Similarly, a curve in V_{n+1} will be called timelike if it has a timelike tangent direction at each of its points. A curve, or an arbitrary submanifold of V_{n+1}, will be called spacelike if *any* vector that is tangent to it is spacelike. Finally, we will call lightlike any line or submanifold of V_{n+1} in which *some* tangent vector has null length and is therefore tangent to the corresponding elementary cone at each of its points.

We immediately recognize that the necessary and sufficient condition for a hypersurface W_n of V_{n+1} to be spacelike is that its normal must be timelike.

A local system of coordinates x^i will be called *adapted* to the normal hyperbolic character of the metric on V_{n+1} if the coordinate curve x^0 = var. is timelike and the coordinate manifold x^0 = const. is spacelike. In such a coordinate system the variable x^0 (which we prefer to consider to be the last one, instead of the first one) will be called the *timelike coordinate*, whereas the others, which will be collectively denoted by x^α, will be called *spacelike coordinates* (we agree once and for all that Greek indices always vary from 1 to n, and reserve for the Latin indices the possibility that they may also assume the value 0). A system of coordinates that are adapted to the character of the metric will thus be called, in the usual terminology of general relativity, a *physically admissible* coordinate system. The adoption of such a coordinate system, which we always intend from now on, notably implies the condition $g_{00} < 0$ and the positive definite character of the ternary quadratic form $g_{\alpha\beta}\, dx^\alpha\, dx^\beta$, with all of the algebraic conditions that this implies:

$$g_{00} < 0, \qquad g_{\alpha\beta}\, dx^\alpha\, dx^\beta > 0 \tag{1.3}$$

(the dx^α are not simultaneously null).

If we arbitrarily choose a coordinate system x^i that is adapted to the normal hyperbolic character of the metric then the congruence C_0 of the timelike curves x^0 = var. will be called the *principal reference congruence* for that system of coordinates. In general relativity, the curves of the principal congruence are interpreted as the world lines of as many ideal reference particles, and which constitute a *physical reference frame* that is associated with the chosen coordinate system (cf., MØLLER [14], pp. 233).

Conversely, if one chooses any congruence C_0 of timelike curves that one pleases then it is always possible, and in an infinitude of ways, to associate an adapted coordinate system with it that has C_0 as its principal congruence.

A coordinate change that leaves the timelike coordinate lines invariant, that is, the principal reference congruence, will be called an *internal* coordinate change of that congruence. (In general relativity, such a change is called, in a more expressive way, internal to the physical frame of reference.) More general internal changes of the form:

$$x^{\alpha'} = x^{\alpha'}(x^1, x^2, \ldots, x^n), \qquad x^{0'} = x^{0'}(x^1, x^2, \ldots, x^n, x^0), \tag{1.4}$$

($\partial x^{0'}/\partial x^0 > 0$) must obviously decompose into the product of a transformation of only the spatial coordinates:

$$x^{\alpha'} = x^{\alpha'}(x^1, x^2, \ldots, x^n), \qquad x^{0'} = x^0, \tag{1.5}$$

with a transformation that changes only the temporal coordinate:

$$x^{\alpha'} = x^{\alpha}, \qquad x^{0'} = x^{0'}(x^1, x^2, \ldots, x^n, x^0). \tag{1.6}$$

In order to facilitate the study of an arbitrary timelike congruence C_0, or the study of the manifold V_{n+1} in relation to that congruence, it is simplest to use coordinates that admit C_0 as their principal congruence.

If we fix a local coordinate system x^i in V_{n+1}, or, for that matter, only its principal reference congruence C_0, then there is a uniquely defined field $\gamma(x)$ of unitary vectors on V_{n+1} that are tangent to all of the x^0 curves and oriented to the future (that is, in the sense of increasing x^0), a field that, in turn, completely characterizes C_0. In adapted coordinates, the components of γ, whether contravariant or covariant, have the values:

$$\begin{cases} \gamma^{\alpha} = 0, & \gamma^0 = 1/\sqrt{-g_{00}} \\ \gamma_i = g_{i0}/\sqrt{-g_{00}}. \end{cases} \tag{1.7}$$

respectively. To the unitary vector γ, which will play an essential part in what follows, we will give the name of *timelike reference vector* at the point x.

In T_x, the space of vectors tangent to V_{n+1} at the point x, we let Θ_x denote the one-dimensional subspace of vectors that are collinear with γ and let Σ_x denote the three-dimensional † subspace that is supplementary to the latter one and orthogonal to γ. By analogy with the case of general relativity, in which T_x has the structure of MINKOWSKI space, to the two subspaces Θ_x and Σ_x, which are mutually orthogonal and supplementary, we give the names of associated *timelike axis* and *spatial platform*, respectively. Often, the suffix x will be understood for both of them.

In conformity with the current definition, in order to avoid confusion with the more general terminology that was introduced in the preceding discussion, we call the vectors of Θ_x *purely timelike* and the vectors of Σ_x *purely spacelike.*

In adapted coordinates, the purely timelike vectors are characterized by having their first n contravariant components null (cf., for example, (1.7)); by contrast, the purely spatial vectors are characterized by having their $(n+1)^{\text{th}}$ covariant component null. In an analogous way, for a tensor of order ≥ 2 an index will be called purely timelike if not one (nulle tutte) of the contravariant components of the tensor that have that index are different from 0. By contrast, an index will be called purely spatial if not one of the covariant components of the tensor that have that index are equal to 0. Such terminology relative to tensors of order higher than one will be justified in what follows.

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† {DHD: apparently, he is assuming that $n = 3$.]

It is somewhat superfluous to add that all of the preceding definitions and terminology are *relative* to the chosen reference frame, or better, its principal congruence C_0.

2. *Natural decomposition of a generic vector. Θ-projection and Σ-projection.* As we pointed out, an arbitrary vector in a vector space E is uniquely decomposable into the sum of vectors that belong to assigned complementary subspaces of E. In particular, any vector $\boldsymbol{V}$ in T_x, the space of vectors tangent to V_{n+1} at the point x, has a uniquely determined decomposition:

$$\boldsymbol{V} = \boldsymbol{A} + \boldsymbol{N}, \tag{2.1}$$

in which $\boldsymbol{A}$ belongs to Θ_x and $\boldsymbol{N}$ to Σ_x. More precisely, one has, as one immediately verifies:

$$\boldsymbol{A} = -(\boldsymbol{V} \cdot \gamma)\,\gamma, \qquad \boldsymbol{N} = \boldsymbol{V} + (\boldsymbol{V} \cdot \gamma)\,\gamma, \tag{2.2}$$

or, in terms of natural components (that is, subordinate to the basis induced in T_x by the coordinate system x^i; cf., the following section no. 3):

$$A_i = -\gamma_i\,\gamma_k\,V^k, \qquad N_i = (g_{ik} + \gamma_i\,\gamma_k)\,V^k = \gamma_{ik}\,V^k, \tag{2.3}$$

in which we have set:

$$\gamma_{ik} = g_{ik} + \gamma_i\,\gamma_k\,. \tag{2.4}$$

In the sequel, this decomposition of $\boldsymbol{V}$ into the sum of a purely timelike vector (parallel to γ) and a purely spacelike vector (orthogonal to γ) will be called the *natural decomposition* of $\boldsymbol{V}$.

The two component vectors $\boldsymbol{A}$ and $\boldsymbol{N}$ will be called the *temporal projection* and the *spatial projection* of $\boldsymbol{V}$, respectively. Moreover, extending to a generic manifold V_{n+1} a terminology that is already in use in general relativity by some (cf., LICHNEROWICZ [13], pp. 7), we will define the spacetime norms of the vectors $\boldsymbol{A}$ and $\boldsymbol{N}$ by:

$$\begin{cases} \|\boldsymbol{V}\|_\Theta = g_{ik}A^iA^k = -\gamma_i\gamma_kV^iV^k & (<0) \\ \|\boldsymbol{V}\|_\Sigma = g_{ik}N^iN^k = \gamma_{ik}V^iV^k & (>0). \end{cases} \tag{2.5}$$

the *temporal norm* and *spatial norm* of the vector $\boldsymbol{V}$, with respect to the timelike reference vector γ, respectively. In (2.5), as in (2.3), there systematically appear two symmetric double tensors $-\gamma_i\,\gamma_k$ and γ_{ik} (defined in (2.4), that the vector $\boldsymbol{V}$ provides, the temporal content and the spatial content, respectively. Each of the terms in (2.3) plays a different role from those in (2.5): in the latter, they play a metric role, whereas in (2.3), when they operate by simple composition, they have the effect of *projecting* the vector $\boldsymbol{V}$ onto Θ_x and Σ_x. In relation to this dual role, we call $-\gamma_i\,\gamma_k$ the *temporal metric tensor*, or *temporal projector* at the point x, while reserving for γ_{ik} the name of *spatial metric tensor*, or *spatial projector*. The effect of the two tensors $-\gamma_i\,\gamma_k$ and γ_{ik} in the case of simple vectors extends to the case of tensors of any order, which ultimately justifies the terminology. We then briefly examine their structure and their principal properties.

3. *Natural basis in T_x and naturally induced basis in Θ_x and Σ_x.* We now apply the operations of spatial and temporal projection to the definition of natural bases in Σ_x and Θ_x.

As one notes, the coordinate system x^i induces a *natural basis* in the tangent vector space T_x , and its vectors (n spacelike and one timelike, which is tangent to the single curve coordinate) are denoted by $\partial_i P$. The contravariant and covariant components of $\partial_i P$ are:

$$(\partial_i P)^h = \delta_i^{\ h}, \quad (\partial_i P)_h = g_{hk}\, \delta_i^{\ k} = g_{hi}, \tag{3.1}$$

respectively, where δ_i^h extends the usual KRONECKER symbol. The preceding notation justifies the observation that if, as is the custom, one denotes the generic vector of the tangent space T_x by $\boldsymbol{d}P \equiv (dx^i)$ it then results that $\boldsymbol{d}P \equiv \partial_i P\, dx^i$, in which ∂_i and $\boldsymbol{d}$ denote partial differentiation and total differentiation, respectively.

In conformity with the preceding definition of a natural basis in T_x, we call the contravariant or covariant components of a vector or tensor relative to the natural basis the *natural components* of one type or the other.

Having said that, it is reasonable that we call the set of n orthogonal projections on Σ_x – which we denote by $\tilde{\partial}_\alpha P$ – the natural basis for the spatial platform Σ_x, which are also the first n vectors $\partial_\alpha P$ of the natural basis for T_x. Due to (3.1), and what was said in no. 2, we immediately recover the n + 1 components of $\overline{\partial}_i P$ in the natural basis for T_x:

$$\begin{cases} (\tilde{\partial}_\alpha P)_i = \gamma_{ik}\delta_\alpha^k = \gamma_{i\alpha} \\ (\tilde{\partial}_\alpha P)^i = g^{ik}\gamma_{h\alpha} = \delta_\alpha^i + \gamma^i\gamma_\alpha. \end{cases} \tag{3.2}$$

In vector form the n vectors of the natural basis in Σ_x can be expressed by means of the n + 1 vectors of the natural basis in T_x in the following way:

$$\tilde{\partial}_\alpha P = \partial_\alpha P + \gamma_a\, \gamma^0\, \partial_0 P\,. \tag{3.3}$$

To anticipate slightly, we say that the expression that appears in the right hand side *formally* coincides with that of a differential operator that shall be introduced later on (no. 6).

As for the one-dimensional temporal subspace Θ_x, we call the vector $\partial_0 P$ its natural basis; that is, the unique timelike vector in the natural basis for T_x. This vector, in turn, is connected with the unitary vector γ that was previously introduced (principal reference vector) by the simple relation:

$$\gamma = \gamma^0\, \partial_0 P. \tag{3.4}$$

Having said this, let $\boldsymbol{V}$ be a generic purely spatial vector, V^i and V_i, its natural components in T_x , and $V^{\tilde{\alpha}}$, $V_{\tilde{\alpha}}$ its natural components in Σ_x ; its purely spatial character implies:

$$\boldsymbol{V} \cdot \gamma \equiv V_i\, \gamma^i \equiv V^i\, \gamma_i = 0\,. \tag{3.5}$$

From a significant simplification of the covariant components and from (3.3), it follows that:

$$V_{\tilde{\alpha}} = \boldsymbol{V} \cdot \tilde{\partial}_\alpha P = \boldsymbol{V} \cdot \partial_\alpha P + \gamma_a\, \gamma^0\, \boldsymbol{V} \cdot \partial_0 P = V_\alpha\,. \tag{3.6}$$

Analogous to the pair of identities $\boldsymbol{V} = V^{\tilde{\alpha}}\tilde{\partial}_\alpha P$, $\boldsymbol{V} = V^i\partial_i P$ and from (3.3), while taking into account (3.6), we deduce that:

$$V^{\tilde{\alpha}} = V^\alpha\,. \tag{3.7}$$

(3.6) and (3.7) may be summarized in the following observation, which is of practical use: *the covariant or contravariant components of a purely spatial vector relative to the natural basis for* Σ_x *can be identified with the first n covariant or contravariant components, respectively, relative to the natural basis for* T_x .

One immediately checks that the covariant natural components of the metric tensor in Σ_x , which are given by the n^2 scalar products $\tilde{\partial}_\alpha P \cdot \tilde{\partial}_\beta P$, can be identified with the $\gamma_{\alpha\beta}$.

As for the metric tensor in Θ_x , which does not seem to present any great interest, it has the unique covariant component given by the scalar product $\partial_0 P \cdot \partial_0 P$, and it is therefore identified with the component g_{00} of the fundamental tensor on V_{n+1} . Its only contravariant component has the value $1/g_{00}$.

4. *Natural decomposition of an arbitrary tensor. Σ-projection and Θ-projection. Purely spatial or temporal character of an index.*

The symbolic decomposition of T_x:

$$T_x = \Sigma_x + \Theta_x \tag{4.1}$$

induces a decomposition of $T_x \otimes T_x$ into a sum of four mutually orthogonal and completely supplementary subspaces:

$$T_x \otimes T_x = \Sigma_x \otimes \Sigma_x + \Sigma_x \otimes \Theta_x + \Theta_x \otimes \Sigma_x + \Theta_x \otimes \Theta_x\,. \tag{4.2}$$

It follows that a general double tensor t_{ij} is uniquely decomposable into a sum of four tensors that are pairwise orthogonal and belong to the four subspaces that were just defined, respectively. Such a decomposition will be called the *natural decomposition* of the tensor t_{ij} and the individual component tensors will be called the *natural projections* of the tensor t_{ij} onto its respective subspaces. Such a projection will be symbolically indicated as follows: $\mathfrak{P}_{\Sigma\Sigma}(t_{ij})$, $\mathfrak{P}_{\Sigma\Theta}(t_{ij})$, $\mathfrak{P}_{\Theta\Sigma}(t_{ij})$, $\mathfrak{P}_{\Theta\Theta}(t_{ij})$, in which $\mathfrak{P}$ generically denotes a projection operator.

The formal procedure for obtaining the various natural projections from a given tensor is much simpler: One operates on each index by means of the convenient projector, γ_{ij} or $-\gamma_i\,\gamma_k$, according to whether that index transforms as a spatial index or a temporal index:

$$\text{(4.3)}\qquad \begin{aligned} &\mathfrak{P}_{\Sigma\Sigma}(t_{ij}) = \gamma_{ir}\gamma_{js}t^{rs}, \qquad \mathfrak{P}_{\Sigma\Theta}(t_{ij}) = -\gamma_{ir}\gamma_j\gamma_s t^{rs},\\ &\mathfrak{P}_{\Theta\Sigma}(t_{ij}) = -\gamma_i\gamma_r\gamma_{js}t^{rs}, \quad \mathfrak{P}_{\Theta\Theta}(t_{ij}) = \gamma_i\gamma_j\gamma_r\gamma_s t^{rs}.\end{aligned}$$

An interesting, if somewhat banal, example is the calculation of the various natural projections of the fundamental tensor g_{ij}:

$$\text{(4.4)}\qquad \begin{aligned} &\mathfrak{P}_{\Sigma\Sigma}(g_{ij}) = \gamma_{ij}, \quad \mathfrak{P}_{\Sigma\Theta}(g_{ij}) = 0,\\ &\mathfrak{P}_{\Theta\Sigma}(g_{ij}) = 0, \quad\ \ \mathfrak{P}_{\Theta\Theta}(g_{ij}) = -\gamma_i\gamma_j.\end{aligned}$$

This confirms that the spatial and temporal projections of g_{ij} coincide with the metric tensors on Σ and Θ, respectively, and that the result of inducing a metric on a subspace that is defined by the basis induced by the projection of the first n vectors of the basis for T_x and successively calculating the various scalar products $\tilde{\partial}_\alpha P \cdot \tilde{\partial}_\beta P$ coincides with the result of the direct projection of g_{ij} .

An index, regardless of its contravariance or covariance, of a generic tensor will be said to have a *purely spatial* character if that tensor does not change when one operates on each index with the spatial projector γ_{ik} . By contrast, an index will be said to have a *purely temporal* character if the tensor is not modified when one operates on each index with the temporal projector $-\gamma_i\,\gamma_k$. In accord with what was already said in no. 1, all of the components of a tensor in which a purely spatial index in a covariant position assumes the value 0 are null. Likewise, all of the components of a tensor that correspond to a purely temporal index in a contravariant position must assume an arbitrary non-zero value.

A double tensor will be called *totally spatial* if it belongs to Σ ⊗ Σ; that is, if both of its indices are purely spatial. On the other hand, it will be called *totally temporal* if it belongs to Θ ⊗ Θ; i.e., if both of its indices are purely temporal.

The preceding may be extended in a rigorous way to tensors of arbitrary order. For them, one speaks of a projection of type $\mathfrak{P}_{\Sigma\Theta\ldots\Sigma}(t_{ij\ldots r})$ with as many indices equal to Σ or Θ as are indices of that tensor.

5. *Some algebraic properties of the projection operation.*

a) When the Σ-*projector* operates on a purely temporal index the result is the null tensor.

b) When the Θ-*projector* operates on a purely spatial index the result is the null tensor.

c) When a complete projection operation, that is, one that operates on all of the indices, operates on the product of two or more tensors (in the sense of contraction), it is divided between the factors, according to the corresponding relationship between their indices: $\mathfrak{P}_{\Sigma\Theta\Sigma}(A_i B_{jr}) = \mathfrak{P}_{\Sigma}(A_i)\ \mathfrak{P}_{\Theta\Sigma}(B_{jr})$.

d) When the natural decomposition is applied to a generic tensor A_{ij} one may write it as:

(5.1) $$A_{ij} = \tilde{A}_{ij} + \tilde{A}_i \gamma_j + \gamma_i \tilde{A}'_j + A\gamma_i\gamma_j$$

in which we have set:

(5.3) $$\tilde{A}_{ij} = \gamma_{ir}\gamma_{js}A^{rs}, \quad \tilde{A}_i = -\gamma_{ir}\gamma_s A^{rs}, \quad \tilde{A}'_{ij} = -\gamma_r\gamma_{js}A^{rs}, \quad A = \gamma_r\gamma_s A^{rs}.$$

A generic double tensor A_{ij} therefore produces, in particular, a totally spatial tensor $\tilde{A}_{ij}$, two purely spatial vectors $\tilde{A}_i$ and $\tilde{A}'_i$, and a scalar A, which corresponds to exactly 16 independent numerical quantities, in all.

If the tensor A_{ij} is symmetric, so is the tensor $\tilde{A}_{ij}$, whereas, the two vectors $\tilde{A}_i$ and $\tilde{A}'_i$ coincide. For a symmetric tensor, $S_{ij} = S_{ji}$, one may write:

(5.4) $$S_{ij} = \tilde{S}_{ij} + \tilde{S}_i\gamma_j + \gamma_i\tilde{S}_j + S\gamma_i\gamma_j \qquad (\tilde{S}_{ij} = \tilde{S}_{ji}) .$$

In the event that the tensor being decomposed is anti-symmetric, $\Omega_{ij} = -\Omega_{ji}$, the resulting tensor $\tilde{\Omega}_{ij}$ is antisymmetric, the two resulting spatial vectors are opposite to each other, and the scalar is null. The natural decomposition of an antisymmetric tensor $\Omega_{ij} = -\Omega_{ji}$ may then be written:

(5.5) $$\Omega_{ij} = \tilde{\Omega}_{ij} + \tilde{\Omega}_i\gamma_j - \gamma_i\tilde{\Omega}_j \qquad (\tilde{\Omega}_{ij} = \tilde{\Omega}_{ji}) .$$

e) The saturation of a purely spatial index with a purely temporal index (which is expected to imply two indices in opposite positions of variance) has zero for its result, regardless of the number and character of the other indices that are involved in the saturation.

II. – TRANSVERSE DERIVATION.

6. *Ordinary transverse derivation.* Assign to the domain D in the manifold V_{n+1} a generic scalar field $f(x^i)$ and consider its gradient grad f, whose covariant components are $\partial_i f$. Furthermore, let $\boldsymbol{d}P = (dx^i)$ be a generic vector field, subject to only the condition that it be, moreover, perpendicular to the vector γ, which then implies that $\gamma_i\, dx^i = 0$, so:

(6.1) $$dx^0 = -\frac{1}{\gamma_0}\gamma_\alpha\, dx^\alpha = \gamma^0\gamma_\alpha\, dx^\alpha.$$

Having said that, the total differential of f relative to the vector $\boldsymbol{dP}$ is expressed as: $df = \partial_i f\, dx^i$. Therefore, taking condition (6.1) into account, one can express it in either of the following ways:

$$df = \left(\partial_\alpha f - \frac{\gamma_\alpha}{\gamma_0}\partial_0 f\right) dx^\alpha \equiv (\partial_\alpha f + \gamma^0 \gamma_\alpha \partial_0 f)\, dx^\alpha , \tag{6.2}$$

in which the summation is only from 1 to n. We introduce, in a natural way, the differential operator, which we denote simply by $\tilde{\partial}_i$, by means of the following definition:

$$\tilde{\partial}_i = \partial_i - \frac{\gamma_i}{\gamma_0}\partial_0 \equiv \partial_i + \gamma_i \gamma^0 \partial_0 , \tag{6.3}$$

an operator that will play an important role in what follows. By means of such operators, which are significant only when $i \neq 0$, the total differential of f relative to a vector $\boldsymbol{dP}$ that is normal to γ, may thus be expressed by the formula:

$$df = \tilde{\partial}_i f\, dx^i \equiv \tilde{\partial}_\alpha f \cdot dx^\alpha \tag{6.3′}$$

which uses the fact that there is only one non-zero contravariant spatial component of $\boldsymbol{dP}$. (6.3′) shows that the $n + 1$ quantities $\tilde{\partial}_i f$, of which the last one ($i = 0$) is always null, are the covariant components of a vector that, for any $f(x^i)$, always remains perpendicular to γ ($\tilde{\partial}_0 f = 0$). For this reason, we call it the *transverse gradient* (in the direction of γ) of the scalar field f; in symbols, $\widetilde{\text{grad}}\, f = (\tilde{\partial}_i f)$.

It is important to point out that the transverse gradient that was just defined is just the projection of the ordinary gradient onto the spatial platform Σ. One has, in fact: $\tilde{\partial}_i f = (\delta_i^h + \gamma_i \gamma^h)$ and therefore:

$$\tilde{\partial}_i f = \mathfrak{P}_\Sigma(\partial_i f). \tag{6.4}$$

All of this justifies the name of *transverse partial derivation* for the operation $\tilde{\partial}_i$ that was defined by (6.3). The term is ultimately justified by the following interpretation: If one presents the formal definition (6.3) at the same time as (3.2) then one immediately recognizes that $\tilde{\partial}_\alpha$ is just the operation of total differentiation with respect to the l^{th} vector of the natural basis induced in Σ_x.

It is almost obvious that the operation $\tilde{\partial}_i$ satisfies most of the formal properties that are true for ordinary partial differentiation; in particular, the rules for the differentiation of a sum, product, quotient, etc.

The operation of transverse partial derivation can be successively applied, in turn, to arbitrary values of the index, as well as that of taking transverse derivatives of second and higher orders. Nevertheless, as one immediately recognizes, the order of the successive derivatives is generally not permutable. We content ourselves with this negative result for now, and reserve until later the examination of the special case in which the product $\tilde{\partial}_\beta \tilde{\partial}_\alpha$ is commutative and the conditions that such commutativity implies for the principal congruence C_0.

For now, we simply recognize that the operator $\tilde{\partial}_\alpha$ is not even commutable with the operator ∂_0, in general.

7. *Longitudinal derivation* $\tilde{\partial}_0$. Once more, consider the vector field grad f and project it orthogonally onto Θ_x at each point of the domain D. The algebraic components of the projected vector relative to the vector basis in $\Theta_x(\partial_0 P)$ will be called the *longitudinal derivative* (in the direction of γ) of the scalar field f and will be denoted by $\tilde{\partial}_0 f$:

(7.1) $$\tilde{\partial}_0 f = \gamma^0 \, \partial_0 f \, .$$

This differs from the simple partial derivative with respect to x^0 by the factor γ^0. Naturally, one also has that the operator $\tilde{\partial}_0$ is not generally commutable with either $\tilde{\partial}_\alpha$ or ∂_0.

It is not pointless to add that the operator $\tilde{\partial}_0$, like the operator $\tilde{\partial}_\alpha$, is invariant under any internal coordinate change of the congruence C_0.

8. *Transverse covariant derivation of a purely spatial vector field.* The operation of transverse derivation that was introduced in the preceding section can be extended in an equally natural way from the case of a scalar field to that of a *purely spatial* vector field.

Let $s(x)$ be a generic purely spatial vector field that satisfies the conditions (which are equivalent):

(8.1) $$s^0 = -\, \gamma_\alpha \, s^\alpha / \gamma^0 \equiv \gamma^0 \, \gamma_\alpha \, s^\alpha ; \qquad s^\alpha = 0 \, .$$

Consider the covariant derivatives $\nabla_h \, s^k$ of $s(x)$ in V_{n+1} and effect the total projection onto Σ_x at any point of V_{n+1}. The purely spatial double tensor – $\mathfrak{P}_{\Sigma\Sigma}(\nabla_i \, s_j)$ – that is so obtained at each point of V_{n+1} will be called the *transverse covariant derivative* of the purely spatial field $s(x)$. Applying the projection procedure that defined in the preceding section (no. 4) and carrying out the calculations, which we will omit, one obtains:

(8.2) $$\mathfrak{P}_{\Sigma\Sigma}(\nabla_i \, s_j) = \tilde{\partial}_i s_j - \frac{1}{2}(\tilde{\partial}_i \gamma_{jh} + \tilde{\partial}_j \gamma_{hi} - \tilde{\partial}_h \gamma_{ij}) s^h \, ,$$

which systematically involves the spatial metric tensor γ_{ij} and the transverse derivative $\tilde{\partial}_i$.

The expression that appears in the right-hand side of (8.2) suggests that we introduce a new type of CHRISTOFFEL symbol, of the first or second type, which is formally constructed like the ordinary symbol, but with the essential substitution of the spatial metric tensor γ_{ij} for the spacetime metric tensor g_{ij} and the partial transverse derivation $\tilde{\partial}_i$ for the ordinary derivation ∂_i:

(8.3)
$$|\widetilde{ij,h}| = \frac{1}{2}(\tilde{\partial}_i \gamma_{jh} + \tilde{\partial}_j \gamma_{hi} - \tilde{\partial}_h \gamma_{ij})$$
$$\left\{ \begin{matrix} \widetilde{h} \\ i\ j \end{matrix} \right\} = g^{hr}\, |\widetilde{ij,r}| = \gamma^{hr}\, |\widetilde{ij,r}|.$$

With this definition, (8.2) takes the form:

(8.4)
$$\mathfrak{P}_{\Sigma\Sigma}(\nabla_i\, s_j) = \tilde{\partial}_i s_j - \left\{ \begin{matrix} \widetilde{h} \\ i\ j \end{matrix} \right\} s_h \equiv \tilde{\nabla}^*_i s_j\,,$$

and may be interpreted as a new type of covariant derivative, which we denote by the symbol $\tilde{\nabla}^*_i s_j$, and we are justified in calling it the *transverse covariant derivative* of the spatial vector field s_j , which we previously denoted by $\mathfrak{P}_{\Sigma\Sigma}(\nabla_i\, s_j)$. With the adopted notation, which is perhaps excessive, the sign * has the purpose of reminding one to use γ_{ij} and the ~ sign, that of using the derivation $\tilde{\partial}_i$.

In the CHRISTOFFEL symbols (8.3), as well as in the components of the transverse covariant derivative (8.4), one immediately recognizes that they reduce to zero identically whenever one of their lower indices assumes the value 0, which conforms to the totally spatial character of the transverse covariant derivative.

We conclude by saying that, just as the operation of transverse ordinary derivation on a scalar field produces a purely spatial vector field (the transverse gradient of f), likewise the operation of transverse covariant derivation on a purely spatial vector field produces a totally spatial double tensor field. Formally, the transverse covariant derivative of a field of purely spatial vectors may be calculated like an ordinary covariant derivative using the CHRISTOFFEL symbols of the first and second type that are formed by starting with the spatial metric tensor γ_{ij} with the systematic use of the $\tilde{\partial}_i$, in place of the ∂_i .

9. *Transverse covariant derivation of a totally spatial tensor of arbitrary order.* The operation of transverse covariant derivation is easily extended to the case of a tensor of arbitrary order, on the condition that it is purely spatial. For example, let s_{ij} be a totally spatial double tensor field ($s_{i0} = 0$, $s_{0i} = 0$). Proceeding in a manner that is perfectly analogous to the one that was described in the preceding section, consider the covariant derivative $\nabla_i\, s_{jm}$ of that field and totally project it onto the space Σ_x at each point. By calculations that are very simple, though somewhat lengthy, and which we omit, we recognize that the desired projection may put into the following form:

(9.1)
$$\mathfrak{P}_{\Sigma\Sigma\Sigma}(\nabla_i\, s_{jm}) = \tilde{\partial}_i s_{jm} - \left\{ \begin{matrix} \widetilde{h} \\ i\ j \end{matrix} \right\} s_{hm} - \left\{ \begin{matrix} \widetilde{h} \\ i\ m \end{matrix} \right\} s_{jh} \equiv \tilde{\nabla}^*_i s_{jm}\,.$$

Once again, we recognize a type of covariant derivative in the right-hand side of (9.1), which we denote by $\tilde{\nabla}^*_i s_{jm}$, which obeys the ordinary formal rules, but with the systematic use of the derivation $\tilde{\partial}_i$ in place of the ∂_i and the CHRISTOFFEL symbols (8.3) in place of the ordinary symbols.

We regard the treatment of the more general case, which would suffice to assure the complete generality of the operation of transverse covariant derivation, as superfluous. It operates on a totally spatial tensor field and results in a new totally spatial tensor field with one more index.

10. *Immediate formal properties of the transverse covariant derivation*. We almost immediately recognize that the transverse covariant derivation enjoys all of the formal properties of the covariant derivation that are true for the ordinary derivation, such as the term-by-term derivation of a sum of totally spatial tensors, and likewise the ordinary rule for the derivation of a product of more than one totally spatial tensors remains valid. One also naturally extends to the transverse covariant derivative the property of being commutable with the operation of saturation, *provided that it operates on two indices that are both purely spatial.*

A theorem that is completely analogous to the RICCI theorem subsists, so we give it the same name:

RICCI THEOREM. – *The transverse covariant derivative of the spatial metric tensor* γ_{ij} *is identically null.*

The proof follows in a manner that is formally similar to the usual proof.

Since this theorem concerns the spatial metric tensor that is naturally induced in Σ_x, it points out a characteristic of the differential operation of transverse covariant derivative that underlines the character of the fundamental spatial tensor.

It is somewhat superfluous to add that the invertibility of two successive transverse covariant derivatives does not persist, in general.

11. – *Formal projection of the CHRISTOFFEL symbols onto* Σ_x. The technique of spatial projection that was applied in the preceding section to tensorial entities may also be more formally applied to other geometrical entities that are non-tensorial, such as the Riemannian connection on V_{n+1}, which is represented in local coordinates by the CHRISTOFFEL symbols of the first and second type. It is truly noteworthy that the result of such a formal projection is, as one easily verifies, the following properties of the modified CHRISTOFFEL symbols, as defined by formula (8.3):

(11.1)
$$\mathfrak{P}_{\Sigma\Sigma\Sigma}(|ij,h|) = |\widetilde{ij,h}|$$
$$\mathfrak{P}_{\Sigma\Sigma\Sigma}\left(\begin{Bmatrix} h \\ i\ j \end{Bmatrix}\right) = \begin{Bmatrix} \widetilde{h} \\ i\ j \end{Bmatrix}.$$

Beyond its noteworthy geometric content, (11.1) has an obvious utility in everyday practice when one is performing calculations.

III. – "TIMELIKE" CONGRUENCES AND THEIR FIRST ORDER DIFFERENTIAL ELEMENTS.

As we said at the beginning of the present work (no. 1), the technique of spatial or temporal projection and the systematic use of the ordinary or covariant transverse derivative finds useful application in the study of congruences in V_{n+1}. We fix our attention on timelike congruences, which have considerable importance in a manifold with a normal hyperbolic metric. The adaptation of the considerations that follow to the case of a generic congruence does not present any difficulties.

Let C_0 be an arbitrary "timelike" congruence in V_{n+1} that assumes the same sense as the principal reference congruence and is adapted to a local coordinate system x^i that is also adapted to the same congruence, such that the curves x^0 = var. are identified with the curves of C_0 . We again make available, if needed, an internal coordinate change of the same congruence.

The congruence C_0 uniquely determines the unitary vector field $\gamma(x)$ that is tangent to all of its curves. We therefore begin the local examination of C_0 with the local examination of the tensors that one obtains from $\gamma(x)$ by derivations of first order, that is, the following three tensors: the antisymmetric tensor $\Omega_{ij} = \partial_i \gamma_j - \partial_j \gamma_i \equiv \nabla_i \gamma_j - \nabla_j \gamma_i$, which we define as the *vorticity tensor* of the congruence, the asymmetric tensor $\nabla_i \gamma_j$, which is the covariant derivative of γ, and the symmetric tensor $K_{ij} = \nabla_i \gamma_j + \nabla_j \gamma_i$, which is called the *Killing tensor* of C_0 .

We now perform the natural decomposition and consider each of its four projections $\mathfrak{P}_{\Sigma\Sigma}$, $\mathfrak{P}_{\Sigma\Theta}$, $\mathfrak{P}_{\Theta\Sigma}$, $\mathfrak{P}_{\Theta\Theta}$. We then examine the geometrical significance of each.

12. – *The four natural projections of the vorticity tensor.* – Define the *vorticity tensor* of a spacetime congruence C_0, which then defines a unitary vector field $\gamma(x)$, to be the antisymmetric double tensor:

$$\Omega_{ij} = \partial_i \gamma_j - \partial_j \gamma_i \equiv \nabla_i \gamma_j - \nabla_j \gamma_i . \tag{12.1}$$

It has a well-known fundamental importance in the relativistic dynamics of fluids (cf., LICHNEROWICZ [13], SYNGE [17], [18]), and, more generally, in the geometry of congruences. We now examine the four natural projections, and will see their noteworthy geometrical significance later on.

$\Sigma\Sigma$ *projection*. – One has:

$$\mathfrak{P}_{\Sigma\Sigma}(\Omega_{ij}) \equiv \tilde{\Omega}_{ij} \tag{12.2}$$
$$= \gamma^0 \partial_i \frac{\gamma_j}{\gamma_0} - \partial_j \gamma_i + \frac{\gamma_j}{\gamma_0} \partial_0 \gamma_i + \gamma_i \gamma_0 \gamma^0 \partial_0 \frac{\gamma_j}{\gamma_0} - \gamma_i \gamma^0 \partial_j \gamma_0 + \gamma_i \gamma^0 \frac{\gamma_j}{\gamma_0} \partial_0 \gamma_0 .$$

Taking into account that $\gamma_0 \gamma^0 = -1$, the association of the second term with the fifth one and the third one with the sixth gives:

(12.3) $$\mathfrak{P}_{\Sigma\Sigma}(\Omega_{ij}) \equiv \tilde{\Omega}_{ij} = \gamma^0 \partial_i \frac{\gamma_j}{\gamma_0} - \gamma_i \gamma^0 \partial_0 \frac{\gamma_j}{\gamma_0} - \gamma_i \partial_0 \frac{\gamma_j}{\gamma_0} + \gamma_j \partial_0 \frac{\gamma_i}{\gamma_0}.$$

Finally, associating the first term with the second one and the third one with the fourth one, one finds:

(12.4) $$\mathfrak{P}_{\Sigma\Sigma}(\Omega_{ij}) \equiv \tilde{\Omega}_{ij} = \gamma_0 \left(\tilde{\partial}_i \frac{\gamma_j}{\gamma_0} - \tilde{\partial}_j \frac{\gamma_i}{\gamma_0} \right).$$

The spatial tensor thus obtained (one immediately checks the annihilation of the covariant components in which the index figures at least once) is an antisymmetric tensor, like Ω_{ij}, which is formally obtained when one substitutes $\frac{\gamma_i}{\gamma_0}$ for γ_i and the operation $\tilde{\partial}_i$ for ∂_i, and multiplies the result by γ_0. This tensor is called the *spatial vorticity tensor*, *or transverse rotation*, of the congruence C_0 (or the vector field $\gamma(x)$).

$\Sigma\Theta$ *projection*. – One has:

$$\mathfrak{P}_{\Sigma\Theta}(\Omega_{ij}) = - \gamma_i^{\ r} \gamma_j \gamma^s \Omega_{rs} = \gamma^0 \Omega_{0i} \gamma_j ,$$

or furthermore, observing that:

$$\gamma^0 \Omega_{0i} = \gamma^r \Omega_{ri} = \gamma^r (\nabla_r \gamma_i - \nabla_i \gamma_r) = \gamma^r \nabla_r \gamma_i = c_i ,$$

in which c_i indicates the *curvature vector* of the curve x^0:

(12.5) $$\mathfrak{P}_{\Sigma\Theta}(\Omega_{ij}) = c_i \gamma_j .$$

Put into this form, one recognizes that the spatial vector, together with the spatial tensor $\tilde{\Omega}_{ij}$ that was previously considered, specifies the vorticity tensor Ω_{ij} and can be identified with the curvature vector of the curve x^0, and is obviously normal to that curve.

$\Theta\Sigma$ *projection*. – In an analogous manner, one recognizes that, in harmony with what was said in no. 5 for a generic antisymmetric tensor, one has:

(12.6) $$\mathfrak{P}_{\Theta\Sigma}(\Omega_{ij}) = - \gamma_i c_j .$$

$\Theta\Theta$ *projection*. – For an arbitrary antisymmetric tensor, such a projection is identically null:

(12.7) $$\mathfrak{P}_{\Theta\Theta}(\Omega_{ij}) = 0 .$$

Summarizing the preceding natural decomposition of the spacetime vorticity tensor Ω_{ij}, we have:

(12.8) $$\Omega_{ij} = \tilde{\Omega}_{ij} + c_i\,\gamma_j - \gamma_i\,c_j\,,$$

in which $\tilde{\Omega}_{ij}$ and c_i indicate the spatial vorticity tensor and the curvature vector of the congruence C_0, respectively.

13. – *The four natural projections of the tensor* $\nabla_i\gamma_j$. – We commence with the observation that the second index of the tensor $\nabla_i\gamma_j$ is a purely spatial index. In fact, taking into account that $-g_{00} = \gamma_0^2$, one has (for any value of the index i):

(13.1) $$\begin{aligned}\nabla_i\gamma_0 &= \partial_i\,\gamma_0 - (i0,\,l)\gamma^l = \partial_i\,\gamma_0 - (i0,\,0)\gamma^0 \\ &= \partial_i\,\gamma_0 - \partial_i\,\gamma_0 = 0\,.\end{aligned}$$

(Moreover, it is quicker if one observes that when γ has a constant norm the resulting vector $\nabla_i\gamma_j\,dx^i$ is orthogonal to γ *for any* dx^i; it follows that the index j in question is purely spatial). From this observation, it follows that in order to obtain the ΣΣ or ΘΣ projections of the tensor $\nabla_i\gamma_j$, it suffices to operate on its first index (with the projectors γ_{ir} or $-\gamma_i\,\gamma_r$, respectively) to show that the resulting ΣΘ and ΘΘ projections are automatically null.

ΣΣ – *projection*. Operating, as we explained, only on the first index, one then has:

(13.2) $$\begin{aligned}\mathfrak{P}_{\Sigma\Sigma}(\nabla_i\gamma_j) &= \gamma_i^h\nabla_h\gamma_j = \nabla_i\gamma_j + \gamma_i\,\gamma^0\nabla_0\gamma_j\,, \\ &= \partial_i\,\gamma_j - (ij,\,0)\gamma^0 + \gamma_i\,\gamma^0\partial_0\gamma_j - \gamma_i\,\gamma^0\,(0j,\,0)\,\gamma^0\,.\end{aligned}$$

Associating the first and third terms, which collectively define $\tilde{\partial}_i\gamma_j$, and developing the CHRISTOFFEL symbols, while also taking into account that $\gamma_0\,\gamma^0 = -1$, $g_{i0} = -\gamma_i\,\gamma_0$, $g_{00} = -\gamma_0^2$, one finds:

$$\mathfrak{P}_{\Sigma\Sigma}(\nabla_i\gamma_j) = \tilde{\partial}_i\gamma_j + \frac{1}{2}\,(\gamma_0\,\partial_i\,\gamma_j + \gamma_j\,\partial_i\,\gamma_0 + \gamma_0\,\partial_j\,\gamma_i + \partial_0\,g_{ij} - \gamma_i\,\partial_j\,\gamma_0)\,\gamma^0.$$

Compare the terms in parentheses with the three terms $\partial_0(\gamma_i\,\gamma_j) - \gamma_i\,\partial_0\,\gamma_j - \gamma_j\,\partial_0\,\gamma_i$, whose sum is obviously null. The first of them combines with the fourth one in parentheses to give simply $\partial_0\gamma_{ij}$. What then remains is:

$$\mathfrak{P}_{\Sigma\Sigma}(\nabla_i\gamma_j) = \tilde{\partial}_i\gamma_j + \frac{1}{2}\,(\gamma_0\,\partial_i\,\gamma_j + \gamma_j\,\partial_i\,\gamma_0 + \gamma_0\,\partial_j\,\gamma_i - \gamma_i\,\partial_0\,\gamma_j - \gamma_j\,\partial_0\,\gamma_i - \gamma_i\,\partial_j\,\gamma_0 + \partial_0\,\gamma_{ij})\,\gamma^0.$$

Between the parentheses, the first and the fourth terms give $\gamma_0\tilde{\partial}_i\gamma_j$, the third and the fifth ones give $\gamma_0\tilde{\partial}_j\gamma_i$, and finally, the second one and the sixth one give $\gamma_j\partial_i\gamma_0 - \gamma_i\tilde{\partial}_j\gamma_0$ (in fact, $\gamma_j\,(\partial_i\,\gamma_0 + \gamma_i\,\gamma^0\partial_0\,\gamma_0) - \gamma_i\,(\partial_j\,\gamma_0 + \gamma_j\,\gamma^0\partial_0\,\gamma_0) \equiv \gamma_j\,\partial_i\,\gamma_0 - \gamma_i\,\partial_j\,\gamma_0$). One needs only to remove the parentheses:

$$\mathfrak{P}_{\Sigma\Sigma}(\nabla_i\gamma_j) = \frac{1}{2}\,\tilde{\partial}_i\gamma_j - \frac{1}{2}\,\tilde{\partial}_j\gamma_i - \frac{1}{2\gamma_0}\gamma_j\tilde{\partial}_i\gamma_0 + \frac{1}{2\gamma_0}\gamma_i\tilde{\partial}_j\gamma_0 + \frac{1}{2}\gamma^0\partial_0\gamma_{ij}\,.$$

Thus, combining the first term with the third one and the second one with the fourth one, one finally arrives at:

(13.3) $$\mathfrak{P}_{\Sigma\Sigma}(\nabla_i\gamma_j) = \frac{1}{2}\gamma_0\left(\tilde{\partial}_j\frac{\gamma_i}{\gamma_0} - \tilde{\partial}_i\frac{\gamma_j}{\gamma_0}\right) + \frac{1}{2}\gamma^0\partial_0\gamma_{ij} = \frac{1}{2}\tilde{\Omega}_{ij} + \frac{1}{2}\gamma^h\,\partial_h\,\gamma_{ij}\,,$$

an expression in which one sees the antisymmetric tensor $\tilde{\Omega}_{ij}$ that we called the spatial vorticity tensor in the preceding section, and a symmetric spatial tensor $\frac{1}{2}\gamma^0\,\partial_0\,\gamma_{ij} \equiv \frac{1}{2}\gamma^h\,\partial_h\,\gamma_{ij}$, that we will examine a little later on (no. 14).

Simply as a check, observe that (13.3) allows one to immediately calculate the ΣΣ projection of the tensor $\Omega_{ij} = \partial_i\,\gamma_j - \partial_j\,\gamma_i = \nabla_i\,\gamma_j - \nabla_j\,\gamma_i$, and one obtains:

$$\mathfrak{P}_{\Sigma\Sigma}(\Omega_{ij}) = \frac{1}{2}(\tilde{\Omega}_{ij} - \tilde{\Omega}_{ji}) = \tilde{\Omega}_{ij}\,,$$

in accord with (12.4).

ΣΘ-*projection*. As we already said, this tensor is null, due to the purely spatial character of the second index:

(13.4) $$\mathfrak{P}_{\Sigma\Theta}(\nabla_i\gamma_j) = 0\,.$$

ΘΣ-*projection*. Operating only on the first index, given the purely spatial character of the second one, one obtains:

$$\begin{aligned}\mathfrak{P}_{\Theta\Sigma}(\nabla_i\gamma_j) &= -\,\gamma_i\,\gamma^r\,\nabla_r\,\gamma^j = -\,\gamma_i\,\gamma^0\,[\partial_0\,\gamma_j - (0j,\,h)\,\gamma^h\,] \\ &= -\,\gamma_i\,\gamma^0\,[\partial_0\,\gamma_j - \partial_i\,\gamma_0\,] \\ &= -\,\gamma_i\,\gamma^r\,\Omega_{rj}\,.\end{aligned}$$

By definition, this results in:

(13.5) $$\begin{aligned}\mathfrak{P}_{\Theta\Sigma}(\nabla_i\gamma_j) &= -\,\gamma_i\,\gamma^r\,\Omega_{rj} \\ &= -\,\gamma_i\,c_j\,,\end{aligned}$$

which coincides with the ΘΣ-projection of the tensor Ω_{ij}.

ΘΘ-*projection*. As we already said, this tensor will also be null, due to the purely spatial character of the second index in $\nabla_i\gamma_j$.

The natural decomposition of the tensor $\nabla_i \gamma_j$ may then be summarized as:

(13.6)
$$\nabla_i \gamma_j = \frac{1}{2}\tilde{\Omega}_{ij} + \frac{1}{2}\gamma^h \partial_h \gamma_{ij} - \gamma_i \gamma^r \Omega_{rj} .$$
$$= \frac{1}{2}(\tilde{\Omega}_{ij} + \gamma^h \partial_h \gamma_{ij}) - \gamma_i c_j .$$

14. – *Natural projection of the Killing tensor:* $K_{ij} \equiv \nabla_i \gamma_j + \nabla_j \gamma_i$. – Another tensor of note that relates to a general congruence C_0, and which is more important, is the symmetric double tensor $K_{ij} = \nabla_i \gamma_j + \nabla_j \gamma_i$, to which it seems quite appropriate to give the name of *Killing tensor* (cf., [10], [6]). As is well known, the identical annihilation of the tensor K_{ij} is the necessary and sufficient condition for the congruence C_0 to define a (one-parameter) group of isometries in V_{n+1} .

The natural projections of this tensor are obtained immediately like the analogous projections of the tensor $\nabla_i \gamma_j$ (cf., the preceding section). Here is what one obtains:

ΣΣ-*projection*. If we apply the results of the preceding section to the two indices of K_{ij} then the formal rules of projection onto Σ give:

(14.1)
$$\mathfrak{P}_{\Sigma\Sigma}(\nabla_i \gamma_j + \nabla_j \gamma_i) \equiv \tilde{K}_{ij} = (\partial_i^h + \gamma_i \gamma^h)\, \nabla_h \gamma_j + (\partial_j^h + \gamma_j \gamma^h)\, \nabla_h \gamma_i =$$
$$= \nabla_i \gamma_j + \nabla_j \gamma_i + \gamma_i \gamma^h \nabla_h \gamma_j + \gamma_j \gamma^h)\, \nabla_h \gamma_i ,$$

and in the last expression one recognizes the symmetric spatial tensor that was introduced in relativity by BORN at the end of 1909 (cf., BORN [1], SYNGE [17], SALZMANN and TAUB [6], and many other authors). Conforming to our general convention we also denote that tensor by $\tilde{K}_{ij}$ and now call it the *Born spatial tensor.* Its vanishing identically, which is less restrictive than the vanishing of the entire Killing tensor, expresses that the motion of the individual fluid of the congruence C_0 along the x^0 curves is *rigid*, according to BORN ([1]), therefore the the spatial distance between two arbitrary infinitely close x^0 lines is constant (cf., SYNGE [17], pp. 36). In the case where it is not identically null, it can be interpreted as the deformation tensor of the fluid C_0, and we should say that such a tensor rests on tentative foundations (HERGLOTZ [9], SYNGE [18], BENVENUTI ([1])) for a relativistic theory of elasticity.

One now comes to a more precise explanation for $\mathfrak{P}_{\Sigma\Sigma}(K_{ij})$ if one immediately recognizes, on the basis of (13.3), and the antisymmetry of the spatial vorticity tensor Ω_{ij} the simple expression:

(14.2)
$$\mathfrak{P}_{\Sigma\Sigma}(K_{ij}) \equiv \tilde{K}_{ij} = \gamma^h \partial_h \gamma_{ij} .$$

In that expression, one observes how the identical vanishing of the BORN tensor is substantially equivalent to the independence of the spatial metric tensor γ_{ij} from x^0 (which is not the same thing as stationarity, which gives the independence of the spacetime metric g_{ij} from x^0).

([1]) P. BENVENUTO, laurea thesis (1956).

As for the other natural projections, one has:

$\Sigma\Theta$-*projection*:

(14.3) $$\mathfrak{P}_{\Sigma\Theta}(K_{ij}) = - c_i \, \gamma_j \, .$$

$\Theta\Sigma$-*projection*:

(14.4) $$\mathfrak{P}_{\Theta\Sigma}(K_{ij}) = -\gamma_i \, c_j \, .$$

$\Theta\Theta$-*projection*:

(14.5) $$\mathfrak{P}_{\Theta\Theta}(K_{ij}) = 0 \, .$$

As in the preceding section, c_i represents the curvature vector (which is orthogonal to γ) of the x^0 curves.

The natural decomposition of the KILLING tensor may be specified definitively like this:

(14.6) $$K_{ij} = \tilde{K}_{ij} - c_i \, \gamma_j - \gamma_i \, c_j \, .$$

15. *Some differential properties of the congruence* C_0. The tensors that were considered in the preceding section, or some of their natural projections, contain noteworthy properties of the congruence C_0 that concern the way that γ varies from one point to the other in V_{n+1}. We treat the properties in question, but it seems to us that the actual formulation gives them a more organic systematization and establishes a natural connection between them.

Normal congruence. A necessary and sufficient condition for C_0 to be a *normal* congruence is the identical annihilation of its spatial vorticity:

(15.1) $$\mathfrak{P}_{\Sigma\Sigma}(\Omega_{ij}) \equiv \tilde{\Omega}_{ij} \equiv \gamma_0 \left(\tilde{\partial}_i \frac{\gamma_j}{\gamma_0} - \tilde{\partial}_j \frac{\gamma_i}{\gamma_0} \right) = 0 \, .$$

This theorem, in an essentially equivalent form, is due to WEYSSENHOFF (cf., [19] and [14], pp. 250). To understand its validity, remember that traditionally (cf., LEVI-CIVITA, [11], pp. 281) the *normality* of a congruence is expressed by means of the identical annihilation of the triple tensor:

(15.2) $$\gamma_i \, \Omega_{jk} + \gamma_j \, \Omega_{ki} + \gamma_k \, \Omega_{ij} = 0 \, ,$$

with the usual meanings for γ_i and Ω_{ij}. It is now simple to recognize the complete equivalence of the conditions (15.1) and (15.2). In fact, if one finally decomposes the tensor Ω_{ij} in (15.2) into its natural components (cf., (12.8)) then one obtains:

$$\gamma_i \, (\tilde{\Omega}_{jk} + c_j \gamma_k - \gamma_j c_k) + \gamma_j \, (\tilde{\Omega}_{ki} + c_k \gamma_i - \gamma_k c_i) + \gamma_k \, (\tilde{\Omega}_{ij} + c_i \gamma_j - \gamma_i c_j) = 0,$$

and thus (15.2) appears to be completely equivalent to the condition:

$$\gamma_i \tilde{\Omega}_{jk} + \gamma_j \tilde{\Omega}_{ki} + \gamma_k \tilde{\Omega}_{ij} = 0 \,. \tag{15.3}$$

If one multiplies both sides of this by γ^i (γ^j or γ^k, resp.) and saturates the mutual index *i* (*j* or *k*, resp.) then it follows that $\tilde{\Omega}_{jk} = 0$, which establishes (15.1). Conversely, from (15.1) it follows immediately that we have (15.3) and thus, the classical condition (15.2).

Geodetic congruence. A necessary and sufficient condition for the congruence C_0 to be a geodetic congruence, that is, for the curves that constitute it to be geodetic, is the nullity of the ΣΘ-projection (and thus the ΘΣ-projection, as well) of its vorticity tensor Ω_{ij}:

$$\mathfrak{P}_{\Sigma\Theta}(\Omega_{ij}) = 0 \,. \tag{15.4}$$

In fact, from (12.5), it follows that $\mathfrak{P}_{\Sigma\Theta}(\Omega_{ij})$ is annulled when and only when c_i is annulled, which is the curvature vector of the x^0 curve.

Normal geodetic congruence. If we combine the preceding two observations then we deduce that the necessary and sufficient condition for the congruence C_0 to be both normal and geodetic is that one identically annihilate its vorticity vector:

$$\Omega_{ij} = 0 \,. \tag{15.5}$$

On the other hand, this condition is satisfied when the line field γ is a gradient field: $\gamma =$ grad f. We are obviously treating a very particular gradient field, one for which any two arbitrary equipotential surfaces are *parallel* surfaces (that is, that the geodetics that are normal to one of them are also normal to the other one, the connecting curve segments between the two surfaces all being geodetics of the same length).

Congruence defining rigid motion. It follows from what was said in no. 14 that, following BORN, the necessary and sufficient condition for the continuous motion that is defined by the congruence C_0 to be rigid is that the distance between two arbitrarily infinitely close x^0 curves, as measured along a common normal, remains constant along those same curves and that the BORN tensor must be null:

$$\mathfrak{P}_{\Sigma\Sigma}(K_{ij}) \equiv \tilde{K}_{ij} \equiv \gamma^0 \, \partial_0 \, \gamma_{ij} = 0 \,. \tag{15.6}$$

The identical annihilation of the ΣΘ-projection (or the ΘΣ-projection) of K_{ij} characterizes the geodetic congruence, just like the identical annihilation of $\tilde{\Omega}_{ij}$.

Rigid geodetic congruence. On the basis of the preceding remarks, these congruences are characterized by the identical annihilation of the entire KILLING tensor:

(15.7) $$K_{ij} \equiv \nabla_i \gamma_j + \nabla_j \gamma_i = 0 \; .$$

On the other hand, as is well known, the condition (15.7) characterizes those congruences that define a group of isometries in V_{n+1} (cf., EISENHART [6], LICHNEROWICZ [13]). We say that the identical annihilation of the KILLING tensor characterizes rigid geodetic motions.

Rigid uniform translatory motion. If we combine the preceding considerations then we see that the identical annihilation of the tensor $\nabla_i \gamma_j$, which implies both the annihilation of K_{ij} and Ω_{ij} , characterizes those motions that are, at the same time, irrotational, rigid, and geodetic:

(15.8) $$\nabla_i \gamma_j = 0 \; .$$

One calls them *uniform translatory motions.*

The preceding seems, to me, to show that the criterion of the natural decomposition of the principal differential tensors that are associated with a timelike congruence suffice to provide a systematic classification of the possible fluid motions. Although I have not arrived at any new results, this more organic systematization does not seem devoid of interest to me.

IV. – ORDINARY AND COVARIANT TRANSVERSE DERIVATION OF THE PARTICULAR TENSOR FIELDS THAT INVOLVE THE DIFFERENTIAL TENSORS THAT CHARACTERIZE THE CONGRUENCE OF REFERENCE.

Recall the explicit calculation that we began in the preceding section 8 of the transverse covariant derivative of the various purely spatial tensor fields, whose explicit expressions acquired (snellezza?) an expressiveness by means of the intervention of the differential tensors that characterize principal reference congruence. In fact, such an explicit intervention allows one to distinguish the expressions that are due to the reference congruence from the ones that are due to the tensorial fields that are derived from it.

As one can show, the purely formal rules that relate to the transverse derivation are therefore subordinate to the choice of principal reference congruence. We give an immediate example.

16. *Conditions for the invertibility of two successive (ordinary) transverse derivations.* We said at the end of no. 7 that two successive transverse derivations are not generally invertible. We now establish under what conditions invertibility exists.

Given an arbitrary numerical function $f(x^i)$ (which may or may not be a scalar), consider – formally – the second transverse derivative $\tilde{\partial}_\beta \tilde{\partial}_\alpha f$ that it defines:

(16.1) $$\tilde{\partial}_\beta \tilde{\partial}_\alpha f = \left(\partial_\beta - \frac{\gamma_\beta}{\gamma_0} \partial_0 \right) \left(\partial_\alpha f - \frac{\gamma_\alpha}{\gamma_0} \partial_0 f \right) =$$

$$=\partial_\beta\partial_\alpha f-\frac{\gamma_\beta}{\gamma_0}\partial_0\partial_\alpha f-\left(\partial_\beta\frac{\gamma_\alpha}{\gamma_0}-\frac{\gamma_\beta}{\gamma_0}\partial_0\frac{\gamma_\alpha}{\gamma_0}\right)\partial_0 f-\frac{\gamma_\alpha}{\gamma_0}\partial_\beta\partial_0 f+\frac{\gamma_\beta\gamma_\alpha}{\gamma_0^2}\partial_0^2 f\ .$$

Analogously, we invert the order of the two derivations and obtain:

(16.2) $$\tilde{\partial}_\alpha\tilde{\partial}_\beta f =$$

$$=\partial_\alpha\partial_\beta f-\frac{\gamma_\alpha}{\gamma_0}\partial_0\partial_\beta f-\left(\partial_\alpha\frac{\gamma_\beta}{\gamma_0}-\frac{\gamma_\alpha}{\gamma_0}\partial_0\frac{\gamma_\beta}{\gamma_0}\right)\partial_0 f-\frac{\gamma_\beta}{\gamma_0}\partial_\alpha\partial_0 f+\frac{\gamma_\beta\gamma_\alpha}{\gamma_0^2}\partial_0^2 f\ .$$

From the difference of the two expressions, one obtains:

(16.3) $$(\tilde{\partial}_\beta\tilde{\partial}_\alpha-\tilde{\partial}_\alpha\tilde{\partial}_\beta)f=\left(\tilde{\partial}_\alpha\frac{\gamma_\beta}{\gamma_0}\partial_0-\tilde{\partial}_\beta\frac{\gamma_\alpha}{\gamma_0}\right)\partial_0 f\ .$$

From this, one concludes that the order of the two successive transverse derivations will not influence the result *if either the reference motion is (spatially) irrotational* or else *if $f(x)$ does not depend upon x^0* (f is stationary along any reference curve x^0 = var.).

17. *Condition for the invertibility of $\tilde{\partial}_\alpha$ and ∂_0.* If one successively apply the two operators $\tilde{\partial}_\alpha$ and ∂_0 to a function $f(x^i)$ one has:

(17.1) $$\tilde{\partial}_\alpha\partial_0 f=\partial_\alpha\partial_0 f-\frac{\gamma_\alpha}{\gamma_0}\partial_0^2 f\ .$$

Applying the two operators in the reverse order, one has instead:

(17.2) $$\tilde{\partial}_\alpha\partial_0 f=\partial_0\partial_\alpha f-\partial_0\left(\frac{\gamma_\alpha}{\gamma_0}\right)\partial_0 f-\frac{\gamma_\alpha}{\gamma_0}\partial_0^2 f\ .$$

One obtains the difference:

(17.3) $$\tilde{\partial}_\alpha\partial_0 f-\tilde{\partial}_\alpha\partial_0 f=\partial_0\left(\frac{\gamma_\alpha}{\gamma_0}\right)\partial_0 f\ .$$

We conclude that the differential operators ∂_0 and $\tilde{\partial}_\alpha$ are commutable only if either f is stationary along C_0 (that is, it does not depend upon x^0), or else the "gravitational potential vector" $\frac{\gamma_\alpha}{\gamma_0}$ is stationary.

18. *Natural projections of $\nabla_i\tau_j$ ($\tau_j\in\Theta_x$).* Let $\tau_j=\tau\gamma_j$ be a generic purely temporal vector field. We consider the covariant derivative $\nabla_i\tau_j$ of it and calculate the four natural projections.

ΣΣ-projection:

(18.1) $$\mathfrak{P}_{\Sigma\Sigma}(\nabla_i \tau_j) = \gamma_i^r \gamma_j^s \partial_r \tau \cdot \gamma_s + \tau \mathfrak{P}_{\Sigma\Sigma}(\nabla_i \gamma_j) = \tau \mathfrak{P}_{\Sigma\Sigma}(\nabla_i \gamma_j) .$$

From this, taking into account (13.3), it follows that:

(18.2) $$\mathfrak{P}_{\Sigma\Sigma}(\nabla_i \tau_j) = \frac{1}{2}\tau(\gamma^h \partial_0 \gamma_{ij} + \tilde{\Omega}_{ij}) = \frac{1}{2}\tau(\tilde{K}_{ij} + \tilde{\Omega}_{ij}) .$$

We observe how the projection that we just calculated depends only on the vector field τ_i for the reference congruence. If it is, for example, both rigid and irrotational then the projection (18.2) is annulled identically, no matter what the purely temporal vector field τ_j.

ΣΘ-projection:

(18.2) $$\mathfrak{P}_{\Sigma\Theta}(\nabla_i \tau_j) = -\gamma_i^r \gamma_j \gamma^s \partial_r \tau \cdot \gamma_s + \tau \mathfrak{P}_{\Sigma\Theta}(\nabla_i \gamma_j) = \tilde{\partial}_i \tau \bullet \gamma_j + \tau \mathfrak{P}_{\Sigma\Theta}(\nabla_i \gamma_j) .$$

If we take (13.4) into account then we conclude:

(18.4) $$\mathfrak{P}_{\Sigma\Theta}(\nabla_i \tau_j) = \tilde{\partial}_i \tau \bullet \gamma_j .$$

ΘΣ-projection:

(18.5) $$\mathfrak{P}_{\Theta\Sigma}(\nabla_i \tau_j) = \gamma_i \gamma^r \gamma_j^s \partial_r \tau \cdot \gamma_s + \tau \mathfrak{P}_{\Theta\Sigma}(\nabla_i \gamma_j) = \tau \mathfrak{P}_{\Theta\Sigma}(\nabla_i \gamma_j) ,$$

and, recalling (13.5):

(18.6) $$\mathfrak{P}_{\Theta\Sigma}(\nabla_i \tau_j) = -\tau \gamma_i c_j .$$

Such a projection is annulled identically for any purely temporal vector field τ_j when C_0 is a geodetic congruence.

ΘΘ-projection:

(18.7) $$\mathfrak{P}_{\Theta\Theta}(\nabla_i \tau_j) = \gamma_i \gamma^r \gamma_j \gamma^s \partial_r \tau \cdot \gamma_s + \tau \mathfrak{P}_{\Theta\Theta}(\nabla_i \gamma_j) .$$

This permits us to conclude:

(18.8) $$\mathfrak{P}_{\Theta\Theta}(\nabla_i \tau_j) = -\gamma^0 \partial_0 \tau \cdot \gamma_i \gamma_j = -\gamma^r \partial_r \tau \cdot \gamma_i \gamma_j .$$

$\mathfrak{P}_{\Theta\Theta}(\nabla_i \tau_j)$ is then annulled when the field τ_j is stationary, that is, when τ_j has constant modulus along any curve of C_0 .

19. *Natural projection of* $\nabla_i s_{jm}$ $(s_{jm} \in \Sigma_x \otimes \Sigma_x)$. We next give the natural decomposition of the triple tensor $\nabla_i s_{jm}$, where s_{jm} is a generic purely spatial vector field ($s_{j0} = 0$, $s_{0m} = 0$). For brevity, we omit the lengthy calculations.

Here are the eight natural projections of $\nabla_i s_{jm}$:

(20.1) $$\mathfrak{P}_{\Sigma\Sigma\Sigma}(\nabla_i s_{jm}) = \tilde{\nabla}^*_i s_{jm} = \tilde{\partial}_i s_{jm} - \begin{Bmatrix} h \\ i\,j \end{Bmatrix} s_{hm} - \begin{Bmatrix} h \\ i\,m \end{Bmatrix} s_{jh} .$$

(20.2) $$\mathfrak{P}_{\Sigma\Sigma\Theta}(\nabla_i s_{jm}) = \frac{1}{2}\tilde{\Omega}_{ik} s_j{}^k \gamma_m + \frac{1}{2}\gamma^0 \partial_0 \gamma_{ik} s_j{}^k \gamma_m .$$

(20.3) $$\mathfrak{P}_{\Sigma\Theta\Sigma}(\nabla_i s_{jm}) = \frac{1}{2}\tilde{\Omega}_{ik} \gamma_j s^k{}_m + \frac{1}{2}\gamma^0 \partial_0 \gamma_{ik} s^k{}_m \gamma_j .$$

(20.4) $$\mathfrak{P}_{\Theta\Sigma\Sigma}(\nabla_i s_{jm}) = -\gamma_i \gamma^0 \partial_0 s_{jm} + 3\gamma_i \gamma_j c_h s^h{}_m + 3\gamma_i \gamma_m c_k s_j{}^k + \\ + \frac{1}{2}\gamma_i \tilde{K}_{jh} s^h{}_m + \frac{1}{2}\gamma_i \tilde{K}_{mk} s_j{}^k + \gamma_i \tilde{\Omega}_{hj} s^h{}_m + \gamma_i \tilde{\Omega}_{km} s_j{}^k .$$

(20.5) $$\mathfrak{P}_{\Theta\Theta\Sigma}(\nabla_i s_{jm}) = -\gamma_i \gamma_j \gamma^r \Omega_{rh} s^h{}_m = -\gamma_i \gamma_j c_h s^h{}_m .$$

(20.6) $$\mathfrak{P}_{\Theta\Sigma\Theta}(\nabla_i s_{jm}) = -\gamma_i c_h s_j{}^h \gamma_m .$$

(20.7) $$\mathfrak{P}_{\Sigma\Theta\Theta}(\nabla_i s_{jm}) = 0 .$$

(20.8) $$\mathfrak{P}_{\Theta\Theta\Theta}(\nabla_i s_{jm}) = 0 .$$

21. – *Concluding considerations.* – The preceding systematic calculations had the objective of carrying out, in an almost automatic way, the following operations:

a) Decomposing an arbitrary vector or tensor in a canonical (natural) way.

b) Calculating the natural projections of the covariant derivative of an arbitrary vector or double tensor field, and then applying the same decomposition to the starting vector or tensor. In fact, this (preventiva?) operation permits one to always (ricondursi?) in the case in which the covariant derivative operates on a purely spatial or purely temporal vector, or a purely spatial double tensor.

The reader will have no difficulty in extending the preceding calculations to the case in which the operation of covariant derivation operates on tensors of order higher than the second.

One may regard the foregoing as an illustration of a technique that, by virtue of its invariant character under more general internal coordinate changes and its formal simplicity, can be of service either to Riemannian geometry, or, more specifically, to general relativity.

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