Excerpted from E. Cesàro, *Lezioni di geometria intrinseca*, from the author-publisher, Naples, 1896. "Sulle equazioni della elasticità negli iperspazii," pp. 257-260.

On the equations of elasticity in hyperspaces

The calculations that **Beltrami** carried out in the paper "Sulle equazioni generali dell'elasticità" can also be done with a certain expediency, and without loss of elegance, for a curved space of as many dimensions as one desires by making use of the notations that we adopted in the first chapter. First, recall (XVII, 6, 7) that if $u_0 = 1$ then the coefficients of elongation and unitary solid dilatation will be given by the formulas:

$$\theta_i = u_{ii} = \frac{\partial u_i}{\partial s_i} + \sum \mathcal{G}_{ij} u_j , \qquad \Theta = \sum \theta_{ii} = \sum \left(\frac{\partial}{\partial s_i} + \mathcal{G}_i \right) u_j .$$

In addition, one has to consider the mutual sliding θ_{ij} of the linear coordinate elements, and the doubled components ϑ_{ij} of the rotation of the medium. Their expressions can be obtained from the formulas:

$$\tfrac{1}{2}(\theta_{ij} + \vartheta_{ij}) = u_{ij} = \frac{\partial u_i}{\partial s_j} - \mathcal{G}_{ji} u_j , \qquad \tfrac{1}{2}(\theta_{ij} - \vartheta_{ij}) = u_{ji} = \frac{\partial u_j}{\partial s_i} - \mathcal{G}_{ij} u_j , \tag{1}$$

which will reduce to just one [XVII, form. (15)], in substance, if one observes that:

$$\theta_{ij} = \theta_{ji} , \qquad \vartheta_{ij} = -\vartheta_{ji} .$$

Given that, when one assumes that:

$$-\tfrac{1}{2}\left(A\Theta^2 + B \sum \vartheta_{ij}^2 \right) \tag{2}$$

is the only effective part of the potential in the formation of the indefinite equations, one will arrive at the equations

$$X_i + A \frac{\partial \Theta}{\partial s_i} + B \sum \left(\frac{\partial}{\partial s_j} + \mathcal{G}_j - \mathcal{G}_{ij} \right) \vartheta_{ij} + 2B\, a_i = 0 \tag{3}$$

by the usual process, which are free of the last term on the left-hand side. That term is the one that one needs to calculate in order for (3) to be the general equations of elasticity for isotropic media in any curved space or hyperspace if one omits the variations of the isotropy constants. Meanwhile, if one follows the process that **Beltrami** used to find formula (4) in his paper then one will obtain the equations:

$$X_i = \left(\frac{\partial}{\partial s_i} + \mathcal{G}_i\right) T_i - \sum \mathcal{G}_{ij} T_j + \sum_{(i)} \left(\frac{\partial}{\partial s_j} + \mathcal{G}_i + \mathcal{G}_{ij}\right) T_{ij}\,, \tag{4}$$

in place of our (3), in which T_i and T_{ij} are the tensions in the (linear and surface, resp.) coordinate elements. The index i that is placed in the final summation sign serves to remind one that one needs to exclude terms with distinct values of i and j from that corresponding sum. Formulas (4) are independent of the geometric nature of the space, as well as the physical constitution of the medium. When that peculiarity is introduced with the isotropy hypothesis, one will have:

$$T_i = -(A - 2B)\,\Theta - 2B\,\theta_i\,, \qquad T_{ij} = -B\,\theta_{ij}\,,$$

and equations (4) will become:

$$X_i + A\frac{\partial \Theta}{\partial s_i} - 2B\frac{\partial}{\partial s_i}\sum_{(i)} \theta_j + 2B\sum \mathcal{G}_{ij}(\theta_j - \vartheta_j) + B\sum_{(i)} \left(\frac{\partial}{\partial s_j} + \mathcal{G}_j + \mathcal{G}_{ij}\right)\theta_{ij} = 0.$$

Now, a comparison with (3) will give immediately, upon observing (1):

$$a_i = -\frac{\partial}{\partial s_i}\sum_{(i)} \theta_j + \sum \mathcal{G}_{ij}(\theta_j - \vartheta_j) + \sum \mathcal{G}_{ij}\left(\frac{\partial u_i}{\partial s_j} - \mathcal{G}_{ij} u_j\right) + \sum_{(i)} \left(\frac{\partial}{\partial s_j} + \mathcal{G}_j\right)\left(\frac{\partial u_j}{\partial s_i} - \mathcal{G}_{ij} u_i\right). \tag{5}$$

Meanwhile:

$$\frac{\partial}{\partial s_i}\sum_{(i)} \theta_j = \sum_{(i)} \frac{\partial^2 u_j}{\partial s_j\,\partial s_i} + \frac{\partial}{\partial s_i}\sum (\mathcal{G}_j - \mathcal{G}_{ij})\,u_j\,.$$

On the other hand, by virtue of the integrability conditions [XVII, form. (16)], one also has:

$$\sum_{(i)} \frac{\partial^2 u_j}{\partial s_j\,\partial s_i} = \sum_{(i)} \left(\frac{\partial}{\partial s_j} + \mathcal{G}_j\right)\frac{\partial u_j}{\partial s_i} + \mathcal{G}_i\,\frac{\partial u_i}{\partial s_i} - \sum \mathcal{G}_{ji}\,\frac{\partial u_j}{\partial s_j} - \sum (\mathcal{G}_i - \mathcal{G}_{ij})\,\frac{\partial u_j}{\partial s_i}\,;$$

hence:

$$\frac{\partial}{\partial s_i}\sum_{(i)} \theta_j = \sum_{(i)} \left(\frac{\partial}{\partial s_j} + \mathcal{G}_j\right)\frac{\partial u_j}{\partial s_i} + \mathcal{G}_i\,\frac{\partial u_i}{\partial s_i} - \sum \mathcal{G}_{ji}\,\frac{\partial u_j}{\partial s_j} + \sum u_j\,\frac{\partial}{\partial s_i}(\mathcal{G}_j - \mathcal{G}_{ij})\,.$$

If one substitutes this in (5) then one will get:

$$a_i = \mathcal{G}_j\left(\theta_i - \frac{\partial u_j}{\partial s_j}\right) - \sum \mathcal{G}_{ji}\left(\theta_j - \frac{\partial u_j}{\partial s_j}\right) - u_i \sum \left(\frac{\partial \mathcal{G}_{ij}}{\partial s_j} + \mathcal{G}_j \mathcal{G}_{ij}\right)$$

$$-\sum\left[\frac{\partial}{\partial s_i}(\mathcal{G}_j-\mathcal{G}_{ij})+\mathcal{G}_{ij}\mathcal{G}_{ji}\right]u_j\,.$$

That proves that a_i is a linear form in the u:

$$a_i=\sum a_{ij}\,u_j\,.$$

If one collects the terms that are multiplied by u_j then one will get:

$$a_{ij}=(\mathcal{G}_j-\mathcal{G}_{ij})\,\mathcal{G}_{ij}\;-\frac{\partial}{\partial s_i}(\mathcal{G}_j-\mathcal{G}_{ij})-\sum\mathcal{G}_{hi}\mathcal{G}_{hj} \tag{6}$$

for $i\neq j$. Moreover:

$$a_{ii}=-\frac{\partial\mathcal{G}_i}{\partial s_i}-\sum\left(\frac{\partial\mathcal{G}_i}{\partial s_j}+\mathcal{G}_j\mathcal{G}_{ij}+\mathcal{G}_{ij}^2\right). \tag{7}$$

Now, we can express the coefficients a by means of the functions Q. However, it is more convenient to introduce the *normal curvature* $\mathcal{N}$ and *geodetic torsion* $\mathcal{T}$, bearing mind the groups (γ) and (δ) of general Codazzi formulas (XVII, 4). Formula (7) can be written in the following way:

$$a_{ii}=-\sum\left(\frac{\partial\mathcal{G}_{ij}}{\partial s_j}+\frac{\partial\mathcal{G}_{ji}}{\partial s_i}+\mathcal{G}_{ij}^2+\mathcal{G}_{ji}^2\right)-\sum(\mathcal{G}_j-\mathcal{G}_{ij})\mathcal{G}_{ij}\,.$$

The second sum is equal to:

$$\sum_j\sum_{h,(i)}\mathcal{G}_{hj}\mathcal{G}_{ij}=\sum_{h,(i)}\sum_j\mathcal{G}_{hj}\mathcal{G}_{ij}=\sum_{j,(i)}\sum_h\mathcal{G}_{ih}\mathcal{G}_{jk}\,.$$

Hence:

$$a_{ii}=-\sum_{(i)}\left(\frac{\partial\mathcal{G}_{ij}}{\partial s_j}+\frac{\partial\mathcal{G}_{ji}}{\partial s_i}+\mathcal{G}_{ij}^2+\mathcal{G}_{ji}^2+\sum\mathcal{G}_{ih}\mathcal{G}_{jh}\right),$$

or, from (γ):

$$a_{ii}=\sum(\mathcal{N}_i\mathcal{N}_j-\mathcal{T}_{ij}^2)\,. \tag{8}$$

Similarly, one can give (6) the form:

$$a_{ij}=\mathcal{G}_{ij}\sum_{(j)}\mathcal{G}_{hi}-\frac{\partial}{\partial s_i}\sum_{(i)}\mathcal{G}_{hj}-\sum\mathcal{G}_{hi}\mathcal{G}_{hj}\;=-\sum_{(i,j)}\left[\frac{\partial\mathcal{G}_{hj}}{\partial s_i}+(\mathcal{G}_{hj}-\mathcal{G}_{ij})\mathcal{G}_{hj}\right],$$

i.e., by virtue of (δ):

$$a_{ij}=-\sum(\mathcal{N}_h\mathcal{T}_{ij}+\mathcal{T}_{ih}\mathcal{T}_{jh})\,. \tag{9}$$

This formula shows that $a_{ij} = a_{ji}$. One is then led to consider the quadratic form:

$$U = \tfrac{1}{2}\sum a_{ij}u_i u_j \ , \tag{10}$$

whose first partial derivatives are precisely the a_i. In order to understand the significance of U, note that one can also arrive at equations (3) by assuming that the effective part of the potential is the expression (2), augmented by $2BU$. That can be expressed by saying that the curvature of space produces a *loss of elastic energy*, as if one part of that energy were expended by the body to overcome the difficulty that it encountered by deforming in a *nonlinear* space. However, it can happen that $U < 0$, and then the elastic energy will be, by contrast, more intense than what one has in a linear space, as if the form of the space is such that it tends to facilitate, rather than contradict, the elastic deformation. In other words, if one imagines the space to be non-rigid in its geometric essence, and on the other hand, one supposes that the matter is endowed with a type of *inertia*, by virtue of which it will always tend to deform *as if it were found in a linear space*, then one can say that the space reacts to that tendency with a force that admits the potential $2BU$.

For example, in the case of a two-dimensional space, one has $a_{11} = a_{22} = K$, $a_{12} = 0$. Hence, $U = \tfrac{1}{2}K(u_1^2 + u_2^2)$, and equations (3) will become:

$$X_1 + A\frac{\partial\Theta}{\partial s_1} - B\frac{\partial\vartheta}{\partial s_2} + 2BKu_1 = 0, \qquad X_2 + A\frac{\partial\Theta}{\partial s_2} + B\frac{\partial\vartheta}{\partial s_1} + 2BKu_2 = 0,$$

and will remain unchanged under deformations of the surface, which one assumes to be flexible, but inextensible. Hence, for a surface, the loss of elastic energy is proportional to the square of the displacement and to the curvature of the surface at the point that one considers. One will have an analogous state of affairs for an arbitrary space. Indeed, imagine that the space is referred to its *system of curvatures*. All of the torsions $\mathcal{T}$ will then be zero, and from (9), one will have that $a_{ij} = 0$, while from (8), one will see that a_{ii} is the sum of the total curvatures of all coordinate surfaces that contain the line q_i . Now, if one represents the projections of the displacements $q_i\ q_j$ onto the surface by u_{ij} , and represents its total curvature by K_{ij} then the equality (10) will become:

$$U = \tfrac{1}{2}\sum K_{ij}\, u_{ij}^2 \ .$$

The loss of elastic energy in an n-dimensional curved space is then equal to the sum of the losses that are due to the $\tfrac{1}{2}n\,(n-1)$ surfaces of curvature.

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