Excerpted from E. Cesàro, *Lezioni di geometria intrinseca*, from the author-publisher, Naples, 1896. "Sull'equilibrio dei fili flessibili ed inestendibili," pp. 254-257.

On the equilibrium of flexible, inextensible strings

Given a string that is completely deformable in an n-dimensional linear space, take the axes to be the tangent, $(n - 1)$-normal, …, principal normal at a moving point of the string. It is assumed to be infinitely thin, but in such a way that each element ds nevertheless has a certain *mass* $q\,ds$. Let X_i be the components along the i axis of the force per unit mass that acts upon $q\,ds$, and let u_i be the projection of the displacement onto that axis. The direction cosines of the element of the string after deformation will obviously be proportional to $ds + \delta u_1$, δu_2, δu_3, …, δu_n, and therefore if one calls the tension per unit length T then one will have:

$$q\,X_i\,ds + \delta\left(T\,\frac{\delta u_i}{ds}\right) = 0$$

for the equilibrium of the external force, if one takes care to append $T\,\delta s$ and $T\,\delta s_i$ when $i = 1$. At the same time, it is important to note that the fundamental formulas (XVI, 4) that relate to the direction $(\alpha_1, \alpha_2, \dots, \alpha_n)$, when written in the form:

$$\frac{\delta\alpha_i}{ds} = \frac{d\alpha_i}{ds} + \frac{\alpha_{i-1}}{\rho_{n-i+2}} - \frac{\alpha_{i+1}}{\rho_{n-i+1}},$$

and if one agrees to set:

$$\alpha_{i+n} = -\,\alpha_i, \qquad \rho_{i+n} = \rho_i, \qquad \frac{1}{\rho_0} = 0,$$

can, as always, still persist when one considers the projections of an arbitrary variable segment onto the axes, instead of α. Indeed:

$$\delta\,p\alpha_i = \alpha_i\,dp + p\,d\alpha_i = d\,p\alpha_i + \left(\frac{p\alpha_{i-1}}{\rho_{n-i+2}} - \frac{p\alpha_{i+1}}{\rho_{n-i+1}}\right) ds\,.$$

One can then write:

$$\delta\left(T\,\frac{\delta u_i}{ds}\right) = d\left(T\,\frac{\delta u_i}{ds}\right) + \frac{T\,\delta u_{i-1}}{\rho_{n-i+2}} - \frac{T\,\delta u_{i+1}}{\rho_{n-i+1}},$$

and the equations of equilibrium will become, in general:

$$q\,X_i + \frac{d}{ds}\left(T\,\frac{\delta u_i}{ds}\right) + \frac{T}{\rho_{n-i+2}}\frac{\delta u_{i-1}}{ds} - \frac{T}{\rho_{n-i+1}}\frac{\delta u_{i+1}}{ds} = 0.$$

Finally, after totally eliminating the δ sign:

$$q\,X_i + \frac{d}{ds}\left[T\left(\frac{du_i}{ds} + \frac{u_{i-1}}{\rho_{n-i+2}} - \frac{u_{i+1}}{\rho_{n-i+1}}\right)\right] + \frac{T}{\rho_{n-i+2}}\frac{du_{i-1}}{ds} - \frac{T}{\rho_{n-i+1}}\frac{du_{i+1}}{ds}$$

$$+\ \frac{T\,u_{i-2}}{\rho_{n-i+3}\,\rho_{n-i+2}} + \frac{T\,u_{i+2}}{\rho_{n-i+1}\,\rho_{n-i}} - Tu_i\left(\frac{1}{\rho_{n-i+2}^2} + \frac{1}{\rho_{n-i+1}^2}\right) = 0.$$

Hence, for i = 1, 2, 3, …, n, if one takes into account all of the conventions that were made, one will arrive at the *intrinsic fundamental equations for the equilibrium of a string* in an n-dimensional linear space:

$$q\,X_1 + \frac{d}{ds}\left[T\left(\frac{du_1}{ds} - \frac{u_n}{\rho_1} + 1\right)\right] - \frac{T}{\rho_1}\left(\frac{du_n}{ds} + \frac{u_1}{\rho_1} + \frac{u_{n-2}}{\rho_2}\right) = 0,$$

$$q\,X_2 + \frac{d}{ds}\left[T\left(\frac{du_2}{ds} - \frac{u_3}{\rho_{n-1}}\right)\right] - \frac{T}{\rho_{n-1}}\left(\frac{du_3}{ds} + \frac{u_2}{\rho_{n-1}} - \frac{u_4}{\rho_{n-2}}\right) = 0,$$

$$q\,X_3 + \frac{d}{ds}\left[T\left(\frac{du_3}{ds} + \frac{u_2}{\rho_{n-1}} - \frac{u_1}{\rho_{n-2}}\right)\right] + \frac{T}{\rho_{n-1}}\left(\frac{du_2}{ds} - \frac{u_3}{\rho_{n-1}}\right) - \frac{T}{\rho_{n-2}}\left(\frac{du_4}{ds} + \frac{u_3}{\rho_{n-2}} - \frac{u_5}{\rho_{n-3}}\right) = 0,$$

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$$q\,X_{n-1} + \frac{d}{ds}\left[T\left(\frac{du_{n-1}}{ds} + \frac{u_{n-2}}{\rho_3} - \frac{u_n}{\rho_2}\right)\right] - \frac{T}{\rho_2}\left(\frac{du_n}{ds} + \frac{u_1}{\rho_1} + \frac{u_{n-1}}{\rho_2}\right) + \frac{T}{\rho_3}\left(\frac{du_{n-2}}{ds} + \frac{u_{n-3}}{\rho_4} - \frac{u_{n-1}}{\rho_3}\right) = 0,$$

$$q\,X_n + \frac{d}{ds}\left[T\left(\frac{du_n}{ds} + \frac{u_1}{\rho_1} + \frac{u_{n-1}}{\rho_2}\right)\right] + \frac{T}{\rho_1}\left(\frac{du_1}{ds} - \frac{u_n}{\rho_1} + 1\right) + \frac{T}{\rho_2}\left(\frac{du_{n-1}}{ds} + \frac{u_{n-2}}{\rho_3} - \frac{u_n}{\rho_2}\right) = 0\,.$$

In particular, for n = 3, if one lets X, Y, Z, u, v, w denote the components of the accelerating force and displacement, resp., and lets ρ and r be the radii of flexure and torsion, resp., then one will get the equations:

$$\left\{\begin{aligned}
&q\,X + \frac{d}{ds}\left[T\left(\frac{du}{ds} - \frac{w}{\rho} + 1\right)\right] - \frac{T}{\rho}\left(\frac{dw}{ds} + \frac{u}{\rho} + \frac{v}{r}\right) = 0,\\
&qY + \frac{d}{ds}\left[T\left(\frac{dv}{ds} - \frac{w}{r}\right)\right] - \frac{T}{r}\left(\frac{dw}{ds} + \frac{u}{\rho} + \frac{v}{r}\right) = 0,\\
&q\,Z + \frac{d}{ds}\left[T\left(\frac{dw}{ds} + \frac{u}{\rho} + \frac{v}{r}\right)\right] + \frac{T}{\rho}\left(\frac{du}{ds} - \frac{w}{\rho} + 1\right) + \frac{T}{r}\left(\frac{dv}{ds} - \frac{w}{r}\right) = 0,
\end{aligned}\right.$$

which were pointed out by **Maggi**. If the string is inextensible then it will be enough to observe that the variation of the element ds results from the relation:

$$(ds + \delta ds)^2 = (ds + \delta u_1)^2 + (\delta u_2)^2 + \ldots + (\delta u_n)^2,$$

from which it will follow that $\delta ds = \delta u_1$ in the case of infinitesimal displacements, in order for one to see that *inextensibility* is always expressed by the equality:

$$\frac{du_1}{ds} = \frac{u_n}{\rho_1}.$$

When one omits the displacements, the equations that were found will reduce to the simpler form:

$$q X_1 + \frac{dT}{ds} = 0, \qquad q X_n + \frac{T}{\rho_1} = 0, \qquad X_2 = X_3 = \ldots = X_{n-1} = 0,$$

and one will see that the string is always arranged in such a way that the osculating plane contain the accelerating force at any point. The equilibrium curve is then planar in the case of forces that emanate from a center. If the accelerating force X has an invariable direction then what was expressed in (II, 1) can be written:

$$\frac{d\varphi}{ds} = \frac{1}{\rho},$$

in which φ is the inclination of the tangent to the string with respect to the direction of X. The first two equations of equilibrium, which are the only ones that we agree to take into account, become:

$$q X \cos \varphi + \frac{dT}{ds} = 0, \quad q X \sin \varphi = \frac{T}{\rho},$$

and when one eliminates X and integrates, it will be easy to deduce that $T \sin \varphi$ keeps a *constant* value T_0 all along the string, in such a way that one has:

$$T = \frac{T_0}{\sin\varphi}, \qquad X = \frac{T_0}{q\rho \sin^2\varphi}.$$

That presents two noteworthy special cases: If the string is *homogeneous* (i.e., q is constant) then the last equation will give:

$$X = \frac{a}{\rho \sin^2\varphi}, \qquad \int X\, ds = -\, a \cot \varphi,$$

after one sets $T_0 = aq$. It then follows that the intrinsic equation of the equilibrium curve will be:

$$\rho = \frac{1}{X}\left[a + \frac{1}{a}\left(\int X\,ds\right)^2\right].$$

Hence, if the string is inhomogeneous (but one can still vary the density from one extreme to the other in such a way that one has *equal resistance* to the action of deformation everywhere) then one will need to set $T = aq$, with a constant, in which case, one can deduce from the second equation of equilibrium that:

$$X = \frac{a}{\rho \sin\varphi}, \qquad \int X\,ds = a \log\tan\frac{\varphi}{2};$$

hence:

$$\rho = \frac{a}{2X}\left(e^{\frac{1}{a}\int X\,ds} + e^{-\frac{1}{a}\int X\,ds}\right).$$

For example, when X is constant (and one can always suppose that $X = 1$ then), as one has for a ponderous string that is fixed at two points and is in equilibrium under the action of gravity, the two preceding intrinsic equations that were obtained will become:

$$\rho = a + \frac{s^2}{a}, \qquad \rho = \frac{a}{2}\left(e^{s/a} + e^{-s/a}\right),$$

which represent the ordinary catenary and the catenary of equal resistance, resp. That explains the reason for the names that are given to those curves (I, 5, *b*, *c*).

One can treat other known questions of mechanics with equal rapidity and simplicity of means, and we encourage the reader to attempt to apply the method that was discussed to the study of the deformations of fibers or material lines that run through an elastic body and consider, in place of tension, the internal forces that act on each element of the fiber in all directions. The formulas that one obtains in that way can offer advantages in the treatment of special problems that are analogous to those of curvilinear coordinates.

Additional note:

The theorem that was stated above (viz., the string is always arranged in such a way that the osculating plane contain the accelerating force at any point) is another way of explaining (XI, 8) why a string that is stretched on a surface will take the form of a geodetic. Indeed, the surface tends to oppose the tendency of the string to rectify with a *normal* reaction F, which must also lie in the osculating plane of the equilibrium curve. It is then such that the osculating plane at each point will be normal to the surface, and therefore a geodetic. In addition, one will see that:

$$\rho\, q\, F = T = constant,$$

i.e., the *reaction*, when computed per unit length, *is proportional to the curvature of the string*, and that will also explain why the reaction is missing from the points of contact between the string and the asymptotes of the surface.