LESSONS

ON

INTRINSIC GEOMETRY

BY

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My glass is not large, but I drink from my glass
(A. DE MUSSET)

Translated by D. H. Delphenich
(with corrections incorporated)

SUMMARY: Intrinsic discussion of plane curves and formulas for their intrinsic analysis – Noteworthy plane curves – Contact and osculation – Roulette, barycenters, barycentric analysis – Systems of plane curves – Skew curves and ruled surfaces – Noteworthy skew curves – Surfaces and congruences – Noteworthy surfaces – Deformations of surfaces – Three-dimensional spaces – Curves in hyperspaces – Hyperspaces – Mechanical applications.

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1896

TO PROFESSORS

P. MANSION & J. NEUBERG

AT THE UNIVERSITIES OF GHENT AND LIEGE

IN GRATITUDE

E. CESÀRO

TO THE READER

I shall collect and coordinate the fundamental formulas for the intrinsic analysis of geometric entities. I shall methodically present the simplest applications, and by the most elementary path, with the only objective of shedding some light upon the power of the intrinsic method and affirming its superiority over all of the procedures in use for the infinitesimal study of geometric phenomena in space. I shall establish and propose a uniform and expressive formalism that permits one to perform the calculations with elegant agility. I shall communicate to my listeners the passion for similar research that is very much alive in me. With that program, I was urged to dictate a brief course of lessons at the University of Naples that I now publish. I hope that it will soon spawn more important and advanced research by the works of the worthy young Italians.

NAPLES, 15 June 1895

E. C.

P. S. (12 May 1896). If this book proves to offer some advantage to Italian mathematical culture then it would be due to my dear and illustrious master, Prof. Valentino CERRUTI at the University of Rome, whose strong moral and material support helped to remove the obstacles that opposed its publication during almost two years of misadventures. I feel that I must also heartily acknowledge my friend Dr. Alfredo Perna for his corrections to the drafts and his accurate and intelligent execution of the figures.

TABLE OF CONTENTS

CHAPTER I

INTRINSIC DISCUSSION OF PLANE CURVES

1. Tangents and normals. – Let M and M' be two points on a plane curve. Fix M and make M' tend to M along the curve. If the line MM' tends to occupy a limiting position then it will take on the name of *tangent*, and the perpendicular to the tangent that is raised at M will be called the *normal* to the curve at M. Always suppose that if one is given M then one can take M' close enough to M for the arc MM' to admit a tangent at any point, and in addition, the angle φ between the tangent at M and a fixed line will always vary in one sense when M is indefinitely close to M'. Under those conditions, it will be clear that the length δs of the arc MM', which is greater than that of the chord MM', will be less than the sum of the distances u and v from M' to the normal and tangent at M, resp. in such a way that one will have:

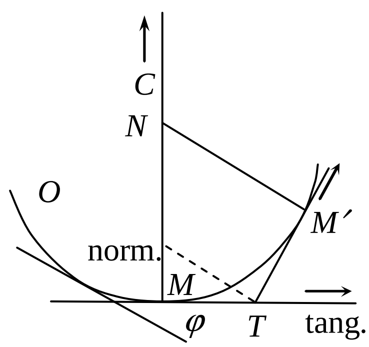
$$\sqrt{u^2+v^2} < \delta s < u + v,$$

and then, by the definition itself of the tangent, when M' tends to M:

$$\lim \frac{v}{u} = 0, \tag{1}$$

one will also have:

$$\lim \frac{\delta s}{u} = 1. \tag{2}$$



From now on, we shall always assume that the independent variable is the length of the arc OM that is contacted in a given sense by starting from an *origin* O that is chosen arbitrarily from the points of the curve in such a manner that it establishes the increment s that it experiences when one passes from M to M' as the principal infinitesimal – i.e., the length δs of the arc MM' – and one will see from (2) that one can replace δs with u in the search for the limits of the ratios. (1) will then show that v is a higher-order infinitesimal. Therefore, from the infinitude of lines that pass through M:

The tangent is characterized by the fact that its distances to two points that are infinitely-close to M are higher-order infinitesimals.

2. Curvature. – Suppose that the point of intersection N of the normals at M and M' tends to a limiting position C when M' tends to M. The point C is called the *center of curvature*, and its distance from M (when measured in a given sense) is the *radius of curvature*, which is habitually represented by ρ. Since it is natural for one to say that the infinitesimal arc MM' is more or less curved according to whether the point N is more or less close to M, resp., one will be led to take $1 / \rho$ to be *the measure of the curvature*. Meanwhile, one has:

$$MC = \lim MN = \lim (v + u \cot \delta\varphi) = \lim \frac{u}{\delta\varphi};$$

hence:

$$\frac{1}{\rho} = \lim \frac{\delta\varphi}{\delta s}. \tag{3}$$

Therefore, the curvature can also be considered to be the limit of the ratio of the angle $\delta\varphi$ to the arc length δs. Obviously, ρ will vary from one point to the other along the curve, in general – i.e., *the curvature is a function of the arc length* – and one quickly sees that knowing that function will be enough to define the *form* of the curve, but not to fix its *position* in the plane. For that reason, the relation:

$$f(s, \rho) = 0$$

between s and ρ that one has at any point of a curve will be called the *intrinsic equation* of that curve.

3. – The angle φ is also a function of s, and the equality (3) then says that the curvature is precisely the derivative of that function. It then follows that if one computes the angle φ by starting from the tangent to the (arbitrary) origin of the arc then one will have:

$$\varphi = \int_0^s \frac{ds}{\rho},$$

provided that the integral has been given a sense. Hence, one can always have that there exists a tangent at any point of the arc OM along the curve that is under consideration, because it is enough to assume that origin O is sufficiently close to M. The function φ has great importance for the discussion of plane curves when they are given by means of their intrinsic equations. If φ becomes infinite when s tends to a finite or infinite limit then the tangent at the corresponding point M will cease to exist. We shall always suppose that this happens only at isolated points.

4. For the moment, transport the origin of the arcs to M and take M' to be close enough to M that any point with an indeterminate tangent is excluded from the arc MM'. If one observes that one has:

$$\frac{du}{ds} = \cos\varphi, \qquad \frac{dv}{ds} = \sin\varphi \tag{4}$$

then **l'Hôpital's** rule will immediately give:

$$\lim \frac{v}{u^2} = \frac{1}{2}\lim\frac{\tan\varphi}{u} = \frac{1}{2\rho} \tag{5}$$

as M' tends to M. Hence, the distance from the tangent M to the points that are infinitely-close to M is generally a second-order infinitesimal. However, it will become an infinitesimal of *higher or lower order* according to whether the curvature is *zero or infinite,* respectively. Now, observe that if one is given the intrinsic equation, and one calculates φ then the integration of (4) will provide u and v as functions of s, and will permit one to construct arbitrary arcs of the curve that do not contain points with indeterminate tangents. It is then proved that the various pieces of a curve, when free of tangents only at their extremes, are determined by the form of the intrinsic equations, and also remain arbitrary in their mutual disposition, just like the situation for the whole curve in the plane.

5. Examples:

a) If the curvature is zero at any point then one will have $\varphi = 0$, $u = s$, $v = 0$, and the line will be *straight*. More generally, one can find a *circumference* of radius a when $\rho = a$. In fact, one will have:

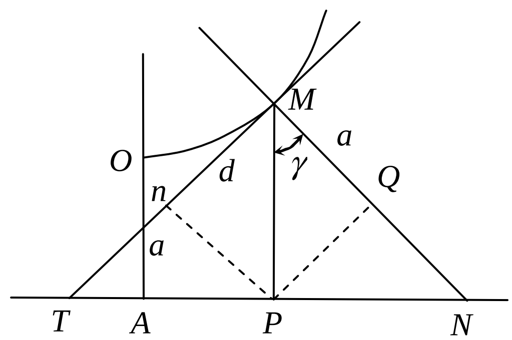
$$\varphi = \frac{s}{a}, \qquad u = a\sin\frac{s}{a}, \quad v = a\left(1 - \cos\frac{s}{a}\right),$$

and one will see immediately that all of the points of the curve are at a distance of a from the unique center of curvature, and that all of the normals are concurrent at it. In particular, one can consider $\rho = 0$ to be the intrinsic equation of an isolated point. Any arbitrary equation that contains only the variable ρ will represent a set of points and circles, real or imaginary.

b) One calls the curve that is represented by the equation:

$$\rho = a + \frac{s^2}{a}$$

a *catenary*.



One has $\rho = a$ at the origin of the arc. Hence, when s increases indefinitely, ρ will also increase to infinity, while:

$$\varphi = \int_0^s \frac{ds}{\rho} = \arctan \frac{s}{a} \tag{6}$$

will increase from 0 to $\pi / 2$. Therefore, the tangents will tend to arrange themselves perpendicularly to their initial positions, and meanwhile they will move to infinity, since:

$$u = a\int_0^\varphi \frac{d\varphi}{\cos\varphi} = a \log \tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right)$$

increases indefinitely with s. Things happen the same way for negative s: The curve is obviously symmetric with respect to the normal at the origin. The parallel to the tangent that goes through the origin at a distance of a in such a way that it does not meet the curve is called the *directrix*. Now:

$$v = a \int_0^\varphi \frac{\sin\varphi}{\cos^2\varphi} d\varphi = \frac{a}{\cos\varphi} - a,$$

and if one then projects M onto the directrix at P then *the projection of MP onto the normal will be constantly equal to a*. In addition, since one has, from (6), $s = a \tan \varphi$, *the projection of MP onto the tangent is equal to the arc OM*. Finally, if one observes that $\rho \cos^2 \varphi = a$ then one will see that *the center of curvature at M is symmetric to the point at which the normal meets the directrix with respect to M.*

c) The *catenary of equal resistance:*

$$\rho = \frac{a}{2}(e^{s/a} + e^{-s/a})$$

resembles the preceding. The calculation of φ leads to the formula:

$$\rho = \frac{a}{\cos\varphi}, \qquad s = a\log\tan\left(\frac{\varphi}{2}+\frac{\pi}{4}\right),$$

from which, one sees that ρ and φ vary with s as they do for the catenary. In addition, the first formula shows that *the projection of the radius of curvature* onto the normal at the origin of the arc *is constant*. One should also note the formulas $u = a\varphi$, v = a log ρ / a, which are obtained by integrating (4).

6. Inflection points and cusps. – Recall formula (5) and observe that when the curvature has a finite, non-zero value at a point M, the curve will be situated entirely on just one side of the tangent (in the vicinity of M), since one can conclude that v preserves the sign of ρ as M' tends to M, no matter what the sign of u. Moreover, suppose that when ρ is measured along the arc when starting from M, it will tend to zero or increase indefinitely like the n^{th} power of s. That will always be the case when an algebraic relation intercedes between s and ρ, in which case, one can confirm that n is rational, in addition. Now, if one supposes that $n < 1$ then one will have:

$$\lim \frac{v}{u^{2-n}} = \frac{1}{2-n}\lim\frac{\tan\varphi}{u^{1-n}} = \frac{1}{(2-n)(1-n)}\lim\frac{s^n}{\rho} \neq 0, \tag{7}$$

and one will again recognize that one will have a *higher-order* contact between the curve and the tangent only when ρ is infinite ($n < 0$) and a *lower-order* contact when ρ is zero ($0 < n < 1$). In either case, the curve will have the usual form around the point in question if n is the quotient of an even number by an odd number. However, when n is the quotient of two odd numbers, c will change sign, along with u – i.e., the curve will cross the tangent at M – and one can also see that more directly by observing that ρ will change sign with s. In that case, the point M will be called an *inflection* or *inflection point*. On the contrary, if n is the quotient of an odd number by an even number then the infinitesimal u will be capable of taking on only positive values, for each of which, v will take on two values with opposite signs. Therefore, the curve will exist on just one side of the normal, and it will admit two branches that separate from the tangent. One then says that one has a *cusp* at M or that M is a *point of regression*. Cusps and inflections can then be presented for only those values of s that are roots of the equations:

$$\rho = 0, \quad \frac{1}{\rho} = 0,$$

and it is useful to note that if one considers only the simple roots then the first equation will yield cusps and the second one, inflections.

7. Asymptotes. – If the function φ tends to a finite limit a as s becomes infinite then the curvature cannot tend to a non-zero limit, since if one assumes the existence of such a limit then since one has:

$$\alpha = \lim \frac{s\varphi}{s} = \lim \left(\varphi + \frac{s}{\rho}\right) = \alpha + \lim \frac{s}{\rho}, \tag{8}$$

one would necessarily see that ρ must increase to infinity with s. One would then have a point of higher-order contact whose coordinates with respect to the tangent and the normal at an arbitrary point (but which one supposes to have been chosen in such a way that φ remains finite) would be obtained by integrating (4):

$$u = \int_0^\alpha \rho \cos \varphi \, d\varphi, \qquad v = \int_0^\alpha \rho \sin \varphi \, d\varphi.$$

It is possible that the preceding integrals are infinite, and the point considered will then be at infinity, but the tangent to the curve at that point will be well-defined in terms of the angle α and the distance from the origin:

$$q = u \sin \alpha - v \cos \alpha = \int_0^\alpha \rho \sin(\alpha - \varphi) \, d\varphi,$$

which can have a well-defined value. The line thus-obtained, which is the limiting position of the tangent to a point M that moves indefinitely along a given branch of the curve, is called an *asymptote.* Now, observe that the final formula tells one the value of the distance q from the asymptote to any point of the curve. If one supposes that it goes back towards the closest point at which φ becomes infinite then the expression for q will transform into:

$$\int_{-\infty}^\alpha \rho \sin(\alpha - \varphi) \, d\varphi = \int_0^\infty \rho \sin \psi \, d\psi$$

when one takes $\alpha - \varphi = \psi$. Hence, if one calculates:

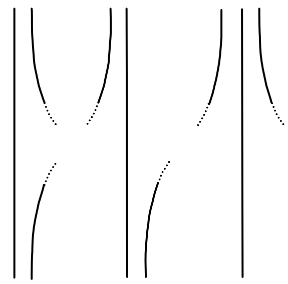
$$\psi = \int_s^\infty \frac{ds}{\rho},$$

and if one determines ρ as a function of ψ then the desired distance will be given by the formula:

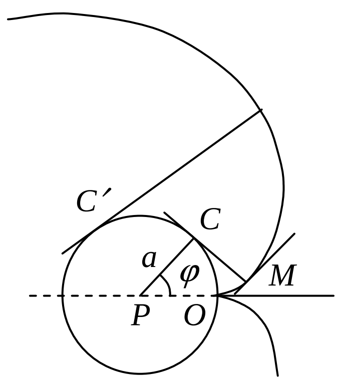
$$q = \int_0^\infty \rho \sin \psi \, d\psi. \tag{9}$$

However, if one supposes, in turn, that the point moves indefinitely along that branch of the curve that was traversed already by the point of contact then it will be clear that α will tend to zero, and one can then also say that q – *i.e., the distance from the moving point on the curve to the asymptote – will tend to zero.* Finally, observe that, by virtue of (8), the asymptote can exist only when ρ is infinitely large of a higher order, along with s.

If one finds that the order is measured by the quotient of an odd number by an even number then one can say that the curve will behave at infinity as it does around an ordinary point, and it will then have two branches that extend to infinity in two senses that give one part of the asymptote. However, if the order is measured by the quotient of an integer (odd or even) by an odd number then the two branches will separate from the asymptote and will extend in two senses or just one sense. In the former case, one will have an *inflectional* asymptote, while in the latter, one will have a *cuspidal* asymptote.



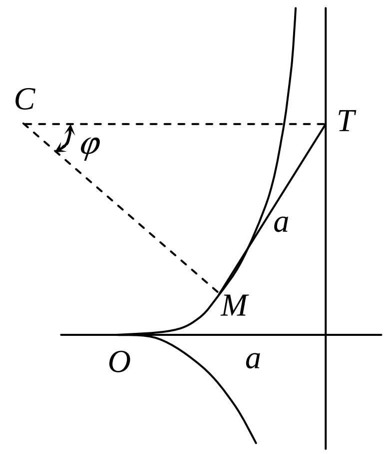
8. Examples:



a) The curve that is represented by the equation $\rho^2 = 2as$ will have a point of regression at the origin. An infinite spiral will then unwind around the origin that that extends to infinity with decreasing curvature. Indeed, one will have $\rho = as$, and one will then see that ρ and φ, which go to zero with s, will increase indefinitely along with s. The distance from any point to the cuspidal tangent will then be:

$$v = a \int_0^{\varphi} \varphi \sin \varphi \, d\varphi = a (\sin \varphi - \varphi \cos \varphi).$$

Hence, the distance from the center of curvature to that line will be $a \sin \varphi$, and then *the centers of curvature will be on a circumference* of radius a. In addition, since MC is precisely equal to the arc OC of the circumference, one can imagine that the curve is described by one of the two ends of an inextensible string that is wrapped around the circumference, while the other end is kept fixed and the string is unwrapped while it is always kept tense. For that reason, the curve considered is given the name of the *involute of the circle.*



b) One calls the curve that is defined by the equation $\rho = a\sqrt{e^{2s/a}} - 1$ the *tractrix*. If one supposes that s is infinitesimal then one will see immediately that the curve behaves around the origin like the involute of the circle that is represented by the equation $\rho = \sqrt{2as}$. However, as s increases to infinity, the angle φ (which always increases) will tend to $\pi / 2$, since one has $\rho = a \tan \varphi$, and since:

$$u = \int_0^{\varphi} \rho \cos \varphi \, d\varphi = a\,(1 - \cos \varphi)$$

tends to a, one will see that an asymptote will exist at a distance a from the cusp and perpendicular to the cuspidal tangent. In addition, from the second formula that was found, *the segment of the tangent that is included between the point of the contact and the asymptote is constantly equal to a*, and the first formula leads to a simple construction of the center of curvature, which shows that *the center of curvature projects to the foot of the tangent on the asymptote.* Finally, in order to take into account the two determinations that ρ has for each value of s, one needs to imagine that the curve is composed of two infinite branches that are symmetric with respect to the common cuspidal tangent.

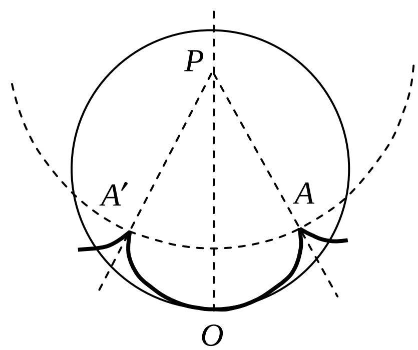
c) The class of curves that are defined by the equation:

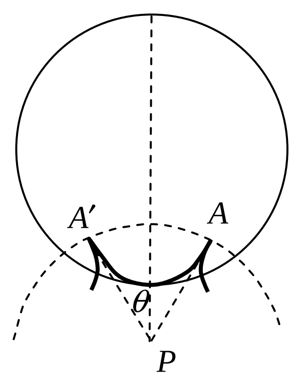
$$\frac{s^2}{a^2} + \frac{\rho^2}{b^2} = 1$$

is very important. In order to satisfy it, take $s = a \sin \theta$, $\rho = b \cos \theta$, in such a way that $\varphi = \frac{a}{b}\theta$. An arc of length $2a$ that is symmetric with respect to the origin is completely contained with the circle of radius b, which touches it at its origin viz., $\theta = 0$) and terminates at two cusps ($\theta = \pm\ \pi / 2$). The coordinates of one cusp with respect to the tangent and the normal at the origin are:

$$u = a\int_0^{\pi/2} \cos\frac{a\theta}{b}\cos\,\theta\,d\theta = -\frac{ab^2}{a^2-b^2}\cos\frac{\pi a}{2b},$$

$$v = a\int_0^{\pi/2} \sin\frac{a\theta}{b}\cos\,\theta\,d\theta = \frac{ba^2}{a^2-b^2} - \frac{ab^2}{a^2-b^2}\sin\frac{\pi a}{2b}.$$

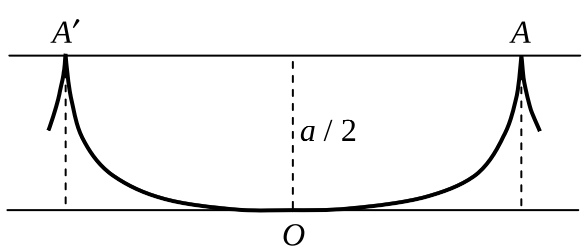




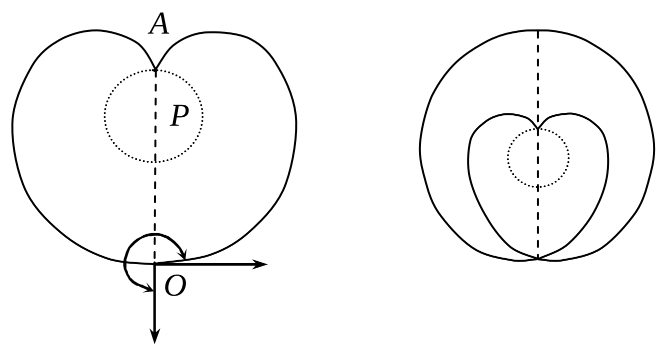
The cuspidal tangents then agree at a point P that is situated at distances of:

$$\frac{ab^2}{a^2-b^2}, \qquad \frac{ba^2}{a^2-b^2}$$

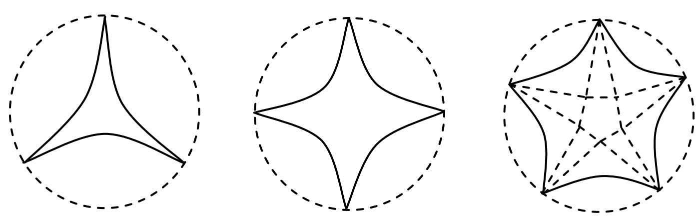
from the cusp and the origin, respectively. The circumference that is described by the center P and radius $ab^2 : (a^2 - b^2)$ is called the *directrix*. In order to take into account the change in sign of ρ when s becomes equal to a, one needs to imagine that the curve is composed of more arcs that are tangent to each other at the cusps; their locus is precisely the directrix. Depending upon whether $a > b$ or $a < b$, the curve will be *external* or *internal* to the directrix, respectively. In the former case, one calls it an *epicycloid*, while in the latter, it is a *hypocycloid*.



Of particular note is the *cycloid*. It is represented by the equation $s^2 + \rho^2 = a^2$, and it is the curve that separates the hypocycloids from the epicycloids, so to speak; its directrices are rectilinear. In general, the number of cusps is infinite, but for the cycloidal lines that one usually considers the ratio $a : b$ is rational, and for them the cusps must fall upon a certain finite number of points that are the vertices of a regular polygon. They can be encountered in the same order in which the directrix follows a point on the circumference that traverses the curve in a continuous way, but it can also happen that the moving point successively reaches the cusps in the other order, and then one will have a *stellate* cycloidal line. The simplest epicycloids are the ones that have just one cusp. In order for the moving point to return to A without encountering any other cusps after starting from the cusp A, it is necessary that the angle between the positive directions of the cuspidal tangent and the tangent at the origin – viz., $\pi a / 2b$ – should be equal to an odd number (≥ 3) times $\pi / 2$. In the simplest case, one will have $a = 3b$, and then one will then have the *cardioid*, which is then represented by the equation $s^2 + 2\rho^2 = constant$. More generally, if n is an odd number then the equation $s^2 + n^2\, 2\rho^2 = constant$ will represent a monocuspidal epicycloid, which will be an even more complicated *stellate cardioid* for $n = 5, 7, 9, \ldots$



The simplest hypocycloids are the ones that have three or four cusps. For them, the angle $\pi - \pi a / b$ is the third or fourth part of 2π, and one will then have $b = 3a$ for the tricuspids, and $b = 2a$ for the others. Hence, the tricuspid is represented by the equation $9s^2 + \rho^2 = constant$. For five cusps, in addition to the hypocycloid $25s^2 + 9\rho^2 = constant$, one also has the stellate hypocycloid $25s^2 + 9\rho^2 = constant$, etc.



d) *Pseudo-cycloids* are the curves that are represented by the equations:

$$s^2 - \rho^2 = a^2, \qquad \rho^2 - s^2 = a^2,$$

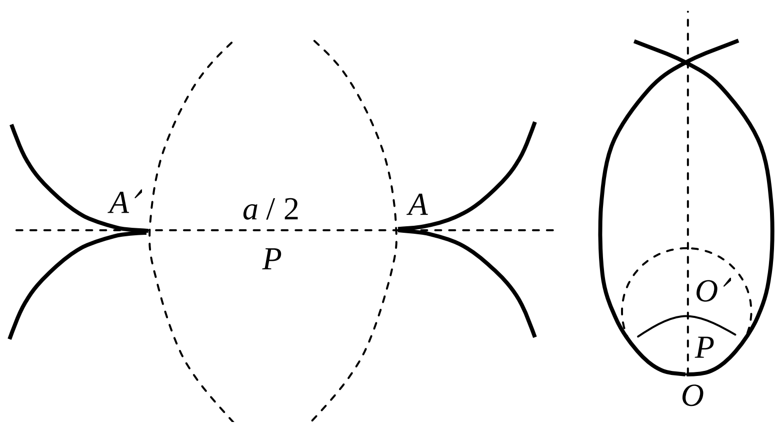
which can be satisfied by taking:

$$s = \frac{a}{2}(e^{\varphi} \pm e^{-\varphi}), \qquad \rho = \frac{a}{2}(e^{\varphi} \mp e^{-\varphi}),$$

in which φ has the usual meaning. The first curve has a cusp ($\varphi = 0$, $s = a$), around which it behaves like the involute of a circle of radius a around its cusps. By contrast, the other one behaves almost like a catenary $\left(\rho = a + \dfrac{s^2}{2a}\right)$ around the origin ($\varphi = 0$, $\rho = a$). As s increases, both of them will unwrap into an infinite spiral that extends to infinity, and they will tend to meet the lines that emanate from a point at an angle of $\pi / 4$. In fact, if one chooses the origin P of the coordinates u and v suitably then one will have:

$$u = \tfrac{1}{2}(s \cos \varphi + \rho \sin \varphi), \qquad v = \tfrac{1}{2}(s \sin \varphi - \rho \cos \varphi)$$

for any point. It will then follow that the cuspidate pseudo-cycloid will meet the radius vectors at an angle that will increase from 0 to $\pi / 4$, while that same angle must decrease from $\pi / 2$ to $\pi / 4$ for the other one. In addition, the *projections of the radius vector PM onto the tangent and normal at M are equal to one-half the arc length and one-half the radius of curvature at M, respectively.*



9. Asymptotic points. – We must further consider the roots of ρ that make the function φ infinite. No tangent will exist at the point M that corresponds to them. However, by hypothesis, one will exist at a point M' that is sufficiently close to M and at all intermediate points. When M' tends to M, the tangent at M', which executes an infinitude of revolutions in one sense, will conclude by becoming indeterminate. The curve will then wind around the point M indefinitely, and for that reason, it is called an *asymptotic point*, although it can also be reached from M' after traversing a finite distance. The coordinates of such a point are calculated by integrating (4), which will give:

$$u = \int_0^\infty \rho \cos \varphi \, d\varphi, \quad v = \int_0^\infty \rho \sin \varphi \, d\varphi, \tag{10}$$

if one assumes that φ is the integration variable, provided that one chooses the origin in such a way that φ remains infinite in the course of integration, which is to say that no other points with indeterminate tangents exist between the origin and the point considered. One rapidly confirms the existence of an asymptotic point at M when one finds that ρ is infinitesimal along with s of an order that is not less than 1, assuming that one measures the arc length from M as the origin. Indeed, one then has:

$$\lim \frac{\varphi}{\log s} = \lim \frac{s}{\rho} \neq 0,$$

instead of (7), and for $n > 1$:

$$\lim \varphi\, s^{n-1} = -\frac{1}{n-1} \lim \frac{s^n}{\rho} \neq 0\ ;$$

i.e., φ will become infinite when s and ρ tends to zero. Hence, one sees (§ **6**) that the tangent is determinate for $n < 1$.

10. Asymptotic circles. – In order to consider all cases of points with indeterminate tangents, one must look for all values of s – finite or infinite – that make the function φ infinite, and that is why one will have:

$$\lim \frac{\varphi}{s} = \lim \frac{1}{\rho},$$

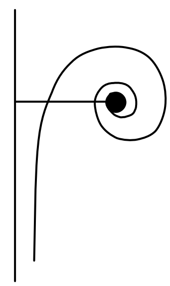
if one measures from the origin at M, and one sees that when the right-hand side exists, finite values will annul ρ, and it will then correspond to an asymptotic point. Only when s increases indefinitely along with φ can one say that ρ tends to a non-zero limit a. Instead of wrapping around a point, the curve will then revolve asymptotically around a circumference of radius a externally or internally according to whether the absolute value of ρ stays greater than or less than its limit, resp. However, one can also say that the curve will end up meandering near the circumference, as ρ does not cease to oscillate near its limit. As a result, one sees that the center of the asymptotic circumference is the limiting position to which the center of curvature at M will tend when M moves indefinitely along the curve from the origin of its arc length. In order to find the coordinates, one then needs to adopt these integrals:

$$u = \int_0^s \cos\varphi\, ds - \rho \sin\varphi, \qquad v = \int_0^s \sin\varphi\, ds + \rho\cos\varphi,$$

instead of (4). As a consequence, one will have:

$$u = -\int_0^\infty \frac{d\rho}{d\varphi} \sin\varphi\, ds, \qquad v = \int_0^\infty \frac{d\rho}{d\varphi} \cos\varphi\, ds \tag{11}$$

in the limit. The asymptotic circles obviously include the asymptotic points as special cases, and one easily verifies by integration by parts, moreover, that formulas (11) reduce to (10) when ρ tends to zero for φ infinite. It is clear *a priori* that it is only near a point that it can happen that the curve then wraps so tightly around the point that it will arrive at it after traversing a finite distance. The asymptotic circle will also be infinitely large when ρ and φ increases indefinitely with s, and one can then express that by saying that the curve wraps around the point at infinity asymptotically. For example, one can say (§ **8**, *a*, *b*) that the involute of the circle and the pseudo-cycloids have an asymptotic point at infinity.



11. Examples:

a) The radius of curvature of the line that is represented by the equation $\rho = a\, e^{s/a}$ increases from zero to infinity when s increases through the entire system of real number. The angle:

$$\psi = \int_0^\infty \frac{ds}{\rho} = \frac{a}{\rho},$$

however, decreases from infinity to zero, and the curve will then admit an asymptotic point ($s = -\infty$) and an asymptote ($s = \infty$) that is at a distance of:

$$q = \int_0^\infty \frac{\sin\psi}{\psi}\, d\psi = \frac{\pi a}{2}$$

from it. The curve is defined geometrically by the property that $\rho\psi = a$. On the circumference that passes through M with its center at C, *the perpendicular that is based at C will sweep out an arc of constant length along the asymptote,* when one starts at M.

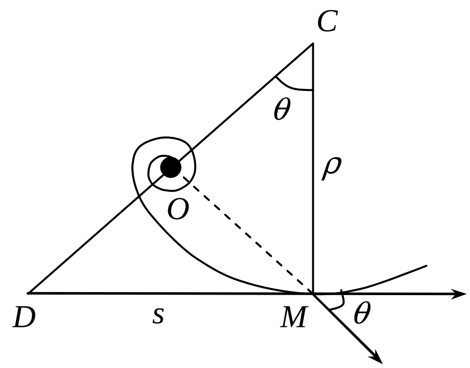
b) A linear equation in s and ρ that actually contains the two variables can always be reduced to the form $\rho = ks$ when one takes the origin of the arc to be the point at which ρ is annulled. The function φ (which is proportional to log s) will become infinite at that point. The origin of the arc will then be an asymptotic point of the curve, such that the radius of curvature will always become larger when one starts from that point, and it will increase indefinitely with the arc length, like φ. Hence, the curve will revolve around the origin in an infinite spiral that extends to infinity, where it will admit another asymptotic point. One gives the name of *logarithmic spiral* to that noteworthy curve, and the asymptotic point at a finite distance is called the *pole* of the spiral. A homogeneous equation between s and ρ represents a set of logarithmic spirals – real or imaginary – and also points or circles. The latter would be the limiting cases of the logarithmic spiral that correspond to the zero or infinite values of k, resp., as one will see better in what follows.

c) The property of the logarithmic spiral is easily deduced from the expressions for u and v relative to the pole when calculated at any point of the curve. One immediately obtains:

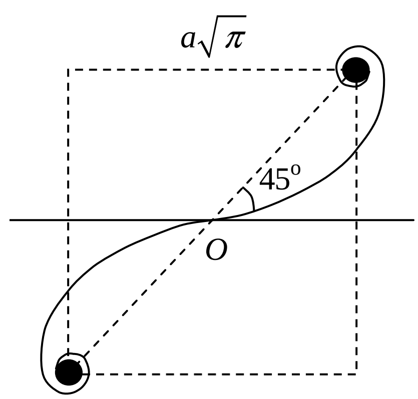
$$u = ks\int_0^{-\infty} e^{k\varphi}\cos\varphi\, d\varphi = -\frac{k^2 s}{1+k^2}, \qquad v = ks\int_0^{-\infty} e^{k\varphi}\ \sin\varphi\, d\varphi = \frac{k^2 s}{1+k^2},$$

so $k = \cot\theta$, if one lets θ denote the acute angle that the spiral makes with the radius vector *OM* and *M*. Hence, *the logarithmic spiral will meet all of the lines that emanate from the pole at a constant angle.* In what follows, we will see that this property characterizes the logarithmic spiral, as long as one excludes the values 0 and $\pi / 2$ from θ, which correspond to the lines and circumferences, resp., and which it will take on at a finite distance from the pole, as well. Meanwhile, one will have:

$$\sqrt{u^2+v^2} = \frac{ks}{1+k^2} = s\sin\theta = \rho\sin\theta.$$



Therefore, *the perpendicular to the radius vector OM that goes through O will meet the normal to the center of curvature and sweep out a segment on the tangent that has a length that is equal to the arc length OM when one starts from M.* This is also (as one sees) a characteristic property of the logarithmic spiral.



d) One calls the curve that is defined by the equation $s\rho = a^2$ a *clothoid*. It will be finite for $s = 0$ when ρ is infinite, and infinite for $s = \pm\infty$ when $\rho = 0$. Hence, the origin of the arc will be an inflection point, and if one starts from it in the two senses then the curvature will always become more pronounced in such a way that the curve will asymptotically wrap around its extremes, which are placed symmetrically with respect to the origin. Since one has:

$$\varphi = \int_0^s \frac{ds}{\rho} = \frac{s^2}{2a^2}, \qquad \rho = \frac{a^2}{s} = \pm\frac{a}{\sqrt{2\varphi}},$$

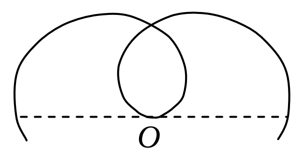
the coordinates of an asymptotic point with respect to the tangent and the normal at the origin will be:

$$u = \frac{a}{\sqrt{2}}\int_0^\infty \frac{\cos\varphi}{\sqrt{\varphi}}\,d\varphi, \qquad v = \frac{a}{\sqrt{2}}\int_0^\infty \frac{\sin\varphi}{\sqrt{\varphi}}\,d\varphi.$$

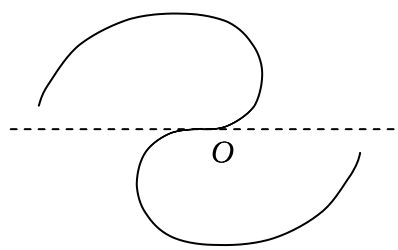
Now, if one observes that:

$$u + iv = \frac{a}{\sqrt{2}}\int_0^\infty \varphi^{-1/2}e^{i\varphi}\,d\varphi = a\sqrt{\frac{i\pi}{2}}$$

then one will find immediately that $u = v = \frac{1}{2}a\sqrt{\pi}$. Hence, the two asymptotic points are opposite vertices of a square that is equivalent to the circle of radius *a*.

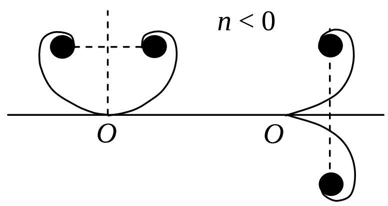


$0 < n < 1$



e) More generally, consider the curve that has its *curvature proportional to a power of the arc length.* Write its equation in the form $\rho = ks^n$, and one will recover the logarithmic spiral for $n = 1$, the clothoid for $n = -1$, and the involute of the circle for $n = 1 / 2$. If one omits the case of $n = 1$ then one will see immediately that φ varies like s^{1-n}, and one agrees to give the coefficient *k*, which can always be assumed to be positive, the form:

$$k = \pm\frac{a^{1-n}}{1-n}.$$

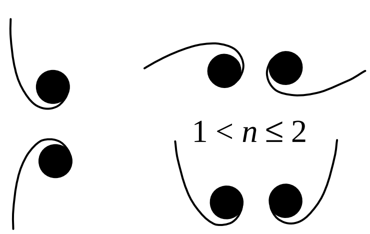


Having done that, when n is between 0 and 1, and it is the quotient of an odd number by an even number, one will have a curve that is analogous to the involute of a circle – i.e., it will be endowed with just one cusp at the origin of arc length, around which it spirals with ever less curvature and wraps asymptotically around the point at infinity. For the other rational values of n that are found between 0 and 1, the curve, while having a lower-order contact with the tangent at the origin, will behave like an ordinary point or experience an inflection, in such a way that its general form will be quite diverse from that of the involute of the circle. For negative n that are equal to the quotient of two odd numbers, one will have a curve that is analogous to the clothoid – i.e., it will be endowed with an inflection at the origin of the arc and two asymptotic points. However, if n is the quotient of an even number by an odd number, or *vice versa*, then the curve, while having higher-order contact with the tangent at the origin, will behave like an ordinary point there or admit a cusp and cease to resemble a clothoid as a whole, although each half of it will resemble half of a clothoid. In any case, if one adopts (10) then one will recognize that the origin is at a distance of:

$$a\ \Gamma\left(\frac{2-n}{1-n}\right)$$

from the asymptotic points, and the line that joins them is inclined from the tangent at the origin by $\dfrac{\pi}{2(1-n)}$. Finally, for $n > 1$, as for the logarithmic spiral ($n = 1$), one will have an asymptotic point at the origin of the arc. However, the asymptotic point at infinity will disappear, since φ will not become infinite along with s. In order to see whether the point at infinity becomes an asymptote, it is enough to adopt formula (8) by setting:

$$\psi = \int_0^\infty \frac{ds}{\rho} = \left(\frac{a}{s}\right)^{n-1}, \qquad \rho = \frac{a}{n-1}\psi^{-\frac{n}{n-1}}.$$

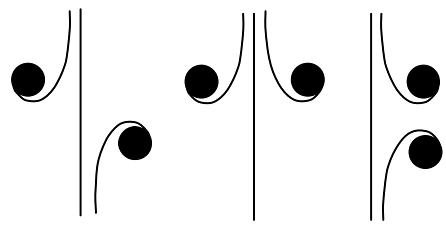


One immediately obtains the distance to the asymptotic point:

$$q = \frac{a}{n-1}\int_0^\infty \psi^{-\frac{n}{n-1}} \sin \psi \, d\psi .$$

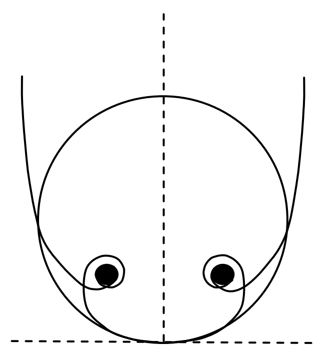
In the vicinity of the upper limit, the integral behaves as if the asymptote were at a finite distance *only for* $n > 2$, and in that case, one will find that:

$$q = a\,\Gamma\left(\frac{2-n}{1-n}\right)\cos\frac{\pi}{2(n-1)} .$$



$n > 2$

When n increases indefinitely, the curve will tend to degenerate into a point and a line that is situated at a distance of $q = a$ from it. For each branch, one will then have: The arc of length a, with an extreme at the asymptotic point, tends to accumulate at that point, while the remaining portion of the curve will extend to the asymptote.



f) The curves that are represented by the equation:

$$\rho = b - \frac{s^2}{a}$$

are analogous to the catenary (§ **5**, *b*), as long as a and b have opposite signs. For $b = -a$, one will recover the catenary, since it is permissible to change the sign of ρ. For $b = k^2 a$, however, one will obtain the *pseudo-catenary*. When s is less than ka in absolute value, one will find that:

$$\varphi = \int_0^s \frac{a\,ds}{k^2a^2 - s^2} = \frac{1}{2k}\log\frac{ka+s}{ka-s} .$$

Therefore, a first arc of length $2ka$ wraps asymptotically around the extremes. It is symmetric with respect to the origin and included completely within the circle of radius k^2a, which touches the curve at it origin. Since one has:

$$s = ka\,\frac{e^{k\varphi} - e^{-k\varphi}}{e^{k\varphi} + e^{-k\varphi}}, \qquad \rho = \frac{4k^2 a}{(e^{k\varphi} + e^{-k\varphi})^2},$$

the coordinates of an asymptotic point with respect to the tangent and the normal to the origin will be:

$$u = 4k^2 a \int_0^\infty \frac{\cos\varphi\, d\varphi}{(e^{k\varphi} + e^{-k\varphi})^2} = \frac{\pi a}{e^{\pi/2k} - e^{-\pi/2k}}, \qquad v = 4k^2 a \int_0^\infty \frac{\sin\varphi\, d\varphi}{(e^{k\varphi} + e^{-k\varphi})^2} = \frac{\pi a}{2(e^{\pi/4k} - e^{-\pi/4k})}.$$

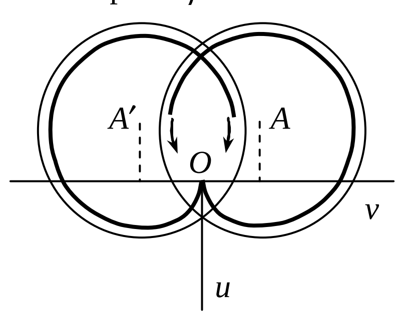
They then locate the asymptotic points, and the other two branches that extend to infinity, analogously to those of the spiral $s^2 = a\rho$. Their points correspond to the values of s that are greater than ka or less than $-ka$. For each branch, the direction of the tangent is determined by the angle:

$$\psi = \int_0^\infty \frac{a\, ds}{s^2 - k^2 a^2} = \frac{1}{2k} \log \frac{s + ka}{s - ka}.$$

The two branches have no asymptotes at a finite distance, since the distance from one asymptote to the corresponding asymptotic point, when given by the formula (9), is:

$$q = 4k^2 a \int_0^\infty \frac{\sin\varphi\, d\varphi}{(e^{k\varphi} - e^{-k\varphi})^2},$$

and on the other hand, one confirms that this integral does not have a finite value, so one will observe that it behaves like $\log\psi$ in the vicinity of the lower limit. Since the two infinite branches are connected with the finite branch at the asymptotic points, this does not result from the preceding discussion (and from what was said at the end of § **4**, it cannot result). However, it is easy to see that around each asymptotic point the curve behaves like the pair of logarithmic spirals $\rho^2 = 4k^2 s^2$ around the common pole.



g) One calls the curve that is defined by the equation:

$$\rho = ka\,\sqrt{1 - e^{-2s/a}}$$

a *pseudo-tractrix*. It coincides with $\rho = k\sqrt{2as}$ in the vicinity of the origin, and therefore the pseudo-tractrix begins to unfold like the involute of a circle of radius k^2a. However, it soon changes its behavior since ρ does not increase indefinitely with increasing s, but it tends to ka. Therefore, the curve will admit two asymptotic circles, which correspond to the two ways of determining ρ. Since one will have:

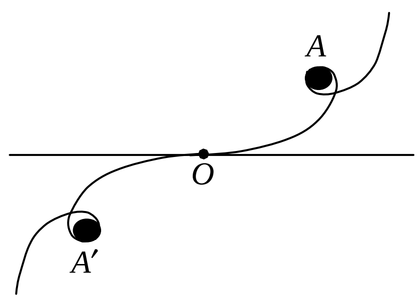
$$\varphi = \frac{1}{2k}\log\frac{ka+\rho}{ka-\rho}$$

upon starting from the cuspidal tangent, formula (11) will give:

$$u = -\,4k^2a\int_0^\infty \frac{\sin\varphi\, d\varphi}{(e^{k\varphi}+e^{-k\varphi})^2} = -\,\frac{\pi a}{2(e^{\pi/4k}+e^{-\pi/4k})},$$

$$v = \pm\, 4k^2a\int_0^\infty \frac{\cos\varphi\, d\varphi}{(e^{k\varphi}+e^{-k\varphi})^2} = \pm\frac{\pi a}{e^{\pi/2k}-e^{-\pi/2k}}$$

for the coordinates of the asymptotic circles. The segment that is traced out along the cuspidal normal by an asymptotic circumference is seen from its center to subtend an angle that cannot be as small as one desires. That angle is a minimum for a value of k that is roughly 0.665, in which case, the distance from the centers of the asymptotic circles to the normal and the cuspidal tangents are fractions of the radius that are close to the values 0.663 and 0.439, resp.



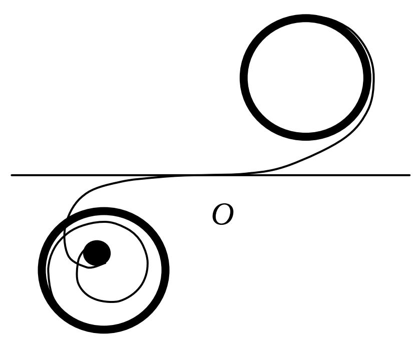
h) *What sort of curves are represented by a quadratic equation in s and ρ* that has no term in ρ^2 ? When $s\rho$ is missing, the equation (if it does not represent a pair of points) will be reducible to the form $\rho = as^2 + 2bs + c$, and will represent the pseudo-catenary or a curve that is analogous to the catenary according to whether $ac - b^2$ is negative or positive, resp. If $s\rho$ is not missing then the equation can always be given the form:

$$\rho = b + ks + \frac{a^2}{s}.$$

φ will increase indefinitely along with s and ρ as long as k is positive. In addition, ρ will also become infinite at the origin of the arcs, and around that point, the curve will behave like a clothoid ($s\rho = a^2$) in the vicinity of the inflection point. The curve will then inflect at the origin, and its two parts will spiral to infinity and tend to coincide with a logarithmic spiral ($\rho = ks$). The radius of curvature will take on the minimum value $a' = 2a\sqrt{k}$ at two well-defined points A and A', and the curve will behave almost like a catenary $\left(\rho = a' + 2k^2 \frac{s}{a'}\right)$ around each of them. However, if k is negative then the curvature will become infinite along with φ at the points A and A'. Those two points will then be asymptotes, and the curve will behave around each of them the way that the pair of spirals $\rho^2 = 4k^2 s^2$ does around their common pole. Finally, if k (but not b) is zero then the equation will represent a curve that is endowed with a pair of asymptotic circles of radius b (one internal, one external), besides an inflection at the origin, since ρ tends to b as s increases indefinitely in absolute value, and on the other hand:

$$\varphi = \frac{s}{b} - \frac{a^2}{b^2}\log\left(1 + \frac{bs}{a^2}\right)$$

will increase indefinitely. In addition, φ also increases to infinity when s tends to the value $-a^2 : b$. One will then have an asymptotic point at a finite distance from the origin, around which the curve behaves as the pair of spirals $a^4 \rho^2 = b^4 s^2$ does around their common pole.



i) Now, if the term ρ^2 *is not missing* then the equation will generally be reducible (for a suitable choice of origin) to one of the following forms:

$$\rho = b + ks \pm k'\sqrt{a^2 - s^2}, \qquad \rho = b + ks \pm k'\sqrt{s^2 - a^2}, \qquad \rho = b + ks \pm k'\sqrt{s^2 + a^2}.$$

In the exceptional case (i.e., when the terms of second degree constitute a perfect square) the equation can, however, be given the form $\rho = b + ks \pm \sqrt{2as}$. As special cases ($b = 0$, $k = 0$), one recovers the cycloid, the two pseudo-cycloids, and the involute of the circle. We leave the task of carrying out the study of all those curves and classifying them by a minimum number of normal types as an exercise for the reader.

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CHAPTER II

FUNDAMENTAL FORMULAS FOR THE INTRINSIC ANALYSIS OF PLANE CURVES

1. – From now on, we shall always assume that the axes are the tangents and the normals to a curve at an arbitrary point M. We consider a point P that moves with M: Let P' be its position when the origin M is transferred to M', and let x and y be the coordinates of P with respect to the axes whose origin is at M, while $x + \delta x$ and $y + \delta y$ are those of P'. In general, the coordinates x and y will vary when the initial axes pass over to the ones whose origin is M'. They will then be functions of s, and it is clear that if M' is infinitely close to M then the coordinates of P' with respect to the axes at the origin M' will be $x + dx$, $y + dy$. Now, if one represents the coordinates of M' with respect to the first axes by u, v and the angle through which the x axis rotates when it passes from the first position to the second one by $\delta\varphi$ then one will have:

$$x + dx = u + (x + \delta x) \cos \delta\varphi - (y + \delta y) \sin \delta\varphi = u + x + \delta x - y\ \delta\varphi,$$

$$y + dy = v + (x + \delta x) \sin \delta\varphi + (y + \delta y) \cos \delta\varphi = v + y + \delta y + x\ \delta\varphi\ ;$$

i.e., if one divides by $\delta s = ds$ and recalls the first three equalities of the preceding chapter then one will get:

$$\frac{\delta x}{ds} = \frac{dx}{ds} - \frac{y}{\rho} + 1, \qquad \frac{\delta y}{ds} = \frac{dy}{ds} + \frac{x}{\rho}. \tag{1}$$

These are the *fundamental formulas*, from which, if one takes δx and δy to be equal to zero (i.e., one supposes that P' coincides with P) then one will immediately derive the important conditions:

$$\frac{dx}{ds} = \frac{y}{\rho} - 1, \quad \frac{dy}{ds} = -\frac{x}{\rho}, \tag{2}$$

which are *necessary and sufficient for the immobility of the point* (x, y). In polar coordinates ($x = r \cos \theta$, $y = r \sin \theta$), those conditions will become:

$$\frac{dr}{ds} = -\cos\theta, \qquad \frac{d\theta}{ds} = -\frac{1}{\rho} + \frac{\sin\theta}{r}. \tag{3}$$

If one would prefer that the line that is defined by its distance q from the origin and the angle φ through which its positive direction rotates in order to make it coincide with that of the tangent should be immobile in the plane then one would need to express the idea that the equation of the line:

$$x \sin \varphi + y \cos \varphi + q = 0$$

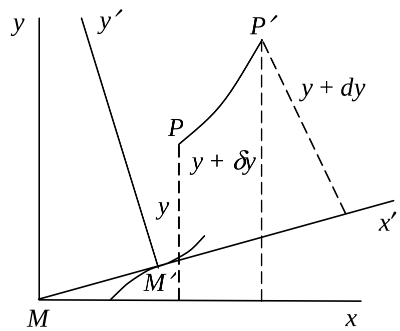
should be satisfied by the infinitude of solutions to (2), and since one will get:

$$(x \cos \varphi - y \sin \varphi)\left(\frac{d\varphi}{ds} - \frac{1}{\rho}\right) - \sin \varphi + \frac{dq}{ds} = 0$$

upon derivation, that relation must split into the (obviously geometrical) other ones:

$$\frac{d\varphi}{ds} = \frac{1}{\rho}, \qquad \frac{dq}{ds} = \sin \varphi, \tag{4}$$

which are necessary and sufficient for the immobility of the line (φ, q).



2. – Before we go any further, let us note some consequences of formulas (3) and (4). Suppose that the distance from M to a fixed point P tends to a finite limit a as s increases to infinity. The coordinates r and θ of P satisfy (3), and since r tends to a, its derivative cannot tend to a limit that is different from zero. Hence, if such a limit exists then the first of (3) must give lim $\theta = \pi / 2$. Now, one can apply the same consideration to θ, and the second formula (3) will then give lim $\rho = a$. Therefore, the circle of radius a and center at P will be an asymptotic circle to the curve. Conversely, when the curve admits an asymptotic circle, the distance from M to the center of the circle cannot tend to a limit that is different from the radius of the circle, since otherwise, a limit to ρ that is different from the first one would exist. If one then observes that the coordinates of the center of curvature are $r = \rho$, $\theta = \pi / 2$, and the first one tends to a then one will see that *the center of curvature will tend to collocate with the center of the asymptotic circle* when r increases to infinity. That observation fills a gap that was left open in the first chapter (§ **10**) and leads to a more precise knowledge of the asymptotic circles. Analogously, the second formula (4) will permit one to make an observation that is useful in the search for asymptotes, since if q tends to the limit zero for s infinite then φ cannot have any other limit; that is to say, *if the tangent admits a limiting position then it will necessarily be the fixed line considered* (cf., I, § **7**).

3. – The following noteworthy fact will emerge from formulas (2) and (4), and it is fundamental to intrinsic geometry: The parameters that serve to fix the points and lines in the plane are functions of s, *whose derivatives are expressed by means of those functions.* (2) will then show that the first derivatives of the coordinates x, y of a fixed point are linear functions of those coordinates, and one can assert the same thing for the successive derivatives, as one will see when one differentiates and applies (2) repeatedly. Having done that, when one seeks the curve that has a given property, one will generally be led to a relation:

$$f(x, y, x', y', \ldots, \varphi, q, \varphi', q', \ldots, s) = 0,$$

which includes a certain number of coordinates for points and lines that are fixed in the plane and which must be true for any s. Now, it is clear that it is enough to differentiate that relation in order to obtain another one between the same coordinates immediately that makes the derivatives of the coordinates disappear, thanks to the immobility condition. In that way, one will discover not only a new property of the unknown curve, but if one continues to differentiate then one will always succeed in constituting a system of relations such that it is possible to eliminate the coordinates x, y, x', … One will then need to integrate the differential equation that results from the elimination in order to find the intrinsic equation of the desired curve.

4. Geometric loci. – Suppose that the coordinates x, y of a point P are known as functions of s. How does one proceed in order to determine the locus of P? The fundamental formulas will immediately provide the values of δx and δy. One can then express the elementary arc length of (P) by $ds'^2 = \delta x^2 + \delta y^2$, that is:

$$s' = \int \kappa \, ds, \tag{5}$$

in which one has set:

$$\kappa^2 = \left(\frac{dx}{ds} - \frac{y}{\rho} + 1\right)^2 + \left(\frac{dy}{ds} + \frac{x}{\rho}\right)^2.$$

If one determines the inclination θ of the tangent at P to the unknown line (P) with respect to the x-axis by means of the formula $\tan\theta = \delta y : \delta x$ then one will observe that the inclinations of the tangent to (P) at P' with respect to the tangents to the line (M) at M and M' will be $\theta + \delta\varphi'$ and $\theta + d\theta$, in such a way that one will have $\delta\varphi' = \delta\varphi + d\theta$, and then if one divides by ds, one will get:

$$\frac{\kappa}{\rho'} = \frac{1}{\rho} + \frac{d\theta}{ds}, \tag{6}$$

as long as the positive sense of the normal at (P) is established in such a way that one rotates the positive direction of the tangent until it coincides with that of the x-axis, and the positive direction of the normal also coincides with that of the y-axis. Finally, it is enough to eliminate s from (5) and (6) to obtain the intrinsic equation for (P).

5. Envelopes. – The equation $f(x, y, s) = 0$ generally represents a simple infinitude of lines, each of which corresponds to a well-defined point along the line (M). The lines that correspond to the infinitely-close points M and M' can have one or more points in common, which can be regarded as fixed under the passage of the origin from M to M'. The coordinates of those points are then obtained by differentiating the equation $f = 0$ and supposing that the conditions (2) are satisfied, so they will then be solutions to the system:

$$f(x, y, s) = 0, \qquad \left(\frac{y}{\rho} - 1\right)\frac{\partial f}{\partial x} - \frac{x}{\rho}\frac{\partial f}{\partial y} + \frac{\partial f}{\partial s} = 0. \tag{7}$$

If x and y are known as functions of s then it will be enough to apply the procedure that was posed in the preceding paragraph in order to obtain the intrinsic equation of the locus of the aforementioned points of intersection. One can call that locus the *envelope* of the lines $f = 0$. Nonetheless, it can happen that (7) reduces to just one equation, and one will then have just one curve, such as, for example, when $f = 0$ is also the equation of (M). That observation will prove useful in what follows.

6. – One easily shows that the *envelope touches all of the lines that are enveloped.* Indeed, let P be a point that is common to two lines $f = 0$ that correspond to two infinitely-close points M and M' on the curve (M). Let θ and θ' be the inclinations of the x-axis with respect to the tangents at P to the envelope and the evolute, resp. Obviously, if one fixes s then the value of $\tan\theta'$ will be given by the ratio $\delta y : \delta x$, which one will get from the relation:

$$\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y = 0.$$

However, θ is determined by the process that was indicated in § **4**, in such a way that one will have:

$$\tan\theta = \left(\frac{dy}{ds} + \frac{x}{\rho}\right) : \left(\frac{dx}{ds} - \frac{y}{\rho} + 1\right), \qquad \tan\theta' = -\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y},$$

in which x and y satisfy (7). Now, if one forms $\tan(\theta - \theta')$ then one will get the expression:

$$\left(\frac{dx}{ds} - \frac{y}{\rho} + 1\right)\frac{\partial f}{\partial x} + \left(\frac{dy}{ds} + \frac{x}{\rho}\right)\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} - \left(\frac{y}{\rho} - 1\right)\frac{\partial f}{\partial x} + \frac{x}{\rho}\frac{\partial f}{\partial y}$$

in the numerator, and that will reduce to:

$$\frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial s} = \frac{df}{ds} = 0,$$

by virtue of (7). Therefore, $\theta = \theta'$.

7. Exercises:

a) *Find the intrinsic equation of a circumference* of radius *a*; i.e., the locus of points that are at a distance *a* from a fixed point (viz., the *center*). One can proceed in the following way: If *x* and *y* are the coordinates of the center then one must always have $x^2 + y^2 = a^2$; differentiate that then and observe (2), so $x = 0$. Therefore, one will have, at the same time, that *all of the normals are concurrent at the center.* If one differentiates again then one will get $y = \rho$; hence, if one substitutes *x* and *y* in the first relation then one will have $\rho = a$.

b) *Find all lines that have constant curvature.* It is enough to observe that when ρ is constant, the conditions (2) will be satisfied by the point ($x = 0$, $y = \rho$). That point is therefore fixed, and the curve will necessarily be a circumference of radius ρ, since all of its points will be found at the *constant* distance ρ from the fixed point $(0, \rho)$.

c) *What curves will meet all of the lines that go through a point at a constant angle* θ? If one applies (3) to the polar coordinates $(r, \pi - \theta)$ of the fixed point then one will get:

$$\frac{dr}{ds} = \cos\theta, \qquad 0 = -\frac{1}{\rho} + \frac{\sin\theta}{r}.$$

One deduces from the first one that $r = s\cos\theta$, if one agrees to measure the arc lengths by starting from the fixed point. If one substitutes that in the second one then one will find that $\rho = s\cot\theta$, which is the equation (I, § **11**, *b*, *c*) of a *logarithmic spiral.*

d) *What is the curve whose normals* (between the points of incidence and the centers of curvature) *bisect a line?* We need to express the idea that the point ($x = 0$, $y = \rho / 2$) describes a line. Now, (1) gives:

$$\frac{\delta x}{ds} = \frac{1}{2}, \qquad \frac{\delta y}{ds} = \frac{1}{2}\frac{d\rho}{ds}.$$

One will then have $\tan\theta = d\rho / ds$, and (6) will show that one must have:

$$\frac{d\theta}{ds} = -\frac{1}{\rho}, \tag{8}$$

which will, moreover, result immediately from the observation that θ will differ from the usual φ only in sign in the case of fixed line. Therefore, one will have, in turn:

$$\tan\theta\frac{d\theta}{ds} = -\frac{1}{\rho}\frac{d\rho}{ds}, \qquad \log\cos\theta = \log\rho + \text{constant}, \qquad \rho = a\cos\theta.$$

If one integrates (8), after inverting the sense in which one measures the arc length and conveniently fixing the origin, then will have:

$$s = \int \rho\, d\theta = a\sin\theta.$$

Therefore, $s^2 + \rho^2 = a^2$, i.e. (I, § **8**, *c*) the stated property characterizes the *cycloid.*

e) *Find a curve that has its centers of curvature symmetric with respect to that curve, such that the normals at its points meet a line.* Now, the point (x = 0, $y = -\rho$) must move along a line. If we repeat the calculations of the preceding exercise then we will get, in turn:

$$\frac{\delta x}{ds} = 2, \qquad \frac{\delta y}{ds} = -\frac{d\rho}{ds}, \quad \tan\theta = -\frac{1}{2}\frac{d\rho}{ds}.$$

Hence:

$$\tan\theta\frac{d\theta}{ds} = \frac{1}{2\rho}\frac{d\rho}{ds}, \qquad \log\cos\theta = -\log\sqrt{\rho} + \text{constant}, \qquad \rho = \frac{a}{\cos^2\theta},$$

and finally, if one inverts the sense in which one measures the arc length:

$$s = \int \rho\, d\theta = a\tan\theta, \qquad \rho = a + \frac{s^2}{a}.$$

That is (cf., I, § **5**, *b*) the equation of a *catenary*.

f) *Find a curve such that the segment of the tangent that is cut out by a line when one starts from the point of contact is constant.* This is treated by expressing the idea that the point ($x = a$, y = 0) describes a line. Now, one has:

$$\frac{\delta x}{ds} = 1, \qquad \frac{\delta y}{ds} = \frac{a}{\rho}, \qquad \tan\theta = \frac{a}{\rho}.$$

When the last equality is put into the form:

$$\cos\theta\frac{d\theta}{ds} = -\frac{1}{a},$$

thanks to (8), one will deduce, in turn:

$$\log \sin \theta = -\frac{s}{a}, \qquad \rho = a \cot \theta = a\sqrt{e^{2s/a} - 1}\ ;$$

i.e., one will get (I, § **8**, *b*) the intrinsic equation of a *tractrix*.

g) *Find a curve whose radius of curvature at any point is equal to the segment that is cut out from the normal by two parallel lines.* In other words, if *a* is the distance between the two lines then the projection of the radius of curvature onto a fixed line must be constantly equal to *a*, in such a way that $a = \rho \sin \varphi$. In order to express the invariability of the direction of the line, one has:

$$\frac{d\varphi}{ds} = \frac{1}{\rho} = \frac{\sin\varphi}{a};$$

hence:

$$\frac{1}{\sin\varphi}\frac{d\varphi}{ds} = \frac{1}{a}, \qquad \log \tan \frac{\varphi}{2} = \frac{s}{a}, \qquad \rho = \frac{a}{\sin\varphi} = \frac{a}{2}(e^{s/a} + e^{-s/a}) .$$

The desired curve is then (I, § **5**, *c*) a *catenary of equal resistance.*

h) What curve has a point in its plane such that *the projection of the radius vector onto the tangent is proportional to the arc length?* We need to have $x = ks$ for a solution (x, y) of (2). Now, the first condition (2) shows that one must also have $y = (k + 1)\,\rho$, and the second one leads to an equation that will give:

$$ks^2 + (k + 1)\,\rho^2 = \text{constant}$$

when integrated. If the constant is taken to be equal to zero then one will obtain a pair of logarithmic spirals. Otherwise, one will have an epicycloid ($k > 0$), a hypocycloid ($k < -1$), or (for *k* found between 0 and – 1) a curve of pseudo-cycloidal type (I, § **8**, *d*). If the equation of the curve is given the form:

$$\frac{s^2}{a^2} + \frac{\rho^2}{b^2} = 1$$

then one will easily find the value of *k* ; one will then have:

$$x = \frac{b^2 s}{a^2 - b^2}, \quad y = \frac{a^2 s}{a^2 - b^2}.$$

In order to show that these are the coordinates of the center *P* of the directrix of the circumference (I, § **8**, *c*), it is enough to observe that *y* is annulled along with *r*, that is to say, that the point that was found is the one at which the cuspidal tangents are concurrent. One will find, in addition, that *x* becomes equal to precisely the radius of the directrix.

i) In order to *find all curves that are similar to a given curve*, it is enough to observe that if r and θ are the polar coordinates of the center of similitude with respect to the given curve then the first coordinate will be multiplied by a constant k, while the other one will remain unaltered with respect to any similar curve. The conditions (3) will then become:

$$k\frac{dr}{ds'} = -\cos\theta, \qquad \frac{d\theta}{ds'} = -\frac{1}{\rho'} + \frac{\sin\theta}{kr};$$

it will then happen that one has $s' = ks$, $\rho' = k\rho$. One will therefore get the curves that are similar to a given curve with s and ρ multiplied by an arbitrary constant in the equations of those curves. In particular, note that if one proceeds in that way for the equations $\rho^2 = 2as$, $s^2 + \rho^2 = a^2$, etc., then one will succeed in only changing the value of a. Hence, the involutes of the circle are all mutually-similar curves, and one can say the same thing about cycloids, catenaries, etc. However, any linear equation in s and ρ will remain essentially unaltered, and the curves that are similar to a given logarithmic spiral can all be superimposed upon that spiral then. In other words, the logarithmic spirals have the peculiar property that *they do not deform when they dilate* (equally in all directions) about an arbitrary point. Dilatations resolve into a translation followed by a rotation around the new position of the pole for them.

j) Just as two (similar) curves can be arranged such that the tangents are *parallel* at two points M and M' along a straight line with a fixed point P, it can also happen that the tangents to two curves at points M and M' that are collinear with P are, on the contrary, *anti-parallel* with respect to MM'. The curves are then called *inverse*, and P is the *center of inversion*. The polar coordinates of P are (r, θ) with respect to the curve (M) and (r', θ') with respect to M'. Obviously, $\theta' = \pi - \theta$ when one agrees that M and M' can traverse the respective curves in the same sense in such a way that they preserve the straight line through P, and the normal to (M') is directed according to the convention that was made in § **4**. Now, if one observes that the inclination of the tangent at (M') with respect to the tangent to (M) is 2θ then formula (6) will give:

$$\frac{\kappa}{\rho'} = \frac{1}{\rho} + 2\frac{d\theta}{ds} = \frac{\sin\theta}{r} + \frac{d\theta}{ds}.$$

On the other hand, the conditions (3) will become:

$$\frac{dr'}{ds} = \kappa\cos\theta, \qquad \frac{\kappa}{\rho'} = \frac{\kappa\sin\theta}{r'} + \frac{d\theta}{ds},$$

and if one compares the last relation with the first one that was obtained then one will see that $\kappa = r' : r$, which would result immediately from a simple geometric consideration, moreover. Having done that, one will have:

$$\frac{dr'}{ds} = \frac{r'}{r} \cos \theta = -\frac{r'}{r} \frac{dr}{ds},$$

and therefore, when one integrates this: $r\, r' = a^2$. The circle of radius a with its center at P is the fundamental circle of inversion: All of the points of intersection of the two curves will necessarily fall along its circumference, since r cannot become equal to a with the same thing being true for r'. Finally, if one substitutes the value of κ in the first relation then one will get:

$$\frac{r}{\rho} + \frac{r'}{\rho'} = 2 \sin \theta,$$

and the geometric interpretation of that equality will show that *the centers of curvature at two corresponding points will also be collinear with the center of inversion.*

k) Inversions do not differ essentially from the transformations with index – 1. The *transformation of index v* consists of making a given point correspond to another point whose multiplier is proportional to the v^{th} power of the multiplier of the first point. Proceeding as for the inversions, one will easily succeed in proving that *the tangents at two corresponding points will be concurrent along the circumference that is determined by those points and the pole*, and one will find the relations:

$$a^{v-1}\, s' = v \int r^{v-1} ds, \qquad a^{v-1}\, \rho' = \frac{v\, r^v \rho}{r + (v-1)\, \rho \sin \theta},$$

from which, one can deduce the intrinsic equation of the transform with index v of any curve when one eliminates s.

l) Given a curve, *find the base with respect to a point*; i.e., the locus of the feet of the perpendiculars to the tangents to the curve that are dropped from a fixed point P. If one applies formulas (1) to the coordinates $x = r \cos \theta$, $y = 0$ of the projection M' of P onto the tangent then one will get:

$$\frac{\delta x}{ds} = \frac{r}{\rho} \sin \theta, \qquad \frac{\delta y}{ds} = \frac{r}{\rho} \cos \theta;$$

hence, $\delta y : \delta x = \cot \theta$. It will then follow that the normal to the base is anti-parallel to PM with respect to the tangents to (M), and it is therefore *divided by one-half the radius vector PM*. In addition, one sees that $\kappa = r : \rho$, and if one is careful to change θ into $\pi / 2 - \theta$ then formula (6) will become:

$$\frac{r}{\rho\rho'} = \frac{1}{\rho} - \frac{d\theta}{ds} = \frac{2}{\rho} - \frac{\sin \theta}{r}.$$

Therefore, the intrinsic equation of the base will result from eliminating s from the equalities:

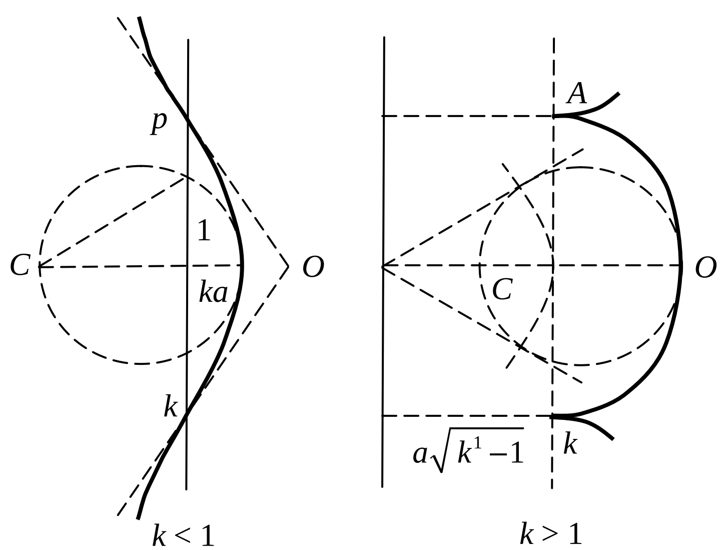
$$s' = \int \frac{r}{\rho} ds, \quad \rho' = \frac{r^2}{2r - \rho \sin\theta}.$$

If the given curve is, for example, a circumference of radius a, and if P belongs to the circumference, in such a way that $\rho = a$, $s = 2a\theta$, $r = 2a \sin\theta$, then the last formula will imply, in turn:

$$s' = 4a \int \sin\theta \, ds = -4a \cos\theta, \qquad \rho' = \frac{4a}{3} \sin\theta, \qquad s'^2 + 9\rho'^2 = 16\, a^2.$$

Therefore (I, § **8**, *c*), the desired base will be a *cardioid*. To conclude, observe that the expression for ρ' leads to the following simple construction for the center of curvature C': *If the projection H of the center of curvature of the given curve onto the radius vector is projected onto the normal at N then the line PN will contain the center of curvature of the foot.* Indeed, if L is the projection of P onto the normal and Q is the point of intersection of the radius vector with LM' (normal to the base) then the transversal $PC'N$ to the triangle LMQ will give:

$$\frac{r - \rho'}{\rho' - r/2} = \frac{LC'}{QC'} = \frac{PM}{PQ} \cdot \frac{LN}{MN} = 2\ \frac{r - \rho \sin\theta}{\rho \sin\theta}, \text{ etc.}$$



m) The curves that have *their curvatures proportional to the length of the segment of the normal that is found between the point of incidence and a fixed line* are important. In other words, the ratio $q\rho$: cos φ has a constant value for those curves. First of all, suppose that it is negative, in such a way that one can write:

$$q = -\frac{a^2}{\rho}\cos\varphi = -a^2\frac{d}{ds}\sin\varphi = -a^2\frac{d^2q}{ds^2},$$

by virtue of (4). It will then follow that if one fixes the origin of the arc length conveniently then $q = ka \cos s / a$, in which k represents an arbitrary constant that can always be assumed to be positive. Derivation will give $\sin\varphi$ the value $-k \sin s / a$, and these three relations:

$$q = ka\cos\frac{s}{a}, \qquad \sin\varphi = -k\sin\frac{s}{a}, \qquad \rho = -\frac{a^2}{\rho}\cos\varphi$$

will be enough to account for the form of the curves. One has $\varphi = 0$, $q = ka$, $\rho = -a : k$ at the origin ($s = 0$). Therefore, the curve will issue from the origin in two senses in parallel to the fixed line, and it will always approach that line more closely as s increases. If $k < 1$ then one can vary s until one has $\pm \pi a / 2$, and one will then have that $\sin\varphi = \mp k$, $q = 0$, $\rho = \infty$, that is to say, the curve terminates by crossing the line of inflection. The line therefore breaks into an infinitude of equal arcs of length πa, each of which is bounded on that line by two inflection points, and the tangents at those points are constructed by observing that its parallel to the chord (of length a) that goes from the origin O will have intersection points of the line with the circumference that is described by the radius of curvature at O for its diameter. If $k > 1$ then s cannot attain $\pm \pi a / 2$, since the value that is associated with $\sin\varphi$ cannot exceed unity. As soon as one has $\sin s / a = \pm 1 / k$ with increasing s, φ will become equal to $\mp\pi/2$, and ρ will be annulled, while q will take on the value $a\sqrt{k^2-1}$. One will then have an infinitude of arcs, each of which is bounded by two cusps, and the locus of the infinitude of cusps is a parallel to the given line. The intrinsic equations can be easily obtained by replacing φ and q with their values in the expression for ρ:

$$\rho = \frac{a}{k} + \frac{1-k^2}{2k}\frac{s^2}{a};$$

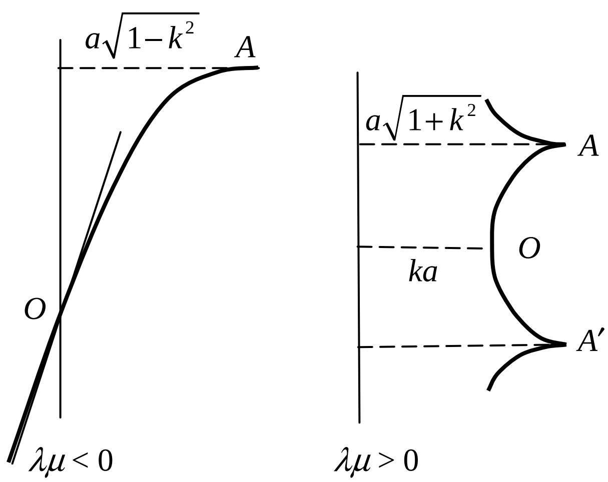
viz., it has the behavior of a catenary of the first kind and a pseudo-catenary of the second kind. The latter will then behave like the evolute of a circle of radius $a : \sqrt{k^2-1}$ in the vicinity of a cusp, while the former, on the contrary, behaves like the clothoid $s\rho = (a^2 / k)\sqrt{1-k^2}$ in the vicinity of an inflection point.

n) If the constant is positive then one will find that q must satisfy the differential equation $q = a^2\, d^2q / ds^2$, and must consequently have the form:

$$q = \lambda\, e^{s/a} + \mu\, e^{-s/a},$$

with λ and μ arbitrary constants. If one gives the value 0 to the constants then one will recover the tractrix. Suppose that λ and μ are not zero then. One will then change the values of the constants by an opportune choice of origin, and make them become equal to each other, at least in absolute value; it is enough to give the origin the displacement $\frac{1}{2} a \log\left(\mp\frac{\mu}{\lambda}\right)$ along the curve. One will then have two types of curve according to whether one takes the upper or lower sign in the formulas:

$$q = \frac{ha}{2}\left(e^{s/a} \mp e^{-s/a}\right), \qquad \sin \varphi = \frac{k}{2}\left(e^{s/a} \pm e^{-s/a}\right), \qquad \rho = \frac{a^2}{q} \cos \varphi.$$



If one sets $s =$ then one will see that the curves of the first kind have their origin on the fixed line and an inflection point there. However, as for the second kind, the origin is not on the line, but it is always the point that is closest to it. The tangent to the curves of the first kind at the origin make an angle with the line that has a sine that is equal to k (and one must then have $k < 1$), while the tangents are parallel to the fixed line for curves of the second kind. Meanwhile, s cannot increase indefinitely for any of those curves since the absolute value of $\sin \varphi$ cannot exceed unity, and s can vary only up to the value:

$$s = a \log \frac{1+\sqrt{1\mp k^2}}{k},$$

and one will then have $q = a\sqrt{1\mp k^2}$, $\varphi = \pm \pi / 2$, $\rho = 0$; i.e., one will have a cusp at which the tangent is perpendicular to the fixed line, and the analogous infinitude of cusps are all on the parallel in the direction of the fixed line at a distance of $a\sqrt{1\mp k^2}$.

8. Parallel curves. – We say that two curves that have the same normals are *parallel*. In order to express the idea that the point (x, y) describes a curve that is parallel to a curve (M), it is enough to set $x = 0$ and $\delta y = 0$. (1) will then become:

$$\frac{\delta x}{ds} = 1 - \frac{y}{\rho}, \qquad \frac{dy}{ds} = 0. \tag{9}$$

The second equality shows that y must have a constant value a, and two parallel curves will therefore be *equidistant,* as well. Conversely, if a curve cuts out segments equal to a on the normals to another one, when starting from the point of incidence, they will be parallel, since when one applies (1) to the point ($x = 0$, $y = a$), one will get $\delta y = 0$. If one observes that $ds' = \delta x$ then the first equality in (9) will give $s' = s - a\varphi$, and since $\theta = 0$, one can infer from (6) that $\rho' = \rho - a$. Therefore, two parallel curves *will have the same centers of curvature*. In order to get the intrinsic equation for the infinitude of parallels to a given curve, it is enough to eliminate s from the equations $s' = s - a\varphi$, $\rho' = \rho - a$. It is useful to observe that the equation of a family of parallel curves can always be given the form:

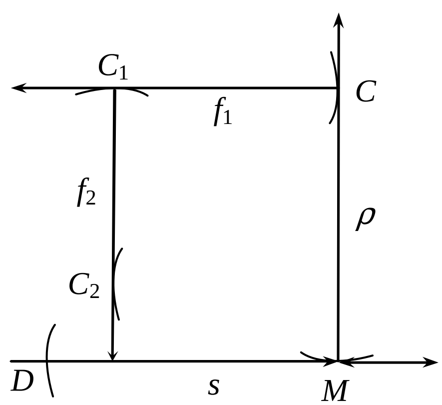
$$s = \int \frac{f(\rho + a)}{\rho + a} \rho \, d\rho, \tag{10}$$

when one determines the function f conveniently. Indeed, for $a = 0$, one will have:

$$s = \int f(\rho) \, d\rho, \qquad \varphi = \int \frac{f(\rho)}{\rho} d\rho;$$

hence:

$$s' = s - a\varphi = \int \left(1 - \frac{a}{\rho}\right) f(\rho) \, d\rho = \int \frac{f(\rho' + a)}{\rho' + a} \rho' \, d\rho'.$$



9. Evolutes and involutes. – One says the *evolute* of a curve to mean the envelope of its normals. Since they are also normals to any other parallel curve, one will see that such a curve is the evolute of an entire family of parallel curves; they are called *involutes* of the curve considered. Having said that, when the equation of the normal ($x = 0$) is differentiated, it will give $y = \rho$. Therefore, *the evolute of a plane curve is the locus of its centers of curvature;* that is an obvious geometrical property (cf., I, § **2**). In order to find the intrinsic equation of the evolute of a given curve (M), one needs to apply (1) to the coordinates $x = 0$, $y = \rho$ of the point C, which is the center of curvature of (M) at M. One will get:

$$\frac{\delta x}{ds} = 0, \qquad \frac{\delta y}{ds} = \frac{d\rho}{ds}; \tag{11}$$

if one distinguishes everything that refers to the evolute by an index 1 then one will have $ds_1 = \delta y = d\rho$. Therefore, if one conveniently fixes the origin of the arc length then $s_1 = \rho$. It will then follow that *any arc of the curve is equal to the difference between the extreme tangents that are bounded between the contact points and any involute.* Hence, it is clear that if an inextensible filament that is initially wrapped along a curve is unwrapped in the plane such a way as to always remain tensed then its points will describe the infinitude of involutes of the curve considered. Finally, if $\theta = \pi / 2$ then one can deduce from (6) that $\rho_1\, ds = \rho\, d\rho$. Therefore, *the intrinsic equation of the evolute of a given curve is obtained by eliminating s from the equations:*

$$s_1 = \rho, \qquad \rho_1 = \rho \frac{d\rho}{ds}. \tag{12}$$

10. – Here, we should observe that when s increases to infinity, if the curve in question gets wrapped asymptotically around a circle of center P then (11) will tend to become the immobility conditions for its center of curvature; hence, it will conform to what we saw in § **2**. In addition, the second of (12) gives $\lim \rho_1 = 0$, and since the function φ that relates to the evolute will increase indefinitely like the one that relates to the given curve (which differs from it by only a constant), the evolute will have P for an asymptotic point. Thus, the centers of the asymptotic circles of a curve are asymptotic points of the evolute. The identical analytical situation will present itself when ρ is a minimum or a maximum at a point will a well-defined tangent. In general, the evolute will experience a regression then. It is easy to account for all of that geometrically (cf., I, §§ **6**, **9**).

11. – The evolute of the evolute of a curve is called the *second* evolute of that curve; the evolute of the second evolute is the *third* evolute, etc. Let s_n and ρ_n be the arc length and radius of curvature, resp., of the n^{th} evolute of (M) at a given point M. If one applies (12) to the $(n-1)^{\text{th}}$ evolute then one will get:

$$s_n = \rho_{n-1}, \qquad \rho_n = \rho_{n-1} \frac{d\rho_{n-1}}{ds_{n-1}} = \rho_{n-1} \frac{ds_n}{ds_{n-1}},$$

in such a way that:

$$\frac{ds_n}{\rho_n} = \frac{ds_{n-1}}{\rho_{n-1}} = \ldots = \frac{ds}{\rho};$$

hence:

$$\rho_n = \rho \frac{d}{ds} \rho_{n-1} . \tag{13}$$

Therefore, if one is given an arbitrary term in the sequence $\rho, \rho_1, \rho_2, \rho_3, \dots$ of functions of s, it is enough to take the derivative and to multiply by ρ in order to obtain the next term. More simply, if one assumes that the independent variable is the function φ that is common to the curve and all of its evolutes then one will see immediately when one takes ds to be $\rho\, ds$ (13), that $\rho_1, \rho_2, \rho_3, \dots$ *are the successive derivatives of r with respect to* φ.

12. – We conclude with an observation that is not devoid of interest. If the centers of curvature $C, C_1, C_2, C_3, \dots$ at a point M tend to a limiting point P then the same thing will be true for all other points of (M), at least in all of a conveniently-determined arc around M. Indeed, assuming the existence of the limiting position P is equivalent to the supposing that the series:

$$x = -\rho_1 + \rho_3 - \rho_5 + \dots, \qquad y = \rho - \rho_2 + \rho_3 + \dots \tag{14}$$

are convergent. The sums of those series are precisely the coordinates of P. Meanwhile, if one applies the formula (13) then one will have:

$$\rho \frac{dx}{ds} = -\rho_2 + \rho_4 - \rho_6 + \dots = y - \rho, \qquad \rho \frac{dx}{ds} = \rho_1 - \rho_3 + \rho_5 - \dots = -x ;$$

i.e., the conditions (1) are satisfied. Hence, the point P will be fixed in the plane of the curve.

13. Exercises:

a) *What is the evolute of the tractrix?* If one writes the equation of the tractrix in the form:

$$\rho^2 + a^2 = a^2 e^{2s/a}$$

and differentiates it then one will get:

$$\rho_1 = a\, e^{2s/a} = a + \frac{\rho^2}{a} = a + \frac{s_1^2}{a} .$$

The desired curve is therefore a catenary.

b) Operating analogously on the equation $k^2 a^2 - \rho^2 = k^2 a^2 e^{-2s/a}$, one will get:

$$\rho_1 = k^2 a\, e^{-2s/a} = k^2 a - \frac{\rho^2}{a} = k^2 a - \frac{s_1^2}{a}.$$

Therefore (I, § **11**, *f*, *g*), *the pseudo-catenary is the evolute of a pseudo-tractrix.*

c) We would like to find the curves that cross an infinitude of equal circumferences with their centers along a line at a constant angle. Obviously, the coordinates of the center of a circle with respect to the tangent and the normal to the corresponding point of the unknown curve have the constant values a and b. One will then deduce from (1) that:

$$\frac{\delta x}{ds} = 1 - \frac{b}{\rho}, \qquad \frac{\delta y}{ds} = \frac{a}{\rho}, \qquad \tan\theta = \frac{a}{\rho - b};$$

hence:

$$s = -\int \rho\, d\theta = \int \frac{a\rho\, d\rho}{(\rho - b)^2 + a^2}.$$

Now, appeal to the formula (10) in order to determine the parallel in the direction of the unknown curve at a distance of b. One finds that:

$$s = \int \frac{a\rho\, d\rho}{\rho^2 + a^2} = \frac{a}{2}\log(\rho^2 + a^2) + \text{constant}, \qquad \text{i.e.,} \qquad \rho = a\sqrt{e^{2s/a} - 1},$$

and the desired curves will then be parallel to a tractrix. They are thus *the infinitude of involutes of the catenary.*

d) The evolute of the logarithmic spiral is obtained immediately by differentiating the intrinsic equation $\rho = ks$. One will find that $\rho_1 = k\rho = ks_1$, and then analogously, $\rho_2 = ks_2$, etc. Therefore, *the logarithmic spiral is equal to all of its evolutes.* In addition, if one observes that one has $\rho_n = k^{n+1} s$ then one will see that for k less than 1 in absolute value, formulas (14) will give:

$$x = -\frac{k^2 s}{1 - k^2}, \qquad y = \frac{ks}{1 - k^2}.$$

The point P that we spoke of in § **12** is then the pole (I, § **11**, *b*) of that curve in the present case.

e) *The cycloid is also equal to its evolute.* Indeed, if one differentiates $s^2 + \rho^2 = a^2$ then one will get $\rho_1 = -s$, and since $s_1 = \rho$, one will have $s_1^2 + \rho_1^2 = a^2$. However, since $s^2 - \rho^2 = a^2$, one will deduce that $s_1^2 - \rho_1^2 = \mp a^2$. Hence (I, § **8**, *d*), two pseudo-cycloids with equal parameters and different types are such that *each of them is the evolute of the other.* It will then follow that *any pseudo-cycloid is equal to its own evolutes of even order.*

f) More generally, it is easy to deduce from the equation $b^2 s^2 + a^2 \rho^2 = a^2 b^2$ that $b_1^2 s_1^2 + a_1^2 \rho_1^2 = a_1^2 b_1^2$, in which, one has set $a_1 = b$, $b_1 = b^2 : a$. Since the equation of the evolute can be obtained from that of the original curve by multiplying s and ρ by $a : b$, one will see that *the epicycloids and the hypocycloids are curves that are similar to their own evolutes.* If one consider the geometric progression that begins with the terms a and b then the n^{th} evolute of the cycloidal line that is defined by the parameters a and b will be defined by the n^{th} and $(n + 1)^{\text{th}}$ terms of the progression. If it converges, which is what happens (I, § **8**, *c*) only for the epicycloid, then the observation that was made in § **12** will be applicable, and one will find that:

$$x = \frac{b^2 s}{a^2 - b^2}, \quad y = \frac{a^2 \rho}{a^2 - b^2}.$$

These are precisely (cf., I, § **7**, *h*) the coordinates of the center of the directrix, at which the successive evolutes of the epicycloid tend to accumulate.

g) Do there exist curves for which *any point and the corresponding center of curvature of the evolute are in a straight line with a fixed point?* More generally, we seek the envelope of the MC_1. The equation of that line is $\rho x + \rho_1 y = 0$. If one differentiates that then one will get $(\rho + \rho_1) y = \rho^2$. Thus, the coordinates of the contact point P of MC_1 with its envelope will be:

$$x = -\frac{\rho\rho_1}{\rho + \rho_2}, \qquad y = \frac{\rho^2}{\rho + \rho_2}, \tag{15}$$

and if one applies the procedure that was pointed out in § **4** to them then that will yield the equation of the evolute in any case. If one wishes that P should be fixed then one would need to express the idea that the conditions (2) are satisfied by the preceding functions x and y; however, it is preferable to continue the derivations and get back to the last equation that was obtained, namely, $(\rho + \rho_2)\, y = \rho^2$. If one keeps in mind the expressions for the coordinates then one will find that $\rho\rho_2 = \rho_1 \rho_2$, and one will see that C_3 belongs to MC_1. Meanwhile, it is easy to convince oneself that the point P belongs to CC_2, i.e., that the curve (C) has the same property as (M), and C_4 will belong to CC_2 then, etc. The curves that one must seek are then the ones whose *successive centers of curvature* at an arbitrary point *M are collocated on two lines* that rotate around the fixed point P when M traverses the curve (M). What are those curves? If one writes the equation $\rho\rho_3 = \rho_1 \rho_2$ in the form $\rho : \rho_1 = \rho_2 : \rho_3$, while recalling the relation (13) and integrating, then one will deduce directly that $\rho_2 = k\rho$; integrating again will give $\rho_1 = ks$, and finally, $\rho^2 - ks^2 =$ constant. Hence, the desired curves are the ones of cycloidal or pseudo-cycloidal type, and the logarithmic spirals. In addition, if one substitutes the values $\rho_1 = ks$, $\rho_2 = k\rho$ then one will get:

$$x = -\frac{ks}{1+k}, \qquad y = \frac{\rho}{1+k},$$

and one will then see (cf., I, § **7**, *h*) that the fixed point P is the center of the directrix. One can also arrive at those curves by posing only the condition that C_1 and C_2 should be on a straight line with M.

h) In order to find the involute of the circle of radius a, it is enough to set $\rho_1 = a$ in the second formula in (12). One gets $\rho\, d\rho = a\, ds$, so $\rho^2 = 2as$, and that will then justify the name that given to the curve that is represented by that equation in the beginning (I, § **8**, *a*). Now, if one would like to find *one* involute of the involute then one would need to substitute the expressions (12) for s_1 and ρ_1 in $\rho_1^2 = 2as_1$, and upon integrating, one would find that ρ^3 is proportional to s^2. In that way, one will be led to assume that the equation of an $(n - 1)^{\text{th}}$ involute of the circle takes the form $\rho^n = k_n\, s^{n-1}$. Meanwhile, thanks to (12), one will find that the intrinsic equation of the evolute of that curve is $\rho^{n-1} = \left(1-\frac{1}{n}\right)^{n-1} k_n\, s^{n-1}$, and on the other hand, it is necessary that $\rho^{n-1} = k_{n-1}\, s^{n-2}$. A comparison of the two equations will permit one to calculate k_n (recalling that $k_1 = a$), and one will then find that:

$$\rho^n = \frac{n!}{n^n}\, a\, s^{n-1}$$

is the equation of the $(n - 1)^{\text{th}}$ involute of the circle of radius a. The successive involutes always tend to increasingly take the form of a logarithmic spiral, since the intrinsic equation will tend to become $\rho = es$ as n increases indefinitely.

i) All of the curves $\rho = ks^n$ have evolutes of the same type. One knows (I, § **11**, *e*) that those curves belong to four different types, according to whether n falls in one of the four intervals that are determined by the end points $-\infty$, 0, 1, 2, ∞. Recall that the clothoid is a curve of the first kind and the involute of the circle is one of the second kind. The curves of the fourth kind are characterized by the presence of asymptotes at finite distances, while those of the third kind extend to infinity, like the curve $a\rho = s^2$, which admits a known (I, § **11**, *a*) involute. If one applies (12) to the equation $\rho = ks^n$ then one will get an analogous equation in which the exponent n will become $n_1 = 2 - 1/n$. It will then follow that the curves of the first kind have evolutes of the fourth kind, and that is explained by observing that the asymptotes arise from the points with higher-order contact than the tangent. The curves of the second type have the evolutes of the same kind or the first; the other ones have evolutes of the third kind. It is the last kind that one always tends to appear when one takes successive evolutes, while making an exception for the involutes of the circle. Indeed, the exponent n for the v^{th} evolute has the value:

$$n_v = \frac{(v+1)\,n - v}{vn - (v-1)},$$

and one will easily see that one has $1 < n_v < 2$ for $v \geq 2$ and for n negative or greater than 1. Only the curves of the second kind have evolutes of all kinds, since the v^{th} evolute is of the third, fourth, first, or second kind according to whether n belongs to one of the intervals that are determined by the end points 0, $\frac{v-2}{v-1}$, $\frac{v-1}{v}$, $\frac{v}{v+1}$, 1, respectively, not including the end points that correspond to the circle and its involutes or to the logarithmic spiral. However, since the intermediate numbers tend to unity when v increases indefinitely, it is obvious that the third kind will eventually prevail in any case. Moreover, one can arrive at that conclusion by observing that when n is found between 0 and 1, it will be enough to take $(v - 1)(1 - n) > 1$, since n_v falls between 1 and 2. Since one will then have $\lim n_v = 1$ in any case, it is clear that the tendency of all evolutes is to take the form of the logarithmic spiral.

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CHAPTER III

NOTEWORTHY PLANE CURVES

1. Conics. – One says *conic* to mean any curve that is represented by an equation of degree two in the coordinates x and y of its points in the usual Cartesian system of immobile axes. When the conic is referred to the tangent and the normal at an arbitrary point M, the constant term will be absent from the equation. Hence, one must have (I, § **1**.4):

$$\lim \frac{y}{x} = 0, \qquad \lim \frac{y}{x^2} = \frac{1}{2\rho}$$

as x tends to zero, so one will see that the equation must also be lacking terms in x, in such a way that it can be given the form:

$$y = \tfrac{1}{2}(\alpha x^2 + \beta y^2 + 2\gamma xy), \tag{1}$$

in which α will represent the curvature at M, for now. One should observe that from the arbitrary choice of origin M, it will generally happen that if x increases indefinitely then y will also be infinitely large of the same order, and therefore the left-hand side of (1) will become negligible compared to the right-hand side, which can always be decomposed into linear factors:

$$\alpha x^2 + \beta y^2 + 2\gamma xy = (\lambda x + \mu y)\,(\lambda' x + \mu' y).$$

Therefore, the infinitude of curves will tend to behave like a pair of lines:

$$\lambda x + \mu y = q, \qquad\qquad \lambda' x + \mu' y = q', \tag{2}$$

which will be conjugate imaginaries when $\Delta = \alpha\beta - \gamma^2$ and positive, real, and distinct when $\Delta < 0$. In the first case, the conic will be called an *ellipse*, while in the second case, it will be called a *hyperbola*. Between the ellipses and hyperbolas, one finds the *parabola*, which is characterized by $\Delta = 0$. One will then call the conic that behaves like a pair of orthogonal lines at infinity an *equilateral hyperbola*. It is characterized by the orthogonality condition of (2), i.e., $\lambda\lambda' + \mu\mu' = 0$, which reduces to $\alpha + \beta = 0$.

2. Asymptotes. – Imagine that one sets x and y equal to the coordinates of the points of the curve in the left-hand sides of (2). The equality between the left-hand sides and the right-hand sides will be destroyed, but it will tend to be reestablished when the point (x, y) recedes indefinitely along the curve when q and q' are set to the limits in the left-hand sides. Since the differences between the left-hand sides and the right-hand sides represent the distances from (x, y) to the two lines, up to finite factors, they will be (II, § **2**) asymptotes of the curve. Meanwhile, if one exhibits the factor x in $\lambda x + \mu y$ and $\lambda' x + \mu' y$ then one will immediately see that if those quantities tend to finite limits when x

increases indefinitely then the ratio $y : x$ will tend to $-\lambda : \mu$ for the first quantity and $-\lambda' : \mu'$ for the second one. Consequently, if one takes into account the relations:

$$\lambda\lambda' = \alpha, \qquad \mu\mu' = \beta, \qquad \lambda\mu' + \mu\lambda' = 2\gamma, \qquad \lambda\mu' - \mu\lambda' = 2i\sqrt{\Delta}$$

then one will find:

$$q = \lim\,(\lambda x + \mu y) = \lim \frac{2y}{\lambda' x + \mu' y} = \frac{2\lambda}{\lambda\mu' - \mu\lambda'} = -\frac{i\lambda}{\sqrt{\Delta}},$$

$$q' = \lim\,(\lambda' x + \mu' y) = \lim \frac{2y}{\lambda x + \mu y} = \frac{2\lambda'}{\mu\lambda' - \lambda\mu'} = \frac{i\lambda'}{\sqrt{\Delta}}.$$

These values will become infinite for the parabola ($\Delta = 0$), and in addition, $\alpha x^2 + \beta y^2 + 2\gamma xy$ and the square of $\lambda x + \mu y$ or $\lambda' x + \mu' y$; i.e., when Δ tends to zero, (2) will tend to represent a pair of coincident lines that get pushed out to infinity.

3. Centers and diameters. – If one solves (2) then one will find that the asymptotes (whether real or imaginary) will always intersect at the real point:

$$x = -\frac{\gamma}{\Delta}, \qquad y = \frac{\alpha}{\Delta}. \tag{3}$$

It is useful to observe that these coordinates satisfy the equations:

$$\alpha x + \gamma y = 0, \qquad \gamma x + \beta y = 1, \tag{4}$$

which one can substitute for (2) as the ones that one appeals to in the search for that point. Now, if (1) is written in the form:

$$2y = (\alpha x + \gamma y)\, x + (\gamma x + \beta y)\, y$$

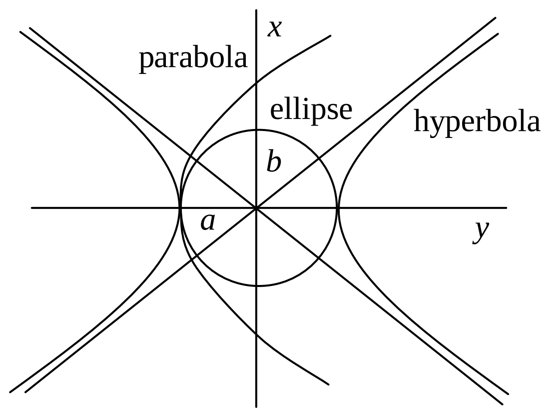
then (4) will show that the left-hand side becomes y, and therefore the equation is not satisfied for the values (3), but it is enough to *double* those values in order to verify the equation. Therefore, the point O, which is defined by the coordinates (3), is a *center* of the curve, which amounts to saying that it divides all of the chords that pass through it in half. In addition, if we fix y arbitrarily then equation (1) will provide two values for x, whose arithmetic mean will be $-\gamma y : \alpha$. Hence, the midpoint of any chord that is parallel to the tangent at M is such that one will have $\alpha x + \gamma y = 0$; i.e., it will satisfy the first of (4), which represents the line OM. Thus, if one calls the locus of midpoints of a system of parallel chords a *diameter* then one will see that *the diameters of a conic are the lines that radiate from the center.*

4. Vertices and axes. – One calls the normal diameters to a conic its *axes* and their points of incidence are the *vertices.* In order for M to be a vertex, from (3), one must have that γ is zero, and then the segment OM will have the length $a = \alpha : \Delta = 1 : \beta$. Meanwhile, if one transports the origin to O, for the moment, then the equation of the conic will become:

$$\alpha x^2 + \beta y^2 + 2\gamma xy = \frac{\alpha}{\Delta}, \tag{5}$$

and when M is a vertex and one sets $b^2 = a : \alpha$, it will reduce to:

$$\alpha x^2 + \beta y^2 = a^2\, b^2. \tag{6}$$



That will permit one to rapidly account for the form of the curve in various cases, and to recognize the existence of two axes (bisectors of the asymptotes) and four vertices, which are all real for the ellipse ($b^2 > 0$) and two real and two imaginary for the hyperbola ($b^2 < 0$). As always, let a denote the positive root of a^2 and let b or $b : i$ denote the positive root of b^2 or $-b^2$, and in the former case, suppose that $a > b$, and switch a with b if that were not true. In both cases, with those conventions, the axis that cuts out the segment $2a$¸ which is always real, from the conic is distinguished from the other one by the name of *focal axis.* Having assumed that, rotate the focal axis by θ and θ', and assume that its new positions are the x and y axes, respectively. Equation (6) transforms into:

$$a^2\,(x \sin\theta + y \sin\theta')^2 + b^2\,(x \cos\theta + y \cos\theta')^2 = a^2\, b^2,$$

and the term xy will be missing if θ is linked to θ' in such a way that it will reduce $a^2 \sin\theta \sin\theta' + b^2 \cos\theta \cos\theta'$ to zero; that is, if one poses the relation:

$$\tan\theta \tan\theta' = -\frac{b^2}{a^2} \tag{7}$$

between θ and θ'. The equation is therefore back to having the form (6), and since for any value that one attributes to one coordinate, the other one will take values that are equal and opposite, one will see that the diameters of a conic can be associated in pairs in such a way that for any pair, *each diameter will bisect the chord that is parallel to the* *other one.* Two such diameters are said to be *conjugate* to each other. One should note that each asymptote is conjugate to itself, and that the only pair of orthogonal conjugate diameters is conjugate to the axes. One excludes the equality of b and a from all of this, since that will be true only when the conic reduces to a circle. It is then pointless to observe that *it is only for the circle and the equilateral hyperbola that any orthogonal pair of diameters will be conjugate to another orthogonal pair,* since in order to have:

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2}, \quad \cot\theta\cot\theta' = -\frac{b^2}{a^2},$$

it would be sufficient that one should have $a^4 = b^4$ (i.e., $a^2 = \pm b^2$), and one would quickly see that if one takes the lower sign then that would define the equilateral hyperbola precisely.

5. – The calculation of the semi-axes a and b follows easily upon observing that when one passes from (5) to (6), an orthogonal form that is defined by the first of the discriminants:

$$\begin{vmatrix} \alpha & \gamma \\ \gamma & \beta \end{vmatrix}, \qquad \begin{vmatrix} \dfrac{\alpha}{b^2\Delta} & 0 \\ 0 & \dfrac{\alpha}{a^2\Delta} \end{vmatrix}$$

will be transformed orthogonally into another one that is defined by the second discriminant. Such a transformation will leave orthogonal invariants unaltered, and one will then have:

$$\alpha + \beta = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\frac{\alpha}{\Delta}, \qquad \alpha\beta - \gamma^2 = \frac{\alpha^2}{a^2b^2\Delta^2},$$

from which, one infers that:

$$a^2 + b^2 = \frac{\alpha(\alpha+\beta)}{\Delta^{3/2}}, \qquad ab = \frac{\alpha}{\Delta^{3/2}}. \tag{8}$$

These formulas show that the *equilateral hyperbola* is characterized by the relation $b = ia$, and that a and b are infinite for the parabola. In addition, if one takes into account the relation:

$$\frac{a^2b^2}{(a^2+b^2)^{3/2}} = \frac{\alpha^{1/2}}{(\alpha+\beta)^{3/2}}, \tag{9}$$

which is obtained by eliminating Δ from (8), then one will see that if one fixes a and b and lets γ^2 increase to $\alpha\beta$ then the ellipse that is defined by the semi-axes a and b will tend to be converted into a parabola, and that will happen in such a way that when the semi-axes increase indefinitely, the left-hand side of (9) will always remain equal to a certain length p, which is called the *parameter* of the parabola. It will then follow that b cannot be infinitely large of order a, since the left-hand side of (9) would then increase to infinity like a. One must then suppose that b is negligible with respect to a, and (9) will then become $\lim (b^2 : a) = p$.

6. Intrinsic equation. – Let the origin move along a curve and differentiate equation (1), which expresses the idea that the immobility conditions are satisfied by x and y, and observe that α, β, γ are functions of s. The equations that one obtains:

$$\left(\alpha-\frac{1}{\rho}\right)x + \gamma y = \frac{1}{2}\left(\frac{d\alpha}{ds}-\frac{2\gamma}{\rho}\right)x^2 + \frac{1}{2}\left(\frac{d\beta}{ds}+\frac{2\gamma}{\rho}\right)y^2 + \left(\frac{d\gamma}{ds}+\frac{\alpha-\beta}{\rho}\right)xy$$

must coincide with (1) (II, § **5**), and therefore $\alpha = 1 / \rho$; hence:

$$\frac{d\alpha}{ds} = \left(\alpha+\frac{2}{\rho}\right)\gamma, \qquad \frac{d\beta}{ds} = \left(\beta-\frac{2}{\rho}\right)\gamma, \qquad \frac{d\gamma}{ds} = \gamma^2 - \frac{\alpha-\beta}{\rho}\ . \tag{10}$$

The first of these formulas immediately gives:

$$\gamma = \frac{\rho}{3}\frac{d}{ds}\frac{1}{\rho} = \frac{d}{ds}\log\rho^{-1/2} = -\frac{\rho_1}{3\rho^2}.$$

The substitution of the values of α and γ in the first equation (4) will lead to the construction that **MacLaurin** pointed out of the center of curvature C_1 of the evolute of a conic. Indeed, the aforementioned equation will become $3\rho\, x = \rho_1\, y$, and one will deduce from this that if the diameter OM meets the normal to the evolute at Q then the segment QC_1 will be divided by the center of curvature of the conic in the ratio of 1 to 3. Return to (10) and observe that one has:

$$\frac{d}{ds}(\alpha+\beta) = (\alpha+\beta)\,\gamma, \qquad \frac{d\Delta}{ds} = 2\gamma\Delta,$$

i.e.:

$$\frac{d}{ds}\log(\alpha+\beta) = \frac{d}{ds}\log\rho^{-1/3}, \qquad \frac{d}{ds}\log\Delta = \frac{d}{ds}\rho^{-1/3}.$$

If one identifies A and B with two arbitrary constants then:

$$\alpha+\beta = A\,\rho^{-1/3}, \qquad \Delta = B\,\rho^{-2/3}. \tag{11}$$

On the other hand:

$$\gamma^2 = -\alpha^2 + \alpha(\alpha+\beta) - \Delta = -\alpha^2(1 - A\,\rho^{2/3} + B\,\rho^{1/3}),$$

or

$$\frac{1}{9}\left(\frac{d\rho}{ds}\right)^2 = -1 + A\,\rho^{2/3} - B\,\rho^{1/3}. \tag{12}$$

The intrinsic equation of the conic is therefore:

$$s = \frac{1}{3}\int \frac{d\rho}{\sqrt{-1 + A\rho^{2/3} - B\rho^{4/3}}}.$$

7. – In order to determine the constants A and B as functions of the semi-axes, recall (§ **4**) that when M is a vertex, the normal will contain the center, and therefore, by virtue of (3), one will have $\gamma = 0$, and the value of a or b will be expressed by:

$$\frac{\alpha}{\Delta} = \frac{\alpha}{B\,\rho^{-2/3}} = \frac{1}{B\,\rho^{1/3}}.$$

As a consequence, if one sets $\rho^{2/3} = 1 : B^2\,z$ in the equation that one obtains by setting the right-hand side of (12) equal to zero then the equation thus-transformed:

$$1 - AB\,z + B^3\,z^2 = 0$$

will admit the roots a^2 and b^2, precisely, and one will then have:

$$a^2 + b^2 = \frac{A}{B^2}, \qquad a^2\,b^2 = \frac{1}{B^3};$$

hence:

$$A = (a^2 + b^2)\,(ab)^{-4/3}, \quad B = (ab)^{-2/3}. \tag{13}$$

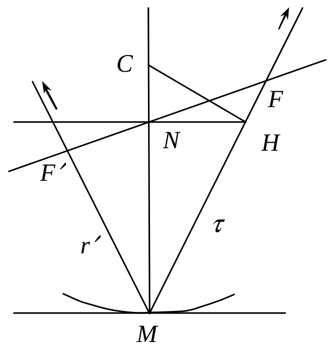
In addition, one notes that the curvature will have the values $B^3\,a^2 = a : b^2$ and $B^3\,b^2 = b : a^2$ on the focal axis and the other axis (but nowhere else), resp. We could have arrived at formula (13) much more rapidly by a different path, since (11) is no different from (8). However, the procedure that we used in order to arrive at (11) has the advantage that it is always applicable, independently of the preliminary knowledge of the properties of the curve. Thanks to (13), the intrinsic equation of the conic will then become:

$$s = \frac{1}{3}\int \frac{d\rho}{\sqrt{\left[1 - \left(\dfrac{b\rho}{a^2}\right)^{2/3}\right]}\sqrt{\left[\left(\dfrac{a\rho}{b^2}\right)^{2/3} - 1\right]}}. \tag{14}$$

What are the particular forms that this equation takes for the parabola and the equilateral hyperbola? In the case of the equilateral hyperbola, one will have $b = ia$, and (13) will give $A = 0$, $B = -a^{-1/3}$. However, one has $B = 0$ for the parabola, and the value of A will be obtained by recalling that the left-hand side of (9) represents the parameter p, in such a way that $A = p^{-2/3}$. Hence, *the intrinsic equations of the parabola and the equilateral hyperbola are:*

$$s = \frac{1}{3}\int \frac{d\rho}{\sqrt{\left(\frac{\rho}{p}\right)^{2/3} - 1}}, \qquad s = \frac{1}{3}\int \frac{d\rho}{\sqrt{\left(\frac{\rho}{a}\right)^{4/3} - 1}}, \tag{15}$$

respectively.



8. Foci and curvature. – Let us look for those curves that are *the sums of the distances from each point to two fixed, constant points.* If F (r, θ) and F (r', θ') are the two points then one must have $r + r' = 2a$, and if one differentiates this, while taking into account the first immobility condition, then one can infer that $\theta + \theta' = \pi$. Therefore, *the normal bisects the angle between the radius vectors.* In addition, from the conditions:

$$\frac{d\theta}{ds} = -\frac{1}{\rho} + \frac{\sin\theta}{r}, \qquad \frac{d\theta'}{ds} = -\frac{1}{\rho} + \frac{\sin\theta'}{r'}, \tag{16}$$

one can deduce:

$$\frac{2}{\rho} = \left(\frac{1}{r} + \frac{1}{r'}\right)\sin\theta \tag{17}$$

when one sums them. If one draws a perpendicular to the normal, which is on FF', through the point N on that normal then that will determine segments MH and MH' along its radius vectors, and the length τ will be given by the formula:

$$\frac{2}{\tau} = \frac{1}{r} + \frac{1}{r'}.$$

Now, from (17), one will have $\tau = \rho \sin\theta$, and therefore *one constructs the center of curvature by* ***once more*** *raising the perpendicular to the radius vector at H until it meets the normal at C.* Meanwhile, (17) will reduce to the form $rr' = a\rho\sin\theta$, and that will give rise to an interesting observation: The radius of curvature of the foot of (M) with respect to F is (II, § **7**, l):

$$\rho' = \frac{r^2}{2r - \rho \sin\theta} = \frac{ar}{2a - r'} = a.$$

Therefore, *the foot of* (M) *with respect to* F (or to F') *is a circumference of radius* a. In other words, the curve (M) can always be considered to be *the envelope of the perpendiculars that are raised from the lines of one sheaf to the points where they meet a circumference.* Turning to (16), observe that upon subtraction, one can also deduce that:

$$2\frac{d\theta}{ds} = \left(\frac{1}{r} - \frac{1}{r'}\right)\sin\theta = \frac{r' - r}{a\rho},$$

and since $r + r' = 2a$, one will have only:

$$r = a\left(1 - \rho\frac{d\theta}{ds}\right), \qquad r' = a\left(1 + \rho\frac{d\theta}{ds}\right),$$

so if one multiplies and integrates:

$$\frac{\rho}{a}\sin\theta = 1 - \rho^2\left(\frac{d\theta}{ds}\right)^2, \qquad s = \int\frac{\rho\,d\theta}{\sqrt{1 - \frac{\rho}{a}\sin\theta}}. \tag{18}$$

On the other hand, if one represents the distance FF' by $2c$ then:

$$4c^2 = r^2 + r'^2 + 2rr'\cos 2\theta = (r + r')^2 - 4rr'\sin^2\theta = 4\,(a^2 - a\rho\sin^2\theta),$$

and as a consequence, $a\,\rho\sin^2\theta$ will constantly keep the value $a^2 - c^2 = b^2$:

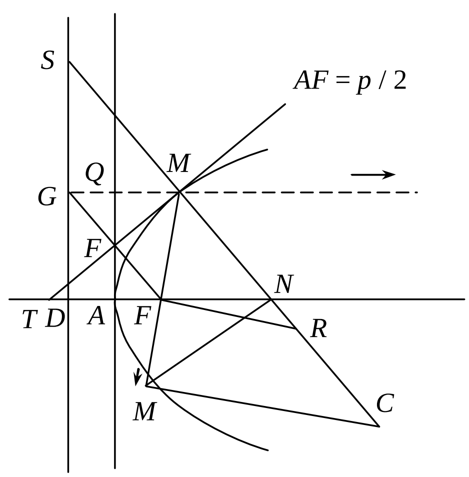
$$\rho = \frac{b^2}{a\sin^2\theta}. \tag{19}$$

Now, if one substitutes θ as a function of ρ in (18) then that will give equation (14), under the hypothesis that $b^2 > 0$. Hence, the property that one will finally obtain belongs to the *ellipses.* In order to get the *hyperbolas*, it is enough to imagine that one repeats the preceding calculations by starting from the relation $r - r' = 2a$, i.e., one supposes that *the difference between the distances to two fixed points remains constant.* One will immediately get $\theta + \theta' = 2\pi$, which amounts to saying that for the hyperbolas, *the tangent bisects the angle between the radius vectors.* The other property remains unaltered. The points F and F' are called *foci* of the conic, $2c$ is *the focal distance*, and the ratio k of c to a is called the *eccentricity* of the conic. Obviously, $k < 1$ for the ellipses, $k > 1$ for the hyperbolas, and in particular, $k = 0, 1, \sqrt{2}$ for the circumference of the circle, the parabola, and the equilateral hyperbola, resp. How are the foci situated with respect to the axes? The normal contains the foci when $\theta = \pi / 2$, and (19) will then show that one has $\rho = b^2 : a$, and one will see (§ **7**) that this can happen only on the focal axis. In addition, the midpoint of FF' will then be found at a distance of $\frac{1}{2}(r + r') = a$ from M, and it will then coincide with the center of the curve. Therefore, *the foci are situated on the focal axes at equal distances from the center.*

9. Application to the parabola:

a) The considerations of the preceding paragraph are applicable to parabola, but the conclusions that are obtained are true for a conic that is defined by arbitrarily-large semi-axes, and consequently tend to be valid in a special case for the parabola, as well. If one fixes the focal axis of an ellipse and a vertex on the axis then one can increase a and b indefinitely in such a way that the ratio $b^2 : a$ tends to p. The other vertices, the center, and a focus F' will then get pushed out to infinity, but the focus F will tend to a limiting position at which its distance from the vertex will be:

$$\lim (a - c) = \lim \frac{b^2}{a+c} = \lim \frac{b^2}{2a} = \tfrac{1}{2}p.$$



The base that one finds for the ellipse has the radius a, and as a becomes infinite, that base will then tend to be converted into a line that must be perpendicular to the axis, by reason of symmetry. Since the foot of the perpendicular to the tangent at the vertex that is based at F is that vertex itself, one can then assert that *the base of a parabola with respect to the focus is the tangent to the vertex.* The perpendicular to the tangent at M that is based at F will meet the tangent at P and the parallel to the axis that is drawn through M at G. If one observes that this parallel is the limit of the radius vector MF' then one will see immediately that *the tangent at M bisects the angle FMG.* Therefore, the triangle FMG, for which the bisector at M is perpendicular to the base, will be isosceles. It follows, among other things, that P bisects the base FG, and therefore if Q is the foot of the perpendicular to the tangent at the vertex A that is based at M then the triangles PQG and PAF will be equal. Hence, $AP = PQ$; i.e., *in order to construct the* *tangent at M, it is sufficient to join M to the midpoint of AP*. In addition, $QG = AF = p$ / 2, and therefore the point G will be on the perpendicular to the axis that is raised from the symmetric image of F with respect to A. The line thus constructed is called the *directrix* of the parabola. One will arrive at a simpler construction of the tangent (or normal) by observing that the segment of the normal MN is equal to the parallel to GF, and that its projection onto the axis will be equal to that of GF – viz., to p. It is then enough to start from the projection of M and measure out a segment of length p along the axis in the sense in which one goes from the vertex to the focus. The end point of that segment will give the normal at M when it is joined to M. Finally, $MF = MG$, and as always that is because FMG is isosceles. Thus, *each point of the parabola is at equal distances from the directrix and the focus.*

b) We now pass on to the construction of the center of curvature. From what was said in § **8**, we need to raise the perpendicular to the normal to the axis with its foot at N until it meets H on the radius vector MF. We then raise the perpendicular to the radius vector at H until it meets C on the normal. Now, if we observe that the triangles MFN, MFH are isosceles then we will see immediately that F divides MH in half. It would therefore be pointless to construct H, since it is enough to raise the perpendicular to the radius vector from F until it meets R in the normal, and the center that we seek will be the point symmetric to M with respect to R. In other words: *The projection of the radius of curvature onto the radius vector is twice that radius.* If S is the point at which the normal meets the directrix then the rectangular triangles MFR, MGS will be equal because they have equal angles at M and on the other hand, we see that $MF = MG$. It then follows that $MR = MS$, and we will then obtain a second construction of the center of curvature for which we can say that the parabola is analogous to the cycloid and the catenary (II, § **7**, *d*, *e*): *The radius of curvature is twice the segment of the normal that is cut out by the directrix when one starts from the point of incidence.*

10. Cassini ovals. – One calls the loci of points that have *a constant product of the distances to two fixed points* **Cassini ovals** or **Cassinioids**. Let F and F' be two fixed points that will be called *foci*, for brevity. Let $2b$ be their distance, O, the midpoint of FF', and let ψ be the inclination of OM with respect to FF'. It is clear that O is the *center* of the curve, since if a point satisfies the definition then the same thing will be true for its symmetric image with respect to O, and by an analogous argument, one can add that the curve will be symmetric with respect to the focal axis and the perpendicular to that axis that is raised from O. Now, if r and θ are the polar coordinates of the center then the distances from the origin M to the foci are given by $\sqrt{r^2 \pm 2br\cos\psi + b^2}$, and the definition of the curve translates into the equality:

$$r^4 - 2b^2 r^2 \cos 2\psi + b^4 = a^4. \tag{20}$$

If one differentiates this then one will get:

$$r^2 \cos \theta = b^2 \cos (2\psi - \theta), \tag{21}$$

while one observes that:

$$\frac{d\psi}{ds} = \frac{d}{ds}(\varphi + \theta) = \frac{1}{\rho} + \frac{d\theta}{ds} = \frac{\sin\theta}{r}.$$

Now, the elimination of r from (20) and (21) will give:

$$a^2 \cos\theta = b^2 \sin 2\psi; \tag{22}$$

(21) will then transform into:

$$r^2 = a^2 \sin\theta + b^2 \cos 2\psi,$$

and from that, along with (20), one will get the formulas:

$$\cos 2\psi = \frac{r^4 - a^4 + b^4}{2b^2 r^2}, \qquad \sin\theta = \frac{r^4 + a^4 - b^4}{2a^2 r^2}. \tag{23}$$

Therefore, up to sign:

$$s = \int \frac{dr}{\cos\theta} = \int \frac{2a^2 r^2 dr}{\sqrt{[(a^2+b^2)^2 - r^4][r^4 - (a^2+b^2)^2]}}. \tag{24}$$

However, if one differentiates (22) and expresses everything as a function of r then one will get:

$$r = \frac{2a^2 r^3}{3r^4 - a^4 + b^4}. \tag{25}$$

11. – The elimination is easy when $a = b$. The Cassinoid will then take the name of *lemniscate.* Formulas (24) and (25) will become:

$$s = \int \frac{2a^2 dr}{\sqrt{4a^4 - r^4}}, \qquad \rho = \frac{2a^2}{3r}, \tag{26}$$

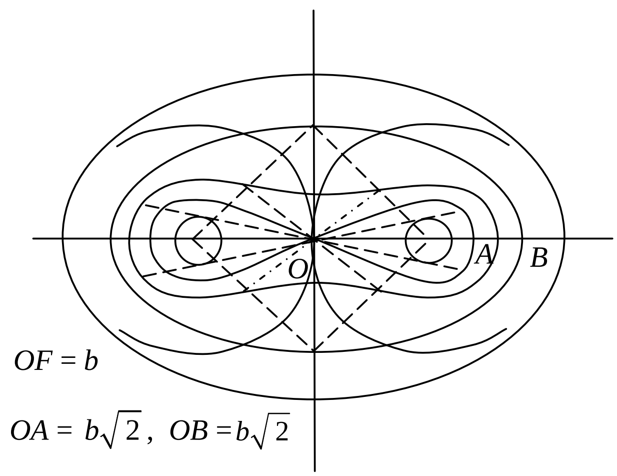
and the elimination of r will give:

$$s = 3\int \frac{d\rho}{\sqrt{\left(\frac{\rho}{c}\right)^4 - 1}},$$

after having set $c = \frac{1}{2}a\sqrt{2}$. That is *the intrinsic equation of the lemniscate.* It shows that ρ increases constantly and indefinitely upon starting from c, its minimum value, while by virtue of (26), r will decrease from the maximum $a\sqrt{2}$ down to zero. On the other hand, formulas (23) will become:

$$\cos 2\psi = \sin\,\theta = \frac{r^2}{2a^2}, \tag{27}$$

and therefore $\psi = \pm\,\pi / 4$ for $r = 0$, and $\theta = \pi / 2$ for $r = a\sqrt{2}$. Thus, if one takes into account the symmetry of the curve with respect to the focal axis then one will see that the lemniscate passes through the center with two branches that intersect at right angles and then go on to meet the focal axis orthogonally at two other points that are situated at distances of $a\sqrt{2}$ from the center. In addition, (27) shows that one always has $2\psi = \pi / 2 - \theta$, so it will follow that *the inclination of the normal from the focal axis is three times that of the radius vector.* That property will permit one to construct the normal at a point when one is given the foci. If one would then like to get the center of curvature then it will be sufficient to observe that by virtue of (27), the second formula (26) will give $r = 3\rho \sin\,\theta$, i.e.: *The projection of the radius of curvature onto the radius vector MO is the third part of MO.* From that property, one can say that the lemniscate is analogous to the parabola (cf., § **9**, *b*) and to the logarithmic spiral.



12. – All of the points of any Cassinioid are at a finite distance. Indeed, in order for the expression (24) to be real, one must have that r^2 is not greater than $a^2 + b^2$, and never less than the absolute value of $a^2 - b^2$. Meanwhile, one sees from the first formula in (23) that $\sin\,\psi$ is zero for $r^2 = a^2 + b^2$, as well as for $r^2 = b^2 - a^2$, and therefore the curve meets the axis at four points when $a < b$ and at only two when $a > b$. The same way of varying r will then show that the curve consists of two equal ovals in the first case and just one closed branch in the second. It will always meet the focal axis at a right angle because $\cos\,\theta$ is annulled, by virtue of (22), along with $\sin\,\psi$. In addition, from (21), one has $r = \pm\,b$ for $\theta = \psi \neq \pi / 2$, and therefore the more-distant points of the axis will belong to the circumference that is described by the focal segment as its diameter. However, one needs to observe that this circumference will not meet the curve if the value of r is not found between the limits that were found before; that will happen when $a > b\sqrt{2}$. If one then sets $r^2 = b^2 - a^2$ then, by virtue of (23), ψ will become equal to $\pi / 2$, and one will then obtain the points of intersection with the perpendicular that is raised from the center of the focal axis. The circumferences of radius a that are described by having those points as centers cut the foci: That will permit one to construct the foci in a way that is analogous to the one that is used for the conic. Also, from the second formula in (23), one will see that the tangent contains the center if $r^4 = b^4 - a^4$, and that can happen only for the Cassinioids with two ovals ($a < b$). The discussion of (25) will then show that the curvature will then attain its minimum value. Finally, on the Cassinioids with just one branch ($a > b$), one will have four bends for $r^4 = (a^4 - b^4) / 2$, since that value of r falls between the limits that were found, and for that, one needs to have $a < b\sqrt{2}$. When $a > b\sqrt{2}$, the Cassinioid is everywhere convex, like the ellipse, and its curvature will vary between the limits:

$$\frac{a^2 - 2b^2}{a^2\sqrt{a^2 - b^2}}, \quad \frac{a^2 + 2b^2}{a^2\sqrt{a^2 + b^2}} .$$

All of those curves are easy to deduce by means of a transformation (II, § **7**, k) of index 1/2 from a pair of circles that are described by the foci as their centers and a radius of $a^2 : b$. The two circles can possibly not meet at all or meet with their centers outside, or finally meet in such a way that each center falls upon the other one. According to the various cases, one will get all forms of the Cassinioids that correspond to the hypotheses:

$$a < b, \qquad b < a < b\sqrt{2}, \qquad b\sqrt{2} < a,$$

respectively. In particular, the lemniscate arises from a pair of tangent equal circles, and the other special Cassinioid ($a = b\sqrt{2}$) will give a pair of circles such that circumference of each circle will pass through the center of the other one.

13. Ribaucour curve and sinusoidal spirals. – We propose to study the curves that have *their radius of curvature proportional to the segment that is cut out from the normal by starting at the point of incidence by the polar to that point with respect to a fixed circle.* We call the circumference of the circle the *directrix*, and its center will be the *pole*. Two particular cases present themselves immediately: The directrix can reduce to the pole, and we will then have the curve that is characterized by the following property: *The projection of the center of curvature onto the radius vector divides that radius in a constant ratio.* Those curves are called *sinusoidal spirals.* For example, the three curves that were cited at the end of § **11** are such things. However, it can happen that the directrix is rectilinear, and we will then have the *Ribaucour curves*; examples of them are the three curves that were mentioned at the bottom of § **9**. *Their radius of curvature is proportional to the segment of the normal that is found between the point of incidence and a fixed line.* Of particular note is the parabola, which belongs to one and the other family by virtue of the two constructions that were found in § **9**. Having assumed that, let R be the radius and let O be the center of the directrix, while x and y are the Cartesian coordinates of O, r and θ are its polar coordinates, and let $(n + 1)\,\rho$ represent the segment of the normal that is found between the point of incidence M and the polar to M with respect to the directrix. The segment that is found between M and the point of contact, which is the cathetus of a rectangular triangle that has hypotenuse r and the other cathetus is equal to R has $(n + 1)\,\rho \sin\theta$ for its projection on the hypotenuse, and therefore:

$$r^2 - R^2 = (n + 1)\,\rho\, y\,. \tag{28}$$

If one differentiates this and takes into account the immobility conditions then one will get:

$$(n + 1)\,\rho\, x - (n + 1)\,\rho_1\, y = 0. \tag{29}$$

Therefore, *the radius vector divides the radius of curvature of the evolute in the constant ratio* – $(n + 1) : 2n$. That is a characteristic property of the curves that we are studying, since the integration of (29) will necessarily lead back to (28) with R an arbitrary constant. Another property is uncovered by considering the circumference that is described along the segment of the normal that is found between M and the polar to M with respect to the directrix. We already know from the elements of geometry that this circumference is *orthogonal to the directrix.* Now, in order to find the envelope, we need to differentiate the equation $x^2 + y^2 = (n + 1)\,\rho\, y$, and we will then get back to (29). The line that this equation represents meets the circumference at M and another point M', in such a way that the envelope will consist of (M) and another curve (M'). Meanwhile, equation (29) will be satisfied by the coordinates of the pole. Therefore, when M traverses the curve, *the line MM' will turn around the pole.* In addition, the tangents to the envelope at M and M', which must (II, § **6**) also touch the evolute circumference at those points, will be anti-parallel with respect to MM'. Therefore (II, § **7**, *j*), *the curve* (M') *is inverse to* (M).

14. Intrinsic equation. – From the immobility condition, one will have:

$$r\,dr = -\,x\,ds = \rho\,dy\,. \tag{30}$$

Dividing by (28) will then give:

$$\frac{r\,dr}{r^2 - R^2} = \frac{dy}{(n+1)\,y}\,;$$

hence:

$$r^2 - R^2 = (n + 1)\,c^2\left(\frac{y}{c}\right)^{\frac{2}{n+1}}, \tag{31}$$

while n, which is finite, will be non-zero and give – 1. Substituting the last result in (28), one will get:

$$\rho = c\left(\frac{y}{c}\right)^{-\frac{n-1}{n+1}}. \tag{32}$$

Now, if one deduces the value of y from this formula, then it will be enough to substitute it in (31) in order to also get x:

$$x = c\left(\frac{\rho}{c}\right)^{-\frac{n-1}{n+1}}\sqrt{(n+1)\left(\frac{\rho}{c}\right)^{\frac{2n}{n-1}} + \frac{R^2}{c^2}\left(\frac{\rho}{c}\right)^{2\frac{n+1}{n-1}} - 1}\,, \qquad y = c\left(\frac{\rho}{c}\right)^{-\frac{n+1}{n-1}}. \tag{33}$$

Finally, from (30), one has that $s = -\int (\rho / x)\, dy$, i.e.:

$$s = \frac{n+1}{n-1}\int \frac{d\rho}{\sqrt{(n+1)\left(\frac{\rho}{c}\right)^{\frac{2n}{n-1}} + \frac{R^2}{c^2}\left(\frac{\rho}{c}\right)^{2\frac{n+1}{n-1}} - 1}}. \tag{34}$$

That is the general intrinsic equation of our line. We can arrive at it more simply by utilizing (29), which gives $s = \frac{n+1}{n-1}\int \frac{y}{x}\, d\rho$, immediately; etc.

15. – Before we go further, let us note some consequences of the formulas (31) and (32). If we pass over the case of a pole at infinity for now then we will easily see that if $n^2 \neq 1$ then *the curve cannot meet the directrix obliquely nor inflect or regress outside of it*, since the curvature will become zero or infinite at the points at which the curve meets the directrix ($r = R$), and only at those points. In addition, the curves that have an index less than – 1 *will not meet the directrix*, and therefore *they will have neither inflections nor cusps.* The curvature *will never be* zero at a finite distance for curves that have indices greater than – 1, but less than 1, and the curvatures of the ones that have indices greater than 1 *will always be finite.* In particular, *the curvature of a sinusoidal spiral cannot be zero or infinite at a finite distance from the pole at outside of that point.* For example, it will follow that a sinusoidal spiral cannot have only one cusp, or only one asymptotic point, or only one inflection point, and the pole will necessarily fall on such a thing. The spirals with indices less than – 1 are devoid of such points because they do not contain the pole. Except for the sinusoidal spirals ($R = 0$), the way that our curves behave in the neighborhood of the directrix is deduced easily from their intrinsic equations. Indeed, when ρ becomes zero or infinite, it will always be the middle terms under the radical that ultimately prevail in equation (34). It will then follow that in the domain of any of its real points of intersection with the directrix of the curve, it will behave as if its equation has the form $\rho^2\, s^{n-1}$ = constant, and according to whether the absolute value of n is greater than unity or less than unity, one can ascribe one of those types (II, § **3**, *i*) of lines that have their curvature proportional to the arc length, and are represented by the clothoid and the involute of the circle, respectively. One can also add (I, § **11**, *e*) that if n is rational, but not equal to the quotient of two odd numbers, then the curve will experience a regression any time that it meets the directrix, and therefore it *will be completely inside of that directrix or completely outside of it.* Obviously, the last assertion will then be true for all curves with index $n < -1$, precisely because those curves do not meet the directrix.

16. Examples:

a) Any value of the index n defines a family of lines that always includes a Ribaucour curve and a sinusoidal spiral. The simplest of them are the ones that are defined by an index of 1: Equation (32) says directly that one is dealing with a family of circles. If a is the distance from the pole to the center of a circle of radius b then one can always choose a point on the circumference of that circle such that the coordinates of the pole are $x = a$, $y = b$, and (28) will then give $R^2 = a^2 - b^2$. Therefore, the directrix is orthogonal to the given circumference, and that is the only case in which a curve meets its own directrix without the curvature becoming zero or infinite. In particular, one can consider any circumference to be a sinusoidal spiral or a Ribaucour curve according to whether the pole is located on that circumference or is pushed out to infinity.

b) The family that is defined by the index – 2 is interesting. It consists of all *conics.* In order to convince oneself of that, it is enough to observe that for $n = -2$, the first property in § **13** will lead to the MacLaurin construction (§ **6**), and in that way, one will also see that the pole is at the center for the conics. After all, equation (34) will reduce to precisely the form (14) when one sets $n = -2$, $c^2 = ab$, $R^2 = a^2 + b^2$, and (33) will become:

$$x = \left(\frac{a^2b^2}{\rho}\right)^{1/3}\sqrt{\left[1-\left(\frac{b\rho}{a^2}\right)^{2/3}\right]\left[\left(\frac{a\rho}{b^2}\right)^{2/3}-1\right]}, \qquad y = \left(\frac{a^2b^2}{\rho}\right)^{1/3}.$$

These are the coordinates of the center, since x will be annulled, while y will become equal to b or a at the vertices – viz., for $\rho = a^2 : b$ or $\rho = b^2 : a$. If one observes the value that is found for R then one will see that *the directrix of a conic is the circumference that circumscribes the rectangle that is constructed from its axes.* Since R is annulled for the equilateral hyperbola and becomes infinite for the parabola, one can add that *the sinusoidal spiral and the Ribaucour curve of index* – 2 *are the equilateral hyperbola and the parabola*, respectively. No matter what the conic, the definition itself of our curves will provide another construction of the center of curvature, since it says that this center *is symmetric to the point at which the normal meets the polar to the point of incidence with respect to the directrix circumference.* Finally, the property that was proved at the end of § **13** will permit us to assert that *the circumference that is symmetric* with respect to the tangents *to the one that is described by the radius of curvature of a conic will meet the directrix circumference orthogonally and envelope another curve that is inverse to the conic in question with respect to the center.*

c) The family that answers to the value 0 for the index is even more interesting. Formula (31) is not appropriate to *any* of the curves in the family, since the choice of the constant is made in such a way that it will cease to be *arbitrary* for $n = 0$. However, it is enough to multiply the left-hand side of (31) by a constant factor k and then, from formulas (31) and (28), one will have $r^2 - R^2 = k\,y^2 = \rho\,y$; hence:

$$x = -\frac{1}{k}\sqrt{(k-1)\rho^2 + k^2R^2}, \qquad y = \frac{\rho}{k}.$$

Moreover, finding all of the curves of index 0 will not require any new calculations, since (29) will become $\rho\, x + \rho_1\, y = 0$, and that will say that *the center of curvature of the evolutes belongs to the radius vector*. Now, one knows (II, § **13**, *g*) that this property characterizes the *cycloidal lines*, which is what one calls all of the lines that are represented by the general intrinsic equation:

$$\rho^2 = \alpha s^2 + 2\beta s + \gamma.$$

One sees that for $\alpha < -1$, this will represent the *hypocycloids*, for $\alpha = 1$, the *cycloids*, for $-1 < \alpha < 0$, the *epicycloids*, for $\alpha = 0$, the *involutes of the circle*, for $\alpha < 0$, two families of *pseudo-cycloidal lines*, which will or will not be cuspidal according to the sign of $\beta^2 - \alpha\gamma$, and will be separate – so to speak – from the *logarithmic spirals*, for which one will have $\beta^2 = \alpha\gamma$. If one observes that:

$$\rho_1 = \alpha s + \beta = \sqrt{\alpha\rho^2 + (\beta^2 - \alpha\gamma)}$$

then (29) will give:

$$x = -\frac{\rho_1}{\rho}\; y = -\frac{1}{k}\sqrt{\alpha\rho^2 + (\beta^2 - \alpha\gamma)}\,,$$

and a comparison of this with the preceding value of x will yield $k = 1 + \alpha$, $k^2 R^2 = \beta^2 - \alpha\gamma$, which is to say that the radius of the directrix circle is given by the formula:

$$R = \frac{\sqrt{\beta \;\; - \alpha\gamma}}{1 + \alpha},$$

and that explains (cf., I, § **8**, *d*) why one of the two families of pseudo-cycloidal lines is devoid of cusps. Indeed, it must belong (§ **15**) to the directrix circumference, which is imaginary when $\beta^2 < \alpha\gamma$. Therefore, *the Ribaucour curve and the sinusoidal spiral of index* 0 *are cycloid and logarithmic spiral*, respectively. If one then observes that when ρ is annulled, one will have $x = -R$, $y = 0$ then one will see that the pole is precisely the point of concurrence of the cuspidal tangents; that is to say, the directrix circumference is precisely the one that is given that name in principle (I, § **8**, *c*). Finally, the property that was pointed out in § **13** will assume a simple form for $n = 0$: *The center of curvature of a cycloidal line at a point M belongs to the polar to M with respect to the directrix circle.* In addition, *the circumference that is described by the radii of curvature of a cycloidal line will cut the directrix orthogonally and envelop a second line that is inverse to the first one.*

17. – If one lets c tend to 0 at infinity according to the value of n then one can do that in such a way that $c^{2n/(n-1)}$ will increase indefinitely with R. However, the ratio of that quantity will have a finite, well-defined value $a^{(n+1)/(n-1)}$ if one sets:

$$R = c\left(\frac{c}{a}\right)^{\frac{n+1}{n-1}}.$$

Equation (34) will then become:

$$s = \frac{n+1}{n-1}\int\frac{d\rho}{\sqrt{\left(\frac{\rho}{a}\right)^{2\frac{n+1}{n-1}}-1}}. \tag{35}$$

That is *the intrinsic equation of the Ribaucour line.* We already know that for $n = 1$, one will find a *circle*, for $n = 0$, a *cycloid*, for $n = -2$, a *parabola*, and, for $n = 0$, (35) will effectively give $s^2 + \rho^2 = a^2$, and for $n = -2$, one will recover the first equation (15). For $n = 3$, and for $n = -5$, one will get the other curves:

$$s = 2\int\frac{d\rho}{\sqrt{\left(\frac{\rho}{a}\right)^{4}-1}}, \qquad s = \frac{2}{3}\int\frac{d\rho}{\sqrt{\left(\frac{\rho}{a}\right)^{4/3}-1}},$$

resp., which are noteworthy because if one wishes to obtain the equations of the lemniscate or the equilateral hyperbola then it will be enough to change s into $2/3\ s$ and $2s$, respectively. For $n = -1/3$, one will get a curve that belongs to a system of parallel curves (II, § **8**) that are represented by the equation:

$$s = -\frac{1}{2}\int\frac{\rho\,d\rho}{\sqrt{(b+\rho)(a-b-\rho)}},$$

and since one will find that $4s^2 + \rho^2 =$ constant for $b = a / 2$, one can say (I, § **8**, *c*) that *the Ribaucour curve with index* $-1/3$ *is parallel to an asteroid.* Finally, if one increases n to infinity then one will get a catenary of equal resistance. However, that is not a Ribaucour line, and that is explained immediately by recalling that equation (34) is linked to the assumption that n is finite.

18. – It is necessary to observe that for the Ribaucour curves, one can even substitute the directrix for the polar of M with respect to the directrix circle. Indeed, note that the latter, and not the directrix, is the line that must be determined on the normal to a segment that is proportional to ρ. However, it is clear that one can just as well say that of the directrix, since it is situated at the mean distance between M and the limit of its polar. Moreover, if q is the segment of the normal that is determined by the perpendicular to OM and situated at a distance $r - R$ from M, and consequently, it will touch the circumference and tend to coincide with it when R increases to infinity, then one will have:

$$r - R = q \sin\theta, \qquad \lim\frac{r}{R} = 1.$$

Therefore, (28) will become:

$$\left(1+\frac{R}{r}\right)q = (n + 1)\,\rho\,;$$

hence, for R infinite, $2q = (n + 1)\,\rho$. In conclusion, observe that formula (32) will give:

$$\rho^{-\frac{n+1}{n-1}} = \lim\,\frac{y}{c^{\frac{2n}{n-1}}} = \lim\,\frac{r}{R}\lim\frac{R\sin\theta}{c^{\frac{2n}{n-1}}} = a^{-\frac{n+1}{n-1}}\sin\,\theta\,;$$

hence:

$$\rho = a\,(\sin\theta)^{-\frac{n-1}{n+1}}, \qquad \lim\,(r - R) = \frac{n+1}{2}a\,(\sin\theta)^{\frac{2}{n+1}}.$$

The discussion of these formulas will show that *the directrix of a Ribaucour line is normal to the curve* at all points (real or imaginary) of lesser or greater contact between the curve and its tangents. *The curves with index n* < -1 *do not meet the directrix*, and are therefore *devoid of inflections and cusps.* For the ones that have index greater than 0 in absolute value, but less than 1, the curvature on the directrix will become zero or infinite, respectively. In the vicinity of the directrix, by virtue of (35), the curve will tend to be represented by the equation $\rho^2\, s^{n-1}$ = constant, and therefore the observations that were made at the end of § **15** can also be applied to the Ribaucour curves. Among them, the curves for which the ratio $\rho : q$ is a whole number ν are important; viz., the curves with index $2/\nu - 1$. Ribaucour has distributed them into four *genera:* For $\nu > 0$, one has the *cycloidal* and *circular* genera, and for $\nu < 0$, one will have the *parabolic* and *catenoid* genera, according to whether ν is even or odd in each case. One notes that the simplest curves of the four types correspond to the values 0, 1, – 2, – 3 of the index (n = 2, 1, – 2, – 1), and are precisely the cycloid, circle, parabola, catenary, resp., which is how the respective genera got their names.

19. – It is enough to set $R = 0$ in (34) to obtain *the intrinsic equation of the sinusoidal spirals:*

$$s = \frac{n+1}{n-1}\int\frac{d\rho}{\sqrt{\left(\frac{\rho}{a}\right)^{\frac{2n}{n-1}}-1}}\,. \tag{36}$$

One notes that if one multiplies s by $1 + 1/n$ then the preceding equation will represent a Ribaucour curve of index $2n - 1$. We already know that for $n = 1$, we will have a *circle*, for $n = 0$, a *logarithmic spiral*, and for $n = -2$, an *equilateral hyperbola*. For $n = -1/2$ and for $n = -2$, equations (36) will reduce to the two equations (15), which represent the *parabola* and the equilateral hyperbola. If one observes that at the vertex of the parabola, the pole will be divided by one-half the radius of curvature and that, on the other hand, p will be the length of that radius then one will see that in the case of the parabola, the pole will be a focus. For $n = 1/2$, equation (36) will give $s^2 + 9\rho^2$ = constant, and for $n = 2$, one will find equation (§ **11**) for the lemniscate. Therefore, the *sinusoidal spirals that are defined by the indices* 1/2 *and* 2 *are the cardioid* (I, § **8**, *c*) *and the lemniscate, resp.* The observations that were made in § **15** will permit us to add that the pole of the first one is at the one cusp that the curve possesses, and the pole of the other will be the center, since the curve is inflected there, as one sees. Finally, for $n = 1/3$, one will get a curve that belongs to a system of parallel curves that are represented by the equation:

$$s = -2\int \frac{\rho\, d\rho}{\sqrt{(b+\rho)(a-b-\rho)}},$$

and since that will become $s^2 + 4\rho^2$ = constant for $b = a / 3$, which is the equation of an epicycloid of two cusps, one will see that *the sinusoidal spiral of index* 1/3 *will be parallel to a certain epicycloid.*

20. – The property that was found in § **13** will assume a simpler form in the case of the sinusoidal spirals. For example: *The circumferences that are tangent to a sinusoidal spiral that leads from the pole cut out segments from the normals that are proportional to the radii of curvature.* That results immediately from the equality of the definition (28), which will become $r = (n + 1)\ \rho \sin\ \theta$ in the present case, and that will reveal another property of the sinusoidal spiral with great facility. For that, let χ and ψ denote the inclinations of the normal and the radius vector, resp., with respect to a fixed normal, and write the aforementioned equality in the form:

$$\frac{1}{\rho} = (n+1)\ \frac{\sin\theta}{r},$$

from which, one will infer upon integrating, after having recalled that:

$$\frac{d\chi}{ds} = \frac{1}{\rho}, \qquad \frac{d\psi}{ds} = \frac{\sin\theta}{r},$$

that $\chi = (n + 1)\ \psi$, and one will immediately recognize the extension of a known (§ **11**) property of the lemniscate in that result. It then follows that *when the radius vector rotates uniformly around the pole, in the same way, the tangent will rotate around the point of contact.* One gives the (much-too-long) name of *curves with proportional inflections* to that property, which was proposed by **Laquière** for the curves that are habitually called sinusoidal spirals, but with twice the impropriety. Now, for $R = 0$, one takes $c : a = (n+1)^{\frac{n-1}{2n}}$, as we already did in the reduction of (34) to (36), and formulas (31) and (33) will give:

$$r = (n+1)\ a \left(\frac{\rho}{a}\right)^{-\frac{1}{n-1}}, \qquad y = (n+1)\ a \left(\frac{\rho}{a}\right)^{-\frac{n+1}{n-1}}.$$

Meanwhile, the intrinsic equation of the sinusoidal spiral shows that ρ cannot be annulled when n is not found between 0 and 1. On the other hand, the first of the two preceding formulas says that when the curvature becomes zero or infinity, the radius r will become infinite or be annulled according to whether n is or is not negative, resp. Therefore, it is not just the spirals with index $n < -1$, as we said in § **15**, *but all the sinusoidal spirals with negative index that have the property that they do not contain their pole.* However, it is always a point of the curve when $n \geq 0$, and since (36) will take the form $\rho = k\, s^{1-n}$ in the neighborhood of the values 0 and ∞ for ρ, one will see (I, § **11**, *e*) that *the sinusoidal spirals with positive index inflect or regress at the pole according to whether the index is the quotient of an even number by an odd number or vice versa, resp., and when the index is the quotient of two odd numbers, they behave as they do at an ordinary point*, although they will have a greater or lesser contact with the tangent according to whether n is less than or greater than 1, resp. One finds the logarithmic spiral ($n = 0$) between those spirals and the ones that do not contain the pole ($n < 0$). While the former have all of their points at a finite distance, the latter extend to infinity, and if one imagines that the index decreases continually then its passage through zero will signal the moment at which the curve abandons its pole in order to expand to infinity. That explains how it happens that the curve is asymptotic to the pole at only that instant. Finally, it is only when $n < 0$ that the radius of curvature can become infinite, and then r and s will also become infinite, while y will tend to zero or infinity according to whether the absolute value of n is greater than or less than 1, resp. It follows that *the only sinusoidal spirals with index* $n < -1$ *are endowed with asymptotes* at a finite distance, and those asymptotes will leave the pole and determine just as many angular regions of equal width π / n. Indeed, when the point is pushed out indefinitely along a branch that admits an asymptote, θ will tend to a multiple of π. Now, if $\lim \theta = -\nu\pi$ then one will have:

$$\lim \psi = \frac{1}{n} \lim\left(\frac{\pi}{2} - \theta\right) = (2\nu + 1)\, \frac{\pi}{2n},$$

and two similar values that correspond to two consecutive values of ν that differ by precisely π / n.

21. Transformations:

a) The sinusoidal spirals are also noteworthy for the great facility by which they allow one to deduce one from another thanks to various transformations. Hence, for example, if one projects the pole at M' onto the tangent to a spiral (M) of index n then one will know that the coordinates of the pole with respect to (M) will be $r' = y$, $\theta' = \theta$, and on the other hand, the radius of curvature will be given (II, § **7**, *l*) by the formula:

$$\rho' = \frac{r^2}{2r - \rho \sin\theta} = \frac{n+1}{2n+1} r = \frac{r'}{(n'+1)\sin\theta'},$$

when one sets $n' = n / (n + 1)$. Therefore, *the pedal of a spiral of index n with respect to pole is a spiral of index n* / (*n* + 1). For example, the pedals of the logarithmic spiral (with respect to the asymptotic point), the parabola (with respect to the focus), the circumference of a circle (with respect to one of its points), and the equilateral hyperbola (with respect to the center) are a logarithmic spiral, a line, a cardioid, and a lemniscate, respectively. The pedal of a cardioid with respect to a point of regression is parallel to an epicycloid, etc. More generally, if n is a whole number then *the spiral of index* 1 / *n will be the* $(n - 1)^{\text{th}}$ *pedal of a circumference of a circle* with respect to one of its points; *the spiral of index* 2 / (2*n* + 1) *will be the* n^{th} *pedal of an equilateral hyperbola* with respect to the center, etc.

b) Similarly, if one applies the transformation (II, **7**, *k*) of index ν after observing that $a^{\nu-1} r' = r^{\nu}$ then one will have:

$$\rho' = \frac{\nu r' \rho}{r + (\nu - 1)\rho \sin\theta} = \frac{\nu r'}{(n + \nu)\sin\theta} = \frac{r'}{(n' + 1)\sin\theta'},$$

when one sets $n' = n / \nu$. Hence, *the transform of index* ν *of a spiral of index n is a spiral of index n* / ν. In particular, the pedal of a spiral of index n can also be deduced from that curve by means of a transformation of index $n + 1$. For $\nu = -1$, one will see that *two sinusoidal spirals with equal and opposite indices are inverse curves.* For example, the following lines are inverses of each other: Two logarithmic spirals, a line and a circle, parabola and cardioid, equilateral hyperbola and lemniscate. For other values of *n*, one will see that the transform of index 2 of the line and the circle will be a parabola and a cardioid, resp., which are also transforms of index 4 of the equilateral hyperbola and lemniscate, resp., etc. In conclusion, observe that all of those curves are easily deduced from the circle by taking the pole to be a point of the circumference and the tangent there to be the polar axis. Indeed, *any sinusoidal spiral of index n is derived from the circle by a transformation of index* 1 / *n.*

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CHAPTER IV

CONTACT AND OSCULATION

1. – When two curves touch at a point M – i.e., when their tangents coincide at M – it can happen that their centers of curvature also coincide. If one transports the origin to M then the two curves can be considered to be represented by the same intrinsic equation in the vicinity of M, up to infinitesimals. In that neighborhood, they are therefore *more than tangent*, since a well-defined intrinsic equation cannot represent just one curve, and since that greater degree of contact will be obvious when the difference $\rho - \rho'$ between the radii of curvature at two well-defined points on two curves for the same value of s is infinitesimal along with s, it is quite natural to assume that the index of contact is more or less restricted to the order of $\rho - \rho'$ as an infinitesimal. Since $\rho - \rho'$ is not infinitesimal in the general case (*simple* contact), we agree to say that the two curves *have contact of order n* when $\rho - \rho'$ is infinitesimal of order $n - 1$, in such a way that simple contact can be said to have order one. In all of this, it is implicitly supposed that ρ and ρ' are finite. We shall also suppose that they are developable in a neighborhood of M in positive integer powers of s, and we shall confine our study of contact at ordinary points in such a way that the reader will not have to confront the many minor difficulties that the complete study of the contact of two curves can present (e.g., inflections, cusps, asymptotic points, etc.), around which the aforementioned form of the development is not always possible.

2. – If one assigns the index 0 to all quantities that are calculated at M then one will have:

$$\rho = \rho_0 + s\left(\frac{d\rho}{ds}\right)_0 + \frac{s^2}{2}\left(\frac{d^2\rho}{ds^2}\right)_0 + \frac{s^3}{6}\left(\frac{d^3\rho}{ds^3}\right)_0 + \ldots,$$

and an analogous equivalence can be written for ρ'. Therefore, in order for $\rho - \rho'$ to be infinitesimal of order $n - 1$, it will be sufficient that one have:

$$\rho_0 = \rho_0', \qquad \left(\frac{d\rho}{ds}\right)_0 = \left(\frac{d\rho'}{ds}\right)_0, \ldots, \qquad \left(\frac{d^{n-2}\rho}{ds^{n-2}}\right)_0 = \left(\frac{d^{n-2}\rho'}{ds^{n-2}}\right)_0, \qquad \left(\frac{d^{n-1}\rho}{ds^{n-1}}\right)_0 \neq \left(\frac{d^{n-1}\rho'}{ds^{n-1}}\right)_0 .$$

On the other hand, the known law [II, form. (13)] for defining the radii of curvature ρ_1, ρ_2, ρ_3, … of the successive developments will easily show that one has:

$$\rho_n = \rho^n \frac{d^n\rho}{ds^n} + \ldots, \qquad \frac{d^n\rho}{ds^n} = \frac{\rho_n}{\rho^n} + \ldots, \tag{1}$$

in which we have neglected to write all of the derivatives of order less than n in the first equality and all of the terms that do not contain ρ_n in the second one. We will then see that ρ_n depends upon just the first n derivatives of ρ, and certainly on the n^{th} one, and that the expression for the n^{th} derivative of ρ will contain ρ_n , but not ρ_{n+1} , ρ_{n+2} , …, etc. Given that, it is clear that the conditions will be found to be equivalent to these other ones:

$$\rho = \rho', \qquad \rho_1 = \rho'_1, \qquad \rho_2 = \rho'_2, \qquad \ldots, \qquad \rho_{n-2} = \rho'_{n-2}, \qquad \rho_{n-1} \neq \rho'_{n-1} \quad \text{(at the point } M\text{)}.$$

Therefore: *In order for two curves to have contact of order n at a given point, it is necessary and sufficient that the first* $n - 1$ *centers of curvature of the one curve should coincide with those of the other one at that point, although the* n^{th} *centers should remain distinct.* It results from that proposition that if two curves have contact of order n then their v^{th} developments will have contact of order $n - v$, and therefore the latter proposition can be stated as: *In order for two curves to have contact of order n, it is necessary and sufficient that their* $(n - 1)^{\text{th}}$ *developments should touch simply.*

3. – It might seem obvious that the curvature of a line will increase in absolute value when one starts from a point the more rapidly that one approaches the tangent at that point, and that of two tangent curves, the one that has the greater curvature at the common point of contact will tend to diverge from the tangent more than the other one. Moreover, that can be justified completely with a more accurate study of the behavior of the two curves in the neighborhood of the contact point (I.4). Now, if n is odd then the ratio of $\rho - \rho'$ to s^{n-1} will eventually take on a definite sign, no matter what the sign of s is, and therefore one of the curves will be inside the other one in the neighborhood of M; obviously, that is the general case. If n is even then s^{n-1} will change sign with s, and one will then have $\rho > \rho'$ on one side of M, while $\rho < \rho'$ on the other, so the two curves will cross. Therefore: *Two curves can cross at the same point at which they touch, but that will be a clear hint that they have higher contact.* Furthermore, if the curves touch without crossing, which will be true in general, one will not always have simple contact, which is how one sees that one can also have higher contact in exceptional cases. However, in those cases, the order will be odd, and since $\rho - \rho'$ keeps a certain sign around zero, one can add that *the difference* $\rho - \rho'$ *will be a minimum or maximum at the point in question.*

4. Osculation. – Fix a point M on a curve and let a second curve $f(s, \rho) = 0$ be given. It is possible to find one or more points M' on the second curve such that as M tends to coincide with M', and the tangent at M' tends to coincide with the tangent at M, their contact, which generally has order one, will however turn out to be of order two. In fact, it is enough to solve the equation of the second curve for s, and replace ρ with the value of the radius of curvature of the first curve at the point M. Instead of giving the curve (M'), one tries to select it from the infinitude of curves that are represented by the intrinsic equation (with $n - 2$ arbitrary parameters):

$$f(s, \rho, a_1, a_2, \ldots, a_{n-2}) = 0, \tag{2}$$

in such a way that the order of contact will prove to be as high as possible. One can also deduce some relations from (2) that have the following form:

$$\begin{aligned} &f_1\,(s, \rho, \rho_1, a_1, a_2, a_3, \ldots, a_{n-1}) = 0, \\ &f_2\,(s, \rho, \rho_1, \rho_2, a_1, a_2, a_3, \ldots, a_{n-1}) = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &f_{n-2}\,(s, \rho, \ldots, \rho_{n-2}, a_1, \ldots, a_{n-1}) = 0 \end{aligned} \tag{3}$$

by $n - 2$ successive differentiations. Like (2), they must be true identically for the curve (M'), but if one replaces $\rho, \rho_1, \ldots, \rho_{n-2}$ with the values that those quantities have at the point M of the first curve then one will define a system of $n - 1$ equations that provide well-defined values for $s, a_1, a_2, a_3, \ldots, a_{n-1}$. Any solution of the system of values for a will serve to select a curve from the infinitude that are represented by equation (2), and the value of s will define a point M' in the curve (M') thus-defined. If that point is made to coincide with M in such a way that the two curves touch there then they will have their first $n - 1$ centers of curvature in common, and in general one will see no reason why the n^{th} center should also coincide so that contact would reach order n, and it is clear *a priori* that one cannot expect a higher degree of contact. However, that does not say that such a contact might not be verified for some singularity that is inherent to the given curve (M). In the general case, one says that among the curves that are defined by equation (2), (M') *osculates* the given curve, and when one finds, in the exceptional cases, that while trying to produce contact of order n between two curves, one finds that they touch even more strictly, one will say that the curve (M') *super-osculates* (M). Therefore: *In any family that is $n - 2$-times infinite, there exists a simple infinitude of curves that have contact of order n with a given curve* (M), and it is only at special points of (M) that one can have an order of contact that exceeds n.

5. Osculating circle. – The preceding considerations are inapplicable when s is missing from (2), and thus, in the case of circles. The infinitude of circles that touch a given curve at a point M will have their centers O on the normal to the curve at M, but as long as O is distinct from C, the contact will be simple, and it will generally become of order two only when O coincides with C. Therefore, among the circles that touch the curve at M, *the osculating circle is the one that has its center at the center of curvature,* Since the contact will then be of order two, *the osculating circle crosses the curve*, in general, at the point of contact, and that is also a property that characterizes it from among the infinitude of circles that touch the curve at the same point. If one imagines that the center O of a tangent circle traverses the normal in a given sense, moving in from infinity and returning to infinity, then the circumference will cross the curve just once, while all of the other positions of O on the curve in the neighborhood of M will either be all internal to the corresponding circumference (which one would obviously have when O goes off to infinity) or all external (as when O tends to coincide with M). Among the simple infinitude of osculating circles to a curve, some of them can super-osculate the curve at special points, and since the contact will then be generally of third order, from the final observation in § **3**, one can assert that $\rho - \rho'$, and therefore ρ, is a minimum or maximum. Conversely, if ρ is a minimum or maximum then the contact must be of higher odd order, and therefore if one imagines that the contact point traverses the curve (always in one sense) then *whenever the osculating circle becomes a minimum or maximum, the contact will rise to order at least three,* and in any case, it will have odd order. The osculating circle will then cease to cross the curve. That is therefore a definite hint that there is super-osculation, which will be obvious whenever the curve is symmetric to the normal in the neighborhood of a point.

6. – Suppose that the curve (M'), rather than being represented by (2), is given by means of its Cartesian equation relative to the tangent and normal at a point M of a given curve, and assume that it is possible to develop y in positive integer powers of x in the neighborhood of M:

$$y = Ax^2 + Bx^3 + Cx^4 + \dots \tag{4}$$

For an opportune determination of the coefficients, that development will agree with any curve that is tangent to the given curve at M, and will agree with that (M), in particular. In order to determine the coefficients that relate to (M), it is enough (II.5) to differentiate (4) and identity the derived equation with (4). One will get:

$$A = \frac{1}{2\rho}, \quad B = \frac{1}{3}\frac{dA}{ds}, \qquad C = \frac{1}{4}\frac{dB}{ds} + \frac{A}{2\rho}, \quad \dots,$$

so:

$$A = \frac{1}{2\rho}, \qquad B = -\frac{\rho_1}{6\rho^3}, \qquad C = \frac{3(\rho^2 + \rho_1^2) - \rho\rho_2}{24\rho^5}, \qquad \dots \tag{5}$$

The very law of formation for those coefficients will show that the v^{th} coefficient will depend upon only the first v radii of curvature, and certainly upon the v^{th} radius. Therefore, the first $n - 1$ coefficients of the development (4) must have the same values for all curves that have a contact of order n with (M) at M. Now, in order to determine the curve (M') that osculates (M), it is enough to substitute the values (5) in (4), and then substitute the development (4) in the equation for (M'), while neglecting the powers of x that have degree higher than n, as well. In that way, one will get an equality that must be satisfied identically, and will then permit one to determine the coefficients in the equation for (M'). Here, one should note that if one writes the development (4) for any curve that has contact of order n with (M) then the first unequal coefficients of the two developments will be the n^{th} coefficients, since they certainly depend upon ρ_{n-1} and $\rho'_{n-1} \neq \rho_{n-1}$. Hence, the difference $y - y'$ of the ordinates is infinitesimal of order $n + 1$; i.e., the two curves *can be regarded as coincident* in the neighborhood of the contact point, *up to infinitesimals of order higher than n*. With that, it will be easy to make the statements in § **3** more precise. Finally, observe that, instead of the unique development (4), it is sometimes preferable to use analogous developments of x and y as functions of s. One will deduce from the relations:

$$\frac{dx}{ds} = \cos\varphi, \qquad \frac{dy}{ds} = \sin\varphi$$

by successive differentiations that:

$$\frac{d^2x}{ds^2} = -\frac{\sin\varphi}{\rho}, \quad \frac{d^3x}{ds^3} = -\frac{\cos\varphi}{\rho^2} + \frac{\rho_1 \sin\varphi}{\rho^3}, \quad \frac{d^4x}{ds^4} = \frac{3\rho_1\cos\varphi}{\rho^4} + \frac{\rho^2 + \rho\rho_2 - 3\rho_1^2}{\rho^5}\sin\varphi, \ldots,$$

$$\frac{d^2y}{ds^2} = \frac{\cos\varphi}{\rho}, \quad \frac{d^3y}{ds^3} = -\frac{\sin\varphi}{\rho^2} - \frac{\rho_1 \cos\varphi}{\rho^3}, \quad \frac{d^4y}{ds^4} = \frac{3\rho_1\sin\varphi}{\rho^4} - \frac{\rho^2 + \rho\rho_2 - 3\rho_1^2}{\rho^5}\cos\varphi, \ldots,$$

so

$$x = s - \frac{s^3}{6\rho^2} + \frac{\rho_1 s^4}{8\rho^4} + \ldots, \qquad y = \frac{s^2}{2\rho} - \frac{\rho_1 s^3}{6\rho^3} - \frac{(\rho^2 + \rho\rho_2 - 3\rho_1^2)\,s^4}{24\rho^3} + \ldots \tag{6}$$

7. Applications:

a) The number of conics is doubly-infinite from the intrinsic viewpoint. It then follows that a conic can be said to osculate a curve when it has a *fourth*-order contact with it. Now, if the equation of the conic:

$$y = \tfrac{1}{2}(\alpha x^2 + \beta y^2 + 2\gamma xy) \tag{7}$$

is substituted in the development (6), while neglecting powers of s of degree higher than four, then one will easily succeed in determining α, β, γ as functions of the radii of curvature ρ, ρ_1, ρ_2 of the given curve (M). Instead of the developments (6), it is perhaps preferable in the present case to adopt just the one development (4) when it is limited to the first three terms and to neglect the powers of x from the fifth and up when one substitutes it in (7). In one way or the other, one will arrive at the following results:

$$\alpha = \frac{1}{\rho}, \qquad \beta = \frac{9\rho^2 + 5\rho_1^2 - 3\rho\rho_2}{9\rho^3}, \qquad \gamma = -\frac{\rho_1}{3\rho^2}. \tag{8}$$

Therefore, *the equation of the osculating conic is:*

$$(3\rho\, x - \rho_1\, y)^2 + (9\rho^2 + 4\rho_1^2 - 3\rho\rho^2)\, y^2 = 18\, \rho^2\, y.$$

b) If one wishes that the contact should be of only *third* order then one will have an infinitude of conics that are always represented by equation (7), in which α and γ have the values (8), while β, which is the only coefficient that depends upon ρ_2 , will remain arbitrary. For $\beta = \gamma^2 : \alpha$, one will get a parabola, and for $\beta = -\alpha$, one will have an equilateral hyperbola. It will follow that *the equations of the osculating parabola and the osculating equilateral hyperbola will be:*

$$(3\rho x - \rho_1 y)^2 = 18\rho^3 y, \qquad x^2 - y^2 - \frac{2\rho_1}{3\rho} xy = 2\rho y,$$

respectively.

c) *In order to determine the lengths of the axes of the osculating conic,* one needs to recall [III. form (8)] that one has:

$$a^2 + b^2 = \frac{\alpha(\alpha+\beta)}{\Delta^2}, \quad ab = \frac{\alpha}{\Delta^{3/2}},$$

in which $\Delta = \alpha\beta - \gamma^2$. If one sets:

$$\mathcal{P} = 9\rho^2 + 4\rho_1^2 - 3\rho\rho_2, \qquad \mathcal{H} = 18\rho^2 + 5\rho_1^2 - 3\rho\rho_2,$$

for brevity, then the values (8) will give:

$$\alpha + \beta = \frac{\mathcal{H}}{9\rho^3}, \qquad \Delta = \frac{\mathcal{P}}{9\rho^4}, \tag{9}$$

and the preceding formula will become:

$$a^2 + b^2 = \frac{9\mathcal{H}\rho^4}{\mathcal{P}^2}, \qquad ab = \frac{27\rho^5}{\mathcal{P}^{3/2}}. \tag{10}$$

That will imply that:

$$a = \frac{3\rho^2}{\mathcal{P}\sqrt{2}}\sqrt{\mathcal{P} + \sqrt{\mathcal{H}^2 - 36\mathcal{P}\rho^2}}, \qquad b = \frac{3\rho^2}{\mathcal{P}\sqrt{2}}\sqrt{\mathcal{P} - \sqrt{\mathcal{H}^2 - 36\mathcal{P}\rho^2}}.$$

One should note that the inner radical is always real since one will have:

$$\mathcal{H}^2 - 36\,\mathcal{P}\,\rho^2 = (5\rho_1^2 - 3\rho\rho_2)^2 + 36\rho^2\rho_1^2$$

identically.

8. Invariants. – For any curve of a family (2) that consists of an n – 2-times infinite number of lines, there will exist a function of the first n radii of curvature that will remain constantly equal to zero all along the curve. Indeed, if one differentiates the last of (3) then one will get another relation:

$$f_{n-1}(s, \rho, \rho_1, \rho_2, \ldots, \rho_{n-1}, a_1, a_2, \ldots, a_{n-2}) = 0,$$

which must be true identically for all of the curves (2). Now, if one eliminates s, a_1, a_2, …, a_{n-2} from the system constructed, while combining the last equation and (2) with the system (3), then one will arrive at a relation:

$$F(s, \rho, \rho_1, \rho_2, \ldots, \rho_{n-1}) = 0, \tag{11}$$

whose left-hand side is precisely the *invariant* of the family (2). Knowing that invariant will permit one to *construct the n^{th} center of curvature at any point when one knows the first $n - 1$ centers*, and the construction that one obtains will characterize the curve (2). Indeed, it is obvious that any invariant will pertain to a well-defined family of curves *whose equation can then be substituted for one's knowledge of that invariant.* In order to convince oneself, one should note that when one substitutes the values (1) in (11), it will become a differential equation of order $n - 1$, and upon integrating, it will give an equation in s and ρ that includes $n - 1$ arbitrary constants. However, of those constants, $n - 2$ of them constitute the system of parameters that appears in equation (2), and the last one is determined by the choice of origin for the arc lengths. However, the n^{th} center is not the only one that can be constructed when one knows the invariant, but all of the successive centers, as well. In fact, one can differentiate (11) indefinitely, and then obtain many relations:

$$F_1(\rho, \rho_1, \rho_2, \ldots, \rho_n) = 0, \quad F_2(\rho, \rho_1, \ldots, \rho_{n+1}) = 0, \quad F_3(\rho, \rho_1, \ldots, \rho_{n+2}) = 0, \ldots, \tag{12}$$

which will yield the values of ρ_n, ρ_{n+1}, ρ_{n+2}, … as functions of $\rho, \rho_1, \rho_2, \ldots, \rho_{n-2}$, successively. Annulling the invariant F at a given point of a curve (M) indicates that the curve (2), which osculates (M) at that point, also has the n^{th} center of curvature in common with it, which is to say that the order of contact exceeds n – i.e., it *super-osculates.* Finally, observe that one eliminates $\rho, \rho_1, \ldots, \rho_{\nu-1}$ from (11) and the first ν relations (12) then one will obtain a relation:

$$F^{(\nu)}(\rho_\nu, \rho_{\nu+1}, \ldots, \rho_{n+\nu-1}) = 0,$$

and one can assert that $F^{(\nu)}(\rho, \rho_1, \ldots, \rho_{\nu-1})$ is the *invariant of the family that is composed of the ν^{th} development* of the curve of the given family (2). In order for a point of a given curve to have contact of order $n + \nu$ with a curve of the family (2), it is necessary and sufficient that $F, F', F'', \ldots, F^{(\nu-1)}$ should be annulled at that point, but not $F^{(\nu)}$.

9. Examples:

a) *The cycloidal lines* (cf., II.13, *g*) constitute a doubly-infinite family of curves that is characterized by the invariant $\rho_1 \rho_2 - \rho \rho_3$. However, if those lines are specialized in such a way that one selects a simple infinitude from the whole family then the invariant cannot contain ρ_3, and one will effectively find that the invariants of the *developments of the circle*, the *logarithmic spiral*, the *cycloids*, the *pseudo-cycloids*, the *tricuspidal hypocycloids*, etc, are ρ_2, $\rho_1^2 - \rho\rho_s$, $\rho + \rho_2$, $\rho - \rho_2$, $9\rho + \rho_2$, etc.

b) The invariants of the *parabola* and the *equilateral hyperbola* are precisely the quantities that were denoted by $\mathcal{P}$ and $\mathcal{H}$, resp. (§ **7**.c), because they are functions of ρ, ρ_1, ρ_2 that are annulled for any parabola and any equilateral hyperbola, respectively, by virtue of (9). In addition, the second formula (10) shows that the osculating conic to a curve will be hyperbolic or elliptic according to whether $\mathcal{P}$ is negative or positive, resp. The values of *s* that annul $\mathcal{P}$ or $\mathcal{H}$ define points on any curve that super-osculate with a parabola or an equilateral hyperbola.

c) In order to find the invariant $\mathcal{C}$ of the whole family of conics, it is enough to differentiate one or the other equality (10). In that way, one will obtain $\mathcal{C}$ in one of the following forms:

$$\mathcal{C} = 8\mathcal{H}\,\rho_1 - 3\rho^2\frac{d\mathcal{H}}{ds} = -3\rho^{14/3}\frac{d}{ds}\frac{\mathcal{H}}{\rho^{3/2}}\,, \qquad \mathcal{C} = 10\mathcal{P}\,\rho_1 - 3\rho^2\frac{d\mathcal{P}}{ds} = -3\rho^{16/3}\frac{d}{ds}\frac{\mathcal{P}}{\rho^{10/2}}\,.$$

If one performs the calculation in one way or the other then one will find that *the invariant of the conic is:*

$$\mathcal{C} = 36\rho^2\rho_1 + 40\rho_1^3 - 45\rho\rho_1\rho_3 + 9\rho^2\rho_3\,.$$

Finally, formulas (10) also permit one to express $\mathcal{C}$ for an arbitrary curve as a function of the semi-axes of the osculating conic:

$$\mathcal{C} = -27\rho^{14/3}\frac{d}{ds}\frac{a^2+b^2}{(ab)^{4/3}}\,, \qquad \mathcal{C} = -27\rho^{16/3}\frac{d}{ds}(a^{2/3}b^{2/3})\,.$$

As one knows that πab measures the area that is enclosed by the ellipse that is defined by the semi-axes *a* and *b*, the final expression will lead one to observe, with **Gravé**, that *the osculating ellipse to a curve* at a point *M cannot maintain a constant area* as *M* traverses the curve. One can add that the sign of $\mathcal{C}$ helps one recognize whether that area increases or decreases. $\mathcal{C}$ will be annulled *when that area becomes a minimum or maximum*, and then *the contact will rise to order at least five.*

d) A noteworthy example is given by the curves that are defined by the invariant:

$$\mathcal{I}(\lambda, \mu) = \lambda\,\rho^2 + (\mu + 1)\,\rho_1^2 - \rho\rho_2$$

for each pair of values of λ and μ. For example, $\mathcal{I}$ (3, 1/3) and $\mathcal{I}$ (6, 2/3) do not differ from $\mathcal{P}$ and $\mathcal{H}$, resp., then. The *catenaries of equal resistance* are characterized by $\mathcal{I}$ (1, 1), the *logarithmic spirals* by $\mathcal{I}$ (0, 0), and more generally $\mathcal{I}$ (0, μ) is the invariant of the curve that has an arc length that is proportional to the $(1-\mu)^{\text{th}}$ power of the radius of curvature. $\mathcal{I}(\lambda, 0)$ is the invariant of the curve that is represented by the equation:

$$s = \int \frac{d\rho}{\sqrt{2\lambda \ln(\rho / a)}},$$

etc. When λ and μ are non-zero, the integration of $\mathcal{I} = 0$ will lead to the intrinsic equations:

$$s = \int \frac{d\rho}{\sqrt{\frac{\lambda}{\mu}\left[\left(\frac{\rho}{a}\right)^{2\mu} - 1\right]}},$$

which represent, in particular [III, form. (35), (36)], the *Ribaucour curve* or the *sinusoidal spirals* of index n according to whether one makes one or the other of the following hypotheses:

$$\lambda = \frac{n-1}{n+1}, \quad \mu = \frac{n+1}{n-1}; \qquad \lambda = \frac{n(n-1)}{(n+1)^2}, \quad \mu = \frac{n}{n-1},$$

resp. If one observes that:

$$\frac{d\mathcal{I}}{ds} = 2\lambda\, \rho_1 - \rho_3 + (2\mu + 1)\, \frac{\rho_1 \rho_2}{\rho}$$

then one will observe that when $2\mu + 1 = 0$, the invariant of the developed curve is $2\lambda\rho - \rho_2$, and since that is the invariant of the cycloidal lines $\rho^2 = 2\lambda s^2 + \ldots$, each of which is, in general the development of an analogous line, one can say that *the curves that are defined by the invariant* $\mathcal{I}(\lambda, -1/2)$ *are certainly parallel to cycloidal lines.* Some examples (cf., III §§ **17**, **19**) are the Ribaucour curve of index – 1/3 and the sinusoidal spiral of index 1/3. For *all* of those curves, the geometric interpretation of the equality $\mathcal{I} = 0$ will lead to the following very simple construction of the third center of curvature when one is given the first two: *If one divides CM in a ratio of* λ *to* $1 - \lambda$*, and* CC_1 *in the ratio of* $(\mu + 1)\,\lambda$ *to* $1 - (\mu + 1)\,\lambda$*, and if the first division point leads to the perpendicular to the line that is conjugate to the second one then that perpendicular will pass through* C_2.

e) One can study the curve that was defined in § **13** of the preceding chapter in an analogous way. Set:

$$\mathbf{P} = (n-1)^2 \rho^2 + (n+1)^2 \rho_1^2 + (n^2 - 1)(\rho_1^2 - \rho\rho_2),$$

$$\mathbf{H} = \frac{n}{n+1}[(n-1)^2\rho^2 + (n+1)^2\rho_1^2] + (n^2-1)(\rho_1^2 - \rho\rho_2) .$$

One will find that the invariant of the preceding curve is:

$$\mathbf{C} = 2n\,\rho_1\,[(2n-1)\,\mathbf{P} - 2\,(n+1)\,\mathbf{H}] + (n-1)\,(n^2-1)\,\rho\,(\rho_1\,\rho_2 - \rho\rho_3) ,$$

and that the directrix circle has its center at the point:

$$x = \frac{(n^2-1)\,\rho^2\rho_1}{\mathbf{P}}, \qquad y = \frac{(n^2-1)^2\rho^3}{\mathbf{P}} \tag{13}$$

and a radius of:

$$R = (n+1)\frac{\rho^2}{\mathbf{P}}\sqrt{-(n+1)\mathbf{H}} .$$

It will follow immediately that **P** and **H** are the invariants of the Ribaucour curve ($R = \infty$) and the sinusoidal spirals ($R = 0$), and that the equation of the directrix circle is:

$$x^2 + y^2 - \frac{2}{\mathbf{P}}(n-1)\,\rho^2\,[(n+1)\,\rho_1\,x + (n-1)\,\rho\,y] + (n-1)^2\,\rho^4 = 0.$$

For the Ribaucour curves, this will reduce to:

$$y = \frac{n+1}{2}\rho - \frac{n+1}{n-1}\frac{\rho_1}{\rho}x . \tag{14}$$

10. Exercises.

a) Determine the *locus of centers of the osculating conics* to a given curve. The coordinates of the center can be deduced from (13) for $n = -2$, or from (3) in the preceding chapter by substituting the values (8) in (9). If one applies the usual procedure (II.4) to these coordinates:

$$x = \frac{3\rho^2\rho_1}{\mathcal{P}}, \quad y = \frac{9\rho^2}{\mathcal{P}} \tag{15}$$

then one will get:

$$\frac{\delta x}{ds} = \frac{\mathcal{C}\rho_1}{\mathcal{P}^2}, \qquad \frac{\delta y}{ds} = \frac{3\mathcal{C}\rho}{\mathcal{P}^2},$$

and one will see that *the tangent passes through M*; one will then have:

$$\kappa = \frac{\mathcal{C}}{\mathcal{P}^2}\sqrt{9\rho^2 + \rho_1^2}, \qquad \rho' = \frac{\mathcal{C}\rho}{\mathcal{P}^3}(9\rho^2 + \rho_1^2)^{3/2} .$$

For example, in the case of the tricuspid hypocycloid ($9s^2 + \rho^2$ = constant), one will find that $\mathcal{P} = 36\, a^2$, $\mathcal{H} = 45a^2$, $\mathcal{C} = -\, 3240\, a^2\, s$. If one then fixes the sense and origin of the arc-length in a convenient way then the preceding formula will give:

$$s' = \frac{15s^2}{4a}, \qquad \rho' = \frac{15s\rho}{8a},$$

and finally $9s'^2 + 4\rho'^2$ = constant. The desired locus will then be a hypocycloid with six cusps, three of which coincide with those of the given curve.

b) Find the *envelope of the directrices of the parabola that osculates* a given curve. For $n = -\,2$, (14) will give the equation of the directrix, which will give:

$$(\rho_2 - 3\rho)\, x + 4\rho_1\, y + 2\rho\rho_1 = 0$$

by differentiation, and from the two equations, one will get $x = 0$, $y = -\,\rho / 2$. Therefore, the directrix *touches its envelope along the normal to the curve.* If one applies the usual fundamental formulas to the coordinates of the point of contact then one will get:

$$\kappa = \frac{\sqrt{9\rho^2 + \rho_1^2}}{2\rho}, \qquad \rho' = \frac{(9\rho^2 + \rho_1^2)^{3/2}}{2\mathcal{P}}. \tag{16}$$

For the tricuspid hypocycloid, one has $\rho' = 3a / 8$. Therefore: *The directrices of the parabola that osculate a tricuspid hypocycloid are all tangent to the directrix circle.*

c) On the contrary, for $\mathcal{H} = 0$, since $\mathcal{P}$ will then reduce to $-\,(9\rho^2 + \rho_1^2)$, formula (16) will become:

$$\rho' = \frac{1}{2}\sqrt{9\rho^2 + \rho_1^2}, \qquad s' = \int \frac{\rho'}{\rho} ds,$$

i.e.:

$$\rho' = \frac{3}{2}\rho\left(\frac{\rho}{a}\right)^{2/3}, \qquad s' = \frac{1}{2}\int \frac{\rho^{2/3} d\rho}{\sqrt{\rho^{1/3} - a^{1/3}}},$$

and upon eliminating ρ, one can infer that:

$$s' = \frac{1}{5}\int \frac{d\rho'}{\sqrt{\left(\frac{2\rho'}{3a}\right)^{1/5} - 1}}.$$

Hence: *The directrices of the parabola that osculate an equilateral hyperbola will touch a sinusoidal spiral of index* – 2 / 3.

d) Find the *locus of the foci of the osculating parabola* to a given curve. If one observes (III.9, *a*) that the focus is symmetric with respect to the tangent to the projection of *M* onto its directrix then one will find that its coordinates are:

$$x = -\frac{3\rho^2\rho_1}{2(9\rho^2+\rho_1^2)}, \qquad y = \frac{9\rho^2}{2(9\rho^2+\rho_1^2)};$$

thus:

$$\frac{\delta x}{ds} = \frac{(9\rho^2-\rho_1^2)\mathcal{P}}{2(9\rho^2+\rho_1^2)^2}, \qquad \frac{\delta y}{ds} = \frac{3\mathcal{P}\rho\rho_1}{(9\rho^2+\rho_1^2)^2}.$$

These formulas say that the normal to the locus of the foci divides *MC* in the ratio of 1 to 3, and that the tangent divides in half the segment that is determined by the directrix on the tangent to (*M*) when one starts from *M*. One will then get:

$$\frac{1}{\kappa} = 2\left(\frac{\mathcal{H}}{\mathcal{P}}-1\right), \qquad \frac{\rho}{\rho'} = 2\left(3\frac{\mathcal{H}}{\mathcal{P}}-5\right).$$

For example, if $9s^2 + \rho^2$ = constant then one will have $s' = 2s$, $\rho' = 2\rho / 3$, $9s'^2 + 25\rho'^2$ = constant. Hence: *The foci to the osculating parabola of a tricuspid hypocycloid belong to an epicycloid that has the same cusps.* Similarly, If $\mathcal{H} = 0$ then the last formula will give $s' = s / 2$, $\rho' = \rho / 10$, and it is easy to deduce that *the osculating parabola of an equilateral hyperbola has its foci on a curve that is defined by the invariant* $\mathcal{I}$ (6/25, 2/3). Finally: *Any curve of the family that is defined by the invariant* $\mathcal{I}$ (– 3/2, – 1/6) *has the foci of its osculating parabola along a straight line.*

e) *Determine the locus of the centers of the osculating equilateral hyperbolas* to a given curve. For $\mathcal{H} = 0$, (15) will give:

$$x = -\frac{3\rho^2\rho_1}{9\rho^2+\rho_1^2}, \qquad y = -\frac{9\rho^3}{9\rho^2+\rho_1^2};$$

one then finds that:

$$\frac{\delta x}{ds} = \frac{(9\rho^2-\rho_1^2)\mathcal{H}}{(9\rho^2+\rho_1^2)^2}, \qquad \frac{\delta y}{ds} = -\frac{6\mathcal{H}\rho\rho_1}{(9\rho^2+\rho_1^2)^2},$$

and one deduces that:

$$\frac{1}{\kappa} = \frac{\mathcal{P}}{\mathcal{H}}-1, \qquad \frac{\rho}{\rho'} = 3\frac{\mathcal{P}}{\mathcal{H}}-5.$$

In particular, if $9s^2 + \rho^2$ = constant then $s' = 5s$, $\rho' = 5s / 7$, $9s'^2 + 49\rho'^2$ = constant, and then *the centers of the osculating equilateral hyperbolas to a tricuspid hypocycloid belong to a stellate epicycloid that has the same cusps.* Finally: *The osculating curves to* *the equilateral hyperbola that have their centers on a straight line are those of the family that is characterized by the invariant* $\mathcal{I}$ (3/2, 1/6).

f) The preceding calculations can be performed in a more general manner on the curves that are defined by the invariant **C**. One then deduces from (13) that:

$$\frac{\delta x}{ds} = -\frac{(n+1)\mathbf{C}\rho_1}{\mathbf{P}^2}, \qquad \frac{\delta y}{ds} = -\frac{(n-1)\mathbf{C}\rho}{\mathbf{P}^2}, \tag{17}$$

and one sees immediately that the tangent to the locus of the centers of the directrix circles will pass through M. The curve that takes the place of the tricuspid hypocycloid is always an hypocycloid ($n < 0$) or an epicycloid ($n > 0$), which are represented by the equation:

$$(n-1)^2 s^2 + (n+1)^2 \rho^2 = a^2. \tag{18}$$

One will have:

$$\mathbf{P} = \frac{2n(n-1)^2}{(n+1)^3} a^2, \qquad \mathbf{H} = \frac{(2n-1)(n-1)^2}{(n+1)^3} a^2, \qquad \mathbf{C} = \frac{4n(2n-1)(n-1)^4}{(n+1)^5} a^2 s$$

for it. When $\mathbf{H} = 0$, (17) will lead to the formulas:

$$\frac{1}{\kappa} = \frac{\mathbf{P}}{\mathbf{H}} - 1, \qquad \frac{\rho}{\rho'} = \frac{n-1}{n+1}\frac{\mathbf{P}}{\mathbf{H}} - 1,$$

which are merely the ones that determine the *locus of poles of the sinusoidal spiral* of index n that osculates a given curve, and if that is (18) then one will find an analogous curve that corresponds to the value $2 - 1/n$ of n.

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CHAPTER V

THE ROULETTE

1. When a curve (M_0) rolls without slipping along a curve (M) that is fixed in the plane, one says that (M_0) *is developed along* (M), that the curve that is described by the points that are rigidly linked with the moving curve is called a *roulette*, and that the fixed curve is the *base* for that roulette. The name "roulette" is not meant to refer to a special curve [since any curve is a roulette when one chooses (M) and (M_0) conveniently], but only to draw one's attention to a mode of generating the curve considered. As to the properties that the roulette enjoys, one can often arrive at them by means of simple and elegant geometric or kinematical considerations that are, however, devoid of the character of analytical uniformity that distinguishes the intrinsic procedures and makes them suitable for the study of infinitesimal geometry. In regard to the tangent and normal that the moving curve has in common with the fixed point at the contact point M, the coordinates x and y of a point P that is rigidly linked with the moving curve must satisfy the immobility condition:

$$\frac{dx}{ds_0} = \frac{y}{\rho_0} - 1, \qquad \frac{dy}{ds_0} = -\frac{x}{\rho_0}. \tag{1}$$

On the other hand, the variations that its coordinates experience in the fixed plane are given (II, § **1**) by the formulas:

$$\frac{\delta x}{ds} = \frac{dx}{ds} + \frac{y}{\rho} + 1, \qquad \frac{\delta y}{ds} = \frac{dy}{ds} - \frac{x}{\rho}, \tag{2}$$

provided that one inverts the sense of the direction of the normal to just the fixed curve. By virtue of the definition, one will then have $ds = ds_0$, and thanks to (1), formula (2) will become:

$$\frac{\delta x}{ds} = \frac{y}{\mathcal{R}}, \qquad \frac{\delta y}{ds} = -\frac{x}{\mathcal{R}}, \tag{3}$$

in which one has set:

$$\frac{1}{\mathcal{R}} = \frac{1}{\rho} + \frac{1}{\rho_0}. \tag{4}$$

Therefore, if one divides the one equation in (3) by the other then one will see that the inclination of the tangent to (P) with respect to the tangent to (M) is given by the formula $\tan \vartheta = -x : y$, that is to say, if r and θ are the polar coordinates of P then $\vartheta = \theta + \pi / 2$; i.e., *the normal to* (P) *at* P *passes through* M. In addition, one also deduces from (3) that the ratio of the elementary arc length of (P) to that of (M) is $\kappa = r : R$; i.e., one has:

$$s' = \int \frac{r}{\mathcal{R}} ds. \tag{5}$$

It is also known (II, § **4**) that if one fixes the positive directions of the tangent and the normal to (P) in such a way that one can rotate both of them until they coincide with the ones that relate to (M) then one will have:

$$\frac{\kappa}{\rho'} = \frac{1}{\rho} - \frac{d\vartheta}{ds}.$$

Meanwhile, the condition:

$$\frac{d\vartheta}{ds} = \frac{d\theta}{ds_0} = -\frac{1}{\rho_0} + \frac{\sin\theta}{r}$$

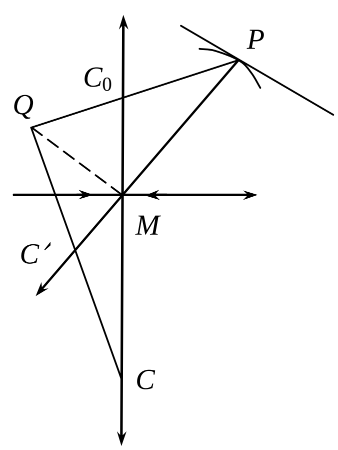
will be satisfied, from the immobility of P in the plane of (M_0). Therefore:

$$\frac{\kappa}{\rho'} = \frac{1}{\mathcal{R}} - \frac{\sin\theta}{r};$$

i.e.:

$$\frac{1}{\rho'} = \frac{1}{r} - \frac{\mathcal{R}y}{r^3}. \tag{6}$$

If one expresses the right-hand sides of (5) and (6) as functions of s_0 (or s) then the elimination of that variable will lead to the intrinsic equation of the roulette in any case.



2. – Formula (6) is also susceptible to a very simple geometric interpretation. If C, C_0, C' are the centers of curvature of the lines (M), (M_0), (P), and if one projects C onto PM at L and lets Q denote the point of PC_0 that is on the perpendicular to PM at M then *the point C' will belong to QC.* Indeed, if N is the point at which QC meets PM then the transversal PQC_0 in the triangle CMN will give:

$$\frac{PN}{PM} = \frac{CC_0}{MC_0}\cdot\frac{QN}{QC} = \frac{\rho+\rho_0}{\rho_0}\cdot\frac{MN}{ML} = \frac{\rho}{\mathcal{R}}\cdot\frac{PN-r}{\rho\sin\theta} = \frac{r}{\mathcal{R}}\cdot\frac{PN-r}{y},$$

from which, one infers, in succession:

$$\frac{PN}{PN - r} = \frac{r^2}{\mathcal{R}\, y}, \qquad PN = \frac{r^3}{r^2 - \mathcal{R}\, y} = \rho' = PC';$$

i.e., N is precisely C'. Therefore, in order to construct C', it is enough to *trace out the line PC_0 until it meets the perpendicular to PM that is raised at M and to join C with the point thus-obtained. That line from C to the latter point will meet PM at C'.*

3. Applications:

a) A circumference of radius τ develops along another one that has radius R. What is the *roulette that is generated by a point P on the circumference of the first circle?* One needs to suppose that:

$$\rho = R, \qquad \rho_0 = \tau, \qquad \theta = \frac{s}{2\tau}, \qquad r = 2\tau \sin\theta$$

in the preceding formulas. If the arc-lengths of the roulette are measured in the opposite sense when starting from the point at which the normal is also normal to the base ($\theta = \pi / 2$) then formulas (5) and (6) will give:

$$s' = \frac{4\tau^2}{\mathcal{R}} \cos\theta, \qquad \rho' = \frac{4\tau^2}{2\tau - \mathcal{R}} \sin\theta,$$

and from (4), one will have:

$$\mathcal{R} = \frac{R\tau}{R+\tau}, \qquad 2\tau - \mathcal{R} = \frac{R + 2\tau}{R+\tau}\tau.$$

Therefore, if one sets:

$$a = \frac{4\tau^2}{\mathcal{R}} = 4\,\frac{\tau}{R}(R + \tau), \qquad b = \frac{4\tau^2}{2\tau - \mathcal{R}} = 4\tau\frac{R+\tau}{R+2\tau}$$

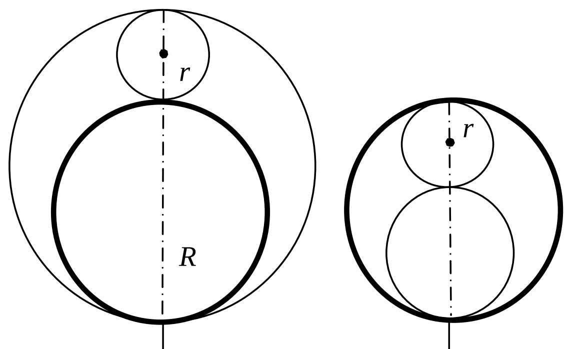
then the equation of the roulette will be:

$$\frac{s'^2}{a^2} + \frac{\rho'^2}{b^2} = 1. \tag{7}$$

Conversely, if one is given that equation then if a and b are the positive roots of a^2 and b^2, resp., then one will have:

$$R = \frac{ab^2}{a^2 - b^2}, \qquad \tau = \pm\frac{1}{2}\frac{ab}{a \pm b}.$$

Therefore, there are two ways that any line (7) can be considered to have been generated by a point on a circumference that develops along the directrix circumference. In particular, *if a circumference develops along a line then any of its points will describe a cycloid*. If $a^2 > b^2$ then one of the values of τ will be positive, like R, while the other will be negative, but its absolute value will be greater than R in such a way that the moving circumference will always be external to the base. However, for $a^2 < b^2$, the two values of τ will have the same sign, which is opposite to that of R, and their absolute values will be less than R. In that case, the moving circles are internal for the fixed circles, and the algebraic sum of the two values of t will be $-R$ in all cases. One therefore has a *hypocycloid* or an *epicycloid* according to whether the generating circle moves *internally* or *externally* to the directrix circumference, resp.

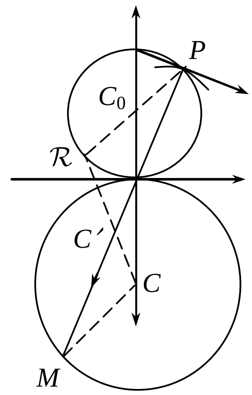


b) One can also arrive at the preceding results with great ease by utilizing the construction in § **2**. It results directly from that construction that *the center of curvature of the roulette* (P) *is found upon the diameter of the fixed circle that passes through the point on the moving circumference that is diametrically-opposite to P*. Having said that, let Q be that point, and let N be the second point at which the normal PM meets the directrix. It is easy to prove that CN is parallel to PQ. Now, if one projects from the point C on the normal to the harmonic quadruple $PC_0Q\infty$ then one will get $PMC'N$. Therefore, C' is the harmonic conjugate to P with respect to MN. It will then follow that *the center of curvature of* (P) *belongs to the polar to P with respect to the directrix circle.* On the other hand, it is known (III, § **16**, *c*) that this property characterizes the cycloidal lines. If one is given such a line by means of equation (7) then one calculates R and τ by recalling that the coordinates of the center of the directrix circle will be:

$$x = \frac{b^2 s}{a^2 - b^2}, \quad y = \frac{a^2 \rho}{a^2 - b^2}, \tag{8}$$

and observing that the first one must take the values $\pm R$ at the cusps ($\rho = 0$, $s = \pm a$), and the second one must have the value $R + 2\tau$ at the origin ($s = 0$, $\rho = \pm b$), in such a way that one will have immediately:

$$R = \frac{ab^2}{a^2 - b^2}, \quad R + 2\tau = \pm \frac{a^2 b}{a^2 - b^2}.$$



c) Now develop a cycloidal line along a line and look for the *roulette that is generated by the center of the directrix circle.* Since ρ is infinite, (4) will imply that one has $\mathcal{R} = \rho_0$, and when (8) is applied to the curve (M_0), one will get:

$$r^2 = \frac{b^4 s_0^2 + a^4 \rho_0^2}{(a^2 - b^2)^2} = a^2 \, \frac{b^4 + (a^2 - b^2)\,\rho_0^2}{(a^2 - b^2)^2}\,;$$

formula (6) will then become:

$$\frac{1}{\rho'} = \frac{(a^2 - b^2)\,r^2 - a^2 \rho_0^2}{(a^2 - b^2)\,r^2} = \frac{a^2 b^4}{(a^2 - b^2)^2\, r^2}\,, \qquad \text{i.e.,} \qquad \rho' = \frac{r^2}{R^2}\,.$$

Similarly, (5) will give:

$$s' = \int \frac{a\,r^2 dr}{\sqrt{(r^2 - R^2)(a^2 r^2 - b^2 R^2)}}\,;$$

hence, if one eliminates *r* from the last of the two formulas then:

$$s' = \int \frac{d\rho'}{\sqrt{\left(1 - \dfrac{b^2}{a^2}\left(\dfrac{\rho'}{R}\right)^{2/3}\right)\left(\left(\dfrac{\rho'}{R}\right)^{2/3} - 1\right)}}\,.$$

Therefore (III, § **7**), the desired roulette is a conic that has its axes proportional to *a* and *b*, resp., and its parameter equal to the radius of the directrix circle. In particular, *when a pseudo-cycloid* (I, § **8**, *d*) *is developed along a line, its pole can describe an equilateral hyperbola.* Another particular case is obtained by observing that equation (7) will change into the known equation for the development of a circle of radius *R* when one transports the origin to a cusp and multiplies *s* and ρ by *b*, and one then makes *a* and *b* increase to infinity in such a way that the ratio of b^2 to *a* tends to *R*. Under those conditions, the last intrinsic equation will tend to represent a parabola, and therefore *if the evolute of a circle is developed along a line then the center of the circle will describe a parabola.*

d) For a sinusoidal spiral of index *n*, it is known [III, form. (28)] that the coordinates of the pole must satisfy the condition $r^2 = (n + 1)\, \rho_0\, y$, and therefore, if one supposes that ρ is infinite, in such a way that $\mathcal{R} = \rho_0$, then formula (6) will give:

$$\rho' = \frac{r^3}{r^2 - \rho_0 y} = \frac{n+1}{n} r .$$

Therefore, the radius of curvature of the roulette that is described by the pole is proportional to the segment of the normal that lies between the roulette and its base. On the other hand, one knows (III, § **18**) that this property characterizes the Ribaucour curve, whose index n' is coupled to *n* by the relation:

$$\frac{2}{n'+1} = \frac{n+1}{n}, \qquad \text{i.e.,} \qquad n' = \frac{n-1}{n+1}.$$

One then finds the following theorem of **Bonnet**:

If a sinusoidal spiral of index n is developed along a line then its pole will describe a Ribaucour curve of index $\frac{n-1}{n+1}$.

For example, if one successively takes $n = 0, -\frac{1}{2}, \frac{1}{2}$, etc. then one will see that if a logarithmic spiral is developed along a line then its pole will describe a catenary. If a cardioid is developed along a line then its cusp will move parallel to an asteroid, etc. Bonnet's theorem can also be deduced from the construction that was given in § **2**, by virtue of which, the line PC_0 will pass through the point of intersection of the perpendiculars to the base and *PM* that go through C' and *M*, resp. Indeed, if *N* is the projection of C_0 onto *PM* then one will have, by the definition of the sinusoidal spirals:

$$n = \frac{PN}{NM} = \frac{PC_0}{QC_0} = \frac{PM}{MC'}, \qquad \text{so} \qquad \frac{PM}{PC'} = \frac{n}{n+1} = \frac{n'+1}{2}, \qquad \text{if} \qquad n' = \frac{n-1}{n+1},$$

and the last proportion defines the Ribaucour curve of index n' precisely.

e) What curve will a sinusoidal spiral develop into when its pole describes a line? In that case, it is ρ' that must become infinite, and therefore one must have:

$$r^2 = \mathcal{R}\, y = (n + 1)\, \rho_0\, y, \qquad \text{so} \qquad \rho_0 = -\frac{n}{n+1} \rho.$$

Therefore, if:

$$s_0 = \frac{n+1}{n-1}\int\frac{d\rho_0}{\sqrt{\left(\frac{\rho_0}{a}\right)^{\frac{2n}{n-1}}-1}}$$

is the equation of the spiral then that of the unknown base will be:

$$s = \frac{n}{n-1}\int\frac{d\rho}{\sqrt{\left(\frac{\rho}{a}\right)^{\frac{2n}{n-1}}-1}}.$$

Now, one should note that this is the equation of a Ribaucour curve whose index n' is coupled with n by the relation:

$$\frac{n'+1}{n'-1} = \frac{n}{n-1}, \qquad \text{i.e.,} \qquad n' = 2n - 1.$$

Therefore, *the line along which a sinusoidal spiral of index n must develop in order for the pole to describe a line is a Ribaucour curve of index* $2n - 1$. Nonetheless, one needs to take care to arrange the two curves in such a way that they have opposite convexities only for n between -1 and 0; in other words:

$$\rho_0 = \frac{n}{n+1}\rho, \qquad \mathcal{R} = \frac{n+1}{2n+1}\rho_0, \qquad \rho' = \frac{2n+1}{2n}r,$$

and the pole describes a Ribaucour line of index $\frac{2n-1}{2n+1}$.

f) The foci of the development of a conic along a line generate important curves, to which one gives the name of *Delaunay curves.* We have seen, moreover (III, § **8**), that the coordinates r and θ of one focus will satisfy the relations:

$$r\,(2a - r) = a\,\rho_0 \sin\theta, \qquad a\,\rho_0 \sin^3\theta = b^2. \tag{9}$$

By virtue of the first one, formula (6) will give:

$$\frac{1}{\rho'} = \frac{1}{r} - \frac{\rho_0 \sin\theta}{r^2} = \frac{1}{r} - \frac{2a-r}{ar} = \frac{1}{a} - \frac{1}{r}. \tag{10}$$

Meanwhile, from the intrinsic equation of the conic, one will have:

$$ds_0 = \frac{ab\,d\rho_0}{3\sqrt{\left(a^2 - (ab\,\rho_0)^{2/3}\right)\left((ab\,\rho_0)^{2/3} - b^2\right)}},$$

and ρ_0 can be expressed as a function of r by eliminating θ from (9). One finds that $(ab\,\rho_0)^{2/3} = r\,(2a - r)$. Formula (5) will then give:

$$s' = \int \frac{ab\,dr}{(2a-r)\sqrt{k^2a^2-(r-a)^2}} \tag{11}$$

when one denotes the eccentricity by k. The elimination of r from (10) and (11) leads to the equation:

$$s' = \int \frac{ab\,dr}{(\rho'-2a)\sqrt{k^2(\rho'-a)^2-a^2}}\,, \tag{12}$$

from which one will infer the *intrinsic equation of the Delaunay curve:*

$$\rho' = a\,\frac{1+k^2-2k\cos\dfrac{s'}{a}}{k\left(k-\cos\dfrac{s'}{a}\right)} \tag{13}$$

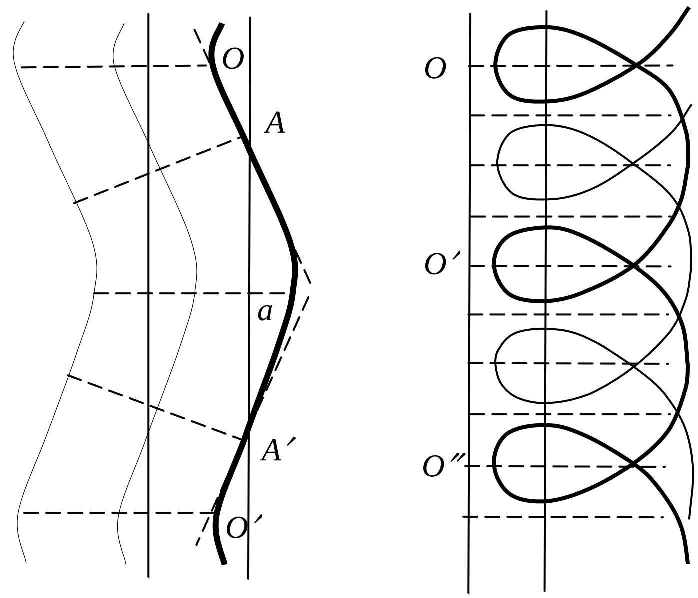
upon integrating. One has curves of two types that are distinguished by whether the positive k is less than or greater than 1 – i.e., whether the generating conic is an ellipse or a hyperbola, resp. The curves of the first type are also called *elliptical catenaries,* while those of the second type are *hyperbolic catenaries.* The case of $k = 1$ (viz., parabola) was considered before in the penultimate application, and furthermore (10) gives $\rho' = -r$ for a infinite, which is a property that is characteristic of the catenary. Therefore, the catenary, properly speaking, presents itself as a limiting curve of separation between the elliptical catenary and the hyperbolic one. The more important property that will be used in what follows is given by formula (10). One knows that if ρ' and r are augmented by a in this formula then one will get $r\,\rho' = a^2$, which is a characteristic property of the curves that were studied before (II, § **7**, *m*). One can also arrive at the same result by observing that the parallels to the curve (12) are represented (II, § **8**) by the equation:

$$s = \int \frac{ab\,\rho\,d\rho}{(\rho+c)(\rho+c-2a)\sqrt{k^2(\rho+c-a)^2-a^2}}\,,$$

which will become:

$$\rho = \frac{a}{k}\sqrt{1+(1-k^2)\tan^2\frac{s}{a}} \tag{14}$$

for $c = a$. In addition, one will get back to equation (12) for $c = 2a$; that is to say, *any Delaunay curve is parallel to an equal curve.*



g) The Delaunay curves are easily discussed (when one would not like to make use of their mode of generation) by availing oneself of formulas (9), (10), and (13), from which one infers that:

$$\cot\theta = \frac{k\sin\dfrac{s'}{a}}{1-k\cos\dfrac{s'}{a}}, \quad y = a\sqrt{1+k^2-2k\cos\frac{s'}{a}}, \tag{15}$$

in which one is cautioned that the radical must always be taken to be positive in the second formula. The tangent becomes parallel to the fixed line when one annuls $\cot\theta$; i.e., for $s' = 0, \pm\pi a, \pm 2\pi a, \ldots$ That therefore takes place on two parallels to the same line, between which the entire curve is found, since the second formula in (15) shows that y will attain its minimum or maximum value for $\cos s' / a = \pm 1$, respectively. The maximum value is $(1 + k)\,a$, and the minimum is $(1 - k)\,a$ or $(k - 1)\,a$ according to whether $k < 1$ or $k > 1$, resp. For the first series of points, formula (13) will give the extreme value $a - a / k$, which can be negative or positive, and for the second series, it will give the other extreme value $a + a / k$, which is always positive. One sees that it is only for $k < 1$ that the curvature will change sign, and that will happen when ρ' becomes infinite; i.e., for $\cos s' / a = k$, by virtue of (13). One sees from (15) that $\cos\theta$ then attains the value k, which is a maximum, and that y will take the value $a\sqrt{1-k^2}$. Therefore, the curve inflects an infinitude of times at the distance $a\sqrt{1-k^2}$ from the fixed line, and that will determine segments that are equal to a on all inflectional normals, which is also apparent from (10), which gives $r = a$ for $\rho' = \infty$. It is then clear that any inflectional normal is also like that for all parallels to the curve, from which, it will follow, in particular, that the intersection points of the aforementioned normals with the fixed line are the inflection points of that curve (14), which, as one sees, is parallel to the curve considered and to another equal curve. Since the two curves are parallel and equal, it is easy to explain that by imagining two equal ellipses that one develops along a line while one keeps the ellipses symmetric with respect to that line. One focus of one ellipse and the opposite focus of the other one will remain constantly collinear with the contact point, by virtue of a known property (III, § **8**) of the ellipse, and they will then generate two Delaunay curves that are equal and parallel. The behavior of our curves will be quite different when $k > 1$. ρ' will then be always positive, while cot θ can increase indefinitely; i.e., the curve is never inflected, and the tangent can become perpendicular to the fixed line. That will happen for $\cos s' / a = 1 / k$, in which case, one will have $y = a\sqrt{k^2 - 1}$ in the second formula (15). Therefore, the parallel that goes through the fixed line at a distance $a\sqrt{k^2 - 1}$ will meet the curve orthogonally at an infinitude of points, and it is clear that the infinite cusps of the curve (14) that belong to the system of lines that are parallel to the curve considered must fall on that parallel.

4. Circle of inflection. – Formula (6) shows that ρ' is infinite for all of the points that satisfy the equality $r = \mathcal{R} \sin \theta$. That equality defines a circle that one calls the *circle of inflection.* Therefore, the *circle of inflection is the locus of points that are points of inflection on their trajectory at a given instant.* If the normal at (M_0) carries the segment $MH = \mathcal{R}$ then the circle of inflection will be described by the diameter MH, and therefore, by virtue of (4), it will be similar, with respect to C_0, to the circle that is described by the diameter MC. It is clear that *the inflectional tangents will be concurrent at the point H* at any instant. Note, in particular, that *any point that moves along a line must constantly belong to the circle of inflection,* and the line that is traversed by the point must constantly pass through H. Conversely, if a point in the development of a curve over an arbitrary base is rigidly coupled with the moving curve and it does not cease to be found on the circle of inflection at any instant then its trajectory will be rectilinear, while the equation $\rho' = \infty$ will define the line. As for the cusps of the infinite trajectory, one sees from (6) that ρ' cannot, in general, be annulled without annulling r. The cusps of the roulette therefore fall upon the base. In exceptional cases, they can be present in the entire plane. That will happen when $\mathcal{R}$ is infinite – i.e., for $\rho_0 = -\rho$. One will then have merely $\kappa = 0$; that is to say, *when the moving curve osculates the fixed curve*, the points that are rigidly coupled with the first one will remain immobile for an instant, and the trajectory will suffer a regression there. However, if $\mathcal{R}$ is annulled (i.e., a cusp of one or the other curve falls upon the contact point) then one will have $\kappa = \infty$, $\rho' = r$; that is to say, the contact point will seem immobile with respect to all points in the plane, and it will be the center of curvature of all their trajectories.

5. Theorems of Steiner and Habich. – For the roulette with a rectilinear base, formulas (5) and (6) will become:

$$s' = \int \frac{r}{\rho_0} ds_0 , \qquad \frac{1}{\rho'} = \frac{1}{r} - \frac{\rho_0 \sin\theta}{r^2} .$$

On the other hand, the values of s and ρ that relate to the pedal of (M_0) with respect to the point P are given (II, § **7**, *l*) by the formulas:

$$s'' = \int \frac{r}{\rho_0} ds_0 , \qquad \frac{1}{\rho''} = \frac{2}{r} - \frac{\rho_0 \sin\theta}{r^2} .$$

Therefore:

$$s'' = s', \qquad \frac{1}{\rho''} - \frac{1}{\rho'} = \frac{1}{r} .$$

The first equality represents **Steiner**'s theorem:

Any arc of a roulette with rectilinear base is equal to the corresponding arc of the pedal of the curve that moves with respect to the generating point.

The second equality leads to **Habich**'s theorem. If one changes the sign of ρ' in order to conform to the conventions that were made in § **1** then one will see that in the development on (P) of the foot of (M_0) with respect to P, the diameter of the circle of inflection will be precisely r. Meanwhile, the coordinates of P with respect to the foot are $r'' = y$, $\theta'' = \theta$, and they satisfy the equality $r'' = r \sin\theta$; that is to say, the point P will constantly belongs to the circle of inflection. Therefore, *if a point P on the development of a curve over a line is fixed in the plane of the curve then it will describe the roulette* (P), *while the pedal of the first curve with respect to P will describe a line with P when it is developed on* (P).

6. Examples:

a) If the curve (M_0) is a sinusoidal spiral of index n then one knows (§ **3**, *d*) that the roulette (P) that is described by the pole will be a Ribaucour curve of index $\frac{n'-1}{n'+1}$, and on the other hand, it is known (III, § **21**, *a*) that the pedal of (M_0) with respect to P is another spiral of index $\frac{n'}{n'+1}$. One can then prove by another method (cf., § **3**, *e*) that if a sinusoidal spiral of index $\frac{n'}{n'+1} = n$ is developed along a Ribaucour curve of index $\frac{n'-1}{n'+1} = 2n - 1$ then the pole of the spiral will describe a line.

b) If (M_0) is a conic then the roulette (P) that is generated by a focus will be a Delaunay curve, and one knows (III, § **8**) that the pedal of the conic with respect to P will be the circumference that is described with the focal axis as its diameter. Conversely, one should note that no matter how one fixes the point P in the plane of any circle, that circle will be the foot of a well-defined conic that has a focus at P. Therefore, the *line on which one needs to develop a circle in order for a given point in its plane to traverse a line will be a Delaunay curve.*

7. – When P is not fixed in the plane of (M_0), one will first of all need to know the trajectory that is described in that plane and the position that is occupied at any instant. In order to do that, it is enough to give the radius of curvature ρ'' of the trajectory and the ratio κ_0 of its elementary arc length to that of (M_0), which are quantities that can consequently be considered to be known functions of s_0. We would like to limit our study to the simplest case of a trajectory that is constantly orthogonal to the radius PM, which is a condition that must represent a constraint between κ_0 and ρ''. In order to abbreviate the calculations somewhat, consider the positions M' and P' of M and P, resp., in the fixed plane after an infinitesimal development of (M_0) on (M), in such a way that $MM' = ds$, $PP' = \kappa\, ds$. The lines PM and $P'M'$ are concurrent at the center of curvature C' of the trajectory of P in the fixed plane, and it is clear that one has:

$$\frac{PP'}{PC'} = \frac{MM'}{MC'} \sin\theta, \qquad \text{i.e.,} \qquad \frac{\kappa}{\rho'} = \frac{\sin\theta}{\rho' - r}.$$

The argument is also valid in the case where the displacement of P is considered in the moving plane, and therefore:

$$\kappa = \frac{\rho' \sin\theta}{\rho' - r}, \quad \kappa_0 = \frac{\rho'' \sin\theta}{\rho'' - r}. \tag{16}$$

Having said that, since the variations of the coordinates of P in the moving plane are the products of the elementary arc length $\kappa_0\, ds_0$ with $\sin\theta$ and $-\cos\theta$, one will have:

$$\frac{\delta x}{ds_0} = \kappa_0 \frac{y}{r}, \qquad \frac{\delta y}{ds_0} = -\kappa_0 \frac{x}{r}.$$

Hence, if one substitutes these values in the fundamental formulas that relate to the moving curve then one will get:

$$\frac{dx}{ds_0} = \left(\frac{\kappa_0}{r} + \frac{1}{\rho_0}\right) y - 1, \qquad \frac{dy}{ds_0} = -\left(\frac{\kappa_0}{r} + \frac{1}{\rho_0}\right) x;$$

thus, the formulas that relate to the fixed curve will become:

$$\frac{\delta x}{ds} = \left(\frac{\kappa_0}{r} + \frac{1}{\mathcal{R}}\right) y, \qquad \frac{\delta y}{ds} = -\left(\frac{\kappa_0}{r} + \frac{1}{\mathcal{R}}\right) x.$$

Therefore, the normal to the roulette also passes through the instantaneous contact point in the present case; i.e., *the two trajectories of P are mutually tangent.* In addition, one sees from the last formula that the ratio of the elementary arc length of the roulette to that of the base is:

$$\kappa = \kappa_0 + \frac{r}{\mathcal{R}}. \tag{17}$$

Given the function κ_0, formula (17) will yield the arc length of the roulette, and the first of (16) will permit one to calculate the curvature.

8. Savary's formula. – If one sets κ and κ_0 in (17) equal to their values in (16) then one will get:

$$\frac{1}{\rho' - r} - \frac{1}{\rho'' - r} = \frac{1}{\mathcal{R}\sin\theta}. \tag{18}$$

That is **Savary**'s important formula, which reduces to (6) for $\rho'' = 0$, and one can always substitute it for the first formula (16) for the determination of ρ', since if one is given κ_0 then the function ρ'' will also be known by means of the second formula (16). Geometrically interpreted, Savary's formula permits one to construct the center of curvature C' of the roulette when one supposes that the center of curvature C'' of the trajectory of the generating point in the moving plane is known. In fact, that says that *the perpendicular to PM and the tangent to* (*M*) *at M go through M and C′, respectively, which are concurrent on HC″.* One can do without *H* by observing that *the line CC′ and C_0C'' are concurrent on the perpendicular to PM that goes through M.* When C'' is coincident with *P*, one will recover the construction that was pointed out in § **2**.

9. Envelopes. – When (M_0) is developed on (*M*), any line that is fixed in the plane of (M_0) will envelop (II, § **5**) a certain line (*P*). Each point *P* can be regarded as a common point to two infinitely-close positions of the line considered; i.e., as fixed in the plane of the curve (M_0) when it is developed infinitely little along (*M*). Therefore (§ **1**), the normal to the envelope that is the trajectory of *P* in the fixed plane passes through *M*; i.e., *the line considered touches its envelope at the foot of the perpendicular that is dropped from M.* Meanwhile, one can then realize the hypothesis of § **7**, i.e., the point *P* moves orthogonally to *PM* and also in the plane of (M_0). Hence, in order to find the coordinates r and θ of *P* by projecting orthogonally from the origin *M* onto the lines that one considers, it is enough to substitute them in formulas (16) and (17) and arrive, in any case, at the *intrinsic equation of the envelope.* One finally applies Savary's formula to that point C'' that is rigidly linked with (M_0), and at a given instant, it coincides with the center of curvature of the moving line at the point where it touches its envelope. If ρ''' is the radius of curvature of the trajectory of C'' then one will need to replace ρ' and ρ'' in (18) with ρ''' and 0, resp., and put $r - \rho''$ in place of r. With that, one succeeds in changing ρ' into $\rho'' + \rho'''$. Therefore, $\rho' = \rho'' + \rho'''$; i.e., the center of curvature of the trajectory of the point considered *coincides with the center of curvature of the envelope of the moving line.*

10. – In order to apply the preceding formulas to the line, one must suppose that $\rho'' = \infty$. Under that hypothesis, the second formula (16) will give $\kappa_0 = \sin\theta$. (17) will then yield κ; and one will get ρ' from (18). One will then obtain:

$$s' = \int\left(\sin\theta + \frac{r}{\mathcal{R}}\right)ds, \qquad \rho' = r + \mathcal{R}\sin\theta, \tag{19}$$

in which r and θ are the coordinates of the projection P of M onto the line considered. Since that is given in the plane of (M_0), r and θ can be expressed as functions of s_0 and s. The intrinsic equation of the roulette (P) will then result from eliminating s in (19).

11. Application. – If a Ribaucour curve of index n is developed along a line then its directrix, which is on the normal to a segment $\frac{1}{2}(n + 1)\,\rho_0$ that starts at M, will touch its envelope at the foot P of the perpendicular that one drops from M, and one will then have $r = \frac{1}{2}(n + 1)\,\rho_0 \sin\theta$. If one substitutes that value in the second formula in (19), in which one has $\mathcal{R} = \rho_0$, then one will find that:

$$\rho' = \frac{n+3}{n+1}r = \frac{2r}{n'+1}, \qquad \text{in which} \qquad n' = \frac{n-1}{n+3},$$

and one will then arrive at the following theorem of **Dubois**:

When a Ribaucour curve of index n is developed along a line, its directrix will envelope a Ribaucour curve of index $\frac{n-1}{n+3}$.

For example, if one sets $n = 1, 0, -2$, etc., then one will find that when a circle is developed along a line, any of its diameters will envelope a cycloid. The directrix of a cycloid that is developed along a line will stay parallel to the tangents to an asteroid. The directrix of a parabola that is developed along a line will stay tangent to a catenary, etc.

12. Circle of regression. – The second formula (19) shows that one has $\rho' = 0$ when $r = -\mathcal{R}\sin\theta$; i.e., when P belongs to the circumference that is symmetric to the circle of inflection (§ **4**) with respect to the tangent to (M) at M. That circumference is called the *circle of regression*, since it is the locus of the cusps that are presented at a given instant along the roulette that is generated (tangentially) by the line in the plane of (M_0). One knows that *all of the cuspidal tangents are concurrent* at the point H', which is symmetric to H with respect to M. It will then follow that if a line that is fixed in the plane of (M_0)

constantly passes through H' then it will *contain a fixed point in the plane of* (M), which is a point that is common to all of the circles of regression. Indeed, the second point P at which the line meets the circle of regression will also be the point of contact of the line with its envelope, and therefore one constantly has $\rho' = 0$ at P, which means that the envelope reduces to the single point P. It is then useful to observe that in the inverse development of (M) along (M_0), the circles of regression and inflection switch with each other. Now, *if a point P in the development of* (M_0) *along* (M) *is fixed in the plane of* (M_0) *and describes a line,* since that line must pass through H (§ **4**), which takes the place of H' in the inverse development, it is clear that *the line considered in the development of* (M) *along* (M_0) *rotates around P.*

13. Examples:

a) One sees (§ **3**, *e*) that if a sinusoidal spiral of index n is developed along a Ribaucour curve of index $n = 2n - 1$ then the pole of the spiral will displace along the directrix of the base. It then follows directly that *if a Ribaucour curve of index n is developed along a sinusoidal spiral then the index* $\frac{1}{2}(n + 1)$ *of its directrix will rotate around the pole of the spiral.* For example, if a line is developed along a catenary then a point of its plane will move in a straight line, and the directrix of the catenary will pass through a fixed point in the inverse development. If a cardioid is developed along a conveniently-chosen cycloid then its cusp will traverse the directrix of the cycloid, and that line will not cease to pass through the cusp of the cardioid in the inverse development. The curve along which one must develop the second pedal of a circle with respect to a point in its plane in order for that point to describe a line will be parallel to an asteroid, and the line will rotate around the point in the inverse development.

b) More generally, if one utilizes Habich's theorem (§ **5**) then one can assert that *the line along which one must develop a curve in order for that line to envelop a point is the pedal of the second curve with respect to P.* Thus, for example, if one supposes that the second curve is a conic with one focus at P then one will find (cf., § **6**, *b*) that *if a Delaunay curve is developed along a conveniently-chosen circle then its base will rotate around a fixed point.*

14. – In conclusion, let us point out the utility that the theory of the roulette has in these lessons, since it provides us with ways of generating curves that were so far known only by their intrinsic equations. For example, we can now account more exactly for the form of the Ribaucour curves that are defined with indices 3 and – 5, and were previously known (III, § **7**) for the resemblance between their intrinsic equations:

$$s = 2\int \frac{d\rho}{\sqrt{\left(\frac{\rho}{a}\right)^4 - 1}}, \qquad s = \frac{2}{3}\int \frac{d\rho}{\sqrt{\left(\frac{\rho}{a}\right)^{4/3} - 1}},$$

resp., and those of the lemniscate and the equilateral hyperbola, resp. One can now say that the first curve is the locus of the center of an equilateral hyperbola that one develops along a line or the envelope of the directrix of a catenary that is developed (externally) along an equal catenary. It is also the curve along which a lemniscate must be developed in order for its center to traverse a line. Similarly, the second equation represents the curve along which one must develop an equilateral hyperbola in order for its center to describe a line. Conversely, when the two curves are developed along a lemniscate and an equilateral hyperbola, respectively, their directrices will rotate around fixed points.

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CHAPTER VI

BARYCENTERS

1. – One can attribute coefficients m_i that one calls *masses* to the points M_i (i = 1, 2, 3, …) that are defined in a plane by coordinates x_i, y_i relative to an arbitrary pair of axes and consider the point G that is defined by the coordinates:

$$x = \frac{\sum \mu_i x_i}{\sum \mu_i}, \quad y = \frac{\sum \mu_i y_i}{\sum \mu_i}. \tag{1}$$

It is clear that any linear transformation that is applied to the coordinates of the points M_i will be repeated identically with the coordinates x, y, and therefore it is enough to exhibit the *uniqueness* of the point (1) – i.e., to show that it is always the same, no matter how one chooses the axes. The point G is called the *barycenter* of the given system of points or masses. In particular, one should note that the barycenter of the system of two masses μ_1 and μ_2, which are assigned to the point M_1 and M_2, resp., is the point that divides $M_1 M_2$ with a ratio that is inverse to μ_1 and μ_2. The barycenter of a system that is composed of more systems of points is also the barycenter of the system, because in each of them, one proposes to place a mass that is equal to the sum of the masses of the corresponding system. That property, along with some others, is easy to deduce from (1). More especially, we would like to consider the case of masses that are distributed continuously along a curve and let $\mu\, ds$ denote the infinitesimal mass that is placed along the element of arc ds. If one wishes to know the law of mass distribution then it is enough to know the function μ of s (which is called the *density*), so the barycenter of any arc is well-defined. The coordinates of that barycenter will be given by the formulas:

$$x\int \mu\, ds = \int \mu\, u\, ds, \qquad y\int \mu\, ds = \int \mu\, v\, ds, \tag{2}$$

in which one supposes that the integrations extend from one endpoint to the other of the arc that one would like to consider, and that u and v represent the coordinates of the points of the curve. One can simplify the hypotheses when the density is supposed to be constant. One will then get the *barycenter*, properly speaking, which is what we shall always speak of in what follows when we do not make other hypotheses explicitly. If one takes the density to be equal (or proportional) to the curvature of the line then one will get the point that is called **Steiner**'s *barycenter of curvature.*

2. – In order to determine the double infinitude of barycenters of all the arcs of a curve, it is enough to know the barycenters of the arcs that have a given endpoint O (which one can always take to be the origin of the arc), since the barycenter of any arc $M_1 M_2$, which is defined by the values s_1 and s_2 of s at the endpoints, divides the rectilinear segment that goes from the barycenter of OM_1 to that of OM_2 in the ratio – $s_2 : s_1$. Therefore, let G be the barycenter of an arc OM, and let x, y be its coordinates with respect to the tangent and normal of the moving endpoint M. In the present case, (2) will reduce to the form:

$$s\,x = \int_0^s u\,ds\,, \qquad\qquad s\,y = \int_0^s v\,ds\,,$$

in which each pair of values u, v satisfies the immobility conditions (II.1), except that ($u = 0$, $v = 0$) refers to the upper limits of the integrals. It first follows that if ρ represents the value of ρ at M more simply then:

$$\frac{d}{ds}\int_0^s u\,ds = \frac{1}{\rho}\int_0^s v\,ds - \int_0^s ds\,, \qquad\qquad \frac{d}{ds}\int_0^s v\,ds = -\frac{1}{\rho}\int_0^s u\,ds\,,$$

i.e.:

$$\frac{d(sx)}{ds} = \frac{sy}{\rho} - s, \qquad\qquad \frac{d(sy)}{ds} = -\frac{sx}{\rho}. \tag{3}$$

The coordinates of G are determined by these equations and the condition that they must go to zero with s, since the barycenter will be O when the arc reduces to the single point O. Those same equations (3) can show us the way that G tends to O with greater precision, since if ρ is not zero at the (arbitrary) origin of the arcs then, by virtue of l'Hôpital's theorem, one will have:

$$\lim\,\frac{x}{s} = \lim\,\frac{sx}{s^2} = \frac{1}{2}\lim\left(\frac{y}{\rho} - 1\right) = -\frac{1}{2}, \qquad\qquad \lim\,\frac{y}{s} = \lim\,\frac{sy}{s^2} = -\frac{1}{3}\lim\frac{x}{s\rho} = \frac{1}{6\rho},$$

and one will see that the equation $x^2 + y^2 = 3/2\,\rho\,y$ will tend to be satisfied in the vicinity of O; i.e., *the barycenter will tend to locate itself on the circumference that one obtains by reducing the osculating circumference* around *O by three-quarters.*

3. – Knowing a *barycentric line* can be very useful – i.e., the curve that is described by the point G when M displaces along the given curve. Obviously, a curve has an infinitude of barycenters, each of which takes the origin to be an arbitrary point of that curve, and it is clear from the observations that were just made that not only does any barycentric line touch the curve at the corresponding origin, but also that its curvature at the contact point is equal to four-thirds of that of the given curve. Furthermore, any curve will belong to the envelope of its barycentric lines, as one will also see more clearly by observing that two arbitrary barycentric lines will meet at the barycenter of the arc that is determined by their origins on the given curve, whose barycenter will tend to locate itself along the curve when the two origins tend to coincide. If the curve is closed then two barycenters will have an infinitude of common points that are barycenters of the infinitude of arcs that are determined by the origins on the curve, and since the difference or sum of two such arcs will always be a multiple of the length of the entire curve, one can add that the intersection of the two barycentric lines will happen above a line that passes through the point Q, which is the barycenter of the entire closed curve and the point that is common to all of the barycentric lines.

4. – Now take under consideration, more generally, the curve that is described by the arbitrary point Γ whose coordinates satisfies (3). By virtue of (3), the formulas immediately give:

$$\frac{\delta x}{ds} = -\frac{x}{s}, \qquad \frac{\delta y}{ds} = -\frac{y}{s},$$

and therefore the tangent to (Γ) at Γ will pass through M – i.e., *the point* Γ *will constantly follow M* – and the ratio of the elementary arc lengths of the two curves will be $\kappa = r : s$. The coordinates r and θ of Γ satisfy the single immobility condition:

$$\frac{d\theta}{ds} = -\frac{1}{\rho} + \frac{\sin\theta}{r},$$

which will result immediately from the observation that the line $M\Gamma$ touches its envelope at Γ, and since one can easily deduce it from (3), moreover. Therefore, (II.4) will give:

$$\frac{\kappa}{\rho'} = \frac{1}{\rho} + \frac{d\theta}{ds} = \frac{\sin\theta}{r}$$

for the calculation of the curvature of (Γ), and therefore the intrinsic equation for (Γ) will result from the elimination of s from the equality:

$$s' = \int \frac{r}{s}\,ds, \qquad \rho' = \frac{r^2}{s\sin\theta}.$$

The second formula gives one the way to construct the center of curvature of (Γ). Draw the segment $MD = s$ along the tangent to (M) in the opposite sense and project D onto the normal to (Γ) at H*: The center of curvature of* (G) *will belong to the perpendicular to MH that is raised at M*. From this fortunate property, in particular, any barycentric line (G) will be characterized among all of the (Γ) by the fact that its passage from O must touch (M) and have the curvature:

$$\frac{1}{\rho'} = \lim_{s=0} \frac{s\sin\theta}{r^2} = \lim\left(\frac{s}{r}\right)^3 \lim\frac{y}{s^2} = \frac{4}{3\rho}.$$

If one is guided by the first property then it will be easy to account for the general form of the barycentric line to any closed curve. The barycentric line that takes the origin at O will pass through the point Q, which is the barycenter of the whole curve, an infinitude of times tangentially to OQ, and its curvature, which will take on an additive constant with every new passage, will conclude by exceeding any limit. The asymptotic point of the barycentric line must therefore be confined indefinitely to the domain of Q, which is nevertheless a line OQ that does not meet the curve an infinitude of times. The tangents to the barycentric line at the infinitude of points where it meets any other line that emanates from Q will intersect at a point, and when the line rotates around Q, the point will describe the closed curve that admits the given barycentric line.

5. – The curve (G) is not indispensible for the determination of the barycenters; it is enough to know *one* arbitrary curve (Γ). Indeed, let ξ and η be the coordinates of Γ, and take:

$$x = \xi + R \cos \theta, \qquad y = \eta + R \sin \theta.$$

If one applies (3) to the points Γ and G then one will get:

$$\frac{d}{ds}(sR \cos \theta) = \frac{sR}{\rho}\sin \theta, \qquad \frac{d}{ds}(sR \sin \theta) = -\frac{sR}{\rho}\cos \theta;$$

therefore:

$$\frac{d(sR)}{ds} = 0, \qquad \frac{d\theta}{ds} = -\frac{1}{\rho}.$$

One sees from the first equality that *sR must be constant*, and the second one says that *the direction ΓG is invariable.* Therefore, if one knows a curve (Γ) and one wishes to determine the entire barycentric line (G) then it will suffice to know only one barycenter G_0. Indeed, if Γ_0 is the point that corresponds to G_0 on the curve (Γ) then it is enough to draw a segment through any point Γ that is parallel to $\Gamma_0 G_0$ and whose length relates to that of $\Gamma_0 G_0$ inversely to the ratio of s to s_0. The endpoint of that segment is precisely G. In particular, one can take the same origin O for G_0, except that Γ_0 will then be at infinity; i.e., (Γ) will have an asymptote whose direction is precisely that of all the segments ΓG. However, the only thing that one knows about the magnitudes of those segments is that they vary from one point to another in inverse proportion to s, and in order to determine them, one needs to remember that when M tends to O while increasing the distance to M indefinitely and G tends to Γ, MG must nonetheless tend to zero.

6. Applications:

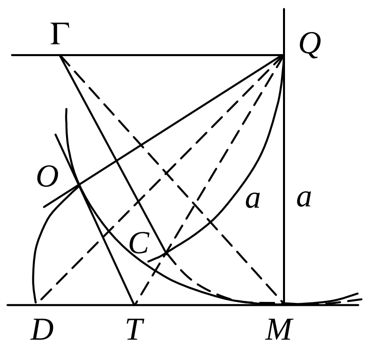
a) In the case of a *circle* ($\rho = a$, $s = a\,\varphi$), when (3) is put into the form:

$$\frac{d(x\varphi)}{d\varphi} = (y - a)\,\varphi, \qquad \frac{d(y\varphi)}{d\varphi} = -x\,\varphi,$$

one can immediately glimpse the solution $y = a$, $x\varphi = -a$. Although one can succeed in finding that the coordinates of G are:

$$x = -\frac{a}{\varphi}(1 - \cos \varphi), \quad y = \theta\left(1 - \frac{\sin\varphi}{\varphi}\right)$$

with an easy integration, from the observations of § **5**, if we are to also be able to determine G then it will be enough for us to know the point Γ that is defined by the coordinates $x = -a / \varphi$, $y = a$. The point Γ is at the intersection of the perpendiculars to the rays QD and QM that go through M and Q. When φ tends to zero, the absolute value of x will increase to infinity, and therefore $M\Gamma$ will tend to touch the circle. Therefore, *the barycenter of the arc OM belongs to the perpendicular to OQ that is based at* Γ. In order to see that, it is enough to construct G, which must also be found on the bisector of the angle OQM, by reasons of symmetry. After all, it is easy to determine the length of ΓG if one observes that it must vary in inverse proportion to φ and that, on the other hand, it will tend to behave like the length of ΓM, which behaves like $a : \varphi$ in a neighborhood of O, in its own right. Therefore, $\Gamma G = a : \varphi = \Gamma Q$; that is to say: *The barycenter also belongs to the circumference that is described by the center* Γ *tangentially to the ray QM*. Note that on that circumference, the arc QG has constant length a. It will then follow that if an infinitely-thin inextensible rod that is fixed *in the neighborhood of Q* is bent into a circular form then its moving extremity will describe the barycenter of the circle whose center is Q and whose origin is at M. Finally, if one considers the similar triangles MOD, QGM with perpendicular homologous sides then one will see that the side OD is also perpendicular to GM, and in that way, one will be led to a much simpler construction of the barycenter: *G belongs to the perpendicular to OD that is based at M*.



b) Let us seek to see if we can satisfy (3) by taking x and y to be proportional to s, and from that hypothesis, those coordinates will certainly define the barycenter, since they are annulled with s. Set $x = \alpha s$, $y = \beta s$, so (3) will give:

$$\frac{\rho}{s} = \frac{\beta}{2\alpha + 1} = -\frac{\alpha}{2\beta},$$

and therefore the curve is a *logarithmic spiral*. Conversely, if one is given a similar curve by means of the equation $\rho = ks$ then one can always get α and β as functions of k from the preceding equations, since one is sure that (3) will be satisfied by $x = \alpha s$, $y = \beta s$. However, that determination is not necessary, since it is enough to put the equations into the form:

$$\frac{\rho}{s} = \frac{y}{2x + s} = -\frac{x}{2y}$$

in order to see that the barycenter of the arc *OM* belongs to the line:

$$s\,x + 2\rho\,y = 0, \qquad 2\rho\,x - s\,(y - \rho) = 0.$$

Meanwhile, it is known (I. § **11**, *c*) that the perpendicular to *OM* that is raised at *O*, which is the pole of the spiral, will meet the normal at *C*, the center of curvature, and the tangent at *D* at a distance *s* from *M*. Now, the geometric interpretation of the last equations will show that *G is the projection of M onto the line that joins C to the midpoint of MD.* Here, we observe that *G*, like *O*, belongs to the circumference that is described by the diameter *MC*.

c) For $x = 0$, (3) will become $y = \rho$, $d\,(sy) / ds = 0$; therefore, $s\rho = a^2$. Consequently, in the case of the *clothoid*, the center of curvature is a point Γ, that is to say, the barycenter relative to the point of inflection can be deduced from the evolute of the curve, thanks to the observations in § **5**. Since $M\,\Gamma$ tends to become the normal at *O* – viz., the inflection point – when *M* tends to *O*, *the barycenter of the arc OM will belong to the perpendicular to the inflexional tangent that is based at the center of curvature at M.* In addition, ΓG varies in inverse proportion to *s* – i.e., proportional to ρ – and since *MG* tends to behave like $\Gamma M - \Gamma G$ in the vicinity of *O*, one will necessarily have that $\Gamma G = \rho$, since otherwise *MG* would exceed any limit, instead of tending to zero. Therefore, *the barycenter of the arc OM also belongs to the circumference that osculates at the endpoint M*. Given that, one knows (§ **2**) that the barycenter of any arc $M_1\,M_2$ will divide the rectilinear segment that goes from the barycenter of OM_1 to that of OM_2 in the ratio $-s_2 : s_1 = -\rho_1 : \rho_2$, and since those barycenters are the endpoints of two parallel radii of the circles that osculate the curve at M_1 and M_2, one will see that *any arc of the clothoid will have its barycenter at a center of similitude of the circles that osculate the endpoints.* It will then be easy to prove that the clothoid is the only curve that has the barycenter of any of its arcs in a straight line with the centers of curvature at the endpoints of those arcs.

d) We would like to look for all of the curves for which *the barycenter of an arc OM belongs to the osculating circumference at M, reduced or amplified by a constant proportion around M*. In other words, we need to be able to find functions *x* and *y* of *s* that are zero for $s = 0$ and satisfy (3) and the equation:

$$x^2 + y^2 = (n+1)\,\rho\,y\,. \tag{4}$$

If one differentiates both sides of this with respect to *s* after having multiplied them by s^2 and observing (3) then one will get:

$$(n-1)\,s\,x = (n+1)\,y\,\frac{d(s\rho)}{ds}. \tag{5}$$

If one then writes that equality in the form:

$$\frac{n-1}{sy}\frac{d(sy)}{ds} + \frac{n+1}{s\rho}\frac{d(s\rho)}{ds} = 0$$

then upon integrating, one will get:

$$(s\,y)^{n-1}\,(s\,\rho)^{n+1} = a^{4n}, \tag{6}$$

as long as $n \neq 0$. For $n = 1$, one will then find the clothoid, to which we would not like to return. The elimination of $s\rho$ from (4) and (60 gives:

$$(s\,x)^2 + (s\,y)^2 = (n+1)\;a^{4n/(n+1)}(sy)^{2/(n+1)}, \tag{7}$$

and since sx and sy are infinitesimal with s, one cannot have $n + 1 > 0$. One must also have $2 / (n + 1) \leq 2$ (i.e., $n \geq 0$), if one does not wish that the absurd equality $x^2 + y^2 = 0$ would be true in the neighborhood of O for real, non-zero values of x and y. Having said that, one sets:

$$s\,\rho = a^2\,t, \qquad \tau = \sqrt{(n+1)\,t^{2n/(n-1)} - 1},$$

for brevity. Formulas (6) and (7) will then give:

$$s\,y = a^2 t^{-\frac{n+1}{n-1}}, \quad s\,x = \pm\tau\, a^2 t^{-\frac{n+1}{n-1}}, \tag{8}$$

and the first of these equalities will show that as s tends to zero, t will tend to zero or increase indefinitely according to whether $n < 1$ or $n > 1$, resp. Finally, if one substitutes the values (8) in (5) and integrates then one will get:

$$s^2 = -2\frac{n+1}{n-1}a^2\int_0^t \frac{dt}{\tau} \quad \text{or} \qquad s^2 = 2\frac{n+1}{n-1}a^2\int_t^\infty \frac{dt}{\tau} \tag{9}$$

in one case or the other, resp. In order to find the intrinsic equations of our curve, it is enough for us to eliminate t from $s\,\rho = a^2\,t$ and one or the other of equations (9). If we would like to examine the behavior of those curves in the neighborhood of the origin then we should observe that when s tends to zero, t will increase to infinity like $t^{\,n/(n-1)}$, so it will follow that the integrals (9) will behave like $t^{\,-1/(n-1)}$; i.e., by virtue of (9) itself, t will tend to zero or increase indefinitely like s^{2-2n} in the cases ($n < 1$, $n > 1$, resp.). Therefore, the curve develops in the neighborhood of O as if its intrinsic equation were $\rho = k\,s^{1-2n}$, that is to say, it will attain a lesser or greater contact with the tangent at that point according to whether $n < 1/2$ or $n > 1/2$, resp. In order to specify that form better, we need to recall the preceding observations (I, § **11**, *e*) and then recall that in any case, the inflection will appear at O, instead. An asymptotic point is possible only in the case of $n = 0$, which shall be examined last, but not least. The final observation of § **2** will permit us to assert that it is only when $n = 1/2$ that the curvature can have a finite, non-zero value at the origin. In that case, the first of (9) will essentially give:

$$s^2 = 6a^2 \int_0^t \frac{t\,dt}{\sqrt{\frac{3}{2} - t^2}} = 6a^2 \left(\sqrt{\frac{3}{2}} - \sqrt{\frac{3}{2} - t^2} \right),$$

and one therefore infers that $s^2 + 36\,\rho^2$ = constant, which is the equation (I, § **8**, *c*) of a *stellate bicuspid epicycloid*. Finally, if one would like to know the curves for which it is possible to have $n = 0$, it is enough to substitute an arbitrary constant for the right-hand side of (6), which will cease to be arbitrary for $n = 0$, and for ease of calculation, one can denote that constant by $1 + 4k^2$. Formulas (4) and (6) will then give:

$$x = -\frac{2k\rho}{1+4k^2}, \qquad y = \frac{\rho}{1+4k^2};$$

if one integrates (5) then one will have:

$$\rho = k\,s + \frac{k'}{s}.$$

Among those curves, it is only the clothoid ($k = 0$) that cannot answer the question, because one has $n = 1$ for it; however, one should note that x and y can be annulled with s, and for that, it is necessary and sufficient that ρ should be annulled; i.e., that one must have $k' = 0$. Therefore, the property that was observed at the end of the penultimate application characterizes the logarithmic spiral.

7. *The search for the barycenters of a curve is always reducible to that of the fixed points in the plane of another curve.* – Indeed, if one sets:

$$s\,x = a\,x_0, \qquad s\,y = a\,y_0, \tag{10}$$

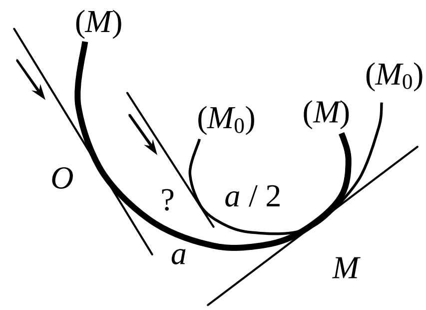
along with:

$$s^2 = 2a\,s_0, \qquad s\,\rho = a\,\rho_0, \tag{11}$$

then (3) will become the known conditions:

$$\frac{dx_0}{ds_0} = \frac{y_0}{\rho_0} - 1, \qquad \frac{dy_0}{ds_0} = -\frac{x_0}{\rho_0}, \tag{12}$$

which guarantee the immobility of the point (x_0, y_0) in the plane of a curve (M_0) whose intrinsic equation will result from the elimination of s from (11). Meanwhile, (11) will establish a correspondence between the points of (M) and those of (M_0), while (10) relates a solution (x, y) of (3) with one (x_0, y_0) of (12), and therefore, it makes a certain *point* in the plane of (M_0) correspond to any *curve* (Γ). In particular, the barycenter that takes the origin to be at O will correspond to the origin of the arcs of (M_0), since from (10), x_0 and y_0 will be annulled with s (and s_0, resp.) if x and y remain finite. That happens (§ **5**) at the origin only for the barycenter.



8. Geometric construction of the barycenters. – The arbitrariness in a permits one to deduce a general construction for the barycenters from formulas (10) and (11). If one must find the barycenter of the arc OM then one can take a to be equal to precisely the length of OM, and that will determine the curve (M_0), thanks to (11). By virtue of (10), if the two curves touch at the corresponding points M and M_0, then the barycenter of OM and the origin of (M_0) will coincide, and the first of (11) will give $s_0 = a / 2$ when $s = a$. It is then enough to measure out an arc along (M_0) that is *one-half of OM* when starting from the origin and to arrange that the other endpoint should touch the arc OM at M: *The origin will be placed at the desired barycenter.* In the choice of the point M_0, one can also be guided by the considerations that the contact of the two arcs *must result at a higher order* (IV, § **1**), while for $s = a$, the second formula in (11) will give $\rho = \rho_0$. Finally, one can deduce a third determination of the point M_0 from the equality:

$$\int_0^a \frac{ds}{\rho} = \int_0^{a/2} \frac{ds_0}{\rho_0},$$

from which, one will see that if one measures the contact of the two arcs in the aforementioned way *then their tangents at the other endpoints will be parallel.*

9. Kinematic construction of the barycenters. – Suppose that the curves (M) and (M_0) are arranged such that they will touch at two corresponding points, and one considers another pair M', M'_0 of similar points in the neighborhood of the contact points. Let C and C_0 be the centers of curvature of the two curves at M. If one observes (11) then one will deduce from (10) that:

$$\frac{x_0}{x} = \frac{y_0}{y} = \frac{s}{a} = \frac{\rho_0}{\rho} = \frac{ds_0}{ds},$$

and therefore the point Γ_0 will be found at the intersection of the line $M\,\Gamma$ with the parallel to $C\,\Gamma$ that goes through C_0 . It will then follow that if the plane of (M_0) experiences a dilatation or a contraction around M that takes C_0 to C then the effect of that deformation will be to transfer the point Γ_0 to Γ and the point M_0 to M, and therefore, the contact between the two curves will persist at corresponding points when one of them rolls along the other. Therefore, if the curve (M) is measured from the origin of the arcs to the contact with the corresponding curve (M_0) at a conveniently-chosen point G, and if that develops on (M) by dilating or contracting around the point of contact in such a way that the two curves constantly preserve second-order contact then *the point G will be the barycenter of the arc OM at any instant.* Hence, one can construct the barycenter of any arc for any given curve in the plane in a kinematically-intelligible way, before one knows another curve, which is very easy to determine, thanks to (11).

10. Examples:

a) In the figure (§ **6**, *a*), which showed how to construct the barycenter of an arc of a circle, the arc of the involute GM, which constantly osculates the fixed circumference at M, appeared clearly, and one got precisely $\rho_0^2 = 2a\, s_0$ from (11) for $\rho = a$, which is the equation for an involute of the circle, and is similar to the one that contains the arc GM. One should also note that the cuspidal tangent will be kept parallel to the tangent OT as long as the cusp does not cease to indicate the barycenter of the arc OM .

b) One sees from (11) that ρ_0 is proportional to s_0 when ρ is proportional to s, which is to say that if (M) is a logarithmic spiral then (M_0) will also be a logarithmic spiral. If the points C and D constructed (cf., § **6**, *b*) relative to the first spiral then what will the analogous points C_0 and D_0 be for the second spiral, which is tangent to the first one at M? Since the contact is of second order, C_0 will coincide with C, and from another observation that was made in § **8**, the arc GM of the second spiral, which rectifies it at D_0M precisely, must be one-half the arc length OM – i.e., of DM – one will see that D_0 is the midpoint of DM. It is also clear now that, just as O is the projection of M onto CD, G *will be the projection of M onto* CD_0 .

c) In the case of a clothoid, the second formula (11) will give ρ_0 = constant, which is to say that if a variable circumference is developed over a clothoid that constantly osculates it then one of its points will describe the barycenter of the clothoid that takes its origin at the inflection point. Otherwise stated, that is the property that was found in § **6**. One can arrive at it more directly and with better precision by utilizing what was said in § **8**. Indeed, after having confirmed that if (M) is a clothoid then (M_0) will be a circle, one can immediately add that it is the osculating circle at M, and one then constructs the barycenter G by drawing an arc MG along the circumference in the negative sense whose length is one-half the arc length MO of the clothoid, or merely upon observing that the normal to the circumference at G and the inflectional normal to the clothoid are parallel.

d) If $\rho = k\, s^n$ is the equation of the curve (M) then (11) will give $\rho_0 = k_0\ s_0^{(n+1)/2}$ for (M_0). One will then recover the preceding results ($n = 0, 1, -1$), and see, in addition, that one requires a $(2n + 1)^{\text{th}}$ involute for the construction of the barycenters of the n^{th} involute of the circle. Similarly, one will find that an asteroid is required for a cycloid, a cycloid for a bicuspidal epicycloid, a cardioid for the stellate epicycloid of § **6**, etc. More generally, any cycloidal line implies an analogous line, in such a way that any vertex of the fixed line will correspond to a cusp of the moving line.

e) Formula (10) and the second of (11) allow one to see immediately that when the curve (M) is defined by a *homogeneous* relation between the radius of curvature and the coordinates of the barycenter, that same relation between the radius of curvature and the coordinates of a fixed point will define the corresponding curve (M_0). For example, it follows that all of the curves that were studied in the application (d) of § **6** will correspond to the *sinusoidal spiral*. Moreover, if one takes $s^2 = 2a\, s_0$ and $\rho_0 = at$ in (9) then one will get precisely (cf., III, § **19**) the equation of the sinusoidal spiral:

$$s_0 = \frac{n+1}{n-1}\int \frac{d\rho_0}{\sqrt{\left(\frac{\rho_0}{a_0}\right)^{2n/(n-1)} - 1}}.$$

The generating point of the barycenter is the *pole* of the spiral, since (III, § **15**) one will have $\rho_0 = 0$ only at the pole. The pole must therefore belong to the spiral, and it must effectively belong to the spiral for $n \geq 0$ (III, § **20**).

11. – Each of the properties that were found so far for certain curves will persist for an arbitrary curve, as long as one varies the density along that curve conveniently. If σ represents the mass that is deposited along the arc OM then formulas (3) must be replaced with:

$$\frac{d(\sigma x)}{ds} = \frac{\sigma y}{\rho} - \sigma, \qquad \frac{d(\sigma y)}{ds} = -\frac{\sigma x}{\rho}, \tag{13}$$

and if one sets:

$$\frac{x_0}{x} = \frac{y_0}{y} = \frac{\sigma}{a} = \frac{\rho_0}{\rho} = \frac{ds_0}{ds}$$

then they will reduce to (12). Therefore, if one imagines that the considerations of § **9** have been repeated then one will always succeed in constructing, for any curve (M) and for a given mass distribution, a curve (M_0) whose development along (M), while dilating or contracting around the contact point in such a way that it preserves a higher-order contact with (M), and during its motion and deformation, it will trace out the barycenter of the mass distribution along that arc of (M) that has been the contact with the moving curve from the beginning. Its equation is found by eliminating s from the relations:

$$a\, s_0 = \int \sigma\, ds, \quad a\, \rho_0 = \sigma\, \rho. \tag{14}$$

The geometric interpretation of the results thus-obtained will provide a construction of the barycenter in the individual cases that will, however, pertain to an arbitrary curve, as long one assumes that σ is a special function of s.

12. Examples:

a) One can give the form $\sigma\rho = a^2$ to the intrinsic equation of any curve by taking μ to be proportional to the derivative of the curvature. The second formula in (14) will then give $\rho_0 = a$, and the barycentric property of the clothoid will then be true for that particular distribution. Hence, *a center of similitude of the osculating circles at the end points of any arc of any curve will be the barycenter of a mass that is distributed along the arc with a density that is proportional to the variation of the curvature.* In particular, if one takes $\sigma = a\,\varphi$ then one will see that a curve that is already known to us (I, § **11**, *c*) is characterized by the property that the barycenter of curvature of any arc and the centers of curvature at the endpoints of that arc must always be in a straight line.

b) Any way by which one can satisfy (13) will yield particular constructions of barycenters. We confine ourselves to pointing out the result of setting:

$$\sigma x = -a\,y_0\,, \quad \sigma y = a\,(x_0 + s) \tag{15}$$

in (13), which will then reduce to the conditions of immobility for the point (x_0, y_0) in the plane (M), as long as one has $\sigma\rho = a\,s$. One then sees, as in § **7**, that the point in question will be the origin of the arcs. Meanwhile, from (15) and the condition that was found, one will deduce that:

$$x\,x_0 + y\,y_0 = -s\,x = \rho\,y_0\,,$$

upon eliminating σ. Therefore, for any curve, *the perpendiculars to OD and OM that are based at M and C, resp., meet at the barycenter of a mass that is distributed along the arc OM with a density that is proportional to the variation of the product of the arc length with the curvature.* In particular, if one considers the curve $\rho = k\,s^n$ then one will find that $\sigma = (n-1)\,a\,\varphi$, and the preceding construction will then yield the barycenter of curvature for that curve. In addition, if one observes that in this case it will result from (14) that $\rho_0 = k_0\,s_0^{1/(2-n)}$ then one will see (II, § **13**, *i*) that the curve (M_0) is an involute of (M).

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CHAPTER VII

BARYCENTRIC ANALYSIS

1. – The notion of barycenter serves as the basis for an elegant method of geometric analysis that we cannot hope to go further into without leaving the field of pure intrinsic geometry. We shall therefore confine ourselves to shedding some light, by way of examples, on some of the simplest and most essential links to the intrinsic analysis of plane curves. First, recall (VI, § **1**) that the barycenter M of the masses μ_1 and μ_2 that are deposited at the points A_1, A_2, resp., belongs to the line $A_1 A_2$ and is such that:

$$\mu_1 \cdot MA_1 + \mu_2 \cdot MA_2 = 0. \tag{1}$$

Any pair of values for μ_1 and μ_2 will then correspond to an infinitude of other pairs (μ_1 , μ_2) that are obtained by multiplying one of them by an arbitrary number, since that will not change (1) in fact. Even better, in order to make a point correspond to just one pair (μ_1 , μ_2), one agrees to set $\mu_1 + \mu_2 = 1$. One can then generate the entire line by varying the distribution of the unit masses between the fundamental points A_1, A_2 . When M *is extended indefinitely* along the line, the ratio of its distance to the fundamental points will tend to unity, and one will see from (1) that *the equality* $\mu_1 + \mu_2 = 0$ *will tend to become valid*. To abbreviate, we say that *one has* $\mu_1 + \mu_2 = 0$ *at the point at infinity*. If N is the point that is specified by the masses that are proportional to $-\mu_1$ and μ_2 then (1) will show that $(MNA_1A_2) = -1$; that is to say, N is the harmonic conjugate of M with respect to the fundamental points. That explains once more why one can say that one has $\mu_1 + \mu_2 = 0$ at infinity if $\mu_1 - \mu_2 = 0$ at the midpoint of $A_1 A_2$, which is conjugate to the point at infinity.

2. – Analogously, if A_1, A_2, A_3 are the vertices of a non-zero triangle (arranged in the order in which they are encountered by starting at one point and traversing the perimeter of the triangle while leaving the area to the left) that has unit masses variously distributed among its vertices then that will give rise to a double infinitude of triples of masses μ_1, μ_2, μ_3, and each triple will give rise to a point (viz., the barycenter) that one can construct by dividing $A_2 A_3$ in the ratio $\mu_3 : \mu_2$ to give L and then dividing A_1L in the ratio $(\mu_2 + \mu_3) : \mu_1$. Conversely, any point in the plane will correspond to a triple of values μ_1, μ_2, μ_3 (viz., the *barycentric coordinates* of the point), such that:

$$\mu_1 + \mu_2 + \mu_3 = 1, \tag{2}$$

and it will correspond to just one such triple, because if the line A_1M divides A_1A_2 at L in the ratio k and M divides A_1L in the ratio k' then one can always, and in just one way, satisfy (2) and the conditions:

$$\mu_3 = k\,\mu_2\,, \qquad \mu_2 + \mu_3 = k'\mu_1\,. \tag{3}$$

From the last one, when k' tends to – 1, one will see that *when M extends indefinitely, the equation:*

$$\mu_1 + \mu_2 + \mu_3 = 0 \tag{4}$$

will tend to be verified.

3. Straight line. – If one takes (2) into account then formula (1) of the preceding chapter will become:

$$x = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 , \qquad y = \mu_1 y_1 + \mu_2 y_2 + \mu_3 y_3 , \tag{5}$$

and then a linear relationship between the Cartesian coordinates x and y can be transformed into a linear *and homogeneous* relationship between the barycentric coordinates. Conversely, if one is given such a relationship then one can always transform it into a linear relationship between the Cartesian coordinates by means of the formulas:

$$a^2 \mu_1 = (y_2 - y_1)\, x - (x_2 - x_1)\, y + (x_2\, y_2 - x_3\, y_2), \qquad \text{etc.}, \tag{6}$$

which are obtained by solving equations (2) and (5) for μ and letting a^2 represent two times the area of the fundamental triangle:

$$a^2 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}. \tag{7}$$

Therefore, any linear equation between the barycentric coordinates will represent a line. For example, if one fixes k then the first equation in (3) will represent that line through A_1 that divides $A_2\, A_3$ in the ratio k. Similarly, when one supposes that k' is constant in the second equality (3), it will become the equation for a parallel to the edge $A_2\, A_3$. That equation can also be written in inhomogeneous form as μ_1 = constant, and one will see that when a point is located parallel to an edge of the fundamental triangle, its barycentric coordinates with respect to the opposite vertex will not change. In particular, the equation for the edge that is opposite to A_i is $\mu_i = 0$. That permits one to directly write down the condition for parallelism of two lines:

$$\alpha_1\, \mu_1 + \alpha_2\, \mu_2 + \alpha_3\, \mu_3 = 0, \qquad \beta_1\, \mu_1 + \beta_2\, \mu_2 + \beta_3\, \mu_3 = 0. \tag{8}$$

In order for them be concurrent with a triple at a point, it is necessary and sufficient that the determinant that is composed of the coefficients of the three equations must be zero, because that is precisely the necessary and sufficient condition for the existence of values for the μ that are not all zero and satisfy the system of three equations. Having said that, saying that two lines are parallel is equivalent to saying that they are concurrent on the line at infinity, and therefore the *condition for parallelism* is expressed by setting the determinant of the system that is composed of equations (4) and (8) to zero, namely:

$$\begin{vmatrix} 1 & \alpha_1 & \beta_1 \\ 1 & \alpha_2 & \beta_2 \\ 1 & \alpha_3 & \beta_3 \end{vmatrix} = 0. \tag{9}$$

4. Distance between two points. – Let $\delta\mu_1$, $\delta\mu_2$, $\delta\mu_3$ be the variations that the barycentric coordinates experience when one passes from M to another arbitrary point M', and calculate the distance R between the two points. Let a_1 , a_2 , a_3 be the lengths of the edges of the triangle. If the segment MM' is parallel to an edge (for example, to $A_2 A_3$) then its length, as well as its sign, can be expressed by $a_1\ \delta\mu_3$, or simply by $-a_1\ \delta\mu_1$, which would result easily from the similarity of the triangles A_1MM', A_1LL'. No matter how M and M' are situated, consider the point of intersection M'' of the parallels to the edges $A_2 A_1$ and $A_3 A_1$ that go through M and M', resp. Obviously, the coordinates of M'' are $\mu_1 - \delta\mu_2$, $\mu_2 + \delta\mu_2$, μ_3 , and therefore two edges of the triangle $MM'M''$ will have lengths $a_1\ \delta\mu_3$ and $-a_3\ \delta\mu_2$. If one now observes that the angle that is opposite to the edge R is equal to the angle A_1 then one will have:

$$R^2 = a_2^2(\delta\mu_3)^2 + a_3^2(\delta\mu_2)^2 + (a_2^2 + a_3^2 - a_1^2)\,\delta\mu_2\,\delta\mu_3;$$

i.e., if $\delta\mu_1 + \delta\mu_2 + \delta\mu_3 = 0$:

$$R^2 = -\ (a_1^2\ \delta\mu_2\ \delta\mu_3 + a_2^2\ \delta\mu_3\ \delta\mu_1 + a_3^2\ \delta\mu_1\ \delta\mu_2)\,. \tag{10}$$

In particular, when the two points are infinitely close, the square of the distance between them will be:

$$ds^2 = -\ (a_1^2\ d\mu_2\ d\mu_3 + a_2^2\ d\mu_3\ d\mu_1 + a_3^2\ d\mu_1\ d\mu_2)\,. \tag{11}$$

5. – We are now in a position to also find the *condition for the perpendicularity* of two lines: Let them be (8), in which one takes the points P and Q to be outside of the point of intersection M. Let the letters ε and η represent the variations of the coordinates when M passes to P and Q. Obviously, ε and η satisfy (8), and in addition, because (2) will be true at any point, one will have:

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0, \qquad \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0.$$

It will then follow that:

$$\frac{\varepsilon_1}{\alpha_2 - \alpha_3} = \frac{\varepsilon_2}{\alpha_3 - \alpha_1} = \frac{\varepsilon_3}{\alpha_1 - \alpha_2}, \qquad \frac{\eta_1}{\beta_2 - \beta_3} = \frac{\eta_2}{\beta_3 - \beta_1} = \frac{\eta_3}{\beta_1 - \beta_2}. \tag{12}$$

Having said that, apply formula (10) to the distances MP, MQ, PQ in the relation $(PQ)^2 = (MP)^2 + (MQ)^2$, which is necessary and sufficient for the perpendicularity. One will then obtains:

$$a_1^2(\varepsilon_2 - \eta_2)(\varepsilon_3 - \eta_3) + \ldots = a_1^2\,\varepsilon_2\,\varepsilon_3 + \ldots + a_1^2\;\eta_2\,\eta_3 + \ldots,$$

i.e.:

$$a_1^2(\varepsilon_2\,\eta_3 + \varepsilon_3\,\eta_1) + a_2^2(\varepsilon_3\,\eta_1 + \varepsilon_1\,\eta_3) + a_3^2(\varepsilon_1\,\eta_2 + \varepsilon_2\,\eta_1) = 0, \tag{13}$$

in which ε and η must be replaced with the proportional quantities (12).

6. Pairs of lines. – If one multiplies the (8) together then one will get a quadratic equation:

$$\sum_{i,j} c_{ij}\mu_i\,\mu_j = 0 \tag{14}$$

with a zero discriminant that must always be satisfied by the two lines, but never outside of them. Conversely, if one is given an equation (14) *whose discriminant:*

$$\Delta = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$$

is zero then one will know from algebra that the equation will be decomposable into two linear equations, so it will then represent a *pair of lines.* What other conditions must the coefficients satisfy in order for the lines to be parallel or perpendicular? If one observes that:

$$c_{11} = 2\alpha_1\,\beta_1, \quad c_{23} = \alpha_2\,\beta_3 + \alpha_3\,\beta_2, \quad \text{etc.,}$$

and consequently, that $\varepsilon_2\,\eta_3 + \varepsilon_3\,\eta_2$ will be proportional to:

$$(\alpha_1 - \alpha_2)\,(\beta_1 - \beta_2) + (\alpha_1 - \alpha_3)\,(\beta_1 - \beta_2) = c_{11} - c_{12} - c_{13} + c_{23}\,,$$

then one will see immediately upon substituting this in (13) that the *condition for perpendicularity* is:

$$\begin{aligned}(a_1^2 - a_2^2 - a_3^2)\,c_{23} + (a_2^2 - a_3^2 - a_1^2)\,c_{31} + (a_3^2 - a_1^2 - a_2^2)\,c_{12} \\ + \; a_1^2\,c_{11} + a_2^2\,c_{22} + a_3^2\,c_{33} = 0.\end{aligned} \tag{15}$$

Similarly, in order to express the idea that the lines are parallel, consider the determinant δ of the left-hand side of (9), and observe that:

$$-\,\delta^2 = \begin{vmatrix} 1 & \alpha_1 & \beta_1 \\ 1 & \alpha_2 & \beta_2 \\ 1 & \alpha_3 & \beta_3 \end{vmatrix} \cdot \begin{vmatrix} 1 & \beta_1 & \alpha_1 \\ 1 & \beta_2 & \alpha_2 \\ 1 & \beta_3 & \alpha_3 \end{vmatrix} = \begin{vmatrix} 1+c_{11} & 1+c_{12} & 1+c_{13} \\ 1+c_{21} & 1+c_{22} & 1+c_{23} \\ 1+c_{31} & 1+c_{32} & 1+c_{33} \end{vmatrix},$$

i.e., if σ represents the sum of the algebraic complements of all elements of Δ then:

$$-\delta^2 = \sigma + \Delta = \sigma.$$

Therefore, the *condition of parallelism* is $\sigma = 0$. In addition, if one is given that (14) has real coefficients then one will see *the lines are real or imaginary according to whether* $\sigma < 0$ *or* $\sigma > 0$, resp.

7. Examples:

a) Let $\mu_3 = k\,\mu_2$ be the equation of the perpendicular that is dropped from A_1 to the opposite edge. One determines k by expressing the idea that the condition (15) is satisfied by the pair of lines $\mu_1\,\mu_3 = k\,\mu_1\,\mu_2$, and one will then find that the equation of the line considered is:

$$(a_3^2 + a_1^2 - a_2^2)\,\mu_2 = (a_1^2 + a_2^2 - a_3^2)\,\mu_3\,.$$

Hence, the perpendiculars that are dropped from the vertices of a triangle to the opposite edges will be concurrent at a point (viz., the *orthocenter*) that is defined by the coordinates that are inversely proportional to the quantities:

$$a_2^2 + a_3^2 - a_1^2,\; a_3^2 + a_1^2 - a_2^2,\; a_1^2 + a_2^2 - a_3^2\,.$$

b) From what was said at the end of § **1**, a pair of lines through A_1 that harmonically divides the opposite side is represented by the equation $\mu_2^2 = k^2\,\mu_2^2$. If one wishes that those lines should be the bisectors of the angle A_1 then one would need to determine k in such a way that the condition (15) is satisfied; i.e., one would need to have $a_2^2 = k^2\,a_2^2$. The three pairs of bisectors of the angles of the fundamental triangle will then be represented by the equations:

$$\frac{\mu_2^2}{a_2^2} = \frac{\mu_3^2}{a_3^2}, \qquad \frac{\mu_3^2}{a_3^2} = \frac{\mu_1^2}{a_1^2}, \qquad \frac{\mu_1^2}{a_1^2} = \frac{\mu_2^2}{a_2^2},$$

and therefore meet at four points that have barycentric coordinates that are proportional to a_1, a_2, a_3 in absolute value. In particular, the three internal bisectors are concurrent at the point (viz., *center of the inscribed circle*) that is defined by the coordinates:

$$\mu_1 = \frac{a_1}{a_1 + a_2 + a_3}, \qquad \mu_2 = \frac{a_2}{a_1 + a_2 + a_3}, \qquad \mu_3 = \frac{a_3}{a_1 + a_2 + a_3}.$$

c) In order to find the points at which the line $\alpha_1\,\mu_1 + \alpha_2\,\mu_2 + \alpha_3\,\mu_3 = 0$ meets the edge $A_2\,A_3$, one needs to set $\mu_1 = 0$, and μ_2 and μ_3 will then be determined from the equations $\mu_1 + \mu_2 = 1$, $\alpha_2\,\mu_3 + \alpha_3\,\mu_2 = 0$. However, if one would like to know the harmonic conjugate of the intersection point with respect to the pair $A_2\,A_3$ then one would have to replace the last equation with $\alpha_2\,\mu_3 - \alpha_3\,\mu_2 = 0$. It would then follow that the point P that is defined by the equality $\alpha_1\,\mu_1 = \alpha_2\,\mu_2 = \alpha_3\,\mu_3$ would be such that each edge of the triangle would be divided harmonically by the line that joins P to the opposite vertex and the line considered. The point P is called the *trilinear pole* of the line, and conversely, the line is the *trilinear polar* of P. If c_1, c_2, c_3 are the barycentric coordinates of an arbitrary point then the barycentric equation of its trilinear polar will be:

$$\frac{\mu_1}{c_1} + \frac{\mu_2}{c_2} + \frac{\mu_3}{c_3} = 0.$$

In particular, it is known that the line at infinity has its trilinear pole at the point $c_1 = c_2 = c_3 = \frac{1}{3}$; it is the point at which the medians are concurrent, and which one calls simply the *barycenter* of the triangle.

8. Conics. – Substituting the values (5) in the Cartesian equation of a conic [III, § **1**] will produce a quadratic relation between the barycentric coordinates that one can always make into a homogeneous one by means of (2). Conversely, when one substitutes the values (6) in any equation (14), that will change it into an equation of degree two between x and y, and it will then represent a conic that degenerates into a pair of lines when the constraint $\Delta = 0$ is imposed upon the coefficients. No matter what the value of Δ might be, observe that the equation that is obtained by setting $c\,(\mu_1 + \mu_2 + \mu_3)^2$ in the right-hand side of (14) will represent a conic for each value of c. The conics that correspond to the infinite values of c behave like the conic (14) at infinity, since (4) will tend to become valid at infinity. One then has parallel asymptotes, and therefore, in order to know a pair of parallel lines to the asymptotes of the conic (14), it is enough to investigate which values of c correspond to a degenerate conic. Meanwhile, one observes that for an arbitrary value of c, the discriminant is:

$$\Delta' = \begin{vmatrix} c_{11}-c & c_{12}-c & c_{13}-c \\ c_{21}-c & c_{22}-c & c_{23}-c \\ c_{21}-c & c_{32}-c & c_{33}-c \end{vmatrix} = \Delta - c\;\sigma.$$

The sum σ' of the algebraic complements of the elements of Δ' is independent of c, because if one imagines returning from Δ' to Δ by joining c to all of the elements of Δ' then one will find that:

$$\Delta = \Delta' + c\;\sigma' = \Delta + c\,(\sigma' - \sigma), \qquad \text{so } \sigma' = \sigma,$$

when one applies the last formula. Having said that, if $\sigma \neq 0$ then the value $c = \Delta : \sigma$ will correspond to a pair of lines that are either parallel to the asymptotes of the conic (14) or coincident with them, and from the final observation of the preceding paragraph, one can state that the aforementioned conic *will be an ellipse or a hyperbola according to* *whether* $\sigma > 0$ *or* $\sigma < 0$. If the coefficients are varied in such a way that σ tends to zero then the two lines will tend to become parallel, and therefore *the condition* $\sigma = 0$ *will characterize the parabola*, since it is (III, § **2**) the only conic that behaves at infinity like a pair of (coincident) parallel lines. When one then annuls Δ along with σ, the parabola will degenerate into a pair of parallel lines. The equilateral hyperbola is characterized by expressing the idea that it has orthogonal asymptotes; i.e., one writes the condition (15) for $c_{ij} - c$, not for the c_{ij} . In doing such substitutions, one will see directly that c disappears, and one will recognize that the same (15) is *necessary and sufficient for equation* (14) *to represent an equilateral hyperbola.*

9. Tangents and normals, poles and polars, centers and asymptotes, pole of a homology:

a) Since the tangent to a curve $f(\mu_1, \mu_2, \mu_3) = 0$ at the point (ν_1, ν_2, ν_3) can be considered to be determined by that point and the infinitely-close point $(\nu_1 + d\nu_1, \nu_2 + d\nu_2, \nu_3 + d\nu_3)$ on the curve, it is clear that its equation is:

$$\begin{vmatrix} \mu_1 & \nu_1 & d\nu_1 \\ \mu_2 & \nu_2 & d\nu_2 \\ \mu_3 & \nu_3 & d\nu_3 \end{vmatrix} = 0, \tag{16}$$

and since $(\nu_1, \nu_2, \nu_3) = 0$, this is merely:

$$\frac{\partial f}{\partial \nu_1} d\nu_1 + \frac{\partial f}{\partial \nu_2} d\nu_2 + \frac{\partial f}{\partial \nu_3} d\nu_3 = 0, \qquad d\nu_1 + d\nu_2 + d\nu_3 = 0.$$

It will then follow that $\nu_2\, d\nu_3 - \nu_3\, d\nu_2$ is proportional to:

$$\nu_1\left(\frac{\partial f}{\partial \nu_1} - \frac{\partial f}{\partial \nu_2}\right) - \nu_3\left(\frac{\partial f}{\partial \nu_3} - \frac{\partial f}{\partial \nu_1}\right) = \frac{\partial f}{\partial \nu_1} - \left(\nu_1 \frac{\partial f}{\partial \nu_1} + \nu_2 \frac{\partial f}{\partial \nu_2} + \nu_3 \frac{\partial f}{\partial \nu_3}\right);$$

(16) will then become:

$$(\mu_1 - \nu_1)\frac{\partial f}{\partial \nu_1} + (\mu_2 - \nu_2)\frac{\partial f}{\partial \nu_2} + (\mu_3 - \nu_3)\frac{\partial f}{\partial \nu_3} = 0.$$

One then establishes the equation for the normal by applying the condition (13). When the function f is homogeneous, the equation of the tangent will reduce to the much simpler form:

$$\mu_1 \frac{\partial f}{\partial \nu_1} + \mu_2 \frac{\partial f}{\partial \nu_2} + \mu_3 \frac{\partial f}{\partial \nu_3} = 0,$$

by virtue of a known theorem of **Euler**'s, and in the case of a conic, that will become:

$$\sum_{i,j} c_{ij}\mu_i \nu_j = 0. \tag{17}$$

b) Due to its bilinearity, the relation (17) translates geometrically into a noteworthy correspondence between the points and lines in the plane. When one fixes the ν, without supposing that (ν_1, ν_2, ν_3) is a point of the conic, the equation will represent a line that one calls *polar* to (ν_1, ν_2, ν_3) with respect to the conic (14), and the point is called the *pole* of that line. Now, if one fixes arbitrary values for the μ in (17) then the ν will satisfy precisely the equation of the polar to (μ_1, μ_2, μ_3), and therefore *the poles of all of the lines that pass through a point will be found on the polar to that point.* It will then follow that the polars of the two points P and P' meet at the pole to PP'. For example, if the vertices of a triangle are the poles and edges of another triangle then the edges of the first one will be the polars of the vertices of the second. Two such triangles are called *mutually conjugate* with respect to the conic. Taking ν to be the successive coordinates of the vertices of the fundamental triangle, one will see that the equations of the edges of the conjugate triangle with respect to the conic (14) are:

$$c_{11}\,\mu_1 + c_{12}\,\mu_2 + c_{13}\,\mu_3 = 0, \quad c_{21}\,\mu_1 + c_{22}\,\mu_2 + c_{23}\,\mu_3 = 0, \quad c_{31}\,\mu_1 + c_{32}\,\mu_2 + c_{33}\,\mu_3 = 0. \tag{18}$$

These will reduce to equations in the edges when (14) is lacking rectangular terms. The triangle will then be conjugate to itself, and the conic will be called *conjugate* to the triangle. Therefore, the infinitude of conics that are represented by the equation:

$$c_1\,\mu_1^2 + c_2\,\mu_2^2 + c_3\,\mu_3^2 = 0$$

are such that each edge of the fundamental triangle is the polar of the opposite vertex. For $\mu_1 = 0$, one will find values of the ratio $\mu_2 : \mu_3$ that are equal only in absolute value. Hence, from the final observation of § **1**, *any conic that is conjugate to a triangle will divide its edges harmonically*. It will then follow from this that any rectilinear segment with one end point at P and the other one on the polar to P with respect to a conic will be divided harmonically by that conic. Indeed, in order to convince oneself of that, it is enough to assume that the vertices of the fundamental triangle are the point P, the point P' at which the polar to P meets the line considered, and the pole to PP'. In other words, *the polar to a point P with respect to a conic is the locus of harmonic conjugates to P on all of the chords that are determined by the conic on the lines that originate at P.*

c) In particular, if one observes (cf., III, § **3**) that the harmonic conjugate to the center is at infinity on each diameter then one will see that *the center of a conic is the pole of the line at infinity.* Hence, if ν_1, ν_2, ν_3 are the coordinates of the center then equation (17) will reduce to (4), and one will then have:

$$c_{11}\,\nu_1 + c_{12}\,\nu_2 + c_{13}\,\nu_3 = c_{21}\,\nu_1 + c_{22}\,\nu_2 + c_{23}\,\nu_3 = c_{31}\,\nu_1 + c_{32}\,\nu_2 + c_{33}\,\nu_3 = 0. \tag{19}$$

If c is the common value of those three quantities then one can also write:

$$(c_{i1} - c)\nu_1 + (c_{i2} - c)\nu_2 + (c_{i3} - c)\nu_3 = 0$$

for i = 1, 2, 3, by virtue of (2). In order for that system to be satisfied by values of the ν that are not all zero, it is enough that its determinant $(\Delta - c\sigma)$ should be zero; i.e., that one should have $c = \Delta : \sigma$. Meanwhile, if σ_i represents the sum of the algebraic complements of the elements of the i^{th} line in Δ then one will always have:

$$c_{i1}\sigma_1 + c_{i2}\sigma_2 + c_{i3}\sigma_3 = \Delta,$$

and that will permit one to see immediately that the preceding equations will be satisfied by setting:

$$\nu_1 = \frac{\sigma_1}{\sigma}, \qquad \nu_2 = \frac{\sigma_2}{\sigma}, \qquad \nu_3 = \frac{\sigma_3}{\sigma}. \tag{20}$$

These are the *coordinates of the center*. As for the asymptotes, we have already seen (§ **8**) that they are parallel to the lines of the pair:

$$\sum_{i,j} c_{ij}\mu_i\,\mu_j = \frac{\Delta}{\sigma},$$

and in order to show that *this is precisely the general equation of the asymptotes*, it is enough to see that it is satisfied by the values (20). Now, one has:

$$\sum_{i,j} c_{ij}\nu_i\,\nu_j = \frac{1}{\sigma^2}\sum_{i,j} c_{ij}\sigma_i\,\sigma_j = \frac{1}{\sigma^2}\sum_i \sigma_i\,\Delta = \frac{\Delta}{\sigma}.$$

One is finally in a position to assert that when one puts c in place of zero in the right-hand side of equation (14), as one varies c, it will represent the infinitude of conics that are asymptotic to just one pair of lines.

d) If one sets $\mu_i = 0$ in the i^{th} equation (18) then one will get a point whose coordinates satisfy the equation:

$$\frac{\mu_1}{c_{23}} + \frac{\mu_2}{c_{31}} + \frac{\mu_3}{c_{12}} = 0, \tag{21}$$

independently of i. Hence, each edge of the fundamental triangle will meet the corresponding edge of the conjugate triangle, and therefore *two triangles that are conjugate with respect to a conic will be homologous*, and the homology axis will be represented by equation (21). That homology can also be established by considering the vertices. The system that is composed of equations (18), when one replaces the i^{th} one with (2), will define the vertex of the fundamental triangle that corresponds to A_i. It will then follow that the coordinates of that vertex are proportional to γ_{i1}, γ_{i2}, γ_{i3}, if γ_{ij}

represents the algebraic complement of c_{ij} in Δ. Hence, the equations of the line that joins the corresponding vertices of the two triangles will be:

$$\mu_2\,\gamma_{13} = \mu_3\,\gamma_{12}\,, \qquad \mu_3\,\gamma_{21} = \mu_1\,\gamma_{23}\,, \qquad \mu_1\,\gamma_{32} = \mu_2\,\gamma_{31}\,,$$

and therefore those lines will concur at the point (viz., the *center of homology*) that is defined by the equality:

$$\mu_1\,\gamma_{23} = \mu_2\,\gamma_{31} = \mu_3\,\gamma_{12}\,. \tag{22}$$

For brevity, that point is called the *pole of the homology* of the conic with respect to the triangle considered.

10. Examples:

a) In order for M (μ_1, μ_2, μ_3) to be the trilinear pole of a line that passes through a given point P (c_1, c_2, c_3), it is necessary and sufficient (§ **7**, *c*) that the condition:

$$\frac{c_1}{\mu_1} + \frac{c_2}{\mu_2} + \frac{c_3}{\mu_3} = 0$$

should be satisfied; i.e., that one will have:

$$c_1\,\mu_2\,\mu_3 + c_2\,\mu_3\,\mu_1 + c_3\,\mu_1\,\mu_2 = 0. \tag{23}$$

That is the most general equation for a conic that is *circumscribed by* the fundamental triangle, because if one desires that (14) should be satisfied by the coordinates of A_i then one must set $c_{ii} = 0$. Therefore, if a line rotates around one of its points then *its trilinear pole with respect to a triangle will describe a conic that is circumscribed by that triangle.* If one then applies (22) then one will see that P is the pole of homology of the conic. However, if one applies (20) then one will find that the center Q of the conic is defined by the coordinates ν_1, ν_2, ν_3, which are proportional to:

$$c_1\,(c_2 + c_3 - c_1), \qquad c_2\,(c_3 + c_1 - c_2), \qquad c_3\,(c_1 + c_2 - c_3)\,.$$

Meanwhile, one observes that:

$$c_2\,\nu_3 + c_3\,\nu_2 = c_3\,\nu_1 + c_1\,\nu_3 = c_1\,\nu_2 + c_2\,\nu_1\,, \tag{24}$$

which would result immediately from (19), moreover. The last relations, by their symmetry, shed light upon the reciprocity constraint that exists between P and Q. It will then follow that the conic that is circumscribed by a triangle can associated with pairs such that for any pair, any of the two conics will be *the locus of the trilinear poles of the diameters of the other one*. The two conics will coincide when the center falls upon the barycenter (§ **7**, *c*) of the triangle. If one would then wish that the circumscribed conic is an equilateral hyperbola then one must express the idea that the condition (15) is satisfied by the coefficients of the equation; i.e., that one has:

$$(a_2^2 + a_3^2 - a_1^2)\,c_1 + (a_3^2 + a_1^2 - a_2^2)\,c_2 + (a_1^2 + a_2^2 - a_3^2)\,c_3 = 0.$$

That equality says that the pole of homology is on the trilinear polar of the orthocenter (§ **7**, *a*), and therefore *all of the circumscribed equilateral hyperbolas are concurrent at the orthocenter.*

b) Since in order for the conic (14) to be *inscribed* in the fundamental triangle, one needs to set, for example, $\mu_1 = 0$ in equation (14), the equality thus-obtained, namely, $c_{22}\,\mu_2^2 + c_{33}\,\mu_3^2 + 2c_{23}\,\mu_2\,\mu_3 = 0$, will have equal roots. Therefore, if we let c_1 , c_2 , c_3 denote the values of the coefficients c_{23} , c_{31} , c_{12} then we must have $c_i^2 c_{ii} = c_{11}\,c_{22}\,c_{33}$, and consequently, if we take care to avoid annulling the discriminant then we will see that $c_i\,c_{ii} = -\,c_1\,c_2\,c_3$, that is to say, that an inscribed conic is represented by the equation:

$$\frac{\mu_1^2}{c_1^2} + \frac{\mu_2^2}{c_2^2} + \frac{\mu_3^2}{c_3^2} - 2\frac{\mu_2\,\mu_3}{c_2\,c_3} - 2\frac{\mu_3\,\mu_1}{c_3\,c_1} - 2\frac{\mu_1\,\mu_2}{c_1\,c_2} = 0,$$

which is ultimately reducible to the very simple form:

$$\sqrt{\frac{\mu_1}{c_1}} + \sqrt{\frac{\mu_2}{c_2}} + \sqrt{\frac{\mu_3}{c_3}} = 0.$$

One easily deduces from (22) that c_1 , c_2 , c_3 are proportional to the coordinates of the pole of homology, and (20) will show that the center is defined by coordinates that are proportional to $c_1\,(c_2 + c_3)$, $c_2\,(c_3 + c_1)$, $c_3\,(c_1 + c_2)$, It will now be easy to answer the question: *What is the locus of the poles of homology of the parabola that is inscribed in or circumscribed by a triangle?* In one and the other case, the coordinates of the pole of homology must be such that the sum of the coordinates of the center proves to be equal to zero, and one must then havc:

$$c_2\,c_3 + c_3\,c_1 + c_1\,c_2 = 0 \qquad \text{or} \qquad c_1^2 + c_2^2 + c_3^2 - 2\,c_2\,c_3 - 2\,c_3\,c_1 - 2\,c_1\,c_2 = 0;$$

i.e., the poles must belong to the circumscribed or inscribed ellipse that has its center (and its pole of homology) at the barycenter of the triangle.

c) The equation of the circumscribed circle is easily deduced from formula (10), which gives:

$$a_1^2(\mu_2 - \nu_2)(\mu_3 - \nu_3) + a_2^2(\mu_3 - \nu_3)(\mu_1 - \nu_1) + a_3^2(\mu_1 - \nu_1)(\mu_2 - \nu_2) + R^2 = 0. \tag{25}$$

It is enough to observe that this equation must reduce to the form (23), if one is to get immediately:

$$a_2^2\,\nu_3 + a_3^2\,\nu_2 = a_3^2\,\nu_1 + a_1^2\,\nu_3 = a_1^2\,\nu_2 + a_2^2\,\nu_1,$$

and to conclude from a comparison of this with (24) that the pole of homology is defined by coordinates that are proportional to the squares of the corresponding edges. Many call such a point a *Lemoine point.* Meanwhile, since the equation of the circumscribed circle must reduce to the form:

$$a_1^2\,\mu_2\,\mu_3 + a_2^2\,\mu_3\,\mu_1 + a_3^2\,\mu_1\,\mu_2 = 0,$$

one will see, even without further appealing to (25), but upon recalling the known relation:

$$4a^4 = 2a_2^2\,a_3^2 + 2a_3^2\,a_1^2 + 2a_1^2\,a_2^2 - a_1^4 - a_2^4 - a_3^4$$
$$= (a_1 + a_2 + a_3)\,(-a_1 + a_2 + a_3)\,(a_1 - a_2 + a_3)\,(a_1 + a_2 - a_3),$$

that the coordinates of the center of the circumscribed circle are given by the formula:

$$4a^4\,\nu_1 = a_1^2\,(a_2^2 + a_3^2 - a_1^2), \qquad 4a^4\,\nu_2 = a_2^2\,(a_3^2 + a_1^2 - a_2^2), \qquad 4a^4\,\nu_3 = a_3^2\,(a_1^2 + a_2^2 - a_3^2).$$

Now, (25) gives:

$$R = \sqrt{a_1^2(\mu_2\,\nu_3 + \mu_3\,\nu_2) + \cdots + a_1^2\,\nu_2\,\nu_3 - \cdots} = \frac{a_1\,a_2\,a_3}{2a^2}.$$

As for the inscribed circle, since (§ **7**, *b*) the coordinates of the center are proportional to a_1, a_2, a_3, it is easy to deduce that those (c_1, c_2, c_3) of the pole of homology are inversely proportional to $a_1 + a_2 - a_3$, $a_3 + a_1 - a_2$, $a_1 + a_2 - a_3$, and one can immediately write down the equation of the circle with that.

11. – Now take the usual moving axes – i.e., the tangent and the normal to a point of an arbitrary curve – and recall that the coordinates of each vertex A_i of the fundamental triangle satisfy the immobility conditions:

$$\frac{dx_i}{ds} = \frac{y_i}{\rho} - 1, \quad \frac{dy_i}{ds} = -\frac{x_i}{\rho}.$$

If μ_1, μ_2, μ_3 are the barycentric coordinates of the moving origin then (6) will become:

$$a^2\,\mu_1 = x_2\,y_3 - x_3\,y_2\,, \quad a^2\,\mu_2 = x_3\,y_1 - x_1\,y_3\,, \quad a^2\,\mu_3 = x_1\,y_2 - x_2\,y_1\,, \tag{26}$$

and if one differentiates them, while taking the preceding conditions into account, then one will infer directly that:

$$\frac{d\mu_1}{ds} = \frac{y_2 - y_3}{a^2}, \qquad \frac{d\mu_2}{ds} = \frac{y_3 - y_1}{a^2}, \qquad \frac{d\mu_3}{ds} = \frac{y_1 - y_2}{a^2}; \tag{27}$$

hence:

$$x_1\,d\mu_1 + x_2\,d\mu_2 + x_3\,d\mu_3 = ds, \qquad y_1\,d\mu_1 + y_2\,d\mu_2 + y_3\,d\mu_3 = 0. \tag{28}$$

Having said that, in order to calculate the length of the *elementary arc*, one can make use of the identity:

$$\begin{aligned} k_2\,k_3\,(\alpha_2\,\beta_3 - \alpha_3\,\beta_2)^2 + k_3\,k_1\,(\alpha_3\,\beta_1 - \alpha_1\,\beta_3)^2 + k_1\,k_2\,(\alpha_1\,\beta_2 - \alpha_2\,\beta_1)^2 \\ = \sum k_i\,\alpha_i^2 \cdot \sum k_i \beta_i^2 - \left(\sum k_i\,\alpha_i\,\beta_i\right)^2, \end{aligned} \tag{29}$$

which we will also appeal to in what follows, and which results immediately from the multiplication of matrices:

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}, \qquad \begin{vmatrix} k_1\,\alpha_1 & k_2\,\alpha_2 & k_3\,\alpha_3 \\ k_1\,\beta_1 & k_2\,\beta_2 & k_3\,\beta_3 \end{vmatrix}.$$

For $\alpha_i = 1$, $\beta_i = x_i$, $k_i = d\mu_i$, if one observes the first equality (28) then the identity (29) will become:

$$(x_2 - x_3)^2\,d\mu_2\,d\mu_3 + (x_3 - x_1)^2\,d\mu_3\,d\mu_1 + (x_1 - x_2)^2\,d\mu_1\,d\mu_2 = -\,ds^2.$$

However, if the β are set equal to the y and one takes the second equality in (28) into account then one will get:

$$(y_2 - y_3)^2\,d\mu_2\,d\mu_3 + (y_3 - y_1)^2\,d\mu_3\,d\mu_1 + (y_1 - y_2)^2\,d\mu_1\,d\mu_2 = 0;$$

upon summing, one will recover formula (11).

12. – The calculation of the *curvature* is just as easy to do. If one differentiates (27) then, by virtue of the immobility conditions, one will get:

$$\frac{d^2\mu_1}{ds^2} = -\,\frac{x_2 - x_3}{a^2\rho}, \qquad \frac{d^2\mu_2}{ds^2} = -\,\frac{x_3 - x_1}{a^2\rho}, \qquad \frac{d^2\mu_3}{ds^2} = -\,\frac{x_1 - x_2}{a^2\rho}. \tag{30}$$

On the other hand, if one takes the reciprocal of the determinant (7) and takes (26) into account then one will find:

$$a^2 = \begin{vmatrix} \mu_1 & x_2 - x_3 & y_2 - y_3 \\ \mu_2 & x_3 - x_1 & y_3 - y_1 \\ \mu_3 & x_1 - x_2 & y_1 - y_2 \end{vmatrix};$$

hence, if one substitutes the values (27) and (30) in the right-hand side then one will arrive at the formula:

$$\frac{1}{\rho} = a^2 \begin{vmatrix} \mu_1 & \dfrac{d\mu_1}{ds} & \dfrac{d^2\mu_1}{ds^2} \\ \mu_2 & \dfrac{d\mu_3}{ds} & \dfrac{d^2\mu_2}{ds^2} \\ \mu_3 & \dfrac{d\mu_3}{ds} & \dfrac{d^2\mu_3}{ds^2} \end{vmatrix}. \tag{31}$$

The intrinsic determination of any curve that is represented by a barycentric equation now proves to be simple. Along with (2), it will constitute a system that permits one to express the μ as functions of just one independent variable t. If one then utilizes the formulas (11) and (31) then one will see that if one knows the functions:

$$\begin{gathered} \kappa = \sqrt{-\left(\frac{a_1^2}{a^2}\frac{d\mu_2}{dt}\frac{d\mu_3}{dt} + \frac{a_2^2}{a^2}\frac{d\mu_3}{dt}\frac{d\mu_1}{dt} + \frac{a_3^2}{a^2}\frac{d\mu_1}{dt}\frac{d\mu_2}{dt}\right)}, \\ W = \begin{vmatrix} \mu_1 & \dfrac{d\mu_1}{dt} & \dfrac{d^2\mu_1}{dt^2} \\ \mu_2 & \dfrac{d\mu_3}{dt} & \dfrac{d^2\mu_2}{dt^2} \\ \mu_3 & \dfrac{d\mu_3}{dt} & \dfrac{d^2\mu_3}{dt^2} \end{vmatrix} \end{gathered} \tag{32}$$

then the intrinsic equation of the curve considered will result from the elimination of t from the equalities:

$$s = a\int \kappa\, dt, \qquad \rho = \frac{a\,\kappa^2}{W}. \tag{33}$$

It is known that thanks to (2), the Wronskian determinant can be written more simply as:

$$W = \frac{d\mu_2}{dt}\frac{d^2\mu_3}{dt^2} - \frac{d\mu_3}{dt}\frac{d^2\mu_2}{dt^2} = \frac{d\mu_3}{dt}\frac{d^2\mu_1}{dt^2} - \frac{d\mu_1}{dt}\frac{d^2\mu_3}{dt^2} = \frac{d\mu_1}{dt}\frac{d^2\mu_2}{dt^2} - \frac{d\mu_2}{dt}\frac{d^2\mu_1}{dt^2}. \tag{34}$$

It facilitates the calculation of W by observing that if the μ are only proportional (but not equal) to the barycentric coordinates, and consequently have a sum $k \neq 1$ then their Wronskian will have the value $k^3\, W$.

13. – If the curve is given by means of the equation $f(\mu_1, \mu_2, \mu_3) = 0$ then one will find directly three quantities that are proportional to the differentials of the μ, since one has:

$$d\mu_1 + d\mu_2 + d\mu_3 = 0, \qquad \frac{\partial f}{\partial \mu_1} d\mu_1 + \frac{\partial f}{\partial \mu_2} d\mu_2 + \frac{\partial f}{\partial \mu_3} d\mu_3 = 0,$$

in which the variables in the partial differentiations with respect to the μ are meant to be free from the constraint (2), for the moment. On the other hand, the coefficient of proportionality depends upon the choice of independent variable t, and one can therefore always determine it in such a way that one has:

$$\frac{d\mu_1}{dt} = \frac{\partial f}{\partial \mu_2} - \frac{\partial f}{\partial \mu_3}, \qquad \frac{d\mu_2}{dt} = \frac{\partial f}{\partial \mu_3} - \frac{\partial f}{\partial \mu_1}, \qquad \frac{d\mu_3}{dt} = \frac{\partial f}{\partial \mu_1} - \frac{\partial f}{\partial \mu_2}. \tag{35}$$

After that, formula (32) will become:

$$a^2 \kappa^2 = a_1^2 \left(\frac{\partial f}{\partial \mu_1}\right)^2 + a_2^2 \left(\frac{\partial f}{\partial \mu_2}\right)^2 + a_3^2 \left(\frac{\partial f}{\partial \mu_3}\right)^2$$

$$+ (a_1^2 - a_2^2 - a_3^2) \frac{\partial f}{\partial \mu_1} \frac{\partial f}{\partial \mu_2} + (a_2^2 - a_3^2 - a_1^2) \frac{\partial f}{\partial \mu_3} \frac{\partial f}{\partial \mu_1} + (a_2^2 - a_1^2 - a_2^2) \frac{\partial f}{\partial \mu_1} \frac{\partial f}{\partial \mu_2}.$$

Similarly, if one substitutes the values (35) in one of (34) then one will get:

$$W = \left(\frac{\partial f}{\partial \mu_3} - \frac{\partial f}{\partial \mu_1}\right) \frac{d^2 \mu_3}{dt^2} - \left(\frac{\partial f}{\partial \mu_1} - \frac{\partial f}{\partial \mu_2}\right) \frac{d^2 \mu_2}{dt^2} = \sum_i \frac{\partial f}{\partial \mu_i} \frac{d^2 \mu_i}{dt^2},$$

and if one introduces the operation:

$$\frac{d}{dt} = \frac{d\mu_1}{dt} \frac{\partial}{\partial \mu_1} + \frac{d\mu_2}{dt} \frac{\partial}{\partial \mu_2} + \frac{d\mu_3}{dt} \frac{\partial}{\partial \mu_3} \tag{36}$$

into the calculations then one can also write:

$$W = \frac{d}{dt} \sum_i \frac{\partial f}{\partial \mu_i} \frac{d\mu_i}{dt} - \sum_i \frac{d\mu_i}{dt} \frac{d}{dt} \frac{\partial f}{\partial \mu_i} = -\sum_i \frac{d\mu_i}{dt} \frac{\partial}{\partial \mu_i} \frac{df}{dt} = -\frac{d^2 f}{dt^2}.$$

Finally, if one observes that when the operation (36) is repeated, it will give:

$$\frac{d^2}{dt^2} = \sum_{i,j} \frac{d\mu_i}{dt} \frac{d\mu_j}{dt} \frac{\partial^2}{\partial \mu_i \, \partial \mu_j}$$

then one will get:

$$W = -\sum_{i,j} \frac{\partial^2 f}{\partial \mu_i \, \partial \mu_j} \frac{d\mu_i}{dt} \frac{d\mu_j}{dt}. \tag{37}$$

14. – When one substitutes the values (35) in the last formula, it will give us W as a function of the first and second partial derivatives of f with respect to the μ. It will simplify noticeably when the function f is *homogeneous*: It is enough to make the first derivatives disappear by means of the known Eulerian relations:

$$(n-1)\, \frac{\partial f}{\partial \mu_i} = \sum_j \mu_j \frac{\partial^2 f}{\partial \mu_i \, \partial \mu_j}, \tag{38}$$

in which n is the degree of f. If one multiplies the two sides of (37) by $(n-1)^2$, in addition, then one will easily see that the second one splits into:

$$(\mu_1 + \mu_2 + \mu_3)^2\, H - \sigma \sum_{i,j} \frac{\partial^2 f}{\partial \mu_i \, \partial \mu_j} \mu_i \, \mu_j ,$$

in which σ represents the sum of the algebraic complements of all elements in the determinant H, which is the Hessian of f with respect to μ. Meanwhile, from (38), one has:

$$\sum_{i,j} \frac{\partial^2 f}{\partial \mu_i \, \partial \mu_j} \mu_i \, \mu_j = n\,(n-1)\, f\,(\mu_1, \mu_2, \mu_3) = 0.$$

Hence:

$$(n-1)^2\, W = H.$$

The second formula in (33) will then finally give:

$$\frac{(n-1)^2}{a^2 \rho} \left[a_1^2 \left(\frac{\partial f}{\partial \mu_1} \right)^2 + \cdots + (a_1^2 - a_2^2 - a_3^2) \frac{\partial f}{\partial \mu_2} \frac{\partial f}{\partial \mu_3} + \cdots \right]^{3/2}$$
$$= \begin{vmatrix} \dfrac{\partial^2 f}{\partial \mu_1^2} & \dfrac{\partial^2 f}{\partial \mu_1 \, \partial \mu_2} & \dfrac{\partial^2 f}{\partial \mu_1 \, \partial \mu_3} \\ \dfrac{\partial^2 f}{\partial \mu_3 \, \partial \mu_1} & \dfrac{\partial^2 f}{\partial \mu_2^2} & \dfrac{\partial^2 f}{\partial \mu_2 \, \partial \mu_3} \\ \dfrac{\partial^2 f}{\partial \mu_3 \, \partial \mu_1} & \dfrac{\partial^2 f}{\partial \mu_3 \, \partial \mu_2} & \dfrac{\partial^2 f}{\partial \mu_3^2} \end{vmatrix}. \tag{39}$$

15. Application to conics. – For conics, one has:

$$n = 2, \qquad f = \tfrac{1}{2}\sum_{i,j} c_{ij}\,\mu_i\,\mu_j\,, \qquad \frac{\partial^2 f}{\partial\mu_i\,\partial\mu_j} = c_{ij}\,, \qquad H = \Delta;$$

hence, (39) will take the form:

$$\rho = \frac{\Phi^{3/2}}{a^2\Delta}, \tag{40}$$

in which the function:

$$\Phi = a_1^2\left(\frac{\partial f}{\partial\mu_1}\right)^2 + \cdots + (a_1^2 - a_2^2 - a_3^2)\frac{\partial f}{\partial\mu_2}\frac{\partial f}{\partial\mu_3} + \ldots$$

is another quadratic form. The discriminant of that form will differ from the determinant:

$$\begin{vmatrix} a_1^2 & \frac{1}{2}(a_3^2 - a_1^2 - a_2^2) & \frac{1}{2}(a_2^2 - a_3^2 - a_1^2) \\ \frac{1}{2}(a_3^2 - a_1^2 - a_2^2) & a_2^2 & \frac{1}{2}(a_1^2 - a_2^2 - a_3^2) \\ \frac{1}{2}(a_2^2 - a_3^2 - a_1^2) & \frac{1}{2}(a_1^2 - a_2^2 - a_3^2) & a_3^2 \end{vmatrix}$$

by only the factor Δ^2. That determinant is zero, because the sum of the elements of any row is zero. With an easy calculation, one will then find that the sum of the algebraic complements of all the elements will be $9a^4$. Hence (§ **6**), the equation $\Phi = 0$ represents a *pair of imaginary lines.* One must then notice that these lines are diameters of the conic, because one has (cf., § **9**, *c*):

$$\frac{\partial f}{\partial\mu_1} = \frac{\partial f}{\partial\mu_2} = \frac{\partial f}{\partial\mu_3}, \qquad \Phi = 0$$

at the center. Meanwhile, consider, along with the first conic, all of the ones that have the same asymptotes and that are represented, as one knows (§ **9**, *c*), by the equation $f = \frac{1}{2}c$. One must subtract c from each c_{ij} . Any first partial derivative of f will then be diminished by c, and one will recognize directly that the function Φ will not be altered by that. Therefore, since, on the one hand, that function cannot be annulled along with f if it is not annulled when ρ is, one will arrive at the conclusion that *all of the conics that are asymptotic to a given pair of lines will have cusps on two common imaginary diameters.* Another interpretation of Φ will result from the following observation: If one fixes the value of ρ arbitrarily in (40) then the equation that one obtains – viz., $\Phi = (a^2\,\rho\,\Delta)^{2/3}$ – represents a conic that meets the given conic at four points, at which r will assume the prescribed value. The conics that correspond to the infinitude of value of r are concentric *ellipses;* their asymptotes are precisely the lines $\Phi = 0$.

16. Symmetric triangular curves. – The curves that are defined by the equation:

$$c_1\,\mu_1^n + c_2\,\mu_2^n + c_3\,\mu_3^n = 0, \tag{41}$$

which were called *symmetric triangles* by **La Gournerie**, are quite interesting because they include the ways that the conics are situated with respect to the fundamental triangle, in the main. Indeed, for $n = 2$, one will find (§ **9**, *b*; § **10**, *a*, *b*) a conic that is *conjugate* to the triangle, for $n = -1$, a *circumscribed* conic, and for $n = 1/2$, an *inscribed* conic. The differentiation of equation (41) will give:

$$c_1\,\mu_1^{n-1}(y_2 - y_1) + c_2\,\mu_2^{n-1}(y_3 - y_1) + c_3\,\mu_2^{n-1}(y_1 - y_2) = 0, \tag{42}$$

by virtue of (27). Hence $c_1\,\mu_1^{n-1}$, $c_2\,\mu_2^{n-1}$, $c_3\,\mu_3^{n-1}$ are proportional to:

$$\mu_2\,(y_1 - y_2) - \mu_3\,(y_3 - y_1) = (\mu_2 + \mu_3)\,y_1 - (\mu_2\,y_2 + \mu_3\,y_3) = y_1,$$

y_2, and y_3, respectively. Having assumed that, when one differentiates (42), one will get:

$$\frac{1}{\rho}[c_1\mu_1^{n-1}(x_2 - x_3) + \cdots] = \frac{n-1}{a^2}[c_1\mu_1^{n-2}(y_2 - y_3)^2 + \cdots],$$

i.e.:

$$-\frac{1}{\rho} = \frac{n-1}{a^4}[\frac{y_1}{\mu_1}(y_2 - y_3)^2 + \cdots], \tag{43}$$

after having observed that from formula (7):

$$(x_2 - x_3)\,y_1 + (x_3 - x_1)\,y_2 + (x_1 - x_2)\,y_3 = -\,a^3.$$

Now set $\alpha = 1$, $\beta = y$, $ky = \mu$ while utilizing the identity (29). Obviously:

$$\sum_i k_i\,\beta_i = \sum_i \mu_i = 1, \qquad \sum_i k_i\,\beta_i^2 = \sum_i \mu_i\,y_i = 0.$$

Hence:

$$\frac{\mu_2\,\mu_3}{y_2\,y_3}(y_2 - y_3)^2 + \frac{\mu_3\,\mu_1}{y_3\,y_1}(y_3 - y_1)^2 + \frac{\mu_1\,\mu_2}{y_1\,y_2}(y_1 - y_2)^2 = -\,1;$$

when one substitutes in (43), will get:

$$\rho = \frac{a^4}{n-1}\frac{\mu_1\,\mu_2\,\mu_3}{y_1\,y_2\,y_3}. \tag{44}$$

Here, we observe that if two curves (41) that correspond to two values n and n' of the exponent touch at a point then their curvatures at that point can be deduced from each other directly, since, as **Jamet** has observed, (44) will give:

$$(n-1)\,\rho = (n'-1)\,\rho'.$$

17. – In order to construct the expression (44), it is useful to adopt polar coordinates (r_i, θ_i) for the vertices A_i. One has $y_i = r_i \sin\theta_i$, and the formulas (26) also give $a^2\mu_1 = -r_2\, r_3 \sin(\theta_2 - \theta_3)$, etc.; if one confines oneself to the case of $n = -1$ then formula (44) will become:

$$\rho = \frac{r_1\, r_2\, r_3}{2a^2}\cdot\frac{\sin(\theta_2-\theta_3)\sin(\theta_3-\theta_1)\sin(\theta_1-\theta_2)}{\sin\theta_1\sin\theta_2\sin\theta_3}.$$

With this, **Fouret**, following **Chasles** and **Mannheim**, was able to easily solve the problem: *Construct the center of curvature at a point M of a conic when one knows three points of the curve and the tangent at M.* However, if the three points are given three tangents then the analogous problem will be immediately reducible to the preceding one. Indeed, for $n = -1$ and $n' = 1/2$, **Jamet**'s theorem will give $4\rho = \rho'$, that is to say, *if one of two tangent conics is circumscribed by a triangle, while the other one is inscribed by it then the curvature of the first one will be four times that of the second one* at the point of contact. If one also considers the case of $n = 2$ then one will find that if C, C', C'' are the centers of curvature at a point M of three conics that touch at M, and the first one is circumscribed by a given triangle, the second one is inscribed by it, while the third one is conjugate to it then C'' will be *symmetric to the midpoint of* MC' *with respect to* M, just as C is symmetric to the midpoint of MC''.

18. Anharmonic curves. – Following **Halphen**, they are the curves for which *the anharmonic ratio of quadruple that is composed of the point M and the points of intersection of the tangent at M with three fixed lines is constant.* If, when one takes the edges of the fundamental triangle, they determine segments t_1, t_2, t_3 along the tangent at M when one starts at M then one can write the problem in the form of the equation:

$$c_1\, t_2\, t_3 + c_2\, t_3\, t_1 + c_3\, t_1\, t_2 = 0, \tag{45}$$

in which c_1, c_2, c_3 are three constants *whose sum is zero.* As one knows, the anharmonic ratio has one of the values:

$$-\frac{c_2}{c_3},\quad -\frac{c_3}{c_1},\quad -\frac{c_1}{c_2},\quad -\frac{c_3}{c_2},\quad -\frac{c_1}{c_3},\quad -\frac{c_2}{c_1}.$$

Meanwhile, one easily calculates the lengths t, and one finds:

$$t_1 = -\frac{a^2\mu_1}{y_2-y_3},\qquad t_2 = -\frac{a^2\mu_2}{y_3-y_1},\qquad t_3 = -\frac{a^2\mu_3}{y_1-y_2},$$

in such a way that (45) will become:

$$\frac{c_1}{\mu_1}(y_2 - y_3) + \frac{c_2}{\mu_2}(y_3 - y_1) + \frac{c_3}{\mu_3}(y_1 - y_2) = 0, \tag{46}$$

i.e., by virtue of (27):

$$\sum_i c_i \frac{d}{ds} \log \mu_i = 0.$$

When this is integrated, it will give the *barycentric equation of the anharmonic curves:*

$$\mu_1^{c_1}\,\mu_2^{c_2}\,\mu_3^{c_3} = \text{constant.} \tag{47}$$

19. – The system that (46) forms with $c_1 + c_2 + c_3 = 0$ gives:

$$\frac{c_1}{\mu_1\,y_1} = \frac{c_2}{\mu_2\,y_2} = \frac{c_3}{\mu_3\,y_3}; \tag{48}$$

hence:

$$c_1 \cot \theta_1 + c_2 \cot \theta_2 + c_3 \cot \theta_3 = 0.$$

In that way, one will find the property that correlates with the one that was given as the definition, namely: *The anharmonic ratio of the quadruple that is composed of any tangent and the lines that join the point of contact to three fixed points is constant.* One then knows how to construct the tangent at any point. In order to construct the center of curvature, one must first show that *the anharmonic curves are a limiting case of the symmetric triangular curves.* When the sum of the c is zero, equation (41) can be written as:

$$\sum_i c_i \frac{\mu_i^n - 1}{n} = \text{constant},$$

after one replaces the right-hand side of (41) with the product n with a constant. Now, since:

$$\lim_{n=0} \frac{\mu^n - 1}{n} = \log \mu,$$

one will recover equation (47) as n tends to zero. Having said that, for $n = 2$ and $n' = 0$, Jamet's theorem will give $\rho = \rho'$, that is to say, *the center of curvature of an anharmonic curve at a point M is symmetric with respect to M to the center of curvature of the conjugate tangent to the fundamental triangle that touches the curve in question at M.*

20. Examples:

a) An interesting example of an anharmonic line is offered by the *potential curve* of a triangle; i.e., the locus of points M that have their barycentric coordinates proportional to the same power n of the corresponding edge. Among those points, one will always find [§ **7**, (10)] the *barycenter* of the triangle ($n = 0$), the *center of the inscribed circle* ($n = 1$), the *Lemoine point* ($n = 2$), etc. In order to fix ideas, always supposes that $a_1 > a_2 > a_3$, and sets:

$$c_1 = \log\frac{a_2}{a_3}, \quad c_2 = \log\frac{a_3}{a_1}, \quad c_3 = \log\frac{a_1}{a_2},$$

for brevity, while observing that $c_1 + c_2 + c_3 = 0$. Having said that, it will result directly from the definition:

$$\frac{\mu_1}{a_1^n} = \frac{\mu_2}{a_2^n} = \frac{\mu_3}{a_3^n} = \frac{1}{a_1^n + a_2^n + a_3^n} \tag{49}$$

that the locus in question will be an anharmonic curve, since one has $\mu_1^{c_1}\,\mu_2^{c_2}\,\mu_3^{c_3} = 1$. Meanwhile, the formulas (49) show that as n increases to infinity, μ_1 will tend to unity, while μ_2 and μ_3 will tend to zero. However, as n tends to $-\infty$, μ_3 will tend to unity, while μ_2 and μ_1 will tend to zero. Hence, the vertices A_1 and A_3 that are opposite to the maximum and minimum edge also belong to the potential curves. How does the curve behave in the vicinity of such points? When M tends to A_1, the line MA_1 will tend to coincide with the tangent at A_1, and one must do likewise with MA_2 or MA_3 in order for the anharmonic ratio of the four lines to keep its value. Hence, the curve must touch one of the edges at the vertex A_1; however, to answer the question that was posed precisely, we need to recall formulas (48), which will become:

$$\frac{y_1}{c_1\,a_1^{-n}} = \frac{y_2}{c_2\,a_2^{-n}} = \frac{y_3}{c_3\,a_3^{-n}} \tag{50}$$

in the present case, and will give directly, in succession:

$$\frac{y_2}{y_3} = \frac{c_2}{c_3}\left(\frac{a_3}{a_2}\right)^n, \qquad \lim_{n=\infty}\frac{y_2}{y_3} = 0.$$

Now, since y_2 cannot exceed a_2, one must have $\lim y_2 = 0$; i.e., the tangent at A_1 is $A_1\,A_2$. One similarly shows that the tangent at A_3 is $A_3\,A_2$. Therefore, *the curve touches the other edges at the end points of the middle edge.* It will then follow that for n infinite, the limit of y_3 will be the distance from A_3 to the opposite edge, just as when n tends to $-\infty$, the limit of y_1 will be the distance from A_1 to the opposite edge; i.e.:

$$\lim_{n=\infty} y_2 = \frac{a^2}{a_3}, \qquad \lim_{n=-\infty} y_1 = \frac{a^2}{a_1}.$$

Now, if one adopts the formulas (50) then one will find that as n increases to infinity:

$$\lim\left(\frac{a_1}{a_3}\right)^n y_1 = \frac{a^2 c_1}{a_3\,c_3}, \qquad \lim\left(\frac{a_2}{a_3}\right)^n y_2 = \frac{a^2 c_2}{a_2\,c_3}, \qquad \lim\left(\frac{a_1\,a_2}{a_3^2}\right)^n y_1\,y_2\,y_3 = \frac{a^6 c_1\,c_1}{a_3^3\,c_3^2}.$$

Similarly, one deduces from (49) that:

$$\lim \mu_1 = \lim\left(\frac{a_1}{a_2}\right)^n \mu_2 = \lim\left(\frac{a_1}{a_3}\right)^n \mu_3 = 1, \qquad \lim\left(\frac{a_1^2}{a_2\,a_3}\right)^n \mu_1\,\mu_2\,\mu_3 = 1.$$

On the other hand, if one ignores the sign then formula (44) will give:

$$\rho = a^4\,\frac{\mu_1\,\mu_2\,\mu_3}{y_1\,y_2\,y_3}$$

for any anharmonic curve; hence, by virtue of the preceding results:

$$\lim\left(\frac{a_1\,a_3}{a_2^2}\right)^n \rho = \frac{a_3^3\,c_3^2}{a^2 c_1\,c_2}. \tag{51}$$

Hence, in general, *the curvature will be zero or infinite* at the vertex A_1; it is zero if $a_2^2 > a_1\,a_3$ and infinite if $a_2^2 < a_1\,a_3$. One shows analogously that the curvature at A_3 will be infinite or zero, respectively, for the same situation. The only exception is the triangles whose edges are in geometric progression. One will have $a_2^2 = a_1\,a_3$, $c_1 = c_3 = -\frac{1}{2}c_2$ for them. It results from the preceding discussion that the radius of curvature at the end points of the middle edge will take values that are proportional to the cubes of the opposite edges, and it is then easy to show that this radius will become proportional to the cube of the middle edge at the barycenter, where one has:

$$\rho = \frac{a_2^3}{2a^2} = \frac{a_1\,a_2\,a_3}{2a^2};$$

that is to say, at the barycenter, the osculating circle will become equal to the circle that is circumscribed by the triangle. Furthermore, in the special case that is being considered, the potential curve will be a conic, since the values that are found for the c will reduce the barycentric equation to the form $\mu_2^2 = \mu_1\,\mu_2$.

b) The potential curve can be prolonged outside of the triangle by attributing imaginary values to n. One changes n into $n + m\sqrt{-1}$, represents argument of $a_i^{m\sqrt{-1}}$ (which is obviously equal to $m \log a$, by θ_i) lets r and θ represent the modulus and argument of the sum $a_1^n + a_2^n + a_3^n$, resp., after one changes n, and observes that (49) will give $r\mu_i = a_i^n\, e^{(\theta_i - \theta)\sqrt{-1}}$. In order for the point (μ_1, μ_2, μ_3) to be real, it is necessary that θ_i

$-\theta$ must be a multiple of π. If one sets $\theta_i - \theta = m_i\,\pi$ then one will have $r\,\mu_i = (-1)^{m_i} a_i^n$; hence:

$$\frac{\mu_1}{(-1)^{m_1} a_1^n} = \frac{\mu_2}{(-1)^{m_2} a_2^n} = \frac{\mu_3}{(-1)^{m_3} a_3^n} = \frac{1}{(-1)^{m_1} a_1^n + (-1)^{m_2} a_2^n + (-1)^{m_3} a_3^n}.$$

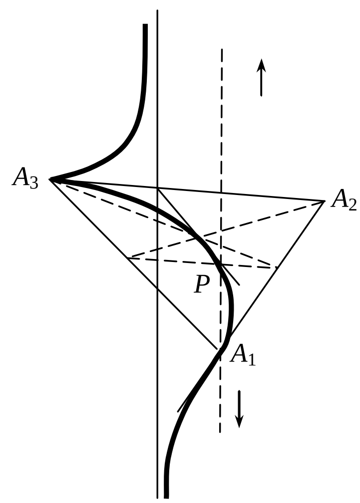
One can always suppose that all of the numbers m_i are even or that *just one of them is odd*. Under the first hypothesis, one gets back to the point M that was defined by (49); under the second one, one will have a new point M' that has a simple correspondence with M: If the odd number is m_ν then *the segment MM' will be divided harmonically by the vertex A_ν and the opposite edge.* How does one determine ν? Observe that:

$$m\,c_1 = \theta_2 - \theta_3 = (m_2 - m_3)\,\pi, \qquad \text{etc.}$$

One then needs, above all, that the mutual ratios of the numbers c should be rational. If that is the case then one can find three integer numbers e_1, e_2, e_3 that are relatively prime and such that $c_i = e_i\,c$, and a relation of the form:

$$a_1^{e_1}\,a_3^{e_3} = a_2^{e_1+e_3} \tag{52}$$

will then exist between the edges of the triangle. Since the numbers e_1, e_2, e_3 are proportional to $m_2 - m_3$, $m_3 - m_1$, $m_1 - m_2$, it is clear that only one of them will be even, and it will be e_ν. It will then follow that the number ν will be known when one is given the relation (52), since one has $\nu = 2$ when e_1 and e_2 are odd, $\nu = 1$ when e_1 is even and e_3 is odd, and $\nu = 3$ when e_1 is odd and e_3 is even. Therefore, when the determination of the numbers e is possible, the curve will admit branches (M') that are external to the triangle that can be deduced from the internal branch (M) by means of a harmonic homology transformation with its pole at the vertex A_ν and its axis through the opposite side. Obviously, for $n = 2$, one will get just one branch (M'), which constitutes a type of oval along with (M). The form of the curve when $\nu = 1$ or $\nu = 3$ is quite different. Let $\nu = 1$, to fix ideas. The line that joins the midpoints of the edges $A_1 A_2$, $A_1 A_3$ meets (M) at a point P, and the point P', which corresponds to P, is at infinity on A_1P. The tangent at P transforms into the tangent at P'; i.e., into an asymptote, which is easily constructed by drawing homology axes that are parallel to PA_1 through the point at which the tangent at P meets $A_2 A_3$. The arcs PA_2 and PA_1 obviously transform into two branches that extend to infinity asymptotically to the line that was just constructed and touch the internal branch at A_3 and A_1, respectively, in such a way that A_3 will have a cusp and A_1 will have an inflection, without the curvature necessarily being infinite at the first point and zero at the second. Moreover, one can gain more precise information about the behavior of the curve at the end points of the middle edge with no difficulty from formulas (32) and (51). The former leads one to write $s = e^{-nc_3}$ in the vicinity of A_1 and $s = e^{-nc_1}$, while (51) can be easily put into the asymptotic form $\rho = k\,e^{n(c_1-c_3)}$. It will then follow that in the vicinity of A_1, the curve behaves as if its intrinsic equation were $\rho = k\,s^{1-(c_1/c_3)}$, and in the vicinity of A_3, it will behave as if the equation were $\rho = k\,s^{1-(c_3/c_1)}$. It is now enough to recall what was said in the first chapter (§ **11**, *c*) in order to complete the discussion. Hence, in summary, we conclude that the potential curve of a triangle can have three very different forms depending upon the relation (52). If that is not verified by any pair of integer numbers e_1 and e_3 then the curve, which is entirely internal to the triangle, will stop abruptly at the end points of the middle edge. If one has a relation (52) between the edges with e_1 and e_3 odd then the potential will be a closed curve. Finally, when e_1 or e_3 is even, the curve will be open and will consist of two branches that start from a cusp and extend to infinity asymptotically to that line. One branch is completely external, while the other, which is initially internal, will be inflected until it becomes external to the triangle. The potential curve will be transcendental in the first case, and algebraic of even or odd degree in the other two cases.



CHAPTER VIII

SYSTEMS OF PLANE CURVES

1. – Consider a continuous function of the points in a plane, namely, a variable u that takes a prescribed value at each point M and varies infinitely little when M suffers an infinitesimal displacement in the plane. If the values that one attributes to u are all real then their number will be simply-infinite, while the number of points in the plane is doubly-infinite. Assigning a constant value to u will then be equivalent to singling out a line in the plane, and changing that value will signify passing from one line to another. It will then follow that any real function of the points in a plane includes the analytical representation of a simply-infinite system of curves whose properties are consequently obtained by geometrically interpreting the properties of the function. It is important to observe that the infinitude of functions of u do not define any new systems of curves: They all represent the unique system that is defined by u, and we shall shortly see that there are no other functions that are capable of representing that system.

2. – If a displacement ds of the point M produces an increment du in the function u then the ratio $du : ds$ will be called the *differential quotient* of u in the direction of the displacement, and it will be represented by $\partial u / \partial s$ when it is referred to a particular direction that one would like to distinguish from the other ones. A function will then have an infinitude of differential quotients at any point, but they will depend upon the quotients relative to *two arbitrary* orthogonal directions in a very simple way, or upon the *single* quotient that relates to *one particular* direction. Indeed, let M be the projection of the end point M' of the segment $MM' = ds$ onto a line that passes through M, and let ds_1 and ds_2 be the lengths of the segments MM'', $M''M'$, resp., in such a way that:

$$\frac{ds_1}{\cos\omega} = \frac{ds_2}{\sin\omega} = ds.$$

When one passes from M to M'', the function will take on the value $u + \frac{\partial u}{\partial s_1} ds_1$. It will then submit to an increment during the passage from M'' to M' that one can consider to be equal to $\frac{\partial u}{\partial s_2} ds_2$, with $\frac{\partial u}{\partial s_2}$ calculated at M, if one disregards higher-order infinitesimals. Now, since $u + du$ is the value of function at M', one will then see that:

$$du = \frac{\partial u}{\partial s_1} ds_1 + \frac{\partial u}{\partial s_2} ds_2 ;$$

hence:

$$\frac{du}{ds} = \cos\,\omega\,\frac{\partial u}{\partial s_1} + \sin\,\omega\,\frac{\partial u}{\partial s_2} . \tag{1}$$

Set:

$$\Delta u = \left(\frac{\partial u}{\partial s_1}\right)^2 + \left(\frac{\partial u}{\partial s_2}\right)^2,$$

for brevity, and consider the direction *MN*, for which one has:

$$\cos \omega_0 = \frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_1}, \qquad \sin \omega_0 = \frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_1}. \tag{2}$$

Formula (1) will give $\sqrt{\Delta u}$ for the value of the differential quotient in that direction. (1) will then become:

$$\frac{du}{ds} = \sqrt{\Delta u} \cdot \cos(\omega - \omega_0),$$

by virtue of (2), and then one will see that *the direction MN is the direction of most rapid variation of u.* However, the differential quotient will be zero in the direction *MT* that is perpendicular to *MN*; i.e., *the function will tend to remain constant.* Therefore, *MT* is the tangent, and consequently, *MN* is the normal to the curve of the system along which *u* keeps the value that it had at *M.*

3. First differential parameter. – One calls the square of the maximum differential quotient of a function *u* at a point – viz., Δu – the *first differential parameter* of that function. It is an invariant because it has (by definition) a significance that is independent of any reference system. In general, the function Δu defines a new system of curves that will coincide with the system that is defined by *u* when it is composed of parallel lines. Indeed, fix two infinitely-close values *u* and $u + du$ for *u*, which define two curves, so the segment that is cut out of the second curve along the normal to the first one when one starts from the point of incidence will be $ds = du : \sqrt{\Delta u}$. Now, since *ds* is constant along the first curve, it is necessary and sufficient that it should not vary when *u* does not vary; i.e., that Δu should depend upon only *u*. Hence, *in order for the system that is defined by the function u to be composed of parallel curves, it is necessary and sufficient that* Δu *should be a function of only u.*

4. – The first differential parameter can be considered to be a particular case of the *mixed differential parameter* of two functions:

$$\Delta(u, v) = \frac{\partial u}{\partial s_1}\frac{\partial v}{\partial s_1} + \frac{\partial u}{\partial s_2}\frac{\partial v}{\partial s_2}.$$

This is also an invariant. Indeed, if one writes formula (2) for the normals to the two curves:

$$u = \text{constant}, \quad v = \text{constant}$$

at a point that is common to those curves then one will find immediately that the angle between them – viz., the angle ψ between the two curves – will be given by the formula:

$$\cos\,\psi = \frac{\Delta(u,v)}{\sqrt{\Delta u \cdot \Delta v}}\,.$$

That will exhibit the invariant significance of $\Delta\,(u,\,v)$, and in addition one will show that *the annulment of the mixed differential parameter of two functions is necessary and sufficient for the orthogonality of the curves that are represented by those functions.* If one then observes that one can also write:

$$\sin\,\psi = \frac{1}{\sqrt{\Delta u \cdot \Delta v}} \begin{vmatrix} \dfrac{\partial u}{\partial s_1} & \dfrac{\partial u}{\partial s_2} \\ \dfrac{\partial v}{\partial s_1} & \dfrac{\partial v}{\partial s_2} \end{vmatrix},$$

and that in order for the right-hand side to be annulled, it is necessary and sufficient that v should be a function of u then one will see that this is also the sufficient (cf., § **1**) *and necessary* condition for the functions u and v to represent the same system of curves.

5. Second differential parameter. – Imagine that the operation:

$$\frac{d}{ds} = \cos\,\omega\,\frac{\partial}{\partial s_1} + \sin\,\omega\,\frac{\partial}{\partial s_2}\,,$$

which provides the differential quotient in an arbitrary direction, is repeated in that direction, which is *assumed to be invariable.* Let:

$$\frac{d^2}{ds^2} = \left(\cos\omega\frac{\partial}{\partial s_1} + \sin\omega\frac{\partial}{\partial s_2}\right)\left(\cos\omega\frac{\partial}{\partial s_1} + \sin\omega\frac{\partial}{\partial s_2}\right).$$

If one sets:

$$\frac{\partial\omega}{\partial s_1} = \mathcal{G}_1\,, \qquad \frac{\partial\omega}{\partial s_2} = -\,\mathcal{G}_2\,, \tag{3}$$

for brevity, then one will get:

$$\frac{d^2}{ds^2} = \cos^2\omega\left(\frac{\partial^2}{\partial s_1^2} + \mathcal{G}_1\frac{\partial}{\partial s_2}\right) + \sin^2\omega\left(\frac{\partial^2}{\partial s_2^2} + \mathcal{G}_2\frac{\partial}{\partial s_1}\right)$$
$$+ \sin\,\omega\cos\,\omega\left(\frac{\partial^2}{\partial s_1\,\partial s_2} + \frac{\partial}{\partial s_2\,\partial s_1} - \mathcal{G}_1\frac{\partial}{\partial s_1} - \mathcal{G}_2\frac{\partial}{\partial s_2}\right).$$

If that operation is also applied in the direction that is defined by the angle $\omega + \pi / 2$, and one then sums the two results then one will see directly that *the sum of the second differential quotients in two directions* (that are fixed in the plane) *will remain constant at a point when the directions vary while remaining orthogonal.* One calls that sum the *second differential parameter* of the function u in question, and it is represented by $\Delta^2 u$. One then has:

$$\Delta^2 = \frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2} + \mathcal{G}_1 \frac{\partial}{\partial s_2} + \mathcal{G}_2 \frac{\partial}{\partial s_1},$$

or:

$$\Delta^2 = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\frac{\partial}{\partial s_1} + \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\frac{\partial}{\partial s_2}. \tag{4}$$

One then calls the functions that have a second differential parameter that is constantly zero *harmonic functions.*

6. Isothermal systems. – One calls any system of curves that is defined by a harmonic function *isothermal,* and one gives the name of the *isometric parameter* of the system to that function. How can one recognize whether the system that is defined by a function u is isothermal? If one does not have $\Delta^2 u = 0$ then that does not mean that the system is not isothermal, but only that u cannot be the isometric parameter. By virtue of the final observation in § **4**, that parameter must by a function of u. Now, in order to apply the operation (4) to $F(u)$, one must observe that:

$$\frac{dF}{ds} = F' \frac{du}{ds}, \qquad \frac{d^2F}{ds^2} = F' \frac{d^2u}{ds^2} + F'' \left(\frac{du}{ds}\right)^2;$$

hence:

$$\Delta^2 F = F' \Delta^2 u + F'' \Delta u. \tag{5}$$

If the system is isothermal then there must exist a function F such that $\Delta^2 F$ is zero, and one will then have:

$$\frac{\Delta^2 u}{\Delta u} = -\frac{F''(u)}{F'(u)};$$

i.e., the ratio of the differential parameters will be functions of only u. Conversely, if it happens that this ratio is found to be equal to a function $f(u)$ then when one substitutes $f(u)$ for the left-hand side of the preceding equality and integrates, one will get:

$$F(u) = \int e^{-\int f\, du} \cdot du .$$

Obviously, if one imagines that this value of F is substituted in (5) then one will see that one must have $\Delta^2 F = 0$ as a result, and therefore the system will be isothermal and admit

F for its isometric parameter. Therefore, *in order for the system of curves that is defined by the function u to be isothermal, it is necessary and sufficient that the ratio of the differential parameters of u should be a function of only u.*

7. Curvilinear coordinates. – Now, consider two systems of curves that are defined by the functions q_1 and q_2 . If they have no link between them – so (§ **4**) the corresponding systems do not coincide – then the number of pairs of values for q_1 and q_2 will be doubly-infinite, like the number of points in the plane, and each of them M can be considered to be represented by the pair of values (viz., *curvilinear coordinates*) of q_1 and q_2, which characterize the curves that pass through M in the respective systems. It does not matter if a pair (q_1, q_2) does or does not correspond to other points or whether one or more points correspond to values of q_1 and q_2 . Nonetheless, we shall suppose (but only because it will make our considerations clearer and more precise) that the points in the plane and the pairs (q_1, q_2) are in one-to-one correspondence with each other. Hence, we can think that just two curves (viz., *coordinate lines*), one from each system, will pass through any point M, and call the line q_1 the one that belongs to the system that is defined by the function q_2 , while the line q_2 belongs to the other system, in such a way that the *line* q_i will always be the one *along which only* q_i *varies.* If $\Delta(q_1, q_2) = 0$ then the two systems will be orthogonal, and we shall always make that hypothesis from now on. In addition, we agree to direct the tangent along each line q_i and to measure the arc length s_i in the sense of increasing q_i . If we then direct the normal to q_1 in the sense of the tangent to q_2 then we must, however, consider the directions of the normal to q_2 and the tangent to q_1 to be *opposite* to each other in order to make the positive directions of the tangent and normal to q_1 coincide with the ones that relate to q_2 . Having said that, from what was said in § **3**, the segment that is found between two infinitely-close lines q_2 will be – $ds_1 = dq_1 : \sqrt{\Delta q_1}$, and we shall represent it by $Q_1\, dq_1$, for brevity. Similarly, a segment $ds_2 = Q_2\, dq_2$ that is equal to $dq_2 : \sqrt{\Delta q_2}$ along the line q_2 will be found between two infinitely-close lines q_1 , in such a way that we will have, by definition:

$$\sqrt{\Delta q_1} = -\frac{1}{Q_1}, \qquad \sqrt{\Delta q_2} = \frac{1}{Q_2}, \tag{6}$$

and from the conventions that were made to begin with, the Q can take on only *positive* values. It is important to observe that the square of the elementary arc length $ds = \sqrt{ds_1^2 + ds_2^2}$ is given by the formula:

$$ds^2 = Q_1^2\, dq_1^2 + Q_2^2\, dq_2^2 ,$$

and since we can replace each q_i with a function of q_i , it is clear that each Q_i can be multiplied by an arbitrary function of the q_i . Obviously:

$$\frac{\partial q_1}{\partial s_1} = \frac{1}{Q_1}, \qquad \frac{\partial q_2}{\partial s_2} = \frac{1}{Q_2}, \qquad \frac{\partial q_1}{\partial s_2} = \frac{\partial q_2}{\partial s_1} = 0. \tag{7}$$

Q_1 and Q_2, like q_1 and q_2, are functions of the point M : *They are therefore functions of the independent variables q_1 and q_2*. One can also say the same thing for arbitrary functions of the points in the plane. The partial derivatives with respect to the q are then coupled in a simple way by the operations that yield the differential quotients that were defined in § **2**, since one has:

$$\frac{\partial}{\partial s_1} = \frac{1}{Q_1}\frac{\partial}{\partial q_1}, \quad \frac{\partial}{\partial s_2} = \frac{1}{Q_2}\frac{\partial}{\partial q_2}. \tag{8}$$

Therefore, the aforementioned quotients *cannot be considered to be true derivatives* if Q_1 is not a function of only q_1 and Q_2 is not a function of only q_2. We will soon see that this will happen only when all of the coordinate lines are straight lines (viz., *Cartesian coordinates*).

8. Fundamental formulas. – Take the x-axis to be the tangent at M (moving origin) to the line q_1 and take the y-axis to be the tangent to the line q_2 (which is normal to q_1). The Cartesian coordinates of a fixed point P must satisfy the immobility condition (II, § **1**) with respect to the first line, namely:

$$\frac{\partial x}{\partial s_1} = \frac{y}{\rho_1} - 1, \qquad \frac{\partial y}{\partial s_1} = -\frac{x}{\rho_1}.$$

The coordinates x and y of P with respect to q_2 are y and $-x$, and therefore the immobility conditions will become:

$$\frac{\partial y}{\partial s_2} = -\frac{x}{\rho_2} - 1, \qquad \frac{\partial x}{\partial s_2} = \frac{y}{\rho_2}.$$

Meanwhile, if one observes that the function ω that was considered in § **5** differs only in sign from the function that was constantly represented by φ in the first chapter, so one will therefore have:

$$\frac{1}{\rho_1} = \frac{\partial \varphi}{\partial s_1} = -\frac{\partial \omega}{\partial s_1} = -\mathcal{G}_1, \qquad \frac{1}{\rho_2} = \frac{\partial \varphi}{\partial s_2} = -\frac{\partial \omega}{\partial s_2} = \mathcal{G}_2.$$

With that, one will see that *the necessary and sufficient conditions for the immobility of the point* (x, y) *are:*

$$\left.\begin{array}{ll} \dfrac{\partial x}{\partial s_1} = -\mathcal{G}_1\, y - 1, & \dfrac{\partial y}{\partial s_1} = \mathcal{G}_1\, x, \\[2ex] \dfrac{\partial y}{\partial s_2} = -\mathcal{G}_2\, x - 1, & \dfrac{\partial x}{\partial s_2} = \mathcal{G}_2\, y. \end{array}\right\} \tag{9}$$

If one wants to know the variations of the coordinates of a point P when the origin passes from the position M to another one M' along a direction that makes an arbitrary angle ω with the x-axis then it will be clear that one will have:

$$\frac{\delta x}{ds} = \left(\frac{\partial x}{\partial s_1} + \mathcal{G}_1\, y + 1\right)\cos\omega + \left(\frac{\partial x}{\partial s_2} - \mathcal{G}_1\, y\right)\sin\omega,$$

$$\frac{\delta y}{ds} = \left(\frac{\partial y}{\partial s_2} + \mathcal{G}_2\, x + 1\right)\sin\omega + \left(\frac{\partial y}{\partial s_1} - \mathcal{G}_1\, x\right)\cos\omega.$$

These are the *fundamental formulas* for the intrinsic analysis of pairs of orthogonal systems of plane curves.

9. Integrability conditions. – Given the functions u and v, we propose to find the *necessary and sufficient condition for the existence of a function f such that one has:*

$$u = \frac{\partial f}{\partial s_1}, \qquad v = \frac{\partial f}{\partial s_2}. \tag{10}$$

From (8), that is equivalent to asking what the condition would be for $Q_1\, u$ and $Q_1\, v$ to be the first partial derivatives of a function $f(q_1, q_2)$, and it is known that this condition is:

$$\frac{\partial Q_2 v}{\partial q_1} = \frac{\partial Q_1 u}{\partial q_2}, \qquad \text{or} \qquad \frac{1}{Q_2}\frac{\partial Q_2 v}{\partial s_1} = \frac{1}{Q_1}\frac{\partial Q_1 u}{\partial s_2},$$

i.e., upon developing:

$$\frac{\partial u}{\partial s_2} - \frac{\partial v}{\partial s_1} = v\frac{\partial \log Q_2}{\partial s_1} - u\frac{\partial \log Q_1}{\partial s_2}.$$

If one substitutes the values (10) in this equality then one will see that for any f, one must satisfy the condition:

$$\frac{\partial^2}{\partial s_1\,\partial s_2} - \frac{\partial^2}{\partial s_2\,\partial s_1} = \frac{\partial \log Q_2}{\partial s_1}\frac{\partial}{\partial s_2} - \frac{\partial \log Q_1}{\partial s_2}\frac{\partial}{\partial s_1}. \tag{11}$$

That is applied to the function x, while taking (9) into account. One first has:

$$\frac{\partial^2 x}{\partial s_1\,\partial s_2} = -\frac{\partial \mathcal{G}_1 y}{\partial s_2} = -\frac{\partial \mathcal{G}_1}{\partial s_2}\, y + \mathcal{G}_1\,(\mathcal{G}_2\, x + 1),$$

$$\frac{\partial^2 x}{\partial s_2\,\partial s_1} = \frac{\partial \mathcal{G}_2 y}{\partial s_1} = \frac{\partial \mathcal{G}_2}{\partial s_1}\, y + \mathcal{G}_1\, \mathcal{G}_2\, x\,;$$

hence, from (11):

$$\left(\frac{\partial\mathcal{G}_1}{\partial s_2}+\frac{\partial\mathcal{G}_2}{\partial s_1}+\mathcal{G}_1\frac{\partial\log Q_1}{\partial s_2}+\mathcal{G}_2\frac{\partial\log Q_2}{\partial s_1}\right)y=\mathcal{G}_1-\frac{\partial\log Q_1}{\partial s_2}. \tag{12}$$

Operating analogously on the function y, one will find that:

$$\left(\frac{\partial\mathcal{G}_1}{\partial s_2}+\frac{\partial\mathcal{G}_2}{\partial s_1}+\mathcal{G}_1\frac{\partial\log Q_1}{\partial s_2}+\mathcal{G}_2\frac{\partial\log Q_2}{\partial s_1}\right)x=\mathcal{G}_2-\frac{\partial\log Q_2}{\partial s_1}. \tag{13}$$

These relations must be true for any values of x and y. For example, if one considers the instant of the passage from a given fixed point P to M then one will have $x = 0$, $y = 0$, and formulas (12) and (13) will give:

$$\mathcal{G}_1=\frac{\partial\log Q_1}{\partial s_2}, \qquad \mathcal{G}_2=\frac{\partial\log Q_2}{\partial s_1}. \tag{14}$$

With that, the integrability condition can be put into the definitive form:

$$\left(\frac{\partial}{\partial s_1}+\mathcal{G}_2\right)v=\left(\frac{\partial}{\partial s_2}+\mathcal{G}_1\right)u,$$

and the condition (11) will become:

$$\left(\frac{\partial}{\partial s_1}+\mathcal{G}_2\right)\frac{\partial}{\partial s_2}=\left(\frac{\partial}{\partial s_2}+\mathcal{G}_1\right)\frac{\partial}{\partial s_1}. \tag{15}$$

10. Lamé's relation. – The curvatures $\mathcal{G}_1$ and $\mathcal{G}_2$ are not independent of each other. Indeed, if one substitutes the values (14) in the equalities (12) and (13) then they will reduce to the single one:

$$\frac{\partial\mathcal{G}_1}{\partial s_2}+\frac{\partial\mathcal{G}_2}{\partial s_1}+\mathcal{G}_1^2+\mathcal{G}_2^2=0, \tag{16}$$

which one calls *Lamé's relation*, and which expresses (as a result of an easy calculation) the *necessary and sufficient condition for the existence of two functions* x_1 *and* x_2 *of* q_1 *and* q_2 *, such that* $dx_1^2+dx_2^2$ *represents the square of the elementary arc length.* When put into the form:

$$-\left(\frac{\partial}{\partial s_1}+\mathcal{G}_2\right)\mathcal{G}_2=\left(\frac{\partial}{\partial s_2}+\mathcal{G}_1\right)\mathcal{G}_1, \tag{17}$$

Lamé's relation says immediately that $\mathcal{G}_1$ and $-\mathcal{G}_2$ are the differential quotients of a function ω, which we already know from (3). One can give another form to (16) by setting the $\mathcal{G}$'s equal to the values (14). If one observes that:

$$\left(\frac{\partial}{\partial s_1}+\mathcal{G}_2\right)u=\frac{1}{Q_1\,Q_2}\frac{\partial Q_2 u}{\partial q_1}, \qquad \left(\frac{\partial}{\partial s_2}+\mathcal{G}_1\right)u=\frac{1}{Q_1\,Q_2}\frac{\partial Q_2 u}{\partial q_2} \tag{18}$$

then the relation (16), after being put into the form (17), will become:

$$\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\right)=0.$$

It is also useful to know the form (which is due to Lamé) that the operation (4) will take, thanks to (18):

$$\Delta^2=\frac{1}{Q_1\,Q_2}\left[\frac{\partial}{\partial q_1}\left(\frac{Q_2}{Q_1}\frac{\partial}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{Q_1}{Q_2}\frac{\partial}{\partial q_2}\right)\right].$$

11. – The operation (4) must give $\Delta^2 u = 0$ for a harmonic function u; i.e., one must have:

$$-\left(\frac{\partial}{\partial s_1}+\mathcal{G}_2\right)\frac{\partial u}{\partial s_1}=\left(\frac{\partial}{\partial s_2}+\mathcal{G}_1\right)\frac{\partial u}{\partial s_2},$$

and therefore $\dfrac{\partial u}{\partial s_2}$ and $-\dfrac{\partial u}{\partial s_1}$ will be the differential quotients of a function v :

$$\frac{\partial u}{\partial s_1}=-\frac{\partial v}{\partial s_2}, \quad \frac{\partial u}{\partial s_2}=\frac{\partial v}{\partial s_1}.$$

Meanwhile, it is known that the systems that are defined by the functions u and v are mutually-orthogonal, since $\Delta(u, v) = 0$. Now, by virtue of (15):

$$\Delta^2 v=\left(\frac{\partial}{\partial s_1}+\mathcal{G}_2\right)\frac{\partial u}{\partial s_1}-\left(\frac{\partial}{\partial s_2}+\mathcal{G}_1\right)\frac{\partial u}{\partial s_2}=0.$$

Therefore, v is harmonic; that is to say: *When one system in a pair of orthogonal systems of plane curves is isothermal, the other one will also be isothermal.* One can also arrive at this theorem by utilizing the rule that was given in § **6** for testing whether a system is isothermal. Indeed, one applies the operation (4) to q_1 and q_2 , while taking (7) into account. Upon making use of (18), one will get:

$$\Delta^2 q_1 = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\frac{1}{Q_1} = \frac{1}{Q_1\,Q_2}\frac{\partial}{\partial q_1}\frac{Q_2}{Q_1}, \qquad \Delta^2 q_2 = \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\frac{1}{Q_2} = \frac{1}{Q_1\,Q_2}\frac{\partial}{\partial q_2}\frac{Q_1}{Q_2};$$

if one recalls (6) then one will have:

$$\frac{\Delta^2 q_1}{\Delta q_1} = \frac{\partial}{\partial q_1}\log\frac{Q_2}{Q_1}, \qquad \frac{\Delta^2 q_2}{\Delta q_2} = \frac{\partial}{\partial q_2}\log\frac{Q_1}{Q_2}, \tag{19}$$

and finally:

$$\frac{\partial}{\partial q_2}\frac{\Delta^2 q_1}{\Delta q_1} + \frac{\partial}{\partial q_1}\frac{\Delta^2 q_2}{\Delta q_2} = 0.$$

This formula shows clearly that if the condition that was stated at the end of § **6** is satisfied for one of two systems then it will be satisfied by the other one. If one chooses isometric parameters for q_1 and q_2 then it will result from (19) that the ratio of Q_1 to Q_2 will be constant, and that conversely, that cannot happen unless q_1 and q_2 are harmonic functions. It will then follow that the isothermal orthogonal pairs of plane curves are characterized by the possibility of making $Q_1 = Q_2$, since one can always multiply any function Q or q by a constant. The elementary arc length is then given by the formula:

$$ds^2 = Q^2(dq_1^2 + dq_2^2),$$

and it is enough to take $dq_1 = dq_2$, since one has $ds_1 = ds_2$ at any point. One can express that by saying that *the curves of any doubly-isothermal orthogonal system divide the plane into infinitesimal squares.*

12. – In order to insure that a double system of orthogonal coordinate lines is isothermal, it is enough to see that one of the systems is isothermal, and if one applies the criterion that was proved in § **6** to one of (19) then one must have:

$$\frac{\partial^2}{\partial q_1\,\partial q_2}\log\frac{Q_1}{Q_2} = 0.$$

With two successive integrations, one will see that *the ratio of the functions Q must be equal to the product of a function of only q_1 by a function of only q_2*. That is the characteristic property of the functions Q in the doubly-isothermal systems. In order to express that in terms of the $\mathcal{G}$'s, observe that if one recalls (14) then:

$$\frac{\partial^2}{\partial q_1\,\partial q_2}\log\frac{Q_1}{Q_2} = Q_1\frac{\partial Q_2\,\mathcal{G}_1}{\partial s_1} - Q_2\frac{\partial Q_1\,\mathcal{G}_2}{\partial s_2} = Q_1\,Q_2\left(\frac{\partial \mathcal{G}_1}{\partial s_1} - \frac{\partial \mathcal{G}_2}{\partial s_2}\right).$$

Hence, *in order for the functions* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ (which are necessarily constrained by Lamé's relation) *to define a doubly-isothermal system, it is necessary and sufficient that one should have:*

$$\frac{\partial \mathcal{G}_1}{\partial s_1} = \frac{\partial \mathcal{G}_2}{\partial s_2}. \tag{20}$$

That is equivalent to saying that the function ω that appears in formula (3) is harmonic, since one has:

$$\Delta^2 \omega = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\mathcal{G}_1 - \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\mathcal{G}_2 = \frac{\partial \mathcal{G}_1}{\partial s_1} - \frac{\partial \mathcal{G}_2}{\partial s_2}.$$

13. Bonnet's formula. – That formula serves to make known the curvature at any point M of the line that passes through M in the system that is defined by a function u. The inclination ω of the tangent to that curve at M is calculated, as we saw in § **2**, by writing:

$$\cos\,\omega\,\frac{\partial u}{\partial s_1} + \sin\,\omega\,\frac{\partial u}{\partial s_2} = 0,$$

from which one infers that:

$$\cos\,\omega = \frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_2}, \qquad \sin\,\omega = -\frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_1}, \tag{21}$$

as long as one takes care to fix the sign in such a way that when one takes formulas (6) and (7) into account, one will have $\omega = 0$ for $u = q_2$ and $\omega = \pi / 2$ for $u = q_1$. Let φ be the inclination of the tangent to the line q_1 with respect to a fixed line; the analogous angles for the line q_2 and the curve considered will be $\varphi + \pi / 2$ and $\varphi + \omega$, respectively. Hence:

$$\frac{\partial \varphi}{\partial s_1} = \frac{1}{\rho_1} = -\,\mathcal{G}_1, \qquad \frac{\partial \varphi}{\partial s_2} = \frac{1}{\rho_2} = \mathcal{G}_2, \qquad \frac{d}{ds}(\varphi + \omega) = \frac{1}{\rho};$$

hence:

$$\frac{1}{\rho} = -\,\mathcal{G}_1 \cos\,\omega + \mathcal{G}_2 \sin\,\omega \frac{d\omega}{ds}. \tag{22}$$

On the other hand, one has:

$$\frac{d\omega}{ds} = \cos\,\omega\,\frac{\partial \omega}{\partial s_1} + \sin\,\omega\,\frac{\partial \omega}{\partial s_2} = \frac{\partial}{\partial s_1}\sin\,\omega - \frac{\partial}{\partial s_2}\cos\,\omega.$$

Therefore:

$$\frac{1}{\rho} = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\sin\omega - \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\cos\omega.$$

Now, if one substitutes the values (21) in this equivalence then one will get:

$$-\frac{1}{\rho}=\left(\frac{\partial}{\partial s_1}+\mathcal{G}_2\right)\left(\frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_1}\right)+\left(\frac{\partial}{\partial s_2}+\mathcal{G}_1\right)\left(\frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_2}\right)$$

$$=\frac{1}{\sqrt{\Delta u}}\left[\left(\frac{\partial}{\partial s_1}+\mathcal{G}_2\right)\frac{\partial u}{\partial s_1}+\left(\frac{\partial}{\partial s_2}+\mathcal{G}_1\right)\frac{\partial u}{\partial s_2}\right]+\frac{\partial u}{\partial s_1}\frac{\partial}{\partial s_1}\frac{1}{\sqrt{\Delta u}}+\frac{\partial u}{\partial s_2}\frac{\partial}{\partial s_2}\frac{1}{\sqrt{\Delta u}},$$

so one will finally have:

$$-\frac{1}{\rho}=\frac{\Delta^2 u}{\sqrt{\Delta u}}+\Delta\left(u,\frac{1}{\sqrt{\Delta u}}\right).$$

Thus, the curvature is known in an explicitly invariant form.

14. Examples:

a) Consider the *constant-angle trajectories* of the lines of a system; i.e., the curves that meet those lines (for example, the line q_1) at the constant angle ω. Obviously, an infinitude of trajectories pass through any point M, each of which corresponds to one value of ω. Formula (22) will give:

$$\frac{1}{\rho}=-\mathcal{G}_1\cos\omega+\mathcal{G}_2\sin\omega,$$

and show that *the center of curvature of any trajectory at each point M will belong to a line* that is represented by the equation:

$$\mathcal{G}_2\,x+\mathcal{G}_1\,y+1=0. \tag{23}$$

Let φ be the angle that the tangent to a coordinate line makes with a fixed line, and consider the system that is defined by the function φ; i.e., the system in which any curve is such that when one is given a point of the other, the directions of the tangents to the coordinate lines will not vary. The equation of the tangent to a curve of the system φ is $\mathcal{G}_1\,x=\mathcal{G}_2\,y$, and therefore the line (23) that corresponds to a given point M will be parallel to the normal to the curve of the system that passes through M. It will then follow that such a curve *will touch the trajectories at their inflection points at a constant angle to the coordinate lines.* The systems that are defined by the functions $\mathcal{G}_1$ and $\mathcal{G}_2$, to which the loci of flexures and cusps of the coordinate lines belong, are also noteworthy. When M is displaced in a direction that makes an angle ω with the x-axis, one will find when one applies formulas (9) to equation (23) and recalls the Lamé relation that the line (23) will touch its envelope along the line:

$$\left(\frac{\partial \mathcal{G}_1}{\partial s_2}\cos\omega - \frac{\partial \mathcal{G}_2}{\partial s_2}\sin\omega\right)x = \left(\frac{\partial \mathcal{G}_1}{\partial s_1}\cos\omega - \frac{\partial \mathcal{G}_2}{\partial s_1}\sin\omega\right)y. \tag{24}$$

That equation will be satisfied for any x and y and a certain value of ω only when the functional determinant of the $\mathcal{G}$ is zero. The double infinitude of lines (23) will then reduce to a simple infinitude, but one needs to observe that this will also occur when a system of $\mathcal{G}$ does not exist – i.e., when one of the fundamental systems that make $\mathcal{G}_1$ or $\mathcal{G}_2$ *constant* is composed of equal circles or lines. In that case, the lines (23) will obviously be the normals to the curve that envelopes the circles or the lines of the system. That explains the known constructions (I, § **11**, *c*; II, § **13**, *c*) of the center of curvature of the logarithmic spiral, the involutes of the catenary, etc. In the general case, one deduces from (24) that when M is displaced along a coordinate line, the lines (23) will touch their envelope along the normals to one of the $\mathcal{G}$ curves.

b) The osculating circles of the coordinate lines at a point M are represented by the equations:

$$x^2 + y^2 + \frac{2}{\mathcal{G}_1}y = 0, \qquad x^2 + y^2 + \frac{2}{\mathcal{G}_2}x = 0.$$

If one differentiates the first one with respect to q_2 and the second one with respect to q_1 and observes (9) then one will get:

$$\mathcal{G}_2\, x + \mathcal{G}_1\, y + y\,\frac{\partial}{\partial s_2}\log \mathcal{G}_1 = 0, \qquad \mathcal{G}_2\, x + \mathcal{G}_1\, y + 1 + x\,\frac{\partial}{\partial s_1}\log \mathcal{G}_2 = 0. \tag{25}$$

Hence, each circumference will touch its envelope along a diameter of the other one. It is then easy to see that, by virtue of the Lamé relation, the two diameters are perpendicular, and that one of the two envelopes is real, while the other one is imaginary. We now propose to find the condition that must be satisfied in order for the osculating circles of the lines q_2 along a line q_1 to constitute a sheaf. Obviously, for that to be true, it is necessary and sufficient that the line that is represented by the second equation in (25) should be fixed in the plane. Meanwhile, if one differentiates that equation with respect to q_1, while taking (9) into account, along with the Lamé relations and its original equation, then one will get:

$$x\left(\frac{\partial}{\partial s_2} + 3\mathcal{G}_1\right)\frac{\partial \mathcal{G}_1}{\partial s_1} = y\,\frac{\partial \mathcal{G}_1}{\partial s_1},$$

and therefore the desired condition is that $\mathcal{G}_1$ should be independent of q_1 ; i.e., that any line q_1 should be a circle. Hence, *the osculating circles to the orthogonal trajectories of any simple infinitude of circumferences along each of them will define a sheaf.* Moreover, that is an immediate consequence of the known theorem: *The circles that are orthogonal to two given circles form a sheaf* whose axis is the common diameter to the two circles. In order to prove the theorem, it is enough to consider two infinitely-close circles from a given system, and to see, in addition, that the axis of the sheaf of osculating circles is the tangent to the locus of centers. It will then follow from that theorem that *any double orthogonal system of circles will necessarily consist of two sheaves.* The axes of the two sheaves – i.e., the lines that are represented by equations (25) – are perpendicular and contain the centers of all circles. One will meet the corresponding circumference at two real points A and A', while the other will meet it at two imaginary points, and at two coincident real points in exceptional cases. With an easy calculation, one will find that if $2a$ is the length of the segment AA' then one will have:

$$\frac{1}{a^2} = \mathcal{G}_1^2 + \frac{\left(\dfrac{\partial \mathcal{G}_1}{\partial s_2}\right)^2}{\dfrac{\partial \mathcal{G}_1}{\partial s_2} - \dfrac{\partial \mathcal{G}_2}{\partial s_1}} = -\,\mathcal{G}_2^2 + \frac{\left(\dfrac{\partial \mathcal{G}_2}{\partial s_1}\right)^2}{\dfrac{\partial \mathcal{G}_1}{\partial s_2} - \dfrac{\partial \mathcal{G}_2}{\partial s_1}}. \tag{26}$$

However, the length of the segment that is cut out from the other line by the corresponding circle is $2a\sqrt{-1}$.

c) We wish to study the *doubly-orthogonal systems of circles* in more detail, which are obviously characterized by the equality:

$$\frac{\partial \mathcal{G}_1}{\partial s_1} = 0, \qquad \frac{\partial \mathcal{G}_2}{\partial s_2} = 0, \tag{27}$$

and consequently they will be *isothermal*, since the condition (20) is satisfied. The calculations that were mentioned in the preceding example can be repeated more rapidly after one observes that from the Lamé condition and the condition (15), one can deduce from (27) that:

$$\frac{\partial^2 \mathcal{G}_1}{\partial s_1\,\partial s_2} = 0, \qquad \frac{\partial^2 \mathcal{G}_1}{\partial s_2\,\partial s_1} = -\,\mathcal{G}_2 \frac{\partial \mathcal{G}_1}{\partial s_1}, \qquad \frac{1}{\mathcal{G}_1}\frac{\partial^2 \mathcal{G}_1}{\partial s_2^2} = \frac{\partial \mathcal{G}_2}{\partial s_1} - 2\frac{\partial \mathcal{G}_1}{\partial s_2},$$

$$\frac{\partial^2 \mathcal{G}_2}{\partial s_2\,\partial s_1} = 0, \qquad \frac{\partial^2 \mathcal{G}_2}{\partial s_1\,\partial s_2} = -\,\mathcal{G}_1 \frac{\partial \mathcal{G}_1}{\partial s_1}, \qquad \frac{1}{\mathcal{G}_2}\frac{\partial^2 \mathcal{G}_2}{\partial s_1^2} = \frac{\partial \mathcal{G}_1}{\partial s_2} - 2\frac{\partial \mathcal{G}_2}{\partial s_1}.$$

If we appeal to these relations then we will arrive quite easily at the fact that the lines (25) are fixed in the plane. We would now like to determine the $\mathcal{G}$, and in order to do that, we suppose that q_1 and q_2 are isometric parameters and recall (§ **11**) that one can always do that in such a way that $Q_1 = Q_2$. (14) will then give:

$$\mathcal{G}_1 = -\frac{\partial}{\partial q_2}\frac{1}{Q}, \qquad \mathcal{G}_2 = -\frac{\partial}{\partial q_1}\frac{1}{Q}, \tag{28}$$

and the Lamé relation will become:

$$\frac{1}{Q}\left(\frac{d\mathcal{G}_1}{dq_2}+\frac{d\mathcal{G}_2}{dq_1}\right)+\mathcal{G}_1^2+\mathcal{G}_2^2=0. \tag{29}$$

When that is differentiated with respect to q_1 and q_2, that will give:

$$\frac{1}{Q}\frac{d^2\mathcal{G}_2}{dq_1^2}=\mathcal{G}_2\left(\frac{d\mathcal{G}_1}{dq_2}-\frac{d\mathcal{G}_2}{dq_1}\right), \qquad \frac{1}{Q}\frac{d^2\mathcal{G}_1}{dq_2^2}=\mathcal{G}_1\left(\frac{d\mathcal{G}_2}{dq_1}-\frac{d\mathcal{G}_1}{dq_2}\right),$$

so

$$\frac{1}{\mathcal{G}_1}\frac{d^2\mathcal{G}_1}{dq_2^2}+\frac{1}{\mathcal{G}_2}\frac{d^2\mathcal{G}_2}{dq_1^2}=0.$$

Since the first term is independent of q_1 and the second one is independent of q_2, we need both of them to be constants, and in that way we will be led to set:

$$\frac{d^2\mathcal{G}_1}{dq_1^2}=-k^2\,\mathcal{G}_1, \qquad \frac{d^2\mathcal{G}_2}{dq_1^2}=-k^2\,\mathcal{G}_2. \tag{30}$$

If we assume that the lines q_1 are the circumferences that pass through A and A' then the line AA' will be a line q_1 that we can always imagine to represented by the equation $q_2 = 0$. In order to satisfy the first equation in (30), we therefore need to take $\mathcal{G}_1 = \lambda \sin k\, q_2$. The other equation can only be satisfied by taking:

$$\mathcal{G}_2 = \mu\, e^{kq_1}+\mu'\, e^{-kq_1},$$

with μ and μ' arbitrary constants, like λ. Meanwhile, we infer from (28) upon integrating that:

$$\frac{k}{Q}=\lambda\cos k\,q_1-(\mu\, e^{kq_1}-\mu'\, e^{-kq_1})+\text{constant}.$$

When one substitutes this in (29), one will then find that the last constant is zero and that the relation $\lambda^2 + 4\mu\mu' = 0$ couples the other ones. If one of the constants μ or μ' is zero (for example, μ') then one must also have $\lambda = 0$, and that will make $\mathcal{G}_1 = 0$, $\mathcal{G}_2 = \mu\, e^{kq_1}$. The *lines* q_1 *form a sheaf of lines, and the lines* q_2 *are concentric circumferences*. If μ and μ' are non-zero then one knows (cf., II, § **7**, *n*) that one can always suppose that $\mu = \pm\mu'$, and in the present case, it is necessary that one should have $\mu = -\mu' = \lambda / 2$. Hence:

$$\mathcal{G}_1=\lambda\sin k\,q_2, \qquad \mathcal{G}_2=\frac{\lambda}{2}(e^{kq_1}-e^{-kq_1}). \tag{31}$$

In the limiting case in which k tends to zero, one will find that $\mathcal{G}_1 = q_2$, $\mathcal{G}_2 = q_1$, which one can deduce directly from (30) for $k = 0$, moreover. In that case, (26) will show that a is zero, and therefore *the two systems are composed of circles that touch two perpendicular line at their common point.* In the general case, it is always legitimate to take $k = 1$, and substituting the values (31) in (26) will give $\lambda a = \pm 1$. Hence:

$$\mathcal{G}_1 = \frac{\sin q_2}{a}, \qquad \mathcal{G}_2 = \frac{e^{q_1} - e^{-q_1}}{2a}.$$

The geometric interpretation of the first formula shows that q_2 is angle AMA' ; the second one leads one to see that q_1 is the logarithm of the ratio $MA' : MA$.

d) Any plane curve gives rise to a double system of curves when any arc is made to correspond to a point in the plane. Indeed, the arcs of a curve are doubly-infinite in number, and each of them is represented by pairs of values ζ_1 and ζ_2 that the arc length s takes at the end points, when one starts measuring it from a fixed origin. A simple way of realizing such a corresponding consists of taking the barycenter G of each arc $A_1 A_2$. The two systems that are defined by the functions ζ then constitute the only system of barycentric lines (VI. § **3**) of the given curve. Two lines pass through any point G (viz., the two barycenters that touch the curve at the end points of the corresponding arc), and we already know that the tangents at G to the two lines are precisely GA_1 and GA_2 . In order to construct a doubly-*orthogonal* system of lines, draw two orthogonal axes through the origin G, which are oriented in such a way that of the three equalities:

$$\int_{\zeta_1}^{\zeta_2} x\,ds = 0, \qquad \int_{\zeta_1}^{\zeta_2} y\,ds = 0, \qquad \int_{\zeta_1}^{\zeta_2} xy\,ds = 0, \tag{32}$$

it is the third one that is true. The differential quotients relative to the new axes will be expressed as follows:

$$\frac{\partial}{\partial s_1} = \frac{\partial \zeta_1}{\partial s_1}\frac{\partial}{\partial \zeta_1} + \frac{\partial \zeta_2}{\partial s_1}\frac{\partial}{\partial \zeta_2}, \qquad \frac{\partial}{\partial s_2} = \frac{\partial \zeta_1}{\partial s_2}\frac{\partial}{\partial \zeta_1} + \frac{\partial \zeta_2}{\partial s_2}\frac{\partial}{\partial \zeta_2}. \tag{33}$$

If one lets D represent the functional determinant of the ζ, which is necessarily non-zero, then one will have simply:

$$\frac{\partial}{\partial \zeta_1} = \frac{1}{D}\left(\frac{\partial \zeta_2}{\partial s_2}\frac{\partial}{\partial s_1} + \frac{\partial \zeta_2}{\partial s_1}\frac{\partial}{\partial s_2}\right), \qquad \frac{\partial}{\partial \zeta_2} = \frac{1}{D}\left(\frac{\partial \zeta_1}{\partial s_1}\frac{\partial}{\partial s_2} - \frac{\partial \zeta_1}{\partial s_2}\frac{\partial}{\partial s_1}\right). \tag{34}$$

It is enough to differentiate the first two relations (32) with respect to the ζ and apply the formulas (34) and (9) to get:

$$\frac{\partial \zeta_1}{\partial s_1} = \frac{D}{\sigma} y_2, \qquad \frac{\partial \zeta_1}{\partial s_2} = -\frac{D}{\sigma} x_2, \qquad \frac{\partial \zeta_2}{\partial s_1} = \frac{D}{\sigma} y_1, \qquad \frac{\partial \zeta_2}{\partial s_2} = -\frac{D}{\sigma} x_1,$$

in which x_i and y_i are the coordinates of A_i , and σ represents the length of the arc $A_1 A_2$. Therefore, if τ once more represents twice the area of the triangle $GA_1 A_2$ then one will have:

$$D = -\frac{D^2}{\sigma^2} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \frac{D^2}{\sigma^2} \tau,$$

and finally, since D cannot always be zero, $\tau D = \sigma^2$. One will also arrive at this result by observing that if ψ is the angle between the two barycenters that cross at G, and if r_i is the length of the edge GA_i then one will have, in succession:

$$\sqrt{\Delta\zeta_1} = \frac{D}{\sigma} r_2, \qquad \sqrt{\Delta\zeta_2} = \frac{D}{\sigma} r_1, \qquad \tau = r_1\, r_2 \sin\psi = \frac{r_1 r_2 D}{\sqrt{\Delta\zeta_1 \cdot \Delta\zeta_2}} = \frac{\sigma^2}{D}.$$

Having said that, (33) and (34) will become:

$$\frac{\partial}{\partial s_1} = \frac{\sigma}{\tau}\left(y_2 \frac{\partial}{\partial \zeta_1} + y_1 \frac{\partial}{\partial \zeta_2} \right), \qquad \frac{\partial}{\partial s_2} = -\frac{\sigma}{\tau}\left(x_2 \frac{\partial}{\partial \zeta_1} + x_1 \frac{\partial}{\partial \zeta_2} \right), \tag{35}$$

$$\frac{\partial}{\partial \zeta_1} = -\frac{1}{\sigma}\left(x_1 \frac{\partial}{\partial s_1} + y_1 \frac{\partial}{\partial s_2} \right), \qquad \frac{\partial}{\partial \zeta_2} = \frac{1}{\sigma}\left(x_2 \frac{\partial}{\partial s_1} + y_1 \frac{\partial}{\partial s_2} \right). \tag{36}$$

So far, the particular orientation of the axes that is defined by the last equality in (32) has yet to be taken into account. Now, if one applies (36) and (8) and sets:

$$\int_{\zeta_1}^{\zeta_2} (x^2 - y^2)\, ds = \kappa\, \sigma$$

then one will find that:

$$\frac{\partial}{\partial \zeta_1} \int_{\zeta_1}^{\zeta_2} xy\, ds = -\, x_1\, y_1 - \kappa(\mathcal{G}_1\, x_1 - \mathcal{G}_2\, y_1),$$

$$\frac{\partial}{\partial \zeta_2} \int_{\zeta_1}^{\zeta_2} xy\, ds = \quad x_1\, y_1 + \kappa(\mathcal{G}_1\, x_2 - \mathcal{G}_2\, y_2);$$

so, from (35):

$$\frac{\partial}{\partial s_1} \int_{\zeta_1}^{\zeta_2} xy\, ds = \frac{\sigma}{\tau} [\kappa\, \tau\, \mathcal{G}_1 - y_1\, y_2\, (x_1 - x_2)],$$

$$\frac{\partial}{\partial s_2} \int_{\zeta_1}^{\zeta_2} xy\, ds = -\frac{\sigma}{\tau} [\kappa\, \tau\, \mathcal{G}_2 - x_1\, x_2\, (y_1 - y_2)].$$

Hence:

$$\mathcal{G}_1 = (x_1 - x_2)\,\frac{y_1\,y_2}{\kappa\tau}, \qquad \mathcal{G}_2 = (y_1 - y_2)\,\frac{x_1\,x_2}{\kappa\tau}. \tag{37}$$

Meanwhile, it is useful to transform the conditions (9) in such a way that the derivatives with respect to the ζ will appear. Thanks to (36), one easily finds that:

$$\frac{\partial x}{\partial\zeta_1} = \left(1 - \frac{yy_1}{\kappa}\right)\frac{x_1}{\sigma}, \qquad \frac{\partial x}{\partial\zeta_2} = -\left(1 - \frac{yy_2}{\kappa}\right)\frac{x_2}{\sigma},$$

$$\frac{\partial y}{\partial\zeta_1} = \left(1 + \frac{xx_1}{\kappa}\right)\frac{y_1}{\sigma}, \qquad \frac{\partial y}{\partial\zeta_2} = -\left(1 + \frac{xx_2}{\kappa}\right)\frac{y_2}{\sigma}.$$

These are the *necessary and sufficient conditions for the immobility of the point* (x, y). One needs to observe that the formulas on the left are also applicable to x_2, y_2, since the point A_2 remains immobile under the partial differentiation with respect to ζ_1. Similarly, the formulas on the right are applicable to the x_1, y_1. If one would then like to know the other derivatives (for example, the derivatives of x_1 and y_1 with respect to ζ_1) then one would need to take the displacement of A_1 into account and to add to the expressions for $\frac{\partial x_1}{\partial\zeta_1}$ and $\frac{\partial y_1}{\partial\zeta_1}$ that are obtained by applying the formulas on the left to the values of $\frac{\delta x_1}{\partial\zeta_1}$ and $\frac{\delta y_1}{\partial\zeta_1}$, resp; i.e., the cosine and sine, resp., of the inclination of the tangent to the curve at A_1 with respect to the x-axis.

e) The curve that gives rise to a barycentric pair of systems of orthogonal lines belongs to one of those systems. Indeed, it is obvious that if one fixes ζ_1 by way of any value of s then if one makes ζ_2 tend to s, one axis will tend to become tangent to the curve, while the other will tend to become normal. If one chooses the first one to be the x-axis then when one takes the relations:

$$\frac{\partial}{\partial\zeta_1} = \frac{\partial}{\partial s} - \frac{\partial}{\partial\sigma}, \qquad \frac{\partial}{\partial\zeta_2} = \frac{\partial}{\partial\sigma}$$

into account, one will see that in order to satisfy the previously-obtained formulas with a series that proceeds in powers of σ, one must take:

$$x_1 = -\frac{\sigma}{2} + \frac{\sigma^3}{48\rho^2} + \frac{\sigma^4}{120}\frac{d}{ds}\frac{1}{\rho^2} + \ldots, \qquad y_1 = \frac{\sigma^2}{12\rho} + \frac{\sigma^3}{30}\frac{d}{ds}\frac{1}{\rho} + \ldots,$$

$$x_2 = \frac{\sigma}{2} - \frac{\sigma^3}{48\rho^2} - \frac{\sigma^4}{80}\frac{d}{ds}\frac{1}{\rho^2} + \ldots, \qquad y_2 = \frac{\sigma^2}{12\rho} + \frac{\sigma^3}{20}\frac{d}{ds}\frac{1}{\rho} + \ldots$$

It is then easy to deduce that:

$$\tau = \frac{\sigma^2}{12\rho} + \frac{\sigma^4}{24}\frac{d}{ds}\frac{1}{\rho} + \ldots, \qquad \kappa = \frac{\sigma^2}{12\rho} - \frac{\sigma^4}{180\rho^2} + \ldots$$

These formulas facilitate the discussion of the geometric facts of the double system in the vicinity of the curve considered. In particular, for $\sigma = 0$, formulas (37) will give:

$$\mathcal{G}_1 = -\frac{1}{\rho}, \qquad \mathcal{G}_2 = -\frac{3}{5}\frac{d}{ds}\log\rho,$$

and will show that not every doubly-orthogonal system can be considered to barycentric with respect to some curve, since one would need to have that along one of the lines that comprise it (for example, along a certain line q_1), the curvature of the line q_2 would take the values:

$$\mathcal{G}_2 = \frac{3}{5}\frac{\partial}{\partial s_1}\log\mathcal{G}_1.$$

Furthermore, it is geometrically clear that the barycentric double systems are very special. For example, a double system of orthogonal circles is not generally barycentric, since it can be true for only one of its circumferences, while it is obvious that the double barycentric system of a circumference is composed of the concentric circumferences and the sheaf of common diameters.

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CHAPTER IX

SKEW CURVES AND RULED SURFACES

1. Fundamental trihedron. – The *tangent* to any curve in three-dimensional space at a given point M is defined (cf., I, § **1**) as it is for a plane curve. The *normals* at M are the infinitude of perpendiculars to the tangent that are raised from M, which will be in a plane that one calls the *normal plane*. One of the infinitude of normals is parallel to the polar axis, while the other one is perpendicular to it; the former is called the *binormal*, while the latter is the *principal normal.* One of the planes that pass through the tangent also contains the binormal, while another contains the principal normal; the former is called the *rectifying plane*, while the latter is the *osculating plane.* Since the polar axis can be considered to be the intersection of two infinitely-close normal planes, one can also say that the *binormal is perpendicular to two infinitely-close tangents.* With that, one would like to just briefly express the idea that the binormal at M is the limiting position of the common perpendicular to the tangents at M and M' when one fixes M and makes M' tend to M.

2. Curvature and intrinsic equations. – One puts the origin of the coordinates at the point M, which moves along a curve while constantly taking the tangent to be the x-axis, the binormal to be the y-axis, and the z-axis to be the principal normal. Consider the trihedron of the axes after the origin has been transferred from M to the infinitely-close position M'. From the observation that was made at the end of the preceding paragraph, it is clear that one will have $\cos(y, x') = 0$. It will then follow that the tangent x' and the binormal y' can be considered to be parallel to the planes zx and zy, respectively. Hence, if $\delta\varphi$ is the angle between two infinitely-close tangents and $\cos(z, x') = d\varphi$ and $\delta\psi$ is the angle between two infinitely-close binormals then one will have $\cos(z, y') = d\psi$, up to higher-order infinitesimals. The ratios of the differentials of φ and ψ to ds (i.e., the limits of the ratios of $\delta\varphi$ and $\delta\psi$ to the arc-length MM' when M' tends to M) measure the curvatures of the line considered at the point M, and the first one is distinguished by the name *flexion*, while the other bears the name of *torsion.* If one then sets $ds = \rho\, d\varphi = r\, d\psi$ then the numbers ρ and r, which are the inverses of the curvatures, measure two lengths that one can call the *radius of flexion* and the *radius of torsion.* The flexion of a skew curve, like that of the plane curve, then consists of the more or less rapid elongation of the curve of the tangents, while the torsion measures, more or less rapidly, the tendency of the curve to leave the osculating plane. Obviously, the plane curves are characterized by the fact that their torsion is zero. Having said that, in order to complete the picture of the direction cosines of the axes with their origin at M' with respect to the ones at the origin M, we would first like to observe that if one neglects higher-order infinitesimals then the cosines of the angles (x, x'), etc., can be regarded as being equal to unity, since one will have, for example:

$$\cos(x, x') = \cos\delta\varphi = 1 - \tfrac{1}{2}(\delta\varphi)^2 + \ldots, \qquad \text{etc.,}$$

so from the perpendicularity of the axes x' and y', x' and z', y' and z', one will have:

$$\cos(x, y') = 0, \qquad \cos(x, z') = -\,d\varphi, \qquad \cos(y, z') = -\,d\psi,$$

and one can then write out the table of the direction cosines in the following way:

$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & 1 & 0 & d\varphi \\ y' & 0 & 1 & d\psi \\ z' & -d\varphi & -d\psi & 1 \end{array} \tag{1}$$

This table shows that in order to discuss the curve in the vicinity of each point, it is enough to know the functions φ and ψ; i.e., it is enough to give ρ and r as functions of s. The equations:

$$f(s, \rho, r) = 0, \qquad g(s, \rho, r) = 0,$$

from which one can infer the values of the curvatures at any point of the curve are called the *intrinsic equations* of the curve. One soon sees that knowing them will also permit one to determine the form of the entire curve uniquely, up to the position that it occupies in space.

3. Fundamental formulas. – Let x, y, z (which are functions of s) be the coordinates of a point P, with respect to the trihedron at the origin M, where P generally moves with M, and let $x + \delta x$, $y + \delta y$, $z + \delta z$ be the coordinates of the point P' that corresponds with M' on the trajectory that P describes. The coordinates of P' with respect to the axes at the origin M' are $x + dx$, y, dy, $z + dz$, and therefore, if one takes the table (1) into account and lets u, v, w denote the (infinitesimal) coordinates of M' then:

$$\begin{aligned} x + \delta x &= u + x + dx - (z + dz)\, d\varphi, \\ y + \delta y &= v + y + dy - (z + dz)\, d\psi, \\ z + \delta z &= w + (x + dx)\, d\varphi + (y + dy)\, d\psi + z + dz\,. \end{aligned} \tag{2}$$

We would like to confine our study to the curves for which it is legitimate to assert, as we can for plane curves, that the limit of the ratio of the arc MM' to the chord when M' tends to M is equal to unity. Since we have:

$$\lim \frac{u}{\sqrt{u^2+v^2+w^2}} = 1, \qquad \lim \frac{v}{\sqrt{u^2+v^2+w^2}} = 0, \qquad \lim \frac{w}{\sqrt{u^2+v^2+w^2}} = 0,$$

by the definition of the tangents, and on the other hand, the hypothesis that was just made will translate into:

$$\lim \frac{\delta s}{\sqrt{u^2+v^2+w^2}} = 1,$$

we will also have:

$$\lim \frac{u}{\delta s} = 1, \quad \lim \frac{v}{\delta s} = 0, \quad \lim \frac{w}{\delta s} = 0.$$

We can therefore suppress v and w in (2) and substitute ds for u; dividing by ds will then give the *fundamental formulas:*

$$\frac{\delta x}{ds} = \frac{dx}{ds} - \frac{z}{\rho} + 1, \qquad \frac{\delta y}{ds} = \frac{dy}{ds} - \frac{z}{r}, \qquad \frac{\delta z}{ds} = \frac{dz}{ds} + \frac{x}{\rho} + \frac{y}{r}. \tag{3}$$

Equations (2) are also applicable to the cosines α, β, γ, which define an arbitrary direction, provided that we suppress u, v, w. It will then follow that we have:

$$\frac{\delta\alpha}{ds} = \frac{d\alpha}{ds} - \frac{\gamma}{\rho}, \qquad \frac{\delta\beta}{ds} = \frac{d\beta}{ds} - \frac{\gamma}{r}, \qquad \frac{\delta\gamma}{ds} = \frac{d\gamma}{ds} + \frac{\alpha}{\rho} + \frac{\beta}{r} \tag{4}$$

for the direction. We finally seek the fundamental formulas that relate to the line. For the coordinates of a straight line, we can assume that its direction cosines α, β, γ and the quantities:

$$\xi = \gamma y - \beta z, \quad \eta = \alpha z - \gamma x, \quad \zeta = \beta x - \alpha y \tag{5}$$

are obviously coupled with the cosines by the identity relation:

$$\alpha\xi + \beta\eta + \gamma\zeta = 0 \tag{6}$$

and are independent of the position of the point (x, y, z) along the line, since they will remain unaltered when x, y, z are replaced with $x + \alpha t$, $y + \beta t$, $z + \gamma t$ for any t. It is now sufficient to apply formulas (3) and (4) to (6) in order to get:

$$\frac{\delta\xi}{ds} = \frac{d\xi}{ds} - \frac{\zeta}{\rho}, \qquad \frac{\delta\eta}{ds} = \frac{d\eta}{ds} - \frac{\zeta}{r} - \gamma, \qquad \frac{\delta\zeta}{ds} = \frac{d\zeta}{ds} + \frac{\xi}{\rho} + \frac{\eta}{r} + \beta. \tag{7}$$

Together with (4), these are the fundamental formulas for the line.

4. – It results from (3) that *in order for the point* (x, y, z) *to be immobile, it is necessary and sufficient that one should have:*

$$\frac{dx}{ds} = \frac{z}{\rho} - 1, \qquad \frac{dy}{ds} = \frac{z}{r}, \qquad \frac{dz}{ds} = -\frac{x}{\rho} - \frac{y}{r}. \tag{8}$$

Fix the origin of the arc length at an arbitrary point of the curve at which the curvatures (which we always suppose to be continuous functions of the arc length) have finite values, and let x, y, z be its coordinates with respect to the fundamental trihedron at an infinitely-close point. Obviously x, y, z are infinitesimals relative to s, and since the conditions (8) must be satisfied, one will have:

$$\lim \frac{x}{s} = \lim \left(\frac{z}{\rho} - 1\right) = -1, \qquad \lim \frac{y}{s} = \lim \frac{z}{r} = 0,$$

upon applying l'Hôpital's theorem. It will then follow that:

$$\lim \frac{z}{s^2} = -\frac{1}{2} \lim \frac{1}{s}\left(\frac{x}{\rho} + \frac{y}{r}\right) = -\frac{1}{2\rho} \lim \frac{x}{s} = \frac{1}{2\rho}.$$

Now, since transferring the origin of the arc length to the point M' $(s + ds)$, which is infinitely close to M, is equivalent to setting $s + ds = 0$, the expressions for the coordinates u, v, w of M' with respect to the axes at the origin M will be obtained by changing s into $-ds$ in the preceding results, in such a way that one has:

$$u = ds, \qquad v = -\frac{ds^3}{6\rho r}, \qquad w = \frac{ds^2}{2\rho}. \tag{9}$$

Hence, the planes that contain the tangent at M are characterized from all of the ones that pass through M by the fact that their distances to the points that are infinitely-close to M are higher-order infinitesimals, and *for just one of them* (viz., the osculating plane) *that distance will be infinitesimal of order at least three.* It will then follow that any sufficiently-small arc that is taken around M will be situated on the same part of any plane that goes through the tangent at M, and the exception is the osculating plane, which *crosses the curve*, in general. When one drops third-order infinitesimals, one can suppose that M' is situated in the osculating plane, and if one also drops those of second-order then one can even consider M' to be situated on the tangent. Consequently, in those questions for which it is proper to drop the higher-order infinitesimals, it as also proper to represent the curve with a polygonal line $MM'M''$, … with infinitely-small edges and consider the tangent to be the line on which an *element* MM' lies, the osculating plane to be the plane that is determined by two *consecutive elements* MM' and $M'M''$, etc. If the fundamental trihedron passes from one position to the successive one then the vertex on the tangent and the edges can rotate around the vertex in such a way they will acquire the direction cosines (1). It will finally result from (9) that if an observer moves along the curve while keeping his head on the positive part of the normal and proceeds in the sense of increasing s then he will see the curve *rise* or *fall* according to whether ρ is *positive* or *negative*, resp., and he will see it turn to the *right* or *left* according to whether r has the same sign as ρ or the opposite sign, resp.

5. – One can now prove that *any pair of intrinsic equations will determine a unique curve*, at least within convenient limits for s, inside of which the curvatures are finite and continuous functions of the arc length. It is known that under those conditions there will always exist a triple of functions x, y, z, and only one of them that satisfies (8) and reduces to a, b, c for $s = 0$. On the other hand, by virtue of (4), *the necessary and sufficient conditions for the invariability of the direction* (α, β, γ) *are:*

$$\frac{d\alpha}{ds} = \frac{\gamma}{\rho}, \qquad \frac{d\beta}{ds} = \frac{\gamma}{r}, \qquad \frac{d\gamma}{ds} = -\frac{\alpha}{\rho} - \frac{\beta}{r}. \tag{10}$$

It is known that if α, β, γ and α', β', γ' are two arbitrary triples of functions that satisfy (1) then one will get, in succession:

$$\frac{d}{ds}(\alpha\alpha' + \beta\beta' + \gamma\gamma') = 0, \qquad \alpha\alpha' + \beta\beta' + \gamma\gamma' = \text{constant}$$

by virtue of (10). It will then follow that if one determines the three triples of functions that are contained in the first of the matrices:

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

in such a way that (10) is satisfied and they take on the homologous values that are depicted in the second matrix then one will constantly have:

$$\alpha_i\,\alpha_j + \beta_i\,\beta_j + \gamma_i\,\gamma_j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases};$$

i.e., the determinant that represents the first matrix is orthogonal. It is now clear that the elements of that determinant are precisely the direction cosines of the axes with their origin at O with respect to the ones with their origin at M, because they give the directions of the aforementioned axes for $s = 0$, and on the other hand, satisfying the conditions (10) is sufficient for the invariability of those directions. Having said that, it is always proper to set:

$$x = x_0 + a\alpha_1 + b\alpha_2 + c\alpha_3, \qquad y = y_0 + a\beta_1 + b\beta_2 + c\beta_3, \qquad z = z_0 + a\gamma_1 + b\gamma_2 + c\gamma_3,$$

and one will then see that if one substitutes these values in (8) and takes (10) into account then the new unknown functions x_0, y_0, z_0 will satisfy (8) and that they must all reduce to zero with s. It is therefore plainly determinate, and it is clear that they represent the coordinates of O with respect to the axes with their origin at M. For each value of s, one can now determine the constants a, b, c in such a way that they represent the coordinates of M with respect to the axes with their origin at O. It is enough (if one imagines the instant that M passes through a fixed point) to set x¸ y, z equal to zero in the preceding relations and solve the system with respect to a¸ b, c:

$$\left.\begin{aligned} a &= -(\alpha_1 x_0 + \beta_1 y_0 + \gamma_1 z_0), \\ b &= -(\alpha_2 x_0 + \beta_2 y_0 + \gamma_3 z_0), \\ c &= -(\alpha_3 x_0 + \beta_2 y_0 + \gamma_3 z_0). \end{aligned}\right\} \tag{11}$$

Hence, the curve that is defined by the given pair of intrinsic equations will be determined uniquely, since the coordinates a, b, c of all of its points with respect to a triple of immobile axes will be known as s varies.

6. – It is interesting to observe that one can rapidly deduce all of the usual theorems of skew curves from (11) by means of the immobility conditions (8) and (10). Indeed, upon differentiating the former equations, one will find that:

$$\frac{da}{ds} = \alpha_1 , \qquad \frac{db}{ds} = \alpha_2 , \qquad \frac{dc}{ds} = \alpha_3 ; \tag{12}$$

hence:

$$\frac{d^2a}{ds^2} = \frac{\gamma_1}{\rho}, \qquad \frac{d^2b}{ds^2} = \frac{\gamma_2}{\rho}, \qquad \frac{d^2c}{ds^2} = \frac{\gamma_3}{\rho}, \tag{13}$$

and another differentiation will give:

$$\frac{d^3a}{ds^3} = \gamma_1 \frac{d}{ds}\frac{1}{\rho} - \frac{\alpha_1}{\rho^2} - \frac{\beta_1}{\rho r}, \qquad \text{etc.} \tag{14}$$

Thus, *when one makes use of immobile axes,* the *first* derivatives of the coordinates with respect to the arc length will be equal to the direction cosines of the *tangent*, and the *second* derivatives will be proportional to the direction cosines of the *principal normal.* For the binormal, one has:

$$\beta_1 = \alpha_3 \gamma_2 - \alpha_2 \gamma_3 = \left(\frac{dc}{ds}\frac{d^2b}{ds^2} - \frac{db}{ds}\frac{d^2c}{ds^2}\right)\rho, \qquad \text{etc.} \tag{15}$$

If one squares and sums (12) then one will get the formula:

$$ds^2 = da^2 + db^2 + dc^2,$$

which permits one to calculate the *arc length* when the coordinates are given as functions of an arbitrary variable. However, if one squares and sums (13) then one will find the formula:

$$\frac{1}{\rho^2} = \left(\frac{d^2a}{ds^2}\right)^2 + \left(\frac{d^2b}{ds^2}\right)^2 + \left(\frac{d^2b}{ds^2}\right)^2,$$

which will give one the *flexion* when s is the independent variable. One operates analogously on (15) with the general formula (i.e., the one that is not linked by a choice of independent variable). If (14) is then multiplied by β_1, β_2, β_3 and summed then if one observes (15), one will get:

$$\frac{1}{r\rho^2} = \begin{vmatrix} \dfrac{da}{ds} & \dfrac{d^2a}{ds^2} & \dfrac{d^3a}{ds^3} \\ \dfrac{db}{ds} & \dfrac{d^2b}{ds^2} & \dfrac{d^3b}{ds^3} \\ \dfrac{dc}{ds} & \dfrac{d^2c}{ds^2} & \dfrac{d^3c}{ds^3} \end{vmatrix}.$$

That formula will help one calculate the *torsion* when one already knows the flexion.

7. Digression on lines:

a) It is known that the angle θ between two directions (α, β, γ) and $(\alpha', \beta', \gamma')$ is given by either of the following formulas:

$$\cos\theta = \alpha\,\alpha' + \beta\,\beta' + \gamma\gamma', \qquad \sin^2\theta = (\beta\gamma' - \gamma\beta')^2 + (\gamma\,\alpha' - \alpha\gamma')^2 + (\alpha\beta' - \beta\alpha')^2.$$

If the directions (α, β, γ) and $(\alpha + \delta\alpha, \beta + \delta\beta, \gamma + \delta\gamma)$ define the infinitesimal angle $\delta\theta$ then the second formula will give:

$$\delta\theta^2 = (\beta\,\delta\gamma - \gamma\,\delta\beta)^2 + (\gamma\,\delta\alpha - \alpha\,\delta\gamma)^2 + (\alpha\,\delta\beta - \beta\,\delta\alpha)^2. \tag{16}$$

However, one can deduce from the first one that:

$$1 - \tfrac{1}{2}\delta\theta^2 + \ldots = \sum \alpha(\alpha + \delta\alpha) = 1 + \sum \alpha\,\delta\alpha = 1 - \tfrac{1}{2}\sum \delta\alpha^2;$$

i.e.:

$$\delta\theta^2 = \delta\alpha^2 + \delta\beta^2 + \delta\gamma^2. \tag{17}$$

b) The cosines λ, μ, ν that determine the direction that is perpendicular to the ones α, β, γ and α', β', γ' satisfy the conditions of perpendicularity $\sum \lambda\alpha = 0$, $\sum \lambda\alpha' = 0$, from which one infers that:

$$\frac{\lambda}{\beta\gamma' - \gamma\beta'} = \frac{\mu}{\gamma\alpha' - \alpha\gamma'} = \frac{\nu}{\alpha\beta' - \beta\alpha'} = \frac{1}{\sin\theta}, \tag{18}$$

if one fixes the positive sense of the direction in such a way that it will coincide with the sense of the z-axis when the given directions coincide with those of the x and y axes, respectively. Obviously, the distance q between two lines is the projection onto a common perpendicular of the rectilinear segment that joins an arbitrary point (x, y, z) of one line to a point (x', y', z') of the other one. It will then follow that:

$$q = \lambda (x' - x) + \mu (y' - y) + \nu (z' - z) \;;$$

i.e., from (18):

$$q \sin \theta = \begin{vmatrix} \alpha & \alpha' & x' - x \\ \beta & \beta' & y' - y \\ \gamma & \gamma' & z' - z \end{vmatrix}, \tag{19}$$

in which the right-hand side can be given the form:

$$\begin{vmatrix} \alpha & \alpha' & x' \\ \beta & \beta' & y' \\ \gamma & \gamma' & z' \end{vmatrix} + \begin{vmatrix} \alpha' & \alpha & x \\ \beta' & \beta & y \\ \gamma' & \gamma & z \end{vmatrix} = -\sum (\alpha \xi' + \alpha' \xi),$$

in such a way that when one recalls the relation (6), one can also write:

$$q \sin \theta = \sum (\alpha' - \alpha)(\xi' - \xi).$$

Therefore, if the two lines are infinitely close and δq represents the distance between them then:

$$\delta q \; \delta\theta = \delta\alpha \; \delta\xi + \delta\beta \; \delta\eta + \delta\gamma \; \delta\zeta. \tag{20}$$

Returning to (19), one observes that if one of the lines is the x-axis then one will have $-q \sin \theta = \xi$, and one will obtain the geometric interpretation of the coordinates ξ, η, ζ in that way.

c) The lines in space are quadruply-infinite in number, since one will always have two relations between the six coordinates of a line. Any other constraint that is imposed upon those coordinates will serve to cut out a triple infinitude of lines from the space that one calls a *complex*. A pair of distinct equations defines a *congruence*, or a *double infinitude of lines*, which one can then consider to be the intersection of two complexes, and it is the analogue of a *surface*, or *double infinitude of points.* Finally, a set of three distinct equations represents a simple infinitude of lines, which obviously constitutes a particular surface that one calls a *ruled surface.* We shall address the ruled surfaces and congruences in what follows. We would like to confine ourselves to exposing some of the simplest complexes, which are represented by a linear equation:

$$a\, \xi + b\, \eta + c\, \zeta + l\, \alpha + m\, \beta + n\, \gamma = 0, \tag{21}$$

and are called *linear complexes* for that reason. Such a complex is called *special* when the coefficients are subject to the constraint:

$$al + bm + cn = 0, \tag{22}$$

by virtue of which, it is proper to consider those coefficients to be proportional to the coordinates of a line, in such a way that (21) will then say that the line will meet all of the ones that belong to the complex. Hence, *the special complex is composed of the infinitude of lines that meet a given line.* That line might be infinitely distant, and that will happen when a, b, c are zero. By virtue of (21), the lines of the complex will all be perpendicular to the direction that is defined by cosines that are proportional to l, m, n (i.e., they are all the lines of a sheaf of parallel planes), and one can therefore indeed say that they meet the common intersections of those planes. For a general linear complex, i.e., when (22) does not exist, it is not possible that a, b, c will all be zero at once, and one can then always consider the direction to be defined by the cosines:

$$\alpha_0 = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad \beta_0 = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad \gamma_0 = \frac{c}{\sqrt{a^2+b^2+c^2}}, \tag{23}$$

and one can assume that a certain line is in that direction whose other coordinates ξ_0, η_0, ζ_0 one would like to determine opportunely. If one observes (21) then one will have from (19) that:

$$-q \sin\theta = \sum \alpha\,\xi_0 + \frac{a\,\xi + b\,\eta + c\,\zeta}{\sqrt{a^2+b^2+c^2}} = \sum \alpha \left(\xi_0 - \frac{l}{\sqrt{a^2+b^2+c^2}}\right), \tag{24}$$

and the right-hand side will reduce to $-p\cos\theta$ when one sets:

$$\xi_0 = \frac{l-ap}{\sqrt{a^2+b^2+c^2}}, \qquad \eta_0 = \frac{m-bp}{\sqrt{a^2+b^2+c^2}}, \qquad \zeta_0 = \frac{n-cp}{\sqrt{a^2+b^2+c^2}}. \tag{25}$$

One then determines the constant p by observing that the relation (6) must be true:

$$a\,(l - ap) + b\,(m - bp) + c\,(n - cp) = 0.$$

It will then follow that:

$$p = \frac{al+bm+cn}{a^2+b^2+c^2}, \tag{26}$$

and the equality (24) will finally reduce to the simpler form:

$$q\tan\theta = p. \tag{27}$$

The line that is defined by the coordinates (23) and (25) is called the *axis* of the complex, and the preceding considerations will show that any linear complex is composed of the lines for which the relation (27) exists between distance to a certain axis and the angle that it makes with each line of the complex.

d) The *bilinear congruences* – or intersections of two linear complexes $\kappa = 0$, $\kappa' = 0$ – also belong to the infinitude of complexes that comprise the sheaf $\kappa + \lambda\,\kappa' = 0$. If one writes out that the condition (22) is satisfied by the coefficients of $\kappa + \lambda\,\kappa'$ then one will find a second-degree equation in λ, and therefore two special complexes will always exist in the that sheaf. Hence, *any linear congruence is composed of the lines that rest upon two certain lines in space.* However, if considers the net of complexes $\kappa + \lambda\,\kappa' + \mu\kappa'' = 0$ in the sheaf then the condition (22) will lead to a relation between *l* and *m*, by virtue of which there will always exist an infinitude of special complexes in that net. Hence, the ruled surface that is the intersection of the complexes $\kappa = 0$, $\kappa' = 0$, $\kappa'' = 0$ will also belong to an infinitude of special complexes; i.e., the lines that comprise it will meet an infinitude of other lines, which can likewise be considered to be generators of the surface. It will then follow that a surface like that can be generated in two very different ways by one line that will successively occupy a simple infinitude of positions in space. That special ruled surface is called a *quadric*, and from the nature of its properties, which are dealt with at length in ordinary analytic geometry, it occupies the same place in the geometry of space that the conic (III, § **1**) does in the geometry of the plane.

e) It is particular noteworthy got us that among the quadrics the *hyperbolic paraboloid* – i.e., the surface that is generated by a line that moves parallel to a plane – will be supported by two given lines. That surface can then be considered to be the intersection of three special complexes:

$$\sum \alpha\,\alpha_0 = 0, \quad \sum (\alpha'\,\xi + \alpha\,\xi') = 0, \quad \sum (\alpha''\,\xi + \alpha\,\xi'') = 0,$$

and therefore its generators (from one system) will also belong to any complex of lines:

$$\sum [(\lambda\alpha' + \mu\alpha'')\,\xi + (\alpha_0 + \lambda\xi' + \mu\xi'')\,\alpha] = 0. \tag{28}$$

The generators of the second system are the axes of the special complexes that are represented by the equation (28) when one establishes a convenient relation between λ and μ ; i.e. :

$$\lambda \sum \alpha'\,\alpha_0 + \mu \sum \alpha''\,\alpha_0 + \lambda\mu \sum (\alpha'\,\xi'' + \alpha''\xi') = 0. \tag{29}$$

Now, since those generators have directions cosines that are equal to linear combinations of cosines α', β', γ' with α'', β'', γ'', one will see that *they are also parallel to a certain plane, like those of the first system.* Therefore, the surface will be generated in the same way as the lines of one or the other system. However, one can generally pose the relation:

$$\lambda \sum \alpha'\,\alpha_0 + \mu \sum \alpha''\,\alpha_0 = 0$$

between λ and μ, which is generally distinct from (29) when the lines that were given to begin with do not intersect, which one supposes when the surface is not planar. The equation (28) will then represent an infinitude of non-special linear complexes, whose axes are parallel to the director plane. Even better, that plane can be fixed in such a way that it contains one of the axes. Hence, the relation (27) must be satisfied, so the generator that is situated in the plane will meet the axis at a right angle, and that situation shows clearly that all of the other axes will necessarily lie in the same plane. If one fixes two *director planes* in that way then their intersection will be called the *axis* of the paraboloid, and the two generators that meet the axis orthogonally will be called *principal*, in order to distinguish them from the other ones is their respective systems. They must meet, precisely because they do not belong to the same system, and their common point will necessarily be on the axis, and it is called the *vertex* of the paraboloid.

f) For an arbitrary generator that is situated at a distance t from the director plane that it relates to, if τ is, in addition, the angle that one finds between that generator and the principle direction then one will have:

$$\tan \tau = \frac{t}{p}, \tag{30}$$

by virtue of (27), since the angle θ is the complement of τ in the case that is being considered. Hence, *the hyperbolic paraboloid can be generated by a line that moves parallel to a plane that rests upon a fixed line and rotates around it according to the law* (30). The fixed line OM is, if one so desires, the principal generator of the second system. Its projection OP onto the first director plane will be, consequently, the principal generator of the first system. Now, if one raises the perpendicular at P, which is the projection of M, to the generator that passes through M, and similarly the perpendicular to OP from the vertex O, then the two lines thus-constructed *will concur at a fixed point F*. Indeed, the consideration of the rectangular triangles OPF and MPF and the equality (30) will give:

$$OF = OP \cdot \cos \tau = t \cot \varphi \cot \tau = p \cot \varphi,$$

if one calls the inclination of OM with respect to the fixed plane φ. Hence (III, § **9**, *c*), *the projections of the generators onto the director plane envelop a parabola* whose focus is at F and whose vertex is at O. Nonetheless, it is known that the envelope of those projections will be the vertex when the principal generators are perpendicular, in which case, the paraboloid will be called *equilateral*. However, if one makes the constants φ and p go to zero in such a way that $p \cot \varphi$ will remain equal to a non-zero constant then the two systems of generators will tend to coincide, and *the paraboloid will tend to degenerate into the system of tangents to a parabola.*

g) In conclusion, observe that the significance of p will result from (30), which gives the equalities:

$$p = \lim_{\tau=0} \frac{t}{\tau}, \qquad p' = \lim_{\tau'=0} \frac{t'}{\tau'}, \tag{31}$$

which relate to the two systems of generators. It is understood that for both systems, one must compute the angles τ and τ' by starting from the principal generators. For example, if the direction $(\alpha', \beta', \gamma')$ is the one that one obtains by projecting $(\alpha_0, \beta_0, \gamma_0)$ onto the second director plane then one will have:

$$\sum \alpha' \alpha_0 = \sin \varphi, \qquad \sum \alpha'' \alpha_0 = \sin \varphi \cos \tau', \qquad \sum \alpha' \alpha'' = \cos \tau',$$

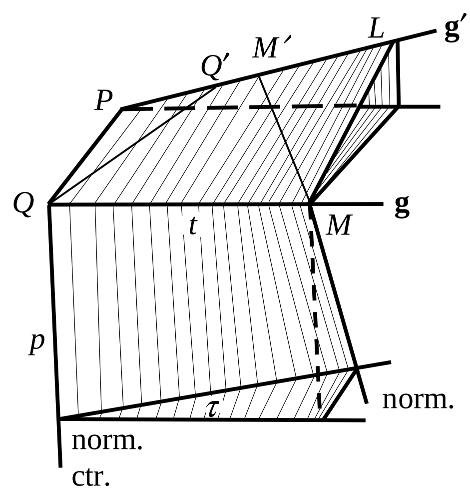
and one must take $\lambda = -\mu \cos \tau'$, in such a way that one will come to the following result:

$$p = -\frac{\lambda\mu t' \sin \tau'}{\lambda^2 + \mu^2 + 2\lambda\mu \cos \tau} = \frac{t'}{\tan \tau'} = p',$$

thanks to formula (26). It is therefore irrelevant whether one takes one or the other system of generators in order to know the value of the constant p. Since one can then say the same thing about the angle φ, it will then result that the two parabolas that are enveloped by the projections of the generators onto the respective director planes will be equal.

h) Various noteworthy surfaces can be generated analogously to the way that hyperbolic paraboloids are generated. If a point moves along a fixed line and one raises a perpendicular to that point in such a way that when the point moves to t, the perpendicular will rotate through an angle τ that is coupled with t by the relation (30) then one will see that the moving line generates an equilateral paraboloid. However, if one substitutes τ for $\tan \tau$ in the left-hand side of (30) then the line will generate another noteworthy surface that one calls a *ruled helicoid with a director plane.* If one puts $\sin 2\tau$ in place of $\tan \tau$ or τ then the surface that is generated will be called a *Plücker cylindroid or conoid,* and it is the locus of the axes of the linear complexes of an arbitrary sheaf.

8. Ruled surfaces. – A ruled surface can be considered (cf., § **4**) to be a simply-infinite succession of *surface elements*, each of which is the strip of the surface that is found between two infinitely-close generators **g** and **g′**. One says that the ruled surface is a *developable* of the planar elements; i.e., the ones for which it is proper to consider **g** and **g′** to be situated in a plane, up to higher-order infinitesimals. That can happen either because the distance δq between the two lines is infinitesimal with respect to their angle $\delta\theta$ or because one constantly has $\delta\theta = 0$; that is to say, the generators are all parallel, in which case the surface will be called a *cylinder*. The point of **g** that is on the common perpendicular to **g** and **g′** will move along with **g′** when one fixes **g** and makes **g′** tend to **g**, and it might be true that it tends to a point Q that is fixed on **g** that is called the *central point* of **g**. The locus of central points is called the *edge of regression* when the surface is developable, and in particular, it is clear that for the cylindrical surfaces, the edge of regression will be completely at infinity. For the other ruled surfaces, which are called *skew*, the locus of central points is called the *line of striction*, instead. If the ratio $\delta q : \delta\theta$ then tends to a limit p when **g′** tends to the fixed position **g** then one will give to p the name of *distributor parameter* along the generator **g** . Obviously, the only ruled surfaces for which one has $p = 0$ on all of the generators are the non-cylindrical developables; however, p is infinite for the cylindrical ones. Now, one traces out an infinitude of curves that meet **g′** at M', M'', … through a point M that is fixed on **g**. When **g′** tends to **g**, the lines MM', MM'', … will tend to touch the aforementioned curve at M, and therefore *the tangents at M to all of the curves in the surface that pass through M will be in a plane* that one calls the *tangent plane* at M, and is the limiting position of the plane that is determined by the point M and the line **g′**. The *normal* to the surface at M is the perpendicular to the tangent plane that goes through M. Having said that, observe that the tangent to all of the curves in the surface along the generator **g** define a linear congruence whose limit is composed of the lines that lie upon **g** and **g′**. In that congruence, the lines that are perpendicular to **g** will then form a hyperbolic paraboloid, and therefore their orientation will be governed by the law (30), in which, by virtue of (31), p is properly the distribution parameter, and t represents the distance QM. Hence, if one knows the plane that touches a skew ruled surface at a point Q of the line of striction then one will also know the tangent plane to any other point M of the generator that passes through A by means of the equality (30). That is the *law of distribution of tangent planes* that was discovered by **Chasles**. If M traverses **g** and goes off indefinitely to Q in one sense or the other then the tangent plane will tend to arrange itself perpendicularly to the central position. The developables behave quite differently. Indeed, when p is zero, one will have $\tau = \pi / 2$, and all of the tangent planes along **g** will coincide with one of them, and that will obviously happen for the cylindrical surfaces, as well. Therefore, among the ruled surfaces, *the developables are characterized by the fact that the tangent plane at each point M will touch the surface along the generator that passes through M.* Finally, observe that it is enough to rotate all of the tangents that are perpendicular to **g** around **g** through $\pi / 2$ in order to make them become normals to the surface, and to see in that way that *the normals to a ruled surface along a generator define a hyperbolic paraboloid.* It will reduce to a system of parallel lines only in the case of developables.



9. Fundamental formulas. – When the distributor parameter of a ruled surface is referred to the fundamental trihedron of an arbitrary curve in space, it will be easy to express it in terms of the variations of the coordinates α, β, γ, ξ, η, ζ of the generators. Indeed, if one divides (20) by (17) then one will obtain:

$$p = \frac{\delta\alpha\,\delta\xi + \delta\beta\,\delta\eta + \delta\gamma\,\delta\zeta}{\delta\alpha^2 + \delta\beta^2 + \delta\gamma^2}. \tag{32}$$

If the curve is chosen from the ones that belong to the surface then ξ, η, ζ will be constantly zero, and formulas (7) will give:

$$\delta\xi = 0, \qquad \delta\eta = -\gamma\,ds, \qquad \delta\zeta = \beta\,ds;$$

hence, if one takes $p\ \delta\theta$ for δq then one will get the (obvious) equivalence:

$$\frac{\delta q}{ds} = \beta\frac{\delta\gamma}{\delta\theta} - \gamma\frac{\delta\beta}{\delta\theta}. \tag{33}$$

The normal to the surface at the central point is perpendicular to **g** and to the perpendicular that is common to **g** and **g′**, and it is defined by direction cosines that are proportional to:

$$\beta\,\delta\gamma - \gamma\,\delta\beta, \qquad \gamma\,\delta\alpha - \alpha\,\delta\gamma, \qquad \alpha\,\delta\beta - \beta\,\delta\alpha.$$

The direction cosines are then proportional to a triple of quantities, the first of which is:

$$(\gamma\,\delta\alpha - \alpha\,\delta\gamma)\,\gamma - (\alpha\,\delta\beta - \beta\,\delta\alpha)\,\beta = \delta\alpha + \tfrac{1}{2}\,\delta\theta^2,$$

namely, $\delta\alpha$, if one ignores the higher-order infinitesimals. Hence, if one takes (17) into account then one will see that *the direction of the central normal is defined by the cosines:*

$$\frac{\delta\alpha}{\delta\theta}, \quad \frac{\delta\beta}{\delta\theta}, \quad \frac{\delta\gamma}{\delta\theta}.$$

Having said that, if one observes that the trihedron that is composed of the parallel to the central normal that goes through M, the tangent to the fundamental curve at the point M, and the perpendicular to **g** through the tangent plane that is raised from M is the rectangle along the third edge then one will immediately find the first of the equalities:

$$\frac{\delta\alpha}{\delta\theta} = \sqrt{\beta^2 + \gamma^2}\ \sin\tau, \qquad \frac{\delta q}{ds} = \sqrt{\beta^2 + \gamma^2}\ \cos\tau; \tag{34}$$

however, the other one is obtained by directing the first edge perpendicular to **g** and **g′**. Meanwhile, the first equality will become:

$$\frac{\delta\alpha}{ds} = \frac{\sqrt{\beta^2 + \gamma^2}}{p}\frac{\delta q}{ds}\sin\tau = \frac{\beta^2 + \gamma^2}{p}\sin\tau\cos\tau, \tag{35}$$

thanks to the second. On the other hand, if one goes back to writing out (33) then one will find:

$$\beta\frac{\delta\gamma}{ds}-\gamma\frac{\delta\beta}{ds}=\frac{1}{p}\left(\frac{\delta q}{ds}\right)^2=\frac{\beta^2+\gamma^2}{p}\cos^2\tau,$$

and one will have just:

$$\beta\frac{\delta\beta}{ds}+\gamma\frac{\delta\gamma}{ds}=-\alpha\frac{\delta\alpha}{ds}=-\frac{\beta^2+\gamma^2}{p}\alpha\sin\tau\cos\tau.$$

Hence:

$$\frac{\delta\beta}{ds}=(-\gamma\cos\tau-\alpha\beta\sin\tau)\frac{\cos\tau}{p}, \quad \frac{\delta\gamma}{ds}=(\beta\cos\tau-\alpha\gamma\sin\tau)\frac{\cos\tau}{p}. \tag{36}$$

Finally, it is enough to set:

$$\frac{1}{r}-\frac{\cos^2\gamma}{p}=\frac{1}{\tau},$$

and substitute the results (35) and (36) in formula (4) to find the *conditions:*

$$\left.\begin{aligned}\frac{d\alpha}{ds}&=\frac{\gamma}{\rho}+\frac{\beta^2+\gamma^2}{2p}\sin 2\tau,\\ \frac{d\beta}{ds}&=\frac{\gamma}{\tau}-\frac{\alpha\beta}{2p}\sin 2\tau,\\ \frac{d\gamma}{ds}&=-\frac{\alpha}{\rho}-\frac{\beta}{\tau}-\frac{\alpha\gamma}{2p}\sin 2\tau,\end{aligned}\right\} \tag{37}$$

which are necessary and sufficient because the functions α, β, γ represent the direction cosines of the generator of a ruled surface with respect to the fundamental trihedron of a curve that is traced out on the surface.

10. – The line of striction is characterized by the condition $t = 0$ or $\tau = 0$, and also by $\delta\alpha = 0$, by virtue of the first equality (34); i.e.:

$$\frac{d\alpha}{ds}=\frac{\gamma}{\rho}.$$

One sees from this that if $\gamma = 0$ then α will be constant, and conversely. We now need to know that what one calls a *geodetic* on an arbitrary surface is a curve whose principal normal at any point is the normal to the surface. Having said that, the last observation permits one to assert, with **Bonnet**, that *if the line of striction is a geodetic then it will meet the generators at a constant angle,* and conversely, *if the line of striction of a ruled surface meets the generators at a constant angle then it will necessarily be a geodetic of* *that surface.* If one then has $\gamma = 0$ and α constant at the same time then one will also have $\delta\alpha = 0$, and therefore *the line of striction is the unique line that can be geodetic and meet all of the generators at a constant angle.* When that line is assumed to be a fundamental curve, formulas (37) will become:

$$\frac{d\alpha}{ds} = \frac{\gamma}{\rho}, \qquad \frac{d\beta}{ds} = \frac{\gamma}{\tau}, \qquad \frac{d\gamma}{ds} = -\frac{\alpha}{\rho} - \frac{\beta}{\tau},$$

which express the idea that the direction (α, β, γ) is invariable with respect to another curve that has the same flexion as the line of striction and a torsion that is equal to $1 / \tau$. In other words, if *one twists* the given line (by means of successive infinitesimal rotations of the osculating plane around the tangents) in such a way that the torsion passes from the value $1 / r$ to $1 / t$, while the flexion of the arc remains invariant, then if the generators are dragged along by the motion, they will remain rigidly coupled with the respective trihedra and conclude by becoming parallel; i.e., the surface will be transformed into a cylinder.

11. Developables. – For developable surfaces, formulas (37) will become:

$$\frac{d\alpha}{ds} = \frac{\gamma}{\rho} + \frac{\beta^2 + \gamma^2}{t}, \qquad \frac{d\beta}{ds} = \frac{\gamma}{\tau} - \frac{\alpha\beta}{t}, \qquad \frac{d\gamma}{ds} = -\frac{\alpha}{\rho} - \frac{\beta}{\tau} - \frac{\alpha\gamma}{t}. \tag{38}$$

However, in order for them to persist when t tends to zero – i.e., (excluding the cylinders) when the edge of regression tends to take the form of a fundamental curve, it is necessary that β and γ must be zero. Hence, *any non-cylindrical developable is the locus of tangents to a skew curve.* In order to account for that in a more direct way, we would first like to observe that the distance δq, when it is not infinitesimal like $\delta\theta$, will be infinitesimal like at least $\delta\theta^2$. Indeed, if one assumes that θ is the independent variable and one returns to writing out formula (20), while recalling that:

$$\delta = d + \tfrac{1}{2} d^2 + \tfrac{1}{6} d^3 + \ldots,$$

then one will get:

$$\delta q \, d\theta = \left(1 + \tfrac{1}{2} d + \tfrac{1}{6} d^2\right) \sum d\alpha \, d\xi - \tfrac{1}{12} \sum d^2\alpha \, d^2\xi + \ldots,$$

and one will see immediately that the right-hand side, from which the infinitesimals of order higher than four have been dropped, is precisely of at least that order, if it is not of order two; that is to say, δq has order three if it does not have order one. Having said that, if one draws a plane through **g** that is parallel to **g′** then it will be clear that the distance from the central point Q' of **g′** to that plane will also be infinitesimal of order at least three, and therefore (§ **4**) the plane thus-constructed will be precisely the one that osculates the edge of regression at Q. In order to prove that **g** is the tangent to that edge of regression, it is enough to see that the distance from **g** to the projection of Q' onto the osculating plane is infinitesimal of higher order, and since that distance is equal to the product of QQ' with an infinitesimal angle, the theorem is thus proved. It is now clear that *the tangent planes to a developable surface are the osculating planes of the edge of regression.*

12. – It is important to observe that any simply-infinite continuum of planes **P** will osculate a curve, and consequently, it will be the totality of all tangent planes to the developable that has that curve for its edge of regression, or, as one would like to say, *any simply-infinite continuum of planes is enveloped by a developable surface.* Indeed, if **g** is the limiting position to which the intersection of the planes **P** and **P′** tends when **P′** tends to the plane **P**, which is supposed to be fixed, then the locus of lines **g** will be a ruled surface, because it admits the plane **P** as its unique tangent plane along **g**. In order to convince oneself of that, it is enough to observe that if M is an point of **g** then the plane M**g′** will tend to coincide with **P**; one can arrive at that conclusion more rigorously by means of calculation. It is known (cf., II, § **5**) that if one differentiates the equation for **P**, which express the idea that x, y, z satisfy the immobility conditions (8) then one must obtain the equation of another plane that passes through **g**. If one fixes the fundamental curve on the surface then it will follow that the two equations must be devoid of independent terms, and for that to happen, it is necessary that the term in x must also be missing from the first. **P** will then contain the tangent, and it will therefore coincide with the tangent plane at the point considered.

13. – *The surfaces that are enveloped by the faces of a fundamental trihedron* are important in the study of skew curves.

a) We have already seen that the osculating planes will envelop the surface that admits that curve as its edge of regression. One has in that a rapid confirmation of the calculations, since when one differentiates the equation of the osculating plane ($y = 0$), one will get $z = 0$; differentiating again will give $x = 0$; that is to say, the generator is the tangent to the curve, and the edge of regression is that curve.

b) From the definition (§ **1**) of the polar axis, the normal planes must envelop precisely the surface that is generated by that line. One gives the name of *polar developable* to that surface. The differentiation of the equation of the normal plane ($x = 0$) will give $z = \rho$, and therefore the polar axis will meet the principal normal at the point (viz., the *center of curvature*) that is situated at the distance ρ from M.

c) Finally, the envelope of the rectifying plane is called the *rectifying surface*, because a curve on that surface will necessarily be a geodetic, and on the other hand, one proves that the geodetic of a surface will single out the shortest path between two arbitrary points on that surface that are not too distantly separated. It will then follow that when one wishes that the surface should map to a plane in the way that was described at the end of § **10**, the curve will transform into a line because it must continue to single out the shortest path between two points in the plane. One can infer a simple confirmation of that fact from direct calculation. If one differentiates the equation of the rectifying plane (z = 0) then one will find immediately that the inclination ε of the generator of the rectifying developable on the tangent to the curve is determined by the formula:

$$\tan \varepsilon = -\frac{r}{\rho}. \tag{39}$$

If one applies formulas (4) to the direction cosines of that generator ($\alpha = \cos \varepsilon$, $\beta = \sin \varepsilon$, $\gamma = 0$) then one will get:

$$\frac{\delta\alpha}{ds} = -\sin\varepsilon\,\frac{d\varepsilon}{ds}, \qquad \frac{\delta\beta}{ds} = \cos\varepsilon\,\frac{d\varepsilon}{ds}, \qquad \frac{\delta\gamma}{ds} = \frac{\cos\varepsilon}{\rho} + \frac{\sin\varepsilon}{r} = 0;$$

hence, $\delta\alpha^2 + \delta\beta^2 + \delta\gamma^2 = ds^2$. Therefore, *the angle between two infinitely-close rectifying generators is* $\delta\varepsilon$. Having said that, if one imagines that one rotates the rectifying plane $\mathbf{g}'x'$ around the generator **g** until it coincides with the preceding plane $\mathbf{g}x$ then it will be clear that the tangent at M' will change until it coincides with the tangent at M; that is to say, two arbitrary consecutive elements (§ **4**) will be in a straight line.

d) It is useful to add the following considerations: When the three generators **g**, **g′**, **g″** of any developable are consecutive in the sense of § **4**, if one assumes that **g″** rotates around **g′** until it is found in the same plane as **g** and **g′** then the point M will not move, and the angle (**g′**, **g″**) will not vary. It will then follow that the edge of regression will keep its flexion at any point unaltered when it becomes planar. Hence, one can make any curve in space correspond point-by-point with a plane curve, in such a way that two arbitrary corresponding arcs are equal, and the flexion at two corresponding points will be the same. In order to do that, it is enough to map the developable of the tangents onto a plane. It is then clear that the intrinsic equation of the plane curve into which an arbitrary curve is transformed is obtained by eliminating the torsion from the intrinsic equations of the skew curve. Any other developable that goes through the curve considered can be mapped into a plane while destroying the torsion, *but at the same time, it will alter the flexion of the curve.* Does there exist a developable such that mapping it onto a plane will also totally destroy the flexion of the curve; *i.e., transform the curve into a line?* Such a surface always exists, and because of what was just said, it will be precisely the one that has been called the rectifying developable.

14. – If one starts with an arbitrary curve in a developable then it will be easy to *determine its edge of regression.* It will be generated by the point ($-t\alpha$, $-t\beta$, $-t\gamma$), and indeed, when one observes (38), an application of the formulas (3) to those coordinates will give:

$$\frac{\delta x}{ds} : \alpha = \frac{\delta y}{ds} : \beta = \frac{\delta z}{ds} : \gamma = \alpha - \frac{dt}{ds}.$$

We then see, in addition, that the arc length of the edge of regression will be:

$$\varepsilon = \int \alpha\, ds - t,$$

and that this curve will reduce to a point when $t = \int \alpha\, ds$. One therefore says that this is an equality that *characterizes the conic surfaces.* In particular, t will be constant for $\alpha = 0$, and therefore the orthogonal trajectories of the generators of a cone are spherical lines. Since one has $\alpha = 0$, one will then have that t = constant, so the condition $t = \int \alpha\, ds$ will be satisfied, and therefore the surface will necessarily be conic. In the general case, when one takes (38) into account, formulas (4) will give:

$$\frac{\delta\alpha}{ds} = \frac{\beta^2 + \gamma^2}{t}, \qquad \frac{\delta\beta}{ds} = -\frac{\alpha\beta}{t}, \qquad \frac{\delta\gamma}{ds} = -\frac{\alpha\gamma}{t}, \tag{40}$$

and that will say what the direction of the principal normal is. It will then follow that the direction cosines a, b, c of the binormal are proportional to 0, γ, $-\beta$, resp.; if one then applies formulas (4) one more time then one will find that:

$$\frac{\delta a}{\delta\alpha} = \frac{\delta b}{\delta\beta} = \frac{\delta c}{\delta\gamma} = \frac{\beta t}{(\beta^2 + \gamma^2)^{3/2}\rho}. \tag{41}$$

It is now enough to square and sum (40) and (41) in order to obtain the radii of curvature of the edge of regression:

$$\rho' = \frac{t}{\sqrt{\beta^2 + \gamma^2}}\left(\frac{dt}{ds} - \alpha\right), \qquad r' = \frac{\beta^2 + \gamma^2}{\beta}\left(\frac{dt}{ds} - \alpha\right). \tag{42}$$

15. Evolutes and involutes. – If (cf., II, § **9**) the tangents to a curve are normals to another curve then former will be called the *evolute* of the latter, and the latter will be called the *involute* of the former. In other words, the orthogonal trajectories of the generators of any developable are the involutes of the edge of regression. Take one of its trajectories to be the fundamental curve. Since the direction cosines ($\alpha = 0$, $\beta = \sin\psi$, $\gamma = \cos\psi$) of the generator must satisfy the conditions (38), one will have:

$$-t\cos\psi = \rho, \qquad \frac{d\psi}{ds} = \frac{1}{r}. \tag{43}$$

The second equality will continue to be satisfied when one adds an arbitrary constant to ψ. It will then follow that *if the generators of a developable are rotated through the same angle around one of their orthogonal trajectories then it will not cease to constitute a developable surface.* However, the first equality says that the z-coordinate of the point of the edge of regression is equal to ρ. That is to say (§ **13**, b), that point will be on the polar axis; i.e., its edge will belong to the polar developable. Hence, the *infinitude of evolutes of a curve are all situated on the polar developable.* In addition, if one observes that by virtue of (40) $\delta\beta$ and $\delta\gamma$ will be zero in the present case then one will see immediately that the rectifying plane of the evolute coincides with the normal plane to the involute. It will then follow that *when the polar developable of a curve is mapped to the plane, all of the evolutes of that curve will be rectifying.* Finally, the curvatures of the evolute are given by formulas (42), and if one takes (43) into account then one will find that:

$$s' = \frac{\rho}{\cos\psi}, \qquad \rho' = \frac{\rho}{\cos\psi}\frac{d}{ds}\frac{\rho}{\cos\psi}, \quad r' = -\frac{\rho}{\sin\psi}\frac{d}{ds}\frac{\rho}{\cos\psi}. \tag{44}$$

When one knows ρ, r, and consequently ψ, as functions of s, if one wishes to find the intrinsic equations of an arbitrary evolute then it will be enough to eliminate s from the preceding equalities. By virtue of the equality $-t = s'$, one can also consider the involutes of skew curves as being described by the points of a flexible, inextensible filament that was originally wrapped along the curve and that one then unrolls while always keeping it tense. It is then easy to explain what results from the first formula in (43) when one observes that r goes to zero with t ; i.e., that *the edge of regression of a developable is the locus of points of regression of the orthogonal trajectories of the generators.*

16. Central axis. – That is what one calls the common perpendicular to two infinitely-close principal normals. That line, which is parallel to two infinitely-close rectifying plane, is parallel to the rectifying generator, and therefore if h is its distance from the point M on the curve then its coordinates will be:

$$\alpha = \cos\varepsilon, \qquad \beta = \sin\varepsilon, \qquad \gamma = 0, \quad \xi = -h\sin\varepsilon, \quad \eta = h\cos\varepsilon, \quad \zeta = 0. \tag{45}$$

ε is given by formula (39), and h is determined by expressing the idea that the line in question will meet the principal normal to the point M', which is infinitely close to M. When one passes from one normal to the other, the coordinates will experience variations that are proportional to $1 / \rho$, $1 / r$, 0, 0, 1, 0, by virtue of (4) and (7). Hence, the condition for them to meet is:

$$\frac{\xi}{\rho} + \frac{\eta}{r} + \beta = 0, \qquad \text{i.e.,} \qquad \frac{1}{\rho} - \frac{\cot\varepsilon}{r} = \frac{1}{h},$$

and finally, if one observes (39) then:

$$h = \frac{\rho r^2}{\rho^2 + r^2}. \tag{46}$$

Therefore, the central axis will meet the principal normal between the point M and the center of curvature and divide the rectilinear segment that terminates at those points in the ratios of $\sin^2\varepsilon$ and $\cos^2\varepsilon$. If one applies formulas (4) and (7) to the coordinates of the central axis then, thanks to formulas (17) and (20), one will find immediately that:

$$\delta\theta = d\varepsilon, \qquad \delta q = dh, \tag{47}$$

and therefore $dh : d\varepsilon$ is the distributor parameter of the surface that is generated by the central axis. That line presents itself in various interesting questions. Thus, for example, when one considers the motion of the fundamental trihedron along the curve, *the central axis in the space that is rigidly fixed with the trihedron will be the instantaneous locus of the points that displace less quickly than all of the other ones.* Indeed, the displacement ds' of a point that is rigidly coupled with the fundamental trihedron is deduced directly from formulas (3) by squaring and summing them and supposing that x, y, z are constants:

$$\frac{ds'^2}{ds^2} = \left(\frac{z}{\rho} - 1\right)^2 + \frac{z^2}{r^2} + \left(\frac{x}{\rho} + \frac{y}{r}\right)^2 .$$

If one then sets the partial derivatives of the right-hand side with respect to x, y, z equal to zero then one will get:

$$\frac{x}{\rho} + \frac{y}{r} = 0, \qquad \left(\frac{z}{\rho} - 1\right)^2 + \frac{z^2}{r^2} = 0 \, ;$$

i.e., $y = x \tan \varepsilon$, $z = h$, where h has the significance that it had in (46) and (39); one will then see that one has $ds = \cos \varepsilon \cdot ds$ on the line thus-found. The preceding property of the central axis is explained by observing that *the normals to the trajectories of all points that are rigidly linked with the fundamental trihedron constitute a linear complex whose axis is the central axis of the trajectory of the origin,* and it is also obvious that it is the central axis of all the other trajectories. Indeed, if the direction (α, β, γ) is that of a normal at (x, y, z) to the trajectory of that point then, by virtue of (3), the perpendicularity condition $\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z = 0$ will become:

$$\alpha\left(1 - \frac{z}{\rho}\right) - \beta\frac{z}{r} + \gamma\left(\frac{x}{\rho} + \frac{y}{r}\right) = 0, \qquad \text{i.e.,} \qquad \frac{\xi}{r} - \frac{\eta}{\rho} + \alpha = 0.$$

That will be (§ **7**, *c*) the equation of a linear complex for which one has $p = -h \cot \varepsilon$, by virtue of (26); that is to say, p is the distributor parameter of the principal normals on the surface, which would result from (33). One will then get precisely the values (45) for the coordinates of the axis from formulas (23) and (25). Moreover, since the fundamental curve can be chosen at random from the trajectories of the points of a rigid system, it is clear that the line that was found is the central axis of any other trajectory. It will then follow that *the principal normals to all of the trajectories are the lines that meet the central axis orthogonally.* One can also easily construct the tangents, binormals, and radii of curvature after one observes that $h \cot \varepsilon$ has a unique value for all trajectories.

CHAPTER X

NOTEWORTHY SKEW CURVES

1. Spherical curves. – A line that is traced out on a sphere has all of its points at a constant distance R from a point O, namely, the center of the sphere. The coordinates of that point with respect to the fundamental trihedron of the curve considered will then be three functions x, y, z that are constantly coupled by the relation:

$$x^2 + y^2 + z^2 = R^2 \tag{1}$$

and satisfy the conditions (8) of the preceding chapter. An initial derivation of (1) will give $x = 0$. Hence, *the normal planes to all of the lines that are traced on a sphere will concur at the center.* The derivation of $x = 0$ will give $z = \rho$. Hence, *the center of curvature at an arbitrary point of a spherical line is obtained by projecting the center of the sphere onto the osculating plane.* Finally, if one differentiates $z = \rho$ again then one will find the value of y, and one will see that the coordinates of the center of the sphere will be:

$$x = 0, \qquad y = -r\frac{d\rho}{ds}, \qquad z = \rho; \tag{2}$$

if one substitutes this in (1) then one will get:

$$R^2 = \rho^2 + \left(r\frac{d\rho}{ds}\right)^2. \tag{3}$$

That equality *characterizes the lines that are traced on a sphere of radius R.* Indeed, if one applies formulas (3) of the preceding chapter to the coordinates (2) then one will find that:

$$\frac{\delta x}{ds} = 0, \qquad \frac{\delta y}{ds} = -\left[\frac{\rho}{r} + \frac{d}{ds}\left(r\frac{d\rho}{ds}\right)\right], \qquad \frac{\delta z}{ds} = 0, \tag{4}$$

and the derivation of (3) will give $\delta y = 0$. Hence, when the condition (3) is satisfied, there will exist a *fixed* point O whose distance from the points of the curve is constantly equal to R; i.e., the curve belongs to a sphere of radius R and center O. One can now add that *the necessary and sufficient condition for a curve to be spherical is:*

$$\frac{\rho}{r} + \frac{d}{ds}\left(r\frac{d\rho}{ds}\right) = 0. \tag{5}$$

2. – For any curve, there always exists a spherical curve through each point M that has the same fundamental trihedron and curvatures that are equal to those of the curve considered. The second curve necessarily belongs to the sphere that has radius R that was given by (3), and its center at the point O that is defined by its coordinates (2). That sphere is said to *osculate* the first curve at the point M. The tangent to the curve (O) at O is determined directly from (4). It is parallel to the binormal to (M) and is assumed to be directed in the inverse sense, so one will have:

$$\frac{ds'}{ds} = \frac{\rho}{r} + \frac{d}{ds}\left(r\frac{d\rho}{ds}\right). \tag{6}$$

Obviously, the tangent in question is the polar axis of (M). In other words, *the locus of centers of the osculating spheres is the edge of regression of the polar developable.* No further calculations will then be required in order to see immediately that the binormal to (O) is parallel to the tangent to (M), and that consequently, *the principal normals to the two curves will be parallel;* we shall assume that they are directed in the opposite sense. In addition, we obviously have:

$$\frac{\rho'}{r} = \frac{r'}{\rho} = \frac{ds'}{ds}. \tag{7}$$

Meanwhile, it is known that the coordinates of M in the osculating plane to (O) are $x = -r\,d\rho/ds$, $y = \rho$. If one substitutes this in (6) then one will get $ds' = (y / r)\,ds - dx$. Hence:

$$\frac{dx}{ds'} = \left(\frac{y}{r} - \frac{ds'}{ds}\right)\frac{ds}{ds'} = \frac{y}{\rho'} - 1, \qquad \frac{dy}{ds'} = -\frac{x}{r}\frac{ds}{ds'} = -\frac{x}{\rho'}.$$

It will then follow that *the point M can be considered to be fixed in the osculating plane to* (O). That will be obvious, moreover, if one observes that when M passes to the successive positions M', etc., in the sense that was indicated above (IX, § **4**), the normal plane to (M) will rotate around O. In other words, the normal planes to the elements MM', $M'M''$, $M''M'''$ concur at O. This makes a fact clearly understandable that is proved in full rigor by calculation, namely, that the osculating sphere at M is the limit of the spheres that pass through M and three other points of the curve when they tend to M.

3. Cylindrical helices. – One calls the curves that meet the generators of a cylindrical surface at a constant angle *cylindrical helices.* It is clear that those curves are the geodetics (IX, § **10**; § **13**, *c*) of similar surfaces because they will transform into lines when that surface is unwrapped onto a plane. If α, β, γ are the direction cosines of the generator then the invariability conditions [IX, form. (10)] must be satisfied for constant α, and one will therefore necessarily have $\gamma = 0$, $\dfrac{\alpha}{\rho} + \dfrac{\beta}{r} = 0$, in such a way that one can set $\alpha = \cos\varepsilon$, $\beta = \sin\varepsilon$, in which ε has the usual significance [IX, form. (39)]. In other words, as was predicted, the rectifying surface is that cylinder. Meanwhile, ε is constant. Conversely, when ε is constant, the invariability conditions will be satisfied by the cosines $\alpha = \cos\varepsilon$, $\beta = \sin\varepsilon$, $\gamma = 0$, and the curve will be traced out on a cylinder and will meet the generators at a constant angle ε; viz., it is a cylindrical helix. Hence, *in order for a curve to be a cylindrical helix, it is necessary and sufficient that the ratio of its curvatures should be constant.*

4. Puiseaux's theorem. – Of all the cylindrical helices, the simplest one is the *circular helix* – i.e., the one that is traced on a circular cylinder with all of its points at an equal distance from a line (viz., the *axis* of the cylinder). It is clear that any normal to the surface will meet the axis at a right angle, and one can then say the same thing about the principal normals of the curve, since it is a geodetic. Hence, all of the central axes (IX, § **16**) will coincide with the axis of the cylinder, which is the common perpendicular to all of the principal normals. It will then follow that the distance h must be constant. Formulas (39) and (46) of the preceding chapter will then give:

$$\rho = \frac{h}{\cos^2\varepsilon}, \qquad r = -\frac{h}{\sin\varepsilon\cos\varepsilon}$$

and show that ρ and r are also constants. Conversely, when ρ and r are constant, ε and h will also be so, and therefore [IX, form. (47)] one has just $\delta\theta = 0$, $\delta q = 0$. The first equality proves that the central axis remains constantly parallel to itself; however, the second one shows that the central axis is not displaced laterally. It can only slide along a line that is fixed in space, and therefore the point M, which stays at a constant distance h from that line, moves on a circular cylinder while describing a curve that meets the generators at the constant angle ε. Hence, *in order for a curve to be a circular helix, it is necessary and sufficient that its curvatures should be constants.*

5. Helices and geodetics of conical surfaces. – One calls the curves that meet the generators of a cone at a constant angle *conical helices.* They are never the geodetics of that surface, because they do not rectify when the surface of the cone is mapped onto the plane, but rather transform (I, § **11**, *c*) into logarithmic spirals. Under the conditions (38) of the preceding chapter, suppose that α is a non-zero constant, which then excludes a case that was considered above (IX, § **14**). One must then have $t = \alpha s$. Hence, after setting $\beta = \sqrt{1-\alpha^2}\sin\psi$, $\gamma = \sqrt{1-\alpha^2}\cos\psi$, those conditions will become:

$$\cos\psi = -\frac{\sqrt{1-\alpha^2}}{\alpha}\frac{\rho}{s}, \qquad \frac{1}{r} = \frac{d\psi}{ds} + \frac{\tan\psi}{s}. \tag{8}$$

If one is given an arbitrary plane curve then one can always twist it without flexing in such a way that it will become a conical helix. The first equality (8) serves to calculate the function ψ. If one then substitutes that in the second equality then one will determine the torsion that one needs to give to the curve at each point in order for it to meet the generators of a cone whose vertex has the coordinates $-t\,\alpha$, $-t\,\beta$, $-t\,\gamma$ at a constant angle. However, for *geodetics*, one needs to set $\alpha = \cos \varepsilon$, $\beta = \sin \varepsilon$, $\gamma = 0$ in the conditions above (38). One will then get the second of the following equalities:

$$\frac{dt}{ds} = \cos \varepsilon, \qquad \frac{d\varepsilon}{ds} = -\frac{\sin \varepsilon}{t}. \tag{9}$$

One knows (IX, § **14**) that the first of these equalities is the one that characterizes conical surfaces. One then deduces, in succession, that:

$$\frac{dt}{t} + \frac{d\varepsilon}{\tan \varepsilon} = 0, \qquad - t = \frac{a}{\sin \varepsilon};$$

that is to say, *the projection of the vertex of the cone onto the normal plane at M will stay at a constant distance a from M.* If one substitutes the last result in the second of (9) and integrates then one will get $sr = \alpha\rho$. Hence, *the torsion of geodetics on conical surfaces will vary like the product of the flexion with the arc length.* That property belongs to no other curve, since if one assumes that it is satisfied then one will see immediately by means of the usual immobility conditions that the point $(- s, a, 0)$ is fixed in space; that is to say, the rectifying plane envelopes a cone.

6. Twisted circles. – A twisted circle is the curve that one obtains by twisting a plane circle without flexing it, in such a way that one of its intrinsic equations will always be r = constant. Under that hypothesis, formula (3) will give $R = r$, and therefore *the osculating spheres of a twisted circle will all be equal to each other.* That property belongs to no other curve. Indeed, differentiating formula (3) will give:

$$R\frac{dR}{ds} = \left[\frac{\rho}{r} + \frac{d}{ds}\left(r\frac{d\rho}{ds}\right)\right] r\frac{d\rho}{ds}.$$

It will then follow that if R is constant and (5) is not satisfied (in which case all of the spheres will coincide in one) then ρ will also be necessarily constant. In addition, formulas (6) and (7) give:

$$s' = R\int \frac{ds}{r}, \qquad \rho' = R, \qquad rr' = R^2.$$

Hence, as **Bouquet** found, *the locus of the centers of curvature of a twisted circle is another twisted circle* that has the same flexion as the first circle and a torsion that varies in inverse proportion to the torsion of the first circle.

7. Exercises:

a) *Can a helix belong to a sphere?* In order for that to be true, the condition (5) would have to be satisfied when one sets $r = -\rho \tan \varepsilon$, with ε constant. In that way, one would obtain, in succession:

$$\frac{d}{ds}\left(\rho \frac{d\rho}{ds}\right) = -\cot^2 \varepsilon, \qquad \rho^2 + s^2 \cot^2 \varepsilon = \text{constant}.$$

Hence (I, § **8**, *c*), the spherical helices are deduced from hypocycloids, epicycloids, and cycloids ($\varepsilon < \pi / 4$, $\varepsilon > \pi / 4$, $\varepsilon = \pi / 4$, resp.) by simple torsion.

b) Do there exist *cylindro-conical helices;* i.e., curves that are helices on either cylinders or cones? One would need for the conditions (8) to be satisfied and to simultaneously have $r = -\rho \tan \varepsilon$, with ε constant. One first observes that (8) can be written in the following form:

$$s \cos \psi = -\frac{\sqrt{1-\alpha^2}}{\alpha}\rho, \qquad \frac{d}{ds}(s \sin \psi) = \frac{\sqrt{1-\alpha^2}}{\alpha}\cot \varepsilon .$$

If one integrates the second of these equations and substitutes the result in the first one then one will find that *the cylindro-conical helices have radii of curvature that are proportional to the arc length*, when measured by starting from the vertex of the cone. Conversely, any curve that is represented by the intrinsic equations $r = ks$, $r = k's$ will be a cylindro-conical helix because if one determines the constants ψ, ε, α thanks to the relations:

$$k = -\cot \psi \cot \varepsilon, \qquad k' = \cot \psi, \qquad \sin \psi = \frac{\sqrt{1-\alpha^2}}{\alpha}\cot \varepsilon$$

then it will be clear that the conditions that were stated to begin with are satisfied, and one can add that the curve *belongs to a circular cone* because the *invariable* direction ($\cos \varepsilon$, $\sin \varepsilon$, 0) of the generators of the cylinder make a *constant* angle with that (α, β, γ) of the radius vector. The cylindro-conical helices are, so to speak, the logarithmic spirals in three-dimensional space. They enjoy the singular property that was pointed out before (II, § **7**, *i*) for those curves that they will not deform when they are subjected to a uniform dilatation around any point in space.

c) What are the *involutes of the cylindrical helices?* Recall the formulas of the preceding chapter, while observing that if the ratio of ρ' to r is assumed to be constant in formulas (44) then it will be necessary that ψ should be constant, and the second formula (43) will then show that the torsion of the involute is zero. Hence, *the involutes of the cylindrical helices are plane curves.* If one proceeds in the opposite sense then one will easily see that there exist no other evolutes of the plane curve. In addition, if one recalls that the binormal to the involute and the rectifying generator of the evolute are parallel then one will see immediately that in the case of helices *the planes of the involutes are perpendicular to the generators of the cylinder.* In the particular case of the circular helix, formulas (44) will then become:

$$\rho \frac{d\rho}{ds} = \rho' \cos^2 \psi = -r' \sin \psi \cos \psi = \frac{\rho' r'^2}{\rho'^2 + r'^2},$$

i.e., $\rho^2 = 2as$, if one calls the radius of the cylinder a. Hence, the *involutes of the circular helices are involutes of the circle.*

d) One of the evolutes of a spherical curve obviously reduces to the center of the sphere. The other one is deduced easily by rotating the radii of the sphere around the curves in the respective normal planes through the same angle. In particular, if one changes ψ into $\psi - \pi / 2$ in the usual formulas (44), in which ψ represents the angle between the principal normal at M and the radius OM, in such a way that $\rho = R \sin \psi$ then one will get the formulas:

$$s' = R \cot \psi, \quad \rho' = \frac{R^2}{2} \frac{d}{ds} (\cot^2 \psi), \quad r' = R^2 \frac{d}{ds} \cot \psi,$$

which define the *edge of regression of the developable that circumscribes the sphere along the given curve.* Since one has $s'r' = R\rho'$, one will see (§ **5**) that the aforementioned edge is a geodetic on a cone. That property will seem obvious when one ponders the fact that the polar developable of a spherical curve is necessarily a cone on which all of the evolutes of the given curve are geodetics (IX, § **15**).

e) *Construct a series of spheres that have their centers along a given line and osculate that curve.* This problem, which was posed by **Jamet**, is easily solved when one observes that by virtue of a known (§ **2**) property, *any line will reduce to a point when its polar developable is mapped to the plane.* With that, one would like to say that when the normal planes are made to coincide with a fixed plane by means of successive rotations of the corresponding polar axes, the points of the curve (which is rigidly fixed in that plane) will ultimately conclude by coinciding in just one point of the fixed plane. In order to solve the Jamet problem, we need to commence by transforming the given curve (O) into a plane curve (O') by altering its torsion. One can then construct a series of spheres that pass through an arbitrary point M in the plane of (O') and have their centers on (O'). It is enough to twist the curve (O') without flexing it until the original configuration (O) is restored. The spheres that are rigidly dragged along with the motion will not cease to osculate that curve, which is the locus of the positions that are occupied by M in osculating planes to (O). Hence, *any series of spheres can be made to osculate an infinitude of curves by suitably deforming the line of centers,* and two arbitrary configurations of that line can always be deduced from each other by a simple torsion.

8. Bertrand curves. – One calls any curve whose curvatures are linearly constrained by:

$$\frac{a}{\rho}+\frac{b}{r}=1 \tag{10}$$

a *Bertrand curve.* In particular, the twisted circles (b = 0) and the curves with constant torsion (a = 0) are Bertrand curves. Helices can also be considered to be Bertrand curves, since if one increases a and b indefinitely in such a way their ratio tends to a limit cot ε then equation (10) will tend to be converted into the characteristic equation of the helix: $r = -\rho \tan \varepsilon$. The Bertrand curves are characterized by the following property: *Their principal normals are principal normals of another curve.* Take the point M_1 on the principal normal of any curve (M) that is at a distance of a from M. If one applies the fundamental formulas to the coordinates (0, 0, a) then one will get:

$$\frac{\delta x}{ds}=1-\frac{a}{\rho}, \quad \frac{\delta y}{ds}=-\frac{a}{r}, \qquad \frac{\delta z}{ds}=\frac{da}{ds}. \tag{11}$$

Now, in order for (M_1) to originally meet the principal normals to (M), one will need to have δz = 0; i.e., a must be constant. Let $\alpha = \cos\theta$, $\beta = \sin\theta$, $\gamma = 0$ be the direction cosines of the tangent to (M_1) at M_1, in such a way that:

$$\frac{a}{\rho}-\frac{a}{r}\cot\theta=1. \tag{12}$$

The fundamental formulas for those directions are:

$$\frac{\delta\alpha}{ds}=-\sin\theta\,\frac{d\theta}{ds}, \qquad \frac{\delta\beta}{ds}=\text{co }\theta\,\frac{d\theta}{ds}, \qquad \frac{\delta\gamma}{ds}=\frac{\cos\theta}{\rho}+\frac{\sin\theta}{r}, \tag{13}$$

and one will see that θ must also be constant in order for the principal normals to (M) to coincide with those of (M_1) at any point. Now, it is known that equation (12) defines a Bertrand curve for which one has $b = -a\cot\theta$. It is clear *a priori* that (M_1) will also be a Bertrand curve for which a and b must have the same values. Moreover, one deduces from formulas (11) and (13) that:

$$\frac{ds_1}{ds}=\frac{\sqrt{a^2+b^2}}{r}, \qquad \frac{a^2+b^2}{\rho_1}=a-b\,\frac{r}{\rho}.$$

Therefore, if one switches the two curves then the ratio $ds : ds_1$ will have the value $\sqrt{a^2+b^2} : r_1$ and one will see that one must have $rr_1 = a^2 + b^2$, which is a property that is known already in the case of twisted circles. One easily verifies that equation (10) also belongs to (M_1). The correspondence between the two curves will become illusory for the curves with constant torsion, since (M) and (M_1) will coincide when a is zero. When one of the curves tends to become a helix, the other will extend to infinity. Finally, it is useful to observe that in the general case, *the fundamental trihedra of the two curves are rigidly coupled to each other.*

9. –The preceding question leads naturally to the study of surfaces that are composed of the principal normals to a curve. If it is taken to be a fundamental curve then formulas (37) of the preceding chapter will be satisfied for $\alpha = \beta = 0$, $\gamma = 1$, and one will then have:

$$\frac{1}{\rho} = -\frac{\sin\tau\cos\tau}{p}, \qquad \frac{1}{r} = \frac{\cos^2\tau}{p},$$

i.e., $\tau = \varepsilon$ and $p = -h \cot \varepsilon$, which was observed before (IX, § **16**) in a different way. In order for *two* lines to exist on the surface that both admit the generators as principal normals, it is necessary that each of them should be a Bertrand curve, and a third one cannot exist, in general. However, if it does exist then an infinitude of other ones must exist that are all the circular helices that are defined by the infinitude of pairs of constant values for ρ and r that satisfy (1) for a given pair of values for a and b. One of the infinitude of helices ($\rho = \infty$, $r = b$) reduces to a line that is the common axis of the infinitude of circular cylinders upon which those helices are traced, and the line of striction of the surface ($\tau = 0$). Meanwhile, since the distributor parameter p has the constant value b, the ratio of the distance to the angle between two arbitrary generators will stay constantly equal to b. Hence (IX, § **7**, *h*), *the surface is a helicoid with a director plane.*

10. – One will arrive at another characteristic property of Bertrand curves when one investigates *whether it can happen that a line that is rigidly linked with the fundamental trihedron of a curve remains normal to the trajectories of its points.* Obviously, such a line must belong to the *complex of normals*, which was found in the last paragraph of the preceding chapter, and therefore the relation:

$$\frac{\xi}{r} - \frac{\eta}{\rho} + \alpha = 0 \tag{14}$$

must intercede between its coordinates, which are assumed to be constants. If the curvatures vary and one varies their ratio at the same time (which generally happens) then (14) will not be satisfied unless one sets $\alpha = 0$, $\xi = 0$, $\eta = 0$. Those equations represent the *normals to the curve and the other perpendiculars to the tangent situated on the rectifying plane.* A first exceptional case presents itself when the curvatures, while varying, preserve a constant ratio, in which case the curve will be a non-circular helix. (14) will then be satisfied by taking $\alpha = 0$, $\xi \cos \varepsilon + \eta \sin \varepsilon = 0$, and therefore *the parallels to the normal plane that meet the generator* are the only lines that answer the question. However, if the helix is circular then the condition (14) will not split, and *any line of the complex of normals* will enjoy the stated property. Furthermore, that case will once more enter the single exceptional case of Bertrand curves. Indeed, when the constraint (10) intercedes between the curvatures, (14) will be satisfied by taking:

$$\xi = -b\,\alpha, \qquad \eta = a\,\alpha, \tag{15}$$

and not otherwise. Of these equations, the second one represents the complex of lines that meet the parallel $\mathbf{g}_1$ to the binormal to (M) that goes through M_1, and one can replace the first one with the equation:

$$a\,\xi + b\eta = 0, \tag{16}$$

which represents the complex of lines that are supported by the parallel $\mathbf{g}$ to the binormal to (M_1) that goes through M. Hence, *those lines that are rigidly coupled with the fundamental trihedron of a Bertrand curve that meet* $\mathbf{g}$ *and* $\mathbf{g}_1$ *will remain normals to the trajectories at all of their points.* In particular, for any twisted circle, the lines that answer the question *will be the ones that are supported by the tangent and the polar axis.* For the non-circular helices, $\mathbf{g}_1$ will be at infinity in the normal plane, and $\mathbf{g}$ will be the generator of the cylinder. However, a circular helix can be considered to be a Bertrand curve in an infinitude of ways, and the infinitude of congruences that one will get in that way constitute precisely the entire complex of normals. Finally, in the case of curves with constant torsion, one can no longer substitute equation (16) for one of the ones in (15). That would make $\eta = 0$, $x = -r\alpha$, and that would not represent the lines that are supported by two distinct lines. That will occur because in the case considered, (M_1) will coincide with (M), and the lines $\mathbf{g}$ and $\mathbf{g}_1$ will tend to coincide with the binormal, in such a way that the ratio of their angle to their distance will tend to a limit that measures the torsion of the curve. The lines that answer the question are therefore *all the tangents to a certain twisted surface along the binormal.*

11. – The Bertrand curves are very special lines that belong to the lines that are defined intrinsically by the equation:

$$\frac{A}{\rho^2} + \frac{B}{r^2} + \frac{C}{\rho r} = \frac{P}{\rho} + \frac{Q}{r}\,. \tag{17}$$

They present themselves as exceptions when one investigates whether *there exist developables among the surfaces that are generated by lines that are connected with the fundamental trihedron.* One knows from the preceding chapter that the (constant) coordinates of those lines satisfy the condition:

$$\delta\alpha\,\delta\xi + \delta\beta\,\delta\eta + \delta\gamma\,\delta\zeta = 0,$$

which will take precisely the form of (17) when one makes use of the fundamental formulas, as long as one sets:

$$\frac{\beta\eta}{A} = \frac{\alpha\xi}{B} = -\frac{\alpha\eta + \beta\xi}{C} = \frac{\alpha\beta}{P} = \frac{\beta^2 + \gamma^2}{Q}. \tag{18}$$

If no constraints of the kind (17) exist between the curvatures then all of the numerators in (18) should be annulled, and the conditions $\alpha = 1$, $\beta = 0$, $\eta = 0$ will consequently be satisfied, which define the *parallels to the tangent that are traced on the rectifying plane.* Hence, they will generally be *the only generators of the developables*, but other similar lines can exist when the curve belongs to the class that is defined by equation (17). Effectively, if not all of the coefficients of that equation are zero then eliminating ξ and η from (18) will give the relations:

$$A\,\alpha^2 + B\,\beta^2 + C\,\alpha\beta = 0, \qquad P\,(\beta^2 + \gamma^2) = Q\alpha\beta, \tag{19}$$

from which, one will see that in general there exist *four other generators of the developables that are parallel to the intersections of a certain quadric cone with a pair of plane.* The cone that has its vertex along the curve will touch the osculating plane along the tangent, and the planes pass through the principal normal. Nonetheless, one knows that if A, B, C are zero then the first equation in (19) will vanish, equation (17) will represent a helix, and the generators of the cone will answer all questions, since one can also infer from (18) that $\xi = 0$, $\eta = 0$.

12. – In the foregoing, in asserting that (19) admits a limited number of common solutions, it is tacitly supposed that the right-hand side of (17) is not a divisor of the left-hand side, in the algebraic sense. In the contrary case, the equation would reduce to the form (10), and would then represent a Bertrand curve. Since one could then attribute arbitrary values to P and Q, as long as they are not both zero, an infinitude of other lines would answer the question that was posed for those curves, *as well as for the helices.* If one writes (19) in the form:

$$(a\alpha + b\beta)\,(P\alpha + Q\beta) = 0, \qquad \alpha\,(P\alpha + Q\beta) = P$$

then one will see immediately that one can satisfy this in two very different ways: viz., by annulling one or the other factor in the left-hand side of the first equation. When one sets the second factor equal to zero, one will get $P = 0$, by virtue of the second equation, and one will then have $\beta = 0$, as well. In addition:

$$A = Pa = 0, \quad B = Qb, \qquad C = Pb + Qa = Qa\,.$$

Having said that, (18) will become:

$$\frac{\xi}{b} = -\frac{\eta}{a} = \frac{\gamma^2}{\alpha}.$$

The numerators will all be zero when that is true of γ and η. One will then get back to the *infinitude of parallels to the tangents that are traced in the rectifying plane.* In the contrary case, the last equations will define a congruence, from which, the condition $\beta = 0$ will single out a hyperbolic paraboloid ϖ that is represented by the equations:

$$a\xi + b\eta = 0, \quad \zeta = -b\gamma, \qquad \beta = 0.$$

However, if one annuls the factor $a\alpha + b\beta$ then one will infer from (18) that:

$$\eta = a\alpha, \qquad \gamma(\zeta + b\gamma) = 0\ .$$

For $\gamma = 0$, one will get *an infinitude of other parallel lines that are situated in a plane that is parallel to the rectifying plane.* If γ is not set equal to zero then the equations:

$$\eta = a\alpha, \qquad \zeta = -b\gamma, \qquad a\alpha + b\beta = 0$$

will define the generators of another paraboloid ϖ_1 that is parallel to a plane that passes through the principal normal. The Bertrand curves are then *characterized by the existence of two hyperbolic paraboloids that are rigidly coupled to the fundamental trihedron and are such that their generators of one system will remain tangent to certain curves in space.* The final observation in § **8** is explained because in the preceding search, we arrived at *two* paraboloids and *two* systems of parallel lines. Indeed, it is clear that the paraboloid ϖ_1 is not, so to speak, the paraboloid ϖ that relates to that Bertrand curve that has its principal normals on the curve in question, from what was said in § **8**. To conclude, observe that the two paraboloids coincide with the curves of constant torsion and degenerate (IX, § **7**, *f*) into two parabolas in the case of twisted circles. One parabola is situated in the osculating plane, and has its vertex on the curve and its focus at the center of curvature. The other one is situated in the normal plane, and has its focus on the curve and its vertex at the center of curvature.

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CHAPTER XI

GENERAL THEORY OF SURFACES

1. Geodetics and asymptotes. – The intrinsic properties of a surface in the neighborhood of each point *M* are strictly linked with those of the curves that pass through *M*. It follows (XV, § **1**) that one will see that the tangents to all such curves belong to a plane that one calls the *tangent plane* to the surface at *M*. The *normal* to the surface – i.e., the perpendicular to the tangent plane that one erects at *M* – is normal to all curves that pass through *M*, and it can be the binormal for some of them and the principal normal for others. The curves that have their binormal coincident with the normal to the surface are called *asymptotes*; the ones that admit that normal as a principal normal (cf., IX, § **10**) are called *geodetics*. In other words, if one considers the developable that is circumscribed by a surface along a given curve – i.e., the envelope of the planes that touch the surface at the points of the curve – then one can say that the developable that is circumscribed along an asymptote will admit that curve for its edge of regression, while the developable that is circumscribed along a geodetic will be the rectifying curve of that curve. In order to better account for the essential difference between the two types of curves, it is useful to materialize the surface by attributing a thickness to it and to imagine, on the other hand, that the curve is like a strip that is cut from the developable of the tangents – i.e., a succession of planes for which one can say (IX, § **4**) that most of them belong to the successive linear elements of the curve. When one wishes to locate a geodetic on a surface, the strip can penetrate normally into the thickness of that surface, while in order to locate an asymptote, it will be enough to draw it upon a surface on which it will lie like a planar strip in its plane. Let ψ be the angle through which the normal to the surface must be rotated clockwise (in the eyes of an observer that is located on the positive part of the tangent) in order to make it coincide with the principal normal. One will soon see that the quantities:

$$\mathcal{N} = \frac{\cos\psi}{\rho}, \quad \mathcal{G} = \frac{\sin\psi}{\rho}$$

which shall be called the *normal curvature* and *geodetic curvature*, resp., will frequently appear in our calculations, and one must always keep in mind that: *The geodetics are characterized by the constant vanishing of the geodetic curvature,* while: *The asymptotes are characterized by the constant vanishing of the normal curvature.* Here, one must note that only straight lines have the property of being both asymptotes and geodetics on any surface.

2. Lines of curvature. – A line that is traced on a surface is called a *line of curvature* if the normals to the surface along that line define a developable. We already saw (IX, § **15**) that in order for that to be true, it is necessary and sufficient that the derivative of ψ with respect to the arc-length must be equal to the torsion of the curve, and if one calls the quantity:

$$\mathcal{T} = \frac{d\psi}{ds} - \frac{1}{r}$$

the *geodetic torsion* then one can state that: *The lines of curvature are characterized by the constant vanishing of the geodetic torsion.* In particular, all of the lines that are traced on a sphere are lines of curvature of that surface because the normals are concurrent with its center, and more particularly, any planar line will be a line of curvature, along with all asymptotes of the plane in which it lies. Turning to the general case, observe that (IX, § **15**): *If a curve is a line of curvature on two surfaces then they will meet each other at a constant angle, so the line of intersection cannot be a line of curvature on one surface without being one on the other.* It will then follow that: *If a line of curvature is planar then its plane will cut the surface at a constant angle*, and one can also say that about any spherical line and the sphere in which it lies. Conversely: *If a plane or sphere cuts a surface at a constant angle then the intersection will be a line of curvature on the surface in question.* Finally, observe that if the angle ψ is constant for a line of curvature then the line will necessarily be planar. In particular ($\psi = 0$): *Any geodetic line of curvature will be planar*, and its plane will cut the surface at a right angle.

3. Fundamental formulas *for the curves traced on a surface.* – For the intrinsic study of a line traced on a surface, it is useful to take the z-axis to be normal to the surface at a moving point M, while keeping the tangent to the curve as the x-axis. If, into the immobility conditions (IX, § **4**):

$$\frac{dx'}{ds} = \frac{z'}{\rho} - 1, \qquad \frac{dy'}{ds} = \frac{z'}{r}, \qquad \frac{dz'}{ds} = -\frac{x'}{\rho} - \frac{y'}{r}$$

that relate to the fundamental trihedron of the curve, one introduces the new coordinates:

$$x = x', \qquad y = y' \cos\psi - z' \sin\psi, \qquad z = y' \sin\psi + z' \cos\psi$$

then one will get the relations:

$$\frac{dx}{ds} = \mathcal{N} z - \mathcal{G} y - 1, \qquad \frac{dy}{ds} = \mathcal{G} x - \mathcal{T} z, \qquad \frac{dz}{ds} = \mathcal{T} y - \mathcal{N} x. \tag{1}$$

Obviously, the fundamental formulas that serve to make known the absolute variations of the coordinates x, y, z of any point that moves with M will be:

$$\left.\begin{aligned}\frac{\delta x}{ds} &= \frac{dx}{ds} - \mathcal{N}z + \mathcal{G}y + 1,\\ \frac{\delta y}{ds} &= \frac{dy}{ds} - \mathcal{G}z + \mathcal{T}z,\\ \frac{\delta z}{ds} &= \frac{dz}{ds} - \mathcal{T}y + \mathcal{N}x,\end{aligned}\right\} \tag{2}$$

and it will be clear that they will also persist when x, y, z have the significance of direction cosines, as long as the constant term is removed from the first one.

4. Theorems of Meusnier and Bonnet. – When the point M passes to an infinitely-close point M', the variation of the coordinates z of any fixed point will have the same value for all curves on the surface that touch at M because it represents the difference between the distances from the fixed point to the planes that touch the surface at M and M', and consequently the same thing will be true for all curves that have the element MM' in common. It will then follow from the third formula in (1) that each of the quantities $\mathcal{T}$ and $\mathcal{N}$ will preserve an invariant value for any curve on the surface that are tangent to Mx at M. In the invariability of $\mathcal{T}$, one finds *Bonnet's theorem*, from which one deduces that *the geodetic torsion of a curve does not differ from the absolute torsion of the tangent geodetic*, up to sign. On the contrary, *Meusnier's theorem* asserts the invariability of $\mathcal{N}$, and has important consequences: First of all, if ψ has the same value ($\neq \pi / 2$) for two tangent curves then the values of ρ must also be equal; i.e., *two curves that are osculated by the same plane at a point of the surface will have equal curvatures* (absolute or geodetic), as long as the common osculating plane is not the tangent plane to the surface. Amongst all curves that touch a given curve at M, one considers the normal planar sections that are made in the surface by the zx-plane. If ρ_0 is its radius of curvature then Meusnier's theorem will give the value $1 / \rho_0$ for $\mathcal{N}$. Hence: *The normal curvature of any line that is traced on a surface is the curvature of the normal planar section that is made in the surface tangentially to that line.* As for the geodetic curvature, note that, by virtue of Meusnier's theorem, in the cylinder that orthogonally projects the curve onto the tangent plane, the curvature of the normal section that is tangent to that curve will be precisely $(\sin \psi) / \rho$. Hence: *The geodetic curvature of a line that is traced on a surface is the curvature of the projection of the curve onto the tangent plane.* Finally, for the invariability of $\mathcal{N}$, one has $\rho = \rho_0 \cos \psi$, and then *the center of curvature of any line on a surface at a point M is obtained by projecting the center of curvature of that normal planar section that touches the curve at M onto the osculating plane of that curve.* That construction will not be valid for the curves that are tangent to the asymptotes, since $\mathcal{N}$ must be zero for the asymptotes in order to annul $\cos \psi$, but ρ will have an arbitrary value. If a curve touches an asymptote at a point without osculating the tangent plane to the surface then $\cos \psi$ will not be zero, and its flexion must then must be zero. However, when the curve is osculated by the tangent plane, its flexion will be capable of taking on any value, which is generally different from the one that the flexion of the tangent asymptote has.

5. – The invariability of $\mathcal{N}$ and $\mathcal{T}$ for all curves that have a given tangent will result even more easily from a study of the normals to the surface in the neighborhood of a point M. When formulas (2), which relate to the directions, are applied to the direction (0, 0, 1), one will find that the normal at M' has direction cosines – $\mathcal{N}\,ds$, $\mathcal{T}\,ds$, 1, and in order for that to be true, it is enough to be sure that each of the quantities $\mathcal{N}$ and $\mathcal{T}$ has just one value for all curves that admit the common element MM'. One sees, in addition, that when one passes from M to M', the angular displacement of the tangent plane to the surface will result from two infinitesimal rotations, one of which (namely, the one that is proportional to the normal curvature) will happen with no rotation of the plane around MM', while the other one, which is proportional to the geodetic torsion, will consist of precisely a rotation around the tangent. One arrives at an interpretation of the geodetic curvature in an analogous manner by considering the direction (1, 0, 0) instead. One will find that the direction of the tangent at M' is defined by the cosines 1, – $\mathcal{G}\,ds$, $\mathcal{N}\,ds$, and then the tangent, while participating in the motion of the tangent plane, will rotate counterclockwise in that plane through an angle of $\mathcal{G}\,ds$, as viewed by an observer that is located on the positive half of the normal. In other words: *the geodetic curvature is proportional to the projection of the angle between two infinitely-close tangents onto the tangent plane,* just as *the geodetic torsion is proportional to the projection of the angular displacement of the normal to the surface onto the normal plane of the curve.* Now, it is clear that $\mathcal{G}$ depends upon not just the tangent at M, as $\mathcal{N}$ and $\mathcal{T}$ do, but upon the tangent at M', as well; that is to say, the value of $\mathcal{G}$ is unique for all curves on the surface that osculate at M with the same plane, but not for all curves that that are mutually tangent at M. That situation must prevail if any point of a surface turns out to be geodetic (viz., $\mathcal{G}$ = 0) in all directions, while the lines of curvature (viz., $\mathcal{T}$ = 0) and the asymptotes (viz., $\mathcal{N}$ = 0) that pass through a given point M are, as we will soon see, limited in number, precisely because fixing the value of $\mathcal{T}$ or $\mathcal{N}$ at M for a curve will signify that one is imposing the same value upon an infinitude of curves that touch the given curve M, no matter what plane osculates it.

6. – The considerations that were made in the first four paragraphs of the Chapter VIII are applicable immediately to the systems of curves that are traced on any surface, and one can then say that the analytical representation of a simply-infinite system of curves on a surface will arise from any function of the points on that surface. If one then defines the differential quotient of the function in any direction and establishes a basic orthogonal system of *curvilinear coordinates* according to § **7** of the cited chapter then the two systems of curves that are defined by functions q_1 and q_2 will lead one to express the distance between two infinitely-close points by means of the formula:

$$ds^2 = Q_1^2\, dq_1^2 + Q_2^2\, dq_2^2\,,$$

in which the Q are functions of the q, and one should recall that the *differential quotients* in the directions of the coordinate lines depend upon *derivatives* with respect to the q in the following way:

$$\frac{\partial}{\partial s_1} = \frac{1}{Q_1}\frac{\partial}{\partial q_1}\,, \qquad \frac{\partial}{\partial s_2} = \frac{1}{Q_2}\frac{\partial}{\partial q_2}\,.$$

Having assumed that, and upon setting:

$$\mathcal{G}_1 = \frac{\partial \ln Q_1}{\partial s_2}\,, \qquad \mathcal{G}_2 = \frac{\partial \ln Q_2}{\partial s_1}\,, \tag{3}$$

the condition:

$$\frac{\partial Q_2 v}{\partial q_1} = \frac{\partial Q_1 v}{\partial q_2}\,,$$

which is necessary and sufficient for the existence of a function f whose differential quotients are the prescribed functions u and v, will transform into:

$$\left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right) u = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right) v\,. \tag{4}$$

It will then follow that one must have:

$$\frac{\partial^2}{\partial s_1\,\partial s_2} - \frac{\partial^2}{\partial s_2\,\partial s_1} = \mathcal{G}_2 \frac{\partial}{\partial s_2} - \mathcal{G}_1 \frac{\partial}{\partial s_1} \tag{5}$$

for any function. This relation is very useful in the intrinsic analysis of surfaces.

7. – Now, establish a system of orthogonal curvilinear coordinates on a surface, such that the Mx and My axes are directed along the tangents to the coordinate lines q_1 and q_2 that pass through M. When M displaces along the line q_1, the conditions:

$$\frac{\partial x}{\partial s_1} = \mathcal{N} z - \mathcal{G} y - 1, \quad \frac{\partial y}{\partial s_1} = \mathcal{G} x - \mathcal{T} z, \qquad \frac{\partial z}{\partial s_1} = \mathcal{T} y - \mathcal{N} x \tag{6}$$

will be necessary for the immobility of the point (x, y, z). In order to find the analogous conditions under the hypothesis that M displaces along the line q_2, one needs to change x and y into y and $-x$, resp., and to attribute the values to $\mathcal{T}$, $\mathcal{N}$, $\mathcal{G}$ that relate to the given line q_2 in such a way that one will have:

$$\frac{\partial x}{\partial s_2} = \mathcal{G}'y + \mathcal{T}'z, \qquad \frac{\partial y}{\partial s_2} = \mathcal{N}'z - \mathcal{G}'x - 1, \qquad \frac{\partial z}{\partial s_2} = -\mathcal{T}'x - \mathcal{N}'y. \tag{7}$$

Here, it is important to notice that when x, y, and z are zero – i.e., when one considers the instant that M passes the fixed position (x, y, z) – formulas (6) and (7) will give:

$$\frac{\partial x}{\partial s_1} = \frac{\partial y}{\partial s_2} = -1, \qquad \frac{\partial x}{\partial s_1} = \frac{\partial y}{\partial s_2} = \frac{\partial z}{\partial s_1} = \frac{\partial z}{\partial s_2} = 0.$$

If we continue to suppose that x, y, z are zero and apply the operation $\partial / \partial s_2$ to (6), while taking (7) into account, then we will get:

$$\frac{\partial^2 x}{\partial s_1\,\partial s_2} = \mathcal{G}, \qquad \frac{\partial^2 y}{\partial s_1\,\partial s_2} = 0, \qquad \frac{\partial^2 z}{\partial s_1\,\partial s_2} = -\mathcal{T}.$$

On the other hand, if we apply the operation $\partial / \partial s_1$ to (7) then we will find, by virtue of (6), that:

$$\frac{\partial^2 x}{\partial s_2\,\partial s_1} = 0, \qquad \frac{\partial^2 y}{\partial s_2\,\partial s_1} = \mathcal{G}', \qquad \frac{\partial^2 z}{\partial s_2\,\partial s_1} = \mathcal{T}'.$$

Having said that, if one is to obtain the values of the geodetic curvatures of the two lines then it is enough to express the idea that the relation (5) must be satisfied by the functions x and y, which is expressed by means of (3), as:

$$\mathcal{G} = \mathcal{G}_1, \qquad \mathcal{G}' = \mathcal{G}_2 .$$

However, if one applies the same relation to z then one will get $\mathcal{T} + \mathcal{T}' = 0$, and that will prove the following theorem of **Bonnet**: *The geodetic torsions of two lines that touch at a right angle are equal and opposite at the point where they meet.*

8. – Observe that if one assumes that a geodetic is a coordinate line (for example, the line q_1) then one will have $\mathcal{G}_1 = 0$, and Q_1 must then be a function of only q_1, by virtue of (3). It will then follow that when one replaces q_1 with a convenient function of q_1, one can always suppose that $Q_1 = 1$, and then the square of the linear element can be expressed by $dq_1^2 + Q_1^2\,dq_2^2$. The distance between two points M and M' of the geodetic considered, when computed along that geodetic, is the absolute difference between the values of q_1 at M and M', and then *two arbitrary orthogonal trajectories of a system of geodetics cut out equal arcs from the infinitude of geodetics,* which is just what happens in the plane for the orthogonal trajectories of any system of lines. That and other properties are due to the fact that *the geodetics are the straight lines of the surface*, so to speak. Indeed, when M goes to M' along a line that is not a geodetic, it will traverse a path of greater length, because q_2 will not be kept constant when it is not along a geodetic. That will explain why *the geodetic symbolizes the shortest path between two points on a surface that are not too distant from each other.* Moreover, that property will seem obvious when one reflects upon the fact that for the situation was given in § **5**, a point that describes an arbitrary curve on a surface will be subjected to a deviation of $\mathcal{G}\,ds$ on that surface at any instant. Now, if one desires that the point should proceed along the shortest path then one will also need to suppose that one constantly has $\mathcal{G} = 0$ if one is to prevent that deviation. In that way, one also explains why *a filament that is tensed on a surface will always arrange itself into a geodetic form*, and that is a remarkable fact that suggests a simple and practical means for characterizing the geodetics on any surface. However, if one materializes the surface then it might happen that at certain points, the filament tends to stretch in space into the form of a line, and one will then need to imagine that one has made just as many holes at those points, in such a way as to permit the filament to traverse the surface in order to rest upon one or the other of its faces. How to allocate the places at which one must puncture the surface *a priori* will result from an obvious observation – i.e., from noticing that the sign of the flexion of the filament will change at those places, in such a way that if one also has $\mathcal{N} = 0$ then the desired points will come from the ones *at which the geodetic touches some asymptote.*

9. Fundamental formulas *for the intrinsic analysis of surfaces*. – With the results of § **7**, if one represents the curvatures $\mathcal{N}$ and $\mathcal{N}'$ in terms of $\mathcal{N}_1$ and $\mathcal{N}_2$ then the conditions (6) and (7) will take on the definitive form:

$$\left.\begin{array}{lll} \dfrac{\partial x}{\partial s_1} = \mathcal{N}_1 z - \mathcal{G}_1 y - 1, & \dfrac{\partial y}{\partial s_1} = \mathcal{G}_1 x - \mathcal{T} z, & \dfrac{\partial z}{\partial s_1} = \mathcal{T} y - \mathcal{N}_1 x, \\ \dfrac{\partial x}{\partial s_2} = \mathcal{G}_1 z - \mathcal{T} z, & \dfrac{\partial y}{\partial s_2} = \mathcal{N}_1 z - \mathcal{G}_2 x - 1 & \dfrac{\partial z}{\partial s_2} = \mathcal{T} x - \mathcal{N}_2 y. \end{array}\right\} \tag{8}$$

These conditions are *necessary for the immobility of the point* (*x*, *y*, *z*), *and also sufficient*, since any infinitesimal displacement of the origin *M* on the surface can always result from two displacements along the coordinate lines. If one wishes to know, more generally, the absolute variations in space of the coordinates of a moving point at *M* when *M* displaces on the surface in the direction that is defined by the angle ω with *Mx* then it will be enough to take, as in (2), the difference between the left-hand and right-hand sides of formulas (8) in order to apply the operation:

$$\frac{\delta}{ds} = \cos\omega\,\frac{\delta}{\partial s_1} + \sin\omega\,\frac{\delta}{\partial s_2}\,. \tag{9}$$

Another set of three relations besides (8) has great importance, since they are necessary and sufficient for (8) to be satisfied by three functions *x*, *y*, *z* of q_1 and q_2. For example, in order for the function z to exist, by virtue of (5), it is necessary and sufficient that one must have:

$$\frac{\partial}{\partial s_2}(\mathcal{T} y - \mathcal{N}_1 x) - \frac{\partial}{\partial s_1}(\mathcal{T} x - \mathcal{N}_2 y) = \mathcal{G}_2 (\mathcal{T} x - \mathcal{N}_2 y)\, \mathcal{G}_1 (\mathcal{T} y - \mathcal{N}_1 x),$$

no matter what x, y, z are, and since the left-hand side can be given the form:

$$\left(\frac{\partial \mathcal{N}_2}{\partial s_1} + \frac{\partial \mathcal{T}}{\partial s_2} + \mathcal{T}\,\mathcal{G}_1 - \mathcal{N}_1 \mathcal{G}_2\right) y - \left(\frac{\partial \mathcal{N}_1}{\partial s_2} + \frac{\partial \mathcal{T}}{\partial s_1} + \mathcal{T}\,\mathcal{G}_2 - \mathcal{N}_2 \mathcal{G}_1\right) x,$$

by virtue of (8), one will see that the five curvatures $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{T}$, $\mathcal{G}_1$, $\mathcal{G}_2$, must satisfy the first two relations of the following three:

$$\begin{cases} \dfrac{\partial \mathcal{N}_2}{\partial s_1} + \dfrac{\partial \mathcal{T}}{\partial s_2} + 2\mathcal{T}\,\mathcal{G}_1 = (\mathcal{N}_2 - \mathcal{N}_1)\,\mathcal{G}_2, \\ \dfrac{\partial \mathcal{N}_1}{\partial s_2} + \dfrac{\partial \mathcal{T}}{\partial s_1} + 2\mathcal{T}\,\mathcal{G}_2 = (\mathcal{N}_2 - \mathcal{N}_1)\,\mathcal{G}_1, \\ \dfrac{\partial \mathcal{G}_1}{\partial s_2} + \dfrac{\partial \mathcal{G}_2}{\partial s_1} + \mathcal{G}_1^2 + \mathcal{G}_2^2 = \mathcal{T}^2 - \mathcal{N}_1 \mathcal{N}_2, \end{cases}$$

and also the set of three that one obtains by operating analogously on x and y. Those are the *Codazzi formulas:* The last one, in particular, bears the name of **Gauss**, and in the planar case, it will reduce to the known Lamé relation (VIII, § **10**).

10. Euler's theorem. – How do the normal curvatures and geodetic torsions vary around a point? Let everything that refers to an arbitrary curve that passes through M tangentially to a line that is inclined from the Mx-axis by ω be distinguished by the index ω. For that curve, the third formula (1) will become:

$$\frac{dz}{ds} = (-x \sin \omega + y \cos \omega)\, \mathcal{T}_\omega - (x \cos \omega + y \sin \omega)\, \mathcal{N}_\omega,$$

and on the other hand, if one observes (8), then one will have:

$$\frac{dz}{ds} = (\mathcal{T} y - \mathcal{N}_1 x) \cos \omega + (\mathcal{T} x - \mathcal{N}_2 y) \sin \omega .$$

Upon identifying them, one will get:

$$\mathcal{N}_\omega \cos \omega + \mathcal{T}_\omega \sin \omega = \mathcal{N}_1 \cos \omega - \mathcal{T}_1 \sin \omega, \tag{10}$$

$$\mathcal{N}_\omega \sin\omega - \mathcal{T}_\omega \cos\omega = \mathcal{N}_2 \sin\omega - \mathcal{T}_1 \cos\omega,$$

and one will infer that:

$$\mathcal{N}_\omega = \mathcal{N}_1 \cos^2\omega - 2\mathcal{T}\cos\omega\sin\omega + \mathcal{N}_2 \sin^2\omega. \tag{11}$$

One should note that such a form can always make the rectangular term vanish (but generally in just one way) by a suitable rotation of the axes around M in the tangent plane. Then and only then will one have $\mathcal{T} = 0$, so the two coordinate lines will be lines of curvature. Hence, as **Monge** found, *only two mutually-perpendicular lines of curvature pass through any point of a surface*. We call them *principal rays of curvature*, and always let R_1 and R_2 represent the radii of curvature of those normal sections that touch the lines of curvature. Of course, when the axes are oriented tangentially to those lines, (11) will become:

$$\mathcal{N}_\omega = \frac{\cos^2\omega}{R_1} + \frac{\sin^2\omega}{R_2}. \tag{12}$$

This important formula by **Euler**, which is a corollary to Meusnier's theorem, shows that if one wishes to know the curvatures of any curve at any point of a surface then it is enough to know them for two lines.

11. It can happen that $R_1 = R_2$ at special points that one calls *umbilics*, and one will then have $\mathcal{T} = 0$ in any direction, while $\mathcal{N}_\omega$ will not depend upon ω; that is to say, *an infinitude of lines of curvature are concurrent at an umbilic*, and the normal curvature will have the same value for all curves that pass through such points. Can a surface be composed of nothing but umbilics? In order to answer that question, assume that the coordinate lines are lines of curvature, and note that the first two Codazzi formulas will then become:

$$\frac{\partial}{\partial s_1}\frac{1}{R_2} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\mathcal{G}_2, \qquad \frac{\partial}{\partial s_2}\frac{1}{R_1} = \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\mathcal{G}_1,$$

which shows that the common value of R_1 and R_2 at any point where one has $R_1 = R_2$ will be a *constant* R. Having said that, it will be enough to observe that the conditions (8) are satisfied by $x = 0$, $y = 0$, $z = R$, to convince oneself that the points of the surface are all at a distance R from a fixed point, i.e., *the only surface upon which every point is umbilic is a sphere*. It will then follow that, other than the sphere (cf., § **2**), *no other surfaces exist for which every line is a line of curvature.*

12. Turning to (12), one should now observe that if R_1 and R_2 have the same sign then $\mathcal{N}_\omega$ will keep the same sign as ω varies, so it will never go to zero. Hence, no real asymptotes will pass through those points, which one calls *elliptical*. However, when R_1 and R_2 have opposite signs, $\mathcal{N}_\omega$ will go to zero in two directions, which are defined by the formula:

$$\tan \omega = \pm\sqrt{-\frac{R_2}{R_1}}\,. \tag{13}$$

Those points are then called *hyperbolic*, and one will see that *two real asymptotes will pass through any hyperbolic point*, and they are inclined the same with respect to the lines of curvature. Meanwhile, just as one deduces (11) from (10), one can also arrive at **Bonnet**'s formula:

$$\mathcal{T}_\omega = \mathcal{T}\cos 2\omega + \tfrac{1}{2}(\mathcal{N}_1 - \mathcal{N}_2)\sin 2\omega, \tag{14}$$

which says how the geodetic torsion varies around each point. If the angle ω is computed by starting from a line of curvature then the law of variation will take the simple form:

$$\mathcal{T}_\omega = \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\sin\omega\cos\omega. \tag{15}$$

In particular, when one sets ω equal to the values that are defined by (13) in (15), one will find that the radius of torsion of an asymptote is equal to $\sqrt{-R_1R_2}$; that is a noteworthy *theorem of Enneper.*

13. Dupin indicatrix. – In order to account for the way that a surface behaves around each of its points, it is useful to recall a geometric representation of formula (12). Along the tangent that is defined by the angle ω, take the segment *MP* that is equal to the square root of the absolute value of the radius of normal curvature. The locus of (real) points *P* is called the *Dupin indicatrix.* The coordinates of *P* in the tangent plane are:

$$x = \frac{\cos\omega}{\sqrt{\pm\mathcal{N}_\omega}}, \quad y = \frac{\sin\omega}{\sqrt{\pm\mathcal{N}_\omega}},$$

and one will then have:

$$\frac{x^2}{R_1} + \frac{y^2}{R_2} = \pm 1 \tag{16}$$

as the equation of the locus of points *P*, which will be either real or imaginary. At the elliptic points of the surface, equation (16) will represent two ellipses (III, § **4**), one of which is real, and is consequently the indicatrix, which will reduce to a circle at the umbilics. However, at the hyperbolic points, equation (16) will represent two real hyperbolas with common centers, axes, and asymptotes. Now, it is clear that the lines of curvature can be defined as the ones that touch an axis of the Dupin indicatrix at any of its points, while the asymptotes touch an asymptote of that indicatrix at each point. That explains why the asymptotes run through the only regions that are composed of hyperbolic points, and how two angular regions are determined around each of them, such that the normal curvature will be positive for one of them, while it will be negative for the other. Meanwhile, if one fixes the origin of the arcs of any curve at M for the moment then formulas (1) will show that one has:

$$\lim \frac{z}{s^2} = \frac{1}{2}\left(\mathcal{T} \lim \frac{dy}{ds} - \mathcal{N} \lim \frac{dy}{ds}\right) = \tfrac{1}{2}\mathcal{N},$$

and z will then have the sign of $\mathcal{N}$. Hence: *In the vicinity of an elliptic point, the surface will be situated completely on one side of the tangent plane.* However: *In the vicinity of a hyperbolic point, the surface will be cut by the tangent plane*, and the line of intersection will have two branches that pass through that point and touch the asymptotes.

14. – *Conjugate tangents* are any two conjugate diameters of the Dupin indicatrix. When the point M displaces along the surface in the direction that is defined by the angle ω with respect to the lines of curvature, the tangent plane ($z = 0$) will rotate around the line that is defined by the equation:

$$\cos \omega \frac{\partial z}{\partial s_1} + \sin \omega \frac{\partial z}{\partial s_2} = 0;$$

i.e., if one observes (8), $y = x \tan \omega'$, then one will have:

$$\tan \omega \tan \omega' = -\frac{R_2}{R_1}.$$

Hence (III, § **4**), two conjugate tangents are such that when a point displaces along one of them, the tangent plane will tend to rotate around the other one. In other words: *The generators of the developable that is circumscribed by a surface along a given curve are conjugate to the tangents of that curve.* In particular, the tangents to the two lines of curvature are mutually conjugate, and any tangent to an asymptotic line will be conjugate to itself. The angle θ that a tangent makes with its conjugate can be determined directly, by virtue of (1), by differentiating the equation of the tangent plane. One will get:

$$\tan \theta = \frac{\mathcal{N}}{\mathcal{T}}, \tag{17}$$

and one will see anew that one has $\theta = 0$ for asymptotes, and $\theta = \pi / 2$ for the lines of curvature.

15. For various questions, it is useful to recall the *spherical representation* of a surface, which consists of putting the points of the surface into correspondence with those of a sphere of radius 1, in such a way that the normals to the two surfaces at the corresponding points will be parallel. If x, y, z are the coordinates with respect to the usual axes of the point of the sphere that corresponds to the origin then the functions x, y, and $z + 1$, which are the coordinates of the center of the sphere, must satisfy the conditions (1), and one will then have:

$$\frac{dx}{ds} = \mathcal{N} z - \mathcal{G} y - 1 + \mathcal{N}, \qquad \frac{dy}{ds} = \mathcal{G} x - \mathcal{T} z - \mathcal{T}, \qquad \frac{dz}{ds} = \mathcal{T} y - \mathcal{N} y\,;$$

formulas (2) will then give:

$$\frac{\delta x}{ds} = \mathcal{N}, \qquad \frac{\delta y}{ds} = -\mathcal{T}, \qquad \frac{\delta z}{ds} = 0.$$

It will then follow that the angle between the tangents to the two curves is $\theta \pm \pi / 2$, where θ has the significance it had in (17). Hence: *The conjugate tangent and the tangent to the spherical image of a curve are perpendicular.* In particular, note that in the spherical representation, *the lines of curvature will not deviate,* while *the asymptotes will deviate by* $\pi / 2$.

16. – Just as we studied the way that the normal curvature and geodetic torsion varied in the infinitude of directions that one can consider around a point in the preceding paragraphs, we would now like to find the law of variation of the geodetic curvature, which does not have (§ **5**) a unique value for all points of the tangent curves at a point and in the same direction either. Let $\partial / \partial s$, $\partial / \partial s'$, $\mathcal{G}$ and $\mathcal{G}'$ be what the operations $\partial / \partial s_1$, $\partial / \partial s_2$, and the curvatures $\mathcal{G}_1$ and $\mathcal{G}_2$, resp., will become when the axes are rotated by ω in the tangent plane. Obviously:

$$\frac{\partial}{\partial s} = \cos\,\omega\,\frac{\partial}{\partial s_1} + \sin\,\omega\,\frac{\partial}{\partial s_2}, \qquad \frac{\partial}{\partial s'} = -\cos\,\omega\,\frac{\partial}{\partial s_1} + \cos\,\omega\,\frac{\partial}{\partial s_2}. \tag{18}$$

Meanwhile, if one applies the second operation (18) to the result of the first one then:

$$\frac{\partial^2}{\partial s\,\partial s'} = \cos^2\,\omega\,\frac{\partial^2}{\partial s_1\,\partial s_2} - \sin^2\,\omega\,\frac{\partial^2}{\partial s_2\,\partial s_1} - \cos\,\omega \sin\,\omega \left(\frac{\partial^2}{\partial s_1^2} - \frac{\partial^2}{\partial s_2^2}\right) + \frac{\partial \omega}{\partial s'}\frac{\partial}{\partial s'}.$$

However, if one applies the first one to the result of the second one then one will get:

$$\frac{\partial^2}{\partial s'\,\partial s} = \cos^2\,\omega\,\frac{\partial^2}{\partial s_2\,\partial s_1} - \sin^2\,\omega\,\frac{\partial^2}{\partial s_1\,\partial s_2} - \cos\,\omega \sin\,\omega \left(\frac{\partial^2}{\partial s_1^2} - \frac{\partial^2}{\partial s_2^2}\right) - \frac{\partial \omega}{\partial s}\frac{\partial}{\partial s}.$$

One then subtracts them and recalls that the relation (5) must be satisfied identically for any pair of orthogonal curves:

$$\left(\mathcal{G}' - \frac{\partial \omega}{\partial s'}\right)\frac{\partial}{\partial s'} - \left(\mathcal{G} + \frac{\partial \omega}{\partial s}\right)\frac{\partial}{\partial s} = \mathcal{G}_2 \frac{\partial}{\partial s_2} - \mathcal{G}_1 \frac{\partial}{\partial s_1}$$
$$= \mathcal{G}_2 \left(\sin \omega \frac{\partial}{\partial s} + \cos \omega \frac{\partial}{\partial s'}\right) - \mathcal{G}_1 \left(\cos \omega \frac{\partial}{\partial s} - \sin \omega \frac{\partial}{\partial s'}\right).$$

Hence:

$$\mathcal{G} + \frac{\partial \omega}{\partial s} = \mathcal{G}_1 \cos \omega - \mathcal{G}_2 \sin \omega, \qquad \mathcal{G}' - \frac{\partial \omega}{\partial s'} = \mathcal{G}_1 \sin \omega + \mathcal{G}_2 \cos \omega. \tag{19}$$

One can also arrive at these relations by applying the process that was applied to z in the beginning of § **10** to the coordinates x or y. One needs to observe that when the formulas:

$$\frac{\delta \alpha}{\partial s_1} = \frac{\partial \alpha}{\partial s_1} - \mathcal{N}_1 \gamma + \mathcal{G}_2 \beta, \qquad \frac{\delta \alpha}{\partial s_2} = \frac{\partial \alpha}{\partial s_2} - \mathcal{G}_1 \beta + \mathcal{T} \gamma$$

are applied to the direction $\alpha = \cos \omega$, $\beta = \sin \omega$, $\gamma = 0$, they will become:

$$\frac{\delta \omega}{\partial s_1} = \frac{\partial \omega}{\partial s_1} - \mathcal{G}_1, \qquad \frac{\delta \omega}{\partial s_2} = \frac{\partial \omega}{\partial s_2} + \mathcal{G}_2.$$

If one recalls (9) then one will have:

$$\frac{\delta \omega}{\partial s} = \frac{\partial \omega}{\partial s} - \mathcal{G}_1 \cos \omega + \mathcal{G}_1 \sin \omega, \qquad \frac{\delta \omega}{\partial s'} = \frac{\partial \omega}{\partial s'} + \mathcal{G}_1 \sin \omega + \mathcal{G}_1 \cos \omega;$$

i.e., by virtue of (19):

$$\mathcal{G} = -\frac{\delta \omega}{\partial s}, \qquad \mathcal{G}' = \frac{\delta \omega}{\partial s'}. \tag{20}$$

These formulas, which follow immediately from the ones that were given for $\mathcal{G}$ in § **5**, moreover, succeed in defining the geodetic curvature of a line on a given surface in the same way that the curvature of a planar line is defined in its plane. In addition, if the curve is considered to be traced on the developable that is circumscribed by the surface then it will be clear that its geodetic curvature will have the same value on the two surfaces, and on the other hand, one will see directly that the given curvature will remain unaltered when the developable is applied to the plane. Hence: *The geodetic curvature of a line that is traced on a surface is equal to the curvature that the line will acquire when the developable that is circumscribed by the surface along that line is applied to the plane.*

17. – We are now in a position to *calculate the flexions of the asymptotes.* Just as Enneper's formula (§ **12**) gives the torsion of that curve, there is an interesting formula by **Bonnet** that will make known the flexion when one is given the principal curvatures.

It is enough to substitute a value of ω that satisfies (13) in the first formula (19), and to attribute the values:

$$\mathcal{G}_1 = \frac{1}{\mathcal{N}_2 - \mathcal{N}_1}\frac{\partial \mathcal{N}_1}{\partial s_2}, \qquad \mathcal{G}_2 = \frac{1}{\mathcal{N}_1 - \mathcal{N}_2}\frac{\partial \mathcal{N}_2}{\partial s_1}$$

that result from the first two Codazzi formulas. In that way, one will obtain, from an easy calculation:

$$\left(\mathcal{G}_1 + \frac{\partial}{\partial s_2}\right)\cos\,\omega = \frac{1}{2\mathcal{N}_1^2}\left(\frac{\mathcal{N}_2}{\mathcal{N}_2 - \mathcal{N}_1}\right)^{3/2}\frac{\partial}{\partial s_2}\frac{\partial \mathcal{N}_1^3}{\partial s_2},$$

$$\left(\mathcal{G}_2 + \frac{\partial}{\partial s_1}\right)\sin\,\omega = \pm\frac{1}{2\mathcal{N}_2^2}\left(\frac{\mathcal{N}_1}{\mathcal{N}_1 - \mathcal{N}_2}\right)^{3/2}\frac{\partial}{\partial s_1}\frac{\partial \mathcal{N}_2^3}{\partial s_1},$$

so when one substitutes these values in:

$$\frac{1}{\rho} = \left(\mathcal{G}_1 + \frac{\partial}{\partial s_2}\right)\cos\,\omega - \left(\mathcal{G}_2 + \frac{\partial}{\partial s_1}\right)\sin\,\omega,$$

one will arrive at *Bonnet's formula:*

$$-\frac{1}{\rho} = \frac{4(-R_1R_2)^{7/3}}{(R_1 - R_2)^{3/2}}\left[\frac{\partial}{\partial s_2}\left(\frac{-R_2}{R_2^3}\right)^{1/3} \mp \frac{\partial}{\partial s_1}\left(\frac{R_1}{-R_2^3}\right)^{1/3}\right].$$

Among the consequences of this formula, we, with **Bonnet**, would like to point out the one that follows when one supposes that the surface is a quadric (IX, § **7**, *d*). Two lines, which can be real or imaginary, will then pass through any point that belongs to the surface, and since the lines are necessarily the asymptotes, the two values of the flexion that are give by the preceding formula must both be zero, and that will require that the ratio $R_1 : R_2^3$ must remain constant along any line q_1, while the ratio $R_2 : R_1^3$ must remain constant along any line q_2. Conversely, if that is the case then that will say that all of the asymptotes are lines, and the surface will then be a quadric. Hence: *The quadrics are characterized by the following property: Along each line of curvature, the corresponding principal curvature will vary proportionally to the cube of the other principal curvature.*

18. Theorems of Laguerre and Darboux. – When the operations (18) are applied to functions that depend upon ω explicitly, it is sometimes useful to exhibit the derivatives with respect to that variable by imagining that the operations $\partial / \partial s_1$ and $\partial / \partial s_2$ are performed under the hypothesis that ω stays constant. If, in addition, one would like to give the left-hand sides the significance of absolute derivatives in space, then one must multiply the derivatives with respect to ω by $\delta\omega / \partial s$ and $\delta\omega / \partial s'$, whose values are given by (20), and not $\partial\omega / \partial s$ and $\partial\omega / \partial s'$, resp. One will then have:

$$\begin{aligned} \frac{d}{ds} &= \cos\omega\,\frac{\partial}{\partial s_1} + \sin\omega\,\frac{\partial}{\partial s_2} - \mathcal{G}\,\frac{\partial}{\partial\omega}, \\ \frac{d}{ds'} &= -\sin\omega\,\frac{\partial}{\partial s_1} + \cos\omega\,\frac{\partial}{\partial s_2} + \mathcal{G}'\,\frac{\partial}{\partial\omega}. \end{aligned} \tag{21}$$

Having assumed that, one easily deduces from formulas (11) and (14) that:

$$\frac{\partial\mathcal{N}_\omega}{\partial\omega} = -2\mathcal{T}_\omega, \qquad \frac{\partial\mathcal{T}_\omega}{\partial\omega} = \mathcal{N}_\omega - \mathcal{N}_{\omega+\pi/2},$$

and if one then suppresses the index ω, which has become superfluous, then:

$$\frac{d\mathcal{N}_\omega}{ds} = \frac{\partial\mathcal{N}}{\partial s} + 2\mathcal{T}\,\mathcal{G}, \qquad \frac{d\mathcal{T}}{ds} = \frac{\partial\mathcal{T}}{\partial\omega} - (\mathcal{N} - \mathcal{N}')\,\mathcal{G}.$$

Now, since the operation $\partial / \partial s$ is supposed to leave ω constant, which is a quantity that can have a unique value for all of the curves that admit a given tangent at the point in question, the same result will always be obvious for all of those curves, and one can then say that, like $\mathcal{N}$ and $\mathcal{T}$, *each of the quantities:*

$$\mathbf{N} = \frac{d\mathcal{N}}{ds} - 2\,\mathcal{T}\,\mathcal{G}, \qquad \mathbf{T} = \frac{d\mathcal{T}}{ds} + (\mathcal{N} - \mathcal{N}')\,\mathcal{G}$$

has just one value for all curves on the surface that touch at a given point. The introduction of **N** and **T** into the calculations will often produce noteworthy simplifications. Here, we shall confine ourselves to pointing out the forms that the first two Codazzi formulas will take when they are written for any pair of orthogonal curves, while H represents the sum of the curvatures $\mathcal{N}$ and $\mathcal{N}'$, as usual, namely:

$$\frac{\partial H}{\partial s} = \mathbf{N} + \mathbf{T}', \qquad \frac{\partial H}{\partial s'} = \mathbf{N} - \mathbf{T}'.$$

19. – If one applies the first or second operation in (18) twice in succession then one will get the relations:

$$\frac{\partial^2}{\partial s^2} = \cos^2\omega\,\frac{\partial^2}{\partial s_1^2} + \sin^2\omega\,\frac{\partial^2}{\partial s_2^2} + \cos\omega\sin\omega\left(\frac{\partial^2}{\partial s_1\,\partial s_2} + \frac{\partial^2}{\partial s_2\,\partial s_1}\right) + \frac{\partial\omega}{\partial s}\frac{\partial}{\partial s'};$$

$$\frac{\partial^2}{\partial s'^2} = \sin^2\omega\,\frac{\partial^2}{\partial s_1^2} + \cos^2\omega\,\frac{\partial^2}{\partial s_2^2} - \cos\omega\sin\omega\left(\frac{\partial^2}{\partial s_1\,\partial s_2} + \frac{\partial^2}{\partial s_2\,\partial s_1}\right) - \frac{\partial\omega}{\partial s'}\frac{\partial}{\partial s}.$$

Summing them gives:

$$\left(\frac{\partial}{\partial s} + \frac{\partial\omega}{\partial s'}\right)\frac{\partial}{\partial s} + \left(\frac{\partial}{\partial s'} - \frac{\partial\omega}{\partial s}\right)\frac{\partial}{\partial s'} = \frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2};$$

i.e., by virtue of (19):

$$\left(\frac{\partial}{\partial s} + \mathcal{G}'\right)\frac{\partial}{\partial s} + \left(\frac{\partial}{\partial s'} + \mathcal{G}\right)\frac{\partial}{\partial s'}$$

$$= \frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2} + (\mathcal{G}_1\sin\omega + \mathcal{G}_2\cos\omega)\frac{\partial}{\partial s} + (\mathcal{G}_1\cos\omega - \mathcal{G}_2\sin\omega)\frac{\partial}{\partial s'}.$$

Meanwhile, the right-hand side can be given the form:

$$\frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2} + \mathcal{G}_1\frac{\partial}{\partial s_2} + \mathcal{G}_2\frac{\partial}{\partial s_1} = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\frac{\partial}{\partial s_1} + \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\frac{\partial}{\partial s_2}.$$

That sheds light upon the invariant character of the operation:

$$\Delta^2 = \left(\frac{\partial}{\partial s} + \mathcal{G}'\right)\frac{\partial}{\partial s} + \left(\frac{\partial}{\partial s'} + \mathcal{G}\right)\frac{\partial}{\partial s'},$$

which is why it is given the name of *second differential parameter*.

20. – It is now easy to extend Bonnet's formula, which was proved already (VIII, § **13**) for the plane to systems of curves that are traced upon an arbitrary surface. *In order to calculate the geodetic curvature of those lines that pass through a given point in the system that is defined by the function u*, we have the first formula in (19), which we agree to write in the following way:

$$\mathcal{G} = \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\cos\omega - \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\sin\omega.$$

If one sets:

$$\cos\omega = \frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_2}, \qquad \sin\omega = -\frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_1} \tag{22}$$

in that formula then one will find immediately *Bonnet's formula:*

$$\mathcal{G} = \frac{\Delta^2 u}{\sqrt{\Delta u}} + \Delta\left(u, \frac{1}{\sqrt{\Delta u}}\right).$$

However, the second formula (19) will give:

$$\mathcal{G}' = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\cos\omega + \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\sin\omega,$$

and if one substitutes the values (22), while taking the conditions (5) into account, then one will find that:

$$\mathcal{G}' = -\frac{\partial\left(u, \dfrac{1}{\sqrt{\Delta u}}\right)}{\partial(s_1, s_2)}.$$

Here, one must notice that the constant vanishing of $\mathcal{G}'$ is necessary and sufficient because, on the one hand, Δu is a function of u, and on the other hand, because the curves of the system are the orthogonal trajectories of a system of geodetics – or, as one can say, for a reason that is easy to explain (cf., § **8**), because they are geodetically parallel. Therefore: *In order for the curves of the system that is defined by the function u to be geodetically parallel, it is necessary and sufficient that* Δu *should be a function of u.*

21. Isothermal systems. – The considerations that were made for plane curves in §§ **6** and **11** of Chapter VIII are immediately applicable to the systems of curves that are traced on an arbitrary surface, and one can then speak of *isothermal systems,* and regard it as having been proved that *in a doubly-orthogonal system, one system of curves cannot be isothermal without the other one being that way, too.* The calculations that were performed in § **12** of that chapter can be assumed to have been repeated here in order to assert that consequently *the condition:*

$$\frac{\partial \mathcal{G}_1}{\partial s_1} = \frac{\partial \mathcal{G}_2}{\partial s_2} \tag{23}$$

is necessary and sufficient for the system of coordinate lines to be isothermal. For example, the condition (23) is satisfied when $\mathcal{G}_1$ and $\mathcal{G}_2$ are constant along the respective lines. It will then follow that: *Any doubly-orthogonal system that is composed of geodetically-constant lines of curvature is isothermal.* In addition, one should note that if the condition (23) is satisfied, and if $\mathcal{G}_1$ remains constant along any line q_1 then $\mathcal{G}_2$ will also be constant along any line q_2 . Hence: *If the lines of one system in an isothermal double system have constant geodetic curvature then the same thing will be true for the lines of the other system.*

22. – Now suppose that one has determined a function g such that one has:

$$\Delta^2 g = \mathcal{N}_1\,\mathcal{N}_2 - \mathcal{T}^2, \tag{24}$$

and observe that Gauss's formula (or the third Codazzi formula) can be written in the following way:

$$\Delta^2 g + \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\mathcal{G}_2 + \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\mathcal{G}_1 = 0;$$

if one gives Δ^2 its expression then:

$$\left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\left(\frac{\partial g}{\partial s_1} + \mathcal{G}_2\right) + \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\left(\frac{\partial g}{\partial s_2} + \mathcal{G}_1\right) = 0.$$

It will then follow that the condition (4) is satisfied by the functions:

$$u = \frac{\partial g}{\partial s_2} + \mathcal{G}_1, \qquad v = -\frac{\partial g}{\partial s_1} - \mathcal{G}_2,$$

and there will then exist a function f such that one can write:

$$\mathcal{G}_1 = \frac{\partial f}{\partial s_1} - \frac{\partial g}{\partial s_2}, \qquad \mathcal{G}_2 = \frac{\partial f}{\partial s_2} - \frac{\partial g}{\partial s_1}. \tag{25}$$

That is therefore a form that one can always give to the functions $\mathcal{G}_1$ and $\mathcal{G}_2$. Conversely, if it happens, in an arbitrary way, that $\mathcal{G}$ has been given in the form (25) then one can be certain that the function g satisfies (24). In order to convince oneself, it is enough to substitute the values (25) in Gauss's formula. Finally, observe that:

$$\Delta^2 f = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\left(\frac{\partial g}{\partial s_2} + \mathcal{G}_1\right) - \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\left(\frac{\partial g}{\partial s_1} + \mathcal{G}_2\right) = \frac{\partial \mathcal{G}_1}{\partial s_1} - \frac{\partial \mathcal{G}_2}{\partial s_2}.$$

Therefore: *In order for the system of coordinate lines to be isothermal, it is necessary and sufficient that the function f should be harmonic.* However, one will see that the reduction of g to a harmonic function is indicative of a considerable specialization of the surface. Moreover, the formula (25) is true for any doubly-orthogonal system, and one will, in fact, deduce from (19), by virtue of (25) itself, that:

$$\mathcal{G} = \frac{\partial}{\partial s}(f - \omega) - \frac{\partial g}{\partial s'}, \qquad \mathcal{G}' = -\frac{\partial}{\partial s'}(f - \omega) - \frac{\partial g}{\partial s},$$

and consequently, if one desires that the system that is defined by the angle ω should be isothermal then one would need to do that in such a way that the function $f - \omega$ was harmonic. Therefore: *The determination of all isothermal systems of a surface depends upon the integration of the equation:*

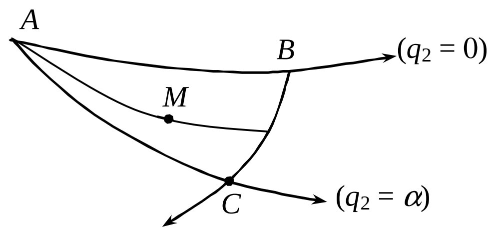
$$\Delta^2 \omega = \frac{\partial \mathcal{G}_1}{\partial s_1} - \frac{\partial \mathcal{G}_2}{\partial s_2}.$$

23. Curvature. – We can infer a means for measuring the curvature of a surface at a given point from considering the orthogonal invariants:

$$H = \mathcal{N} + \mathcal{N}' = \frac{1}{R_1} + \frac{1}{R_2}, \qquad K = \mathcal{N}\mathcal{N}' - \mathcal{T}^2 = \frac{1}{R_1 R_2}$$

of the quadratic form (11). The *mean curvature* at M is the arithmetic mean of the normal curvatures of all curves in the surface that pass through M, if one imagines that the orientations of those curves are distributed equally around M. If one associates any curve with one that is perpendicular to it then the sum of the normal curvatures will remain constantly equal to H when the pair of tangents rotates around M, and it will then be clear the desired mean is $H / 2$. Therefore: *The mean curvature is measured by* $H / 2$; i.e., by *one-half the sum of the principal curvatures.* One says *total curvature* – or simply *curvature* – of the surface to mean the limit of a certain ratio that is analogous to the one that one considers in order to measure the curvature of a plane curve. One takes a *linear element* on such a curve and constructs the normals at its boundary points. The ratio of the *angle* between the normals to the *length* of the element tends to measure the curvature of the line considered at the point M when the element tends to reduce to the single point M. Analogously, in order to measure the curvature of a surface at a point M, **Gauss** imagined a *surface element* around M and constructed the normals to the surface along the contour to the element. The solid *angle* that is subtended by those normals, divided by the *area* of the element will tend to measure the curvature at M when the element tends to reduce to M. By definition, that solid angle will then be measured by the area that is cut out from a spherical surface of radius 1 by the cone of rays that are parallel to the line considered. Therefore, the curvature is measured by the ratio of the area of the spherical image of the surface element to the area of that element. Suppose that it is the rectangle that is constructed from the elementary arcs ds_1 and ds_2 of the lines of curvature that pass through M. Since those lines are not deviant in the spherical representation, the spherical image of the rectangle will be another rectangle, and from what we saw in § **15**, its edges will be $\mathcal{N}_1\, ds_1 = ds_1 : R_1$ and $\mathcal{N}_2\, ds_2 = ds_2 : R_2$ (since $\mathcal{T} = 0$), which results from simple geometric considerations, moreover. Therefore, $ds_1\, ds_2$ and $K\, ds_1\, ds_2$ will be the areas of the two rectangles, and *the total curvature will then be measured by K – i.e. by the product of the principal curvatures.* If the first rectangle is constructed from any two orthogonal curves then one will arrive at that result a little less rapidly, but one will succeed in exhibiting the invariant character of the expression $\mathcal{N}\mathcal{N}' - \mathcal{T}^2$, in addition. Here, one should observe that knowing that character will permit one to immediate state Enneper's theorem, which was proved at the end of § **12**. In fact, one has $\mathcal{N} = 0$, $\mathcal{T} = -1 / r$ for the asymptotes and consequently $K = -1 / r^2$. Hence, at any point, *the total curvature (with the opposite sign) is equal to the square of the torsion of the asymptote.* To conclude, we shall point out the form that **Gauss** gave to the expression for K. It is enough to substitute the values (3) in the third Codazzi formula in order to obtain (cf., VIII, § **10**):

$$K = -\frac{1}{Q_1 Q_2}\left[\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\right)\right].$$



24. – Consider a *geodetic triangle* – i.e., the figure *ABC* that is determined by three geodetics on a surface – and let α, β, γ be the internal angles of that triangle. In order to determine the area σ of the image of *ABC* in the spherical representation, take the lines q_1 to be the outgoing geodetics at the vertex *A*, and assume that the coordinates q_1 and q_2 of any point *M* are the geodetic distance *AM* and the angle that geodetic *AM* makes with *AB*, resp. Among the lines q_2 that are orthogonal trajectories of the outgoing geodetics at the vertex *A*, the ones that are infinitely close to *A* can be considered to be situated in the plane that touches the surface at *A*, and their elementary arc-lengths $Q_2\, dq_2$ can then coincide with $q_1\, dq_2$, up to higher-order infinitesimals. In other words:

$$\lim_{q_1=0}\frac{Q_2}{q_1} = 1, \qquad \lim_{q_1=0}\frac{\partial Q_2}{\partial q_1} = 1.$$

Having said that, observe that by virtue of Gauss's formula, one will have:

$$\sigma = \iint K\, Q_1\, Q_2\, dq_1\, dq_2 = -\iint \frac{\partial^2 Q_2}{\partial q_1^2}\, dq_1\, dq_2\,,$$

in which the integration extends over the entire area that is enclosed by *ABC*. On the other hand, the first formula in (19) will give:

$$\frac{\partial \omega}{\partial s} = -\mathcal{G}_2 \sin \omega, \qquad \text{i.e.,} \qquad d\omega = -\mathcal{G}_2\, Q_2\, dq_2 = -\,dq_2\,.$$

Now, turning to σ, if one first performs the integration along a geodetic that is defined by the values of q_2 that are found between 0 and α, and if one can vary q_2 from 0 to α, then by virtue of the preceding observation, one will get:

$$\sigma = \int_0^{\alpha}\left(1-\frac{\partial Q_2}{\partial q_1}\right)dq_2 = \alpha + \int_B^C \omega,$$

in which ω – viz., the inclination of the geodetic BC from the lines q_1 – has the value $\pi - \beta$ at B and the value γ at C. Therefore:

$$\sigma = \alpha + \gamma - (\pi - \beta) = \alpha + \beta + \gamma - \pi.$$

In particular, when K is constant, σ will represent the area of ABC, multiplied by K. Therefore, on a surface of constant curvature (which we will speak of in the following chapter), *the area of a geodetic triangle will be proportional to the excess of the sum of its angles over two right angles.* It follows that this sum will be greater than, less than, or equal to two right angles according to whether the curvature is positive, negative, or zero, resp.

25. Mappability. – In this first attempt, we would like to limit ourselves to a few comments about the mappability of one surface onto another. If it is possible to establish a correspondence between the points of the two surfaces such that the geodetic distance between two points, when taken at random on one surface, is equal to the geodetic distance between the points that they correspond to on the other one then one will say that the two surfaces are *mappable* to each other, because a material that one images to be woven from flexible, inextensible filaments that are stretched in all directions on one of them can obviously be mapped to the other one without the filaments (cf., § **8**) being torn or broken; i.e., without the material being folded or torn. In other words, the surfaces that can be mapped onto a given surface can be considered to be the infinitude of configurations that they can assume when they are flexed without stretching or contracting any of their parts. If the elementary arc-length on a surface, when referred to an arbitrary system of orthogonal curvilinear coordinates, is given by the formula $ds^2 = Q_1^2\, dq_1^2 + Q_2^2\, dq_2^2$ then one must be able to find a system of orthogonal curvilinear coordinates on any surface that can be mapped to the given surface such that the elementary arc-length can be represented by the same formula, and it is obvious that one will find precisely the necessary and sufficient condition for the mappability of the two surfaces in that possibility. It will then follow that if one writes down Gauss's formula for the two mappable surfaces at two corresponding points then one must find the same value for K. Hence: *In order for two surfaces to be mappable to each other, it is necessary that the corresponding points must have the same curvature.* In other words, when a flexible, inextensible surface is deformed in space, there is something that does not vary at each point, namely, the total curvature.

26. Evolutes and developments. – The properties of the evolutes of plane curves compel us to deal with them by analogy with the locus of centers of principal curvature of any surface, which is a locus that obviously composed to two sheets, one of which is generated by the center C_1, while the other is generated by the center C_2. Each sheet can also be considered to be the locus of the edges of regression of the infinitude of developables that cut the given surface orthogonally along the lines of curvature of one system. The two sheets constitute what one calls the *evolute* of the proposed surface and take the name of the *development* with respect to the evolute. Consider the first sheet –

i.e., the one that is generated by the point C_1 – whose coordinates are $x = 0$, $y = 0$, $z = R_1$. When M displaces along a line of curvature q_1, the fundamental formulas will give:

$$\frac{\delta x}{\partial s_1} = 0, \qquad \frac{\delta y}{\partial s_1} = 0, \qquad \frac{\delta z}{\partial s_1} = \frac{\partial R_1}{\partial s_1}, \tag{26}$$

and C_1 will then displace along the normal, which could have been foreseen, since C_2 then traverses the edge of regression of the developable that is composed of the normals to the surface along the given line q_1 . However, when M displaces along the line q_2, one will find that:

$$\frac{\delta x}{\partial s_2} = 0, \qquad \frac{\delta y}{\partial s_2} = \frac{l}{R_2}, \qquad \frac{\delta z}{\partial s_2} = \frac{\partial R_1}{\partial s_2},$$

in which l represents the length of the segment C_1C_2 . Therefore, for an arbitrary displacement of M that is defined by the inclination ω of the line q_r, from formulas (9), one will have:

$$\frac{\delta x}{ds} = 0, \qquad \frac{\delta y}{ds} = \frac{l}{R_2}\sin\omega, \qquad \frac{\delta z}{ds} = \frac{dR_1}{ds}. \tag{27}$$

From the first of these equalities, one will see that *the tangent plane to one sheet of the developable is normal to the corresponding line of curvature,* from which, it will follow immediately that the tangent planes to the two sheets at two corresponding points are always mutually perpendicular. In addition, (27) shows that if one wishes that C_1 should displace perpendicularly to the normal then M must move in such a way that R_1 will remain constant. In other words, those lines correspond to the edges of regression of the developables of the normals along the lines of curvature of one system, so *their orthogonal trajectories will correspond to the developments along which the corresponding radius of curvature will remain constant.* Now, it is natural to assume that those edges and their orthogonal trajectories are coordinate lines on the surface (C_1). Hence, at C_1, the axes x' and y' are parallel to the z and y axes, respectively, and the z'-axis must point in the opposite direction to the x-axis. Having said that, in order to find all of the fundamental curvatures that relate to the surface (C_1), it is enough say that the immobility conditions for the point (x, y, z) that are satisfied with respect to the trihedron of the surface (M) are also satisfied with respect to the trihedron of (C_1) of the new coordinates:

$$x' = z - R_1, \qquad y' = y, \qquad z' = -x. \tag{28}$$

First of all, in order to find the relations between the new and the old differential quotients, one should note that it will result from the formulas (26) and (27) that:

$$ds_1' = \frac{\partial R_1}{\partial s_1} ds_1, \qquad ds_2' = \frac{l}{R_2}\sin\omega\, ds,$$

in which ω is defined by the conditions:

$$\frac{dR_1}{ds} = 0, \qquad \text{i.e.,} \qquad \cos\omega \frac{\partial R_1}{\partial s_1} + \sin\omega \frac{\partial R_1}{\partial s_2} = 0.$$

It follows that:

$$\frac{\partial R_1}{\partial s_1}\frac{\partial}{\partial s_1'} = \frac{\partial}{\partial s_1}, \qquad \frac{l}{R_2}\frac{\partial R_1}{\partial s_1}\frac{\partial}{\partial s_2'} = \frac{\partial R_1}{\partial s_1}\frac{\partial}{\partial s_2} - \frac{\partial R_1}{\partial s_2}\frac{\partial}{\partial s_1}. \tag{29}$$

If the first of those operations is applied to the third coordinate (28) then one will get:

$$\frac{\partial R_1}{\partial s_1}\frac{\partial z'}{\partial s_1'} = -\frac{\partial x}{\partial s_1} = -\mathcal{N}_1\, z + \mathcal{G}_1\, y + 1,$$

while on the other hand, if one distinguishes everything that refers to the surface (C_1) by a prime then one must have:

$$\frac{\partial z'}{\partial s_2'} = \mathcal{T} y - \mathcal{N}_1' x' = \mathcal{T}'\, y - \mathcal{N}_1' z + \mathcal{N}_2' R_1;$$

therefore, upon identification:

$$\frac{\mathcal{N}_1'}{\mathcal{N}_1} = \frac{\mathcal{T}'}{\mathcal{G}_1} = \frac{1}{\dfrac{\partial R_2}{\partial s_1}}.$$

Hence, $\mathcal{N}_1'$ and $\mathcal{T}'$ will be known when the principal curvatures of the development are known, since from the Codazzi formula, one will have:

$$\mathcal{G}_1 = \frac{R_2}{lR_1}\frac{\partial R_1}{\partial s_2}, \qquad \mathcal{G}_2 = -\frac{R_1}{lR_2}\frac{\partial R_2}{\partial s_1}. \tag{30}$$

Analogously, when one considers the equalities:

$$\frac{\partial R_1}{\partial s_1}\frac{\partial y'}{\partial s_1'} = \frac{\partial y}{\partial s_1} = \mathcal{G}_1\, x, \qquad \frac{\partial y'}{\partial s_1'} = \mathcal{G}_1' x' - \mathcal{T}'\, z' = \mathcal{T}'\, x + \mathcal{G}_1' z - \mathcal{G}_1' R_1,$$

one will recover the value of $\mathcal{T}'$, and get that $\mathcal{G}_1' = 0$, in addition. Thus: *The edges of regression of the developable of normals to a surface are geodetics on the evolute surface.* If the first operation (29) is also applied to x' then one will arrive at only a confirmation of the preceding results. Now, apply the second operation in (29) to x':

$$\frac{l}{R_2}\frac{\partial R_1}{\partial s_1}(\mathcal{T}'\, x + \mathcal{G}_2'\, y) = \frac{\partial R_1}{\partial s_2}\left(\mathcal{N}_2 x + \frac{\partial R_1}{\partial s_1}\right) - \frac{\partial R_1}{\partial s_1}\left(\mathcal{N}_2 y + \frac{\partial R_1}{\partial s_2}\right).$$

One will then get a value for $\mathcal{T}'$ that formulas (30) show will be equal to the preceding one. In addition, one will find that $\mathcal{G}_2' = -1 / l$, i.e., that C_2 *is the center of geodetic curvature of that curve* R_1 = constant *that* C_1 *passes through on the first sheet*, just as, conversely, C_1 is the center of geodetic curvature at C_2 of the curve of the system that is defined on the second sheet by the condition R_2 = *constant*. Finally, if the second operation (29) is applied to y' or to z' then one will find that:

$$\frac{l}{R_2}\frac{\partial R_1}{\partial s_1}\mathcal{N}_2' = \mathcal{G}_2\frac{\partial R_1}{\partial s_1} + \mathcal{G}_1\frac{\partial R_1}{\partial s_2},$$

and one can then also express $\mathcal{N}_2'$ in terms of just the functions R, by virtue of (30).

27. – Conversely, any surface can be associated with an infinitude of other ones, each of which, along with the given surface, constitutes the evolute of an entire system of parallel surfaces. Indeed, if one wishes that the point $(-t, 0, 0)$ should displace normally to Mx arbitrarily in the direction of displacement for M on the given surface then the fundamental formulas will show that it is necessary and sufficient that one must have $\partial t : \partial q_1 = Q_1$, $\partial t : \partial q_2 = 0$; i.e., t, and consequently Q_1, must be functions of only the parameter q_1, and the condition $\mathcal{G}_1 = 0$ will then be satisfied on the lines q_1, which was already found to be necessary in the preceding paragraph. Hence: *In order for the lines of one congruence to be normal to a surface, it is necessary and sufficient that they should be the tangents to a simple infinitude of geodetics to another surface.* Meanwhile, if one takes $Q_1 = 1$ then one must have that $t = s_1$ + constant, and therefore an infinitude of points of Mx that are at a constant distance from each other and describe an infinitude of surfaces whose normals are all tangents to the given surface. Obviously, if one considers the tangents to a line q_1 then they will be incident on lines of curvature on an infinitude of parallel surfaces, and the point of contact will define the centers of curvature at any instant. One will then have that *any surface can be considered to be one of the two sheets of the evolute of one system of parallel surfaces for each of its simply-infinite systems of curvilinear geodetics.* The other sheet is said to be *complementary* to the first one, and it will clearly result from what was said that *any surface will admit an infinitude of complementary surfaces, each of which corresponds to a simply-infinite system of curvilinear geodetics that are traced arbitrarily on the surface in question.* Now, the theorem that was proved at the end of the preceding paragraph can be stated by saying that *the complementary surface that corresponds to a given system of geodetics is the locus of centers of geodetic curvature of the orthogonal trajectories to those geodetics.* In conclusion, we say that one can mechanically trace the development of a given surface, in such a way that everything will be similar to what we already know for the plane or skew curves. One imagines a material that is woven with inextensible filaments that are stretched over the surface and intersected at a right angle by an infinitude of other complementary deformable filaments, in such a way that all parts of the material can adhere to the surface without folding or tearing. Having said that, if one unrolls the material from the surface, while always taking care to maintain its tension in the directions of the inextensible filaments then it will be clear that the other filaments, which describe the infinitude of parallel developables that correspond to the system of geodetics (cf., § **8**), will be the original filaments of the surfaces.

CHAPTER XII

EXERCISES ON SURFACES

1. – In order to *determine the edge of regression of the developable that is circumscribed* by a surface along a given line, we need to recall [XI, form. (17)] that the generator of that developable is defined in the tangent plane at M by means of the angle $\theta = \arctan(\mathcal{N} : \mathcal{T})$. We then find the distance t from M to the corresponding point of the edge of regression by differentiating the equation $y = x \tan\theta$ and expressing the idea that the coordinates of that point (viz., $-t\cos\theta$, $-t\sin\theta$, 0) satisfy the immobility condition [XI, form. (1)]. In that way, one gets:

$$t = \frac{\sin\theta}{\mathcal{G} - \frac{d\theta}{ds}}. \tag{1}$$

It is then easy to calculate the arc length and the curvatures by the usual process, thanks to formulas (2) of the previous chapter. For the lines of curvature, formula (1) shows that $-t$ reduces to the radius of geodetic curvature, and in that case, it is clear, moreover, that the edge that we seek is an evolute of the curve, in such a way that t must vary, in general, because it represents precisely the arc length of the evolute. However, if t is constant (and that can happen when $\mathcal{G}$ is constant) then that edge will reduce to a point; i.e., the circumscribed developable is a conic surface whose generators cross the curve in question at a right angle. One then arrives at the following proposition of **Brioschi**, which generalizes a known theorem (XI, § **2**):

Any line of curvature that has constant geodetic curvature will belong to a sphere that cuts the surface orthogonally.

We return to (1) in order to observe that the lines along which the circumscribed surface is cylindrical (these lines are interesting in that they separate the part of the surface that are illuminated in a sheaf of parallel rays from the one that remains in the shadow) are characterized by the equality $\mathcal{G} = d\theta : ds$, which we can put into a form that expresses the geodetic curvature at any point M as a function of the quantity (XI, § **18**) that remain invariant for all of the curves in M that are tangent to the curve in question. Indeed, if we calculate the derivative of θ then we will transform the preceding condition into this one:

$$K\mathcal{G} = \mathcal{T}N - \mathcal{N}T.$$

Therefore, if we are given the tangent then we can also determine the osculating plane at any point.

2. – The property of N that was just recalled permits one to *calculate the curvature at a point M of the intersection of a surface with the tangent plane at M.* One knows that this curve has two branches that are tangents to the asymptotes. Meanwhile, if one distinguishes everything that refers to one asymptote by an index then one will have:

$$\mathcal{N} = 0, \qquad \mathcal{T} = -\frac{1}{r_0}, \qquad \mathcal{G} = \frac{1}{\rho_0}, \qquad N = \frac{2}{\rho_0\, r_0}.$$

On the other hand, the geodetic torsion of all curves that are tangent to the asymptote will have (XI, § **4**) just one value, and therefore, *at the point M*:

$$\frac{d\psi}{ds} = \frac{1}{r} - \frac{1}{r_0}.$$

Hence, if the curve is the intersection of the surface with the tangent plane, in particular, in which case, one will have $\psi = \pi / 2$, $1 / r = 0$, then one also have:

$$\frac{d\mathcal{N}}{ds} = \cos y\,\frac{d}{ds}\frac{1}{\rho} - \frac{\sin\psi}{\rho}\frac{d\psi}{ds} = \frac{1}{\rho\, r_0}, \qquad N = \frac{d\mathcal{N}}{ds} - 2\mathcal{T}\mathcal{G} = \frac{3}{\rho\, r_0}.$$

If one equates the two values of N then one will find that $\rho = 3\rho_0 / 2$, in general, and one will discover the elegant theorem of **Beltrami** from this:

The curvature of each branch of the intersection of a surface with the tangent plane at a hyperbolic point is equal to 3/2 *of the curvature of the asymptotic tangent to the branch in question at that point.*

Here, recalling what was said at the end of § **4** of the preceding chapter, we direct our attention to the example of lines that touch and osculate at one point, but do not have equal flexion at the contact point.

3. – Any curve in a plane (XI, § **2**) is asymptotic to all lines of curvature, in such a way that K is zero because $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{T}$ are zero. More generally, the curvature of any developable surface is zero because a line will pass through any point of that surface along which the normals to the surface will form a plane. That line is therefore a line of curvature, and on the other hand (XI, § **1**), it is also an asymptote, from which, it will follow that since $\mathcal{N}$ and $\mathcal{T}$ are zero for any rectilinear generator, one will have $K = 0$ at any point. *Do there exist other surfaces with zero curvature?* If the surface is indeed referred to its lines of curvature then $\mathcal{T}$ must constantly be zero, and that must also be true for one of the $\mathcal{N}$ – for example, $\mathcal{N}_1$. Meanwhile, the second Codazzi formula (XI, § **9**) will give $\mathcal{N}_2\, \mathcal{G}_1 = 0$. If that condition is satisfied by taking $\mathcal{N}_2 = 0$ then a known formula

(XI, § **12**) will show that one has $\mathcal{T}_\omega = 0$ for any ω – i.e., all lines are lines of curvature – and therefore (XI, **11**), the surface will necessarily be spherical. More particularly, it is also *planar*, since a sphere of finite radius does not admit real asymptotes. If one does not have $\mathcal{N}_2 = 0$ then one must suppose that $\mathcal{G}_1 = 0$. Hence (XI, § **1**), any line q_1 that is geodetic and asymptotic will necessarily be a *straight line*; that is to say, the surface will be a ruled surface. It cannot be twisted, since otherwise (IX, § **8**), the generators would not be lines of curvature. Hence, *the only surfaces with zero curvature are the plane and the developable surfaces.*

4. – Can a surface admit two systems of geodetics that cut at a constant angle? If the geodetics of a system are taken to be coordinate lines q_1 then one will have $\mathcal{G}_1 = 0$, and the first formula (19) of the preceding chapter will show that one must also have $\mathcal{G}_2 = 0$, in such a way that any other system of trajectories of the curves of the given system will be composed of geodetics. That will happen on the plane and on developables. Meanwhile, Gauss's formula gives $K = 0$, and one can then state the following theorem of **Liouville**:

Two systems of geodetics of a non-developable surface cannot cut at a constant angle.

5. – *What is the curvature of a ruled surface?* If α, β, γ are the direction cosines of the generator with respect to the fundamental trihedron of any curve on the surface then the angle ψ between the principal normal and the normal to the surface (which is perpendicular to the tangent and the generator) will be given by the relation $\beta \sin \psi + \gamma \cos \psi = 0$, from which, when one differentiates and takes some known conditions [IX, form. (37)] into account, one will deduce, in succession, that:

$$\frac{d\psi}{ds} = \frac{1}{\tau} + \frac{\alpha\beta}{(\beta^2 + \gamma^2)\,\rho}, \qquad \mathcal{T} + \alpha\,\mathcal{N} = -\frac{\cos^2\tau}{p}.$$

Now, if one assumes that the generator is a fundamental line then it will be clear that one must have $\mathcal{N} = 0$, and therefore [IX, form. (30)]:

The distributor parameter of a twisted ruled surface represents the radius of geodetic torsion of the generators along the line of striction (up to sign).

Since $K = -\mathcal{T}^2$, one will then see that *the absolute value of the curvature of a ruled surface along the line of striction is the inverse of the square of the distributor parameter.* However, the curvature will vary along a generator like $\cos^4 \tau$, in such a way that it is annulled at infinity. That explains the fact that any ruled surface will admit an *asymptotic developable*, which will be the envelope of the planes that go through the generators perpendicular to the central planes, and therefore it will always behave like a developable at infinity. Now, that we know the value of $\mathcal{T}$ and that $\mathcal{N}$ and $\mathcal{G}$ are zero, the second Codazzi formula will immediately give the geodetic curvature of any orthogonal trajectory of the generators:

$$\mathcal{G}' = -\frac{1}{2}\frac{\partial}{\partial s}\log \mathcal{T} = -\frac{\partial}{\partial s}\log\cos\tau = \frac{t}{t^2+p^2}.$$

Since $\mathcal{G}'$ will not be annulled at a finite distance unless it is annulled for $t = 0$, one will see (cf., IX, § **16**) that *the line of striction is the locus of points at which the geodetic curvatures of the orthogonal trajectories to the generators are zero.*

6. Surfaces of rotation. – One gives that name to the surface that is generated by a plane curve that rotates around a fixed line in its plane that one calls the *axis*. The generating curve in each of the infinitude of its positions is called a *meridian*, and the circle that is described by any point of a meridian is called a *parallel*. Therefore, the meridians and the parallels are the sections that are made in the surface by the planes that pass through the axis and the perpendiculars to the axes, resp. One sees no reason why the normal to the surface at an arbitrary point M should be situated on one side of the plane of the meridian that passes through M, rather than the other one, and that is why that normal will always coincide with the normal to the meridian at M. Hence, the meridians are both lines of curvature and geodetics of the surface. It will then follow that if one assumes that the lines q_1 are meridians then one will have:

$$\mathcal{N}_1 = \frac{1}{\rho_1}, \qquad \mathcal{G}_1 = 0, \qquad \mathcal{T} = 0. \tag{2}$$

The other system of lines of curvature is composed of the orthogonal trajectories to the meridians (i.e., the parallels) as one can see directly, moreover, by observing that the normals to the surface along each parallel will concur on the axis. If φ is the inclination of the axis with respect to the tangents to the meridians along a parallel of radius q then it will be clear that φ and q will be functions of only $q_1 = s_1$, and it is known (II, § **1**) that one has:

$$\frac{d\varphi}{ds_1} = \frac{1}{\rho_1}, \qquad \frac{dq}{ds_1} = \sin\varphi. \tag{3}$$

Finally, observe that since ψ is $\pi - \varphi$ in the present case, the definitions that were given at the beginning of the preceding chapter will give

$$\mathcal{N}_2 = -\frac{\cos\varphi}{q}, \quad \mathcal{G}_2 = \frac{\sin\varphi}{q}. \tag{4}$$

One then sees that the principal radii of curvature are:

$$R_1 = \rho_1\,, \qquad R_2 = -\frac{q}{\cos\varphi}\ .$$

That is to say, the principal centers of curvature at any point M are the centers of curvature of the meridian that passes through M and that point on the axis that is on the normal to the surface that is raised at M. If one takes (3) into account then it is easy to verify that the values (2) and (4) satisfy the Codazzi relations. Having said that, a known formula [XI, form. (19)] will give the value of the geodetic curvature in any direction:

$$\mathcal{G} = -\,\mathcal{G}_2 \sin\,\omega - \frac{\partial\omega}{\partial s}.$$

On the other hand, one will have:

$$\mathcal{G}_2 = \frac{\sin\varphi}{q} = \frac{1}{q}\frac{dq}{ds_1} = \frac{1}{q\cos\omega}\frac{\partial q}{\partial s}\,,$$

by virtue of (3). Hence:

$$\mathcal{G} = -\frac{1}{q\cos\omega}\frac{\partial}{\partial s}(q \sin\,\omega). \tag{5}$$

For $\mathcal{G} = 0$, one will find the following theorem of **Clairaut:**

Any geodetic of a surface of rotation will meet the meridians at an angle whose sine will value from one meridian to the other in proportion to the curvature of the parallel.

7. – *Determine the asymptotes of a surface of rotation;* i.e., if one is given the intrinsic equation $f(s, \rho) = 0$ of the meridian then find the intrinsic equations of the asymptote. One first knows that the inclination ω of that curve with respect to the meridians is given (XI, § **12**) by the formula:

$$\tan^2\,\omega = -\,\frac{R_2}{R_1} = \frac{q}{\rho\cos\varphi}\,, \tag{6}$$

in which q and φ are functions of s that one can get from the given intrinsic equation by means of (3). It will then follow that ω is a function of only the variable s, in such a way that it is enough to know one asymptote in order to know all of them. Having said that, if one appeals to (5) then it will be easy to calculate the arc length and flexion of the asymptotes:

$$s_0 = \int \frac{ds}{\cos\omega}\,, \qquad \frac{1}{\rho_0} = -\frac{1}{q}\frac{d}{ds}(q \sin\,\omega). \tag{7}$$

The torsion is then given immediately by Enneper's theorem:

$$\frac{1}{r_0^2} = -\frac{1}{R_1 R_2} = \frac{\cos\varphi}{q\rho}, \qquad r_0 = \rho \tan \omega. \tag{8}$$

Consider, for example, the surface that is generated by a Ribaucour line of index n that rotates around its directrix. In order for the surface to have real asymptotes, one needs to suppose that $n < -1$ because only then will one have that the curve constantly revolves with its convexity towards the axis. Meanwhile, for $q = -\frac{1}{2}(n + 1)\,\rho \cos \varphi$, the formula (6) will give $\tan^2 \omega = -\frac{1}{2}(n + 1)$; that is to say, *the asymptotes meet the meridians at a constant angle.* Formulas (7) and (8) will then give:

$$s_0 = \frac{s}{\cos\omega}, \qquad \rho_0 = \frac{\sin\omega}{\cos^2\omega}\rho \cot \varphi, \quad r_0 = \rho \tan \omega.$$

The most notable of all cases is that of the catenary ($n = -3$). Hence, the surface is called a *catenoid*, and it is characterized among the surfaces of rotation by the fact that $\omega = \pi/4$; i.e., by the fact that *the asymptotes also form a doubly-orthogonal system.* From the last formula, if one recalls that the equation of the catenary is $\rho = a + s^2 / a$ then one will get that *the asymptotes of the catenoid are defined by the intrinsic equations:*

$$\rho = s + \frac{2a^2}{s}, \quad r = a + \frac{s^2}{2a}.$$

8. – In order to study the *surfaces that can be mapped* (XI, § **25**) *onto surfaces of rotation*, one needs to observe that one can always give the square of the elementary arc length on those surfaces the form $ds^2 + q^2\, d\theta^2$, in which ds and $q\, d\theta$ represent the elementary arc lengths of the meridians and parallels of radius q, resp. Whenever one succeeds in establishing a system of coordinate lines on a surface such that the square of the elementary arc length takes the form $dq_1^2 + f^2(q_1)\, dq_2^2$, one can assert that the surface can be mapped to a surface of rotation, and one will know, in addition, that under the effective map of one surface to the other one, the lines q_1 will stretch along the meridians and the q_2 will stretch along the parallels. One will also find an infinitude of surfaces of rotation onto which the given surface can be mapped, since one can take $q_1 = s$, $q_1 = \theta : k$, $q = k\, f(s)$, if k represents an arbitrary constant. In order to know what those surfaces are, it is enough to *determine the intrinsic equation of the meridian*, and one can arrive at that easily when one differentiates the equality $q = k\, f(s)$, while taking (3) into account:

$$\sin \varphi = k\, f'(s), \qquad \cos \varphi = k\, \rho\, f''(s).$$

Hence, *the desired equation is:*

$$\rho = \frac{\sqrt{1 - k^2 f'^2(s)}}{k\, f''(s)}. \tag{9}$$

The two surfaces that correspond to the values k and $k' > k$ can be mapped to each other. However, it is easy to see that in order to cover the first one completely, it is enough to use a part of the second one that is terminated by two meridians.

9. Surfaces with constant total curvature. – We already have one example of these important surfaces in the ruled surfaces, since we see that we will have $K = 0$ at any point of a developable. If the constant value of K is not zero then the surface cannot be ruled, since the curvature of the ruled surface, which is non-zero on the line of striction, will be bent to zero at infinity. However, there exist an infinitude of surfaces with positive or negative constant curvature among the surfaces of rotation, and we can determine all of them. Indeed, in order for $R_1 R_2$ to be constant, *the curvature of the meridian must be proportional to the segment of the normal that is found between the point of incidence and the axis of rotation.* Hence, the meridian will belong to a class of curves that was studied before in the second chapter (§ **7**, *m*, *n*). We need to distinguish the surfaces of constant *positive* curvature (which have form that was pointed out above) from the surfaces with constant *negative* curvature, among which, the *pseudo-sphere* is particularly noteworthy; i.e., the surface is generated by a tractrix (I, § **8**, *b*) that rotates around its asymptote. That surface separates, so to speak, the two types of surfaces of negative curvature, just as the *sphere* separates the two types of surfaces with positive curvature. Its asymptotes are determined directly by recalling that if a is the (constant) segment of the tangent that is cut from the axis upon starting from the contact point then we will have:

$$\sin \varphi = -e^{-s/a}, \qquad q = a \sin \varphi, \qquad \rho = a \cot \varphi,$$

in such a way that (6) will give $\omega = \pm \varphi$; we can then deduce from (7) and (8) that:

$$s_0 = a \log \cot \frac{\varphi}{2}, \qquad \rho_0 = \frac{a}{2 \sin \varphi}, \qquad r_0 = a .$$

Therefore, *the asymptotes of the pseudo-sphere are defined by the intrinsic equations:*

$$\rho = \frac{a}{4}(e^{s/a} + e^{-s/a}), \qquad r = a.$$

Each of these curves touch the maximum parallel ($\varphi = \pi / 2$), and when extended to infinity ($\varphi = 0$), they will tend to coincide with the axis around which they turn indefinitely. One can also deduce that from the easily-proved fact that any pair of meridians will cut out equal arcs from any asymptote and the maximum parallel. One needs to note that this parallel is a singular line on which several propositions of the general theory will cease to be valid, and in particular, the asymptotic lines will be bent more that was indicated (§ **2**) by Beltrami's theorem at their points of contact with that parallel.

10. – The surfaces that were found in the preceding paragraph are also important because they provide all of the surfaces of constant curvature by just bending without extending or contracting. Indeed, one soon sees that the condition of equality of the curvatures at corresponding points, which was already (XI, § **25**) found to be *necessary* for the one surface to be mapped onto the other one, is also *sufficient* when one treats the surfaces of constant curvature. In other words: *Any surface with constant curvature can be mapped to any other surface that has the same curvature.* In order to prove that, take the line q_1 from the geodetics of the surface and observe that Gauss's formula will reduce to:

$$\frac{\partial^2 Q_2}{\partial q_1^2} + KQ_2 = 0. \tag{10}$$

If the lines q_1 are chosen as in § **24** of the preceding chapter – i.e., they are concurrent at real point – then one must have:

$$\lim_{q_1=0} \frac{Q_2}{q_1} = 1, \qquad \lim_{q_1=0} \frac{\partial Q_2}{\partial q_1} = 1. \tag{11}$$

Equation (10) cannot be satisfied for $K = 0$ under these conditions unless one takes $Q_2 = q_1$, and the square of the elementary arc length will present itself in the form $dq_1^2 + q_1^2\, dq_2^2$. However, that is precisely the form that ds^2 takes in the plane when one makes use of polar coordinates. Hence, if one recalls the proposition that was obtained in § **3**, one can state the following theorem:

In order for a surface to be mapped to the plane, it is enough that it should be developable.

The developables are therefore the infinitude of forms that a flexible and inextensible plane can take in space. However, if one supposes that K has a positive value $1 / a^2$ then one must take $Q_2 = a \sin q_1 / a$ in order to satisfy equation (10) and the conditions (11). The square of the elementary arc length will then take the form $dq_1^2 + a^2 \sin^2 \frac{q_1}{a} dq_2^2$, which is unique for all surfaces whose curvature is $1 / a^2$, among which, one will find the sphere of radius a, as would also result from equation (9), moreover, which will become $\rho = a$ for $k = 1$ when one takes $f(s) = a \sin (s / a)$. Hence, *all surfaces with curvature* $1 / a^2$ *can be mapped to the sphere of radius* a. In other words, any surface of constant positive curvature can be obtained by deforming a flexible and inextensible sphere. Finally, suppose that K has the value $- 1 / a^2$. In order to satisfy (10) in a more general way, one will then need to take:

$$Q_2 = \varphi(q_1)\, e^{q_1/a} + \psi(q_2)\, e^{-q_1/a},$$

with φ and ψ arbitrary functions that can always be arranged in such a way that the surface proves to be mappable onto a surface of rotation by either taking one of them equal to zero and incorporating the other one into dq_2, or by taking them to be equal to each other, or also by setting $\varphi = -\psi$. One will then arrive at the following forms for Q_2:

$$\frac{a}{2}(e^{q_1/a} - e^{-q_1/a}), \qquad a\,e^{-q_1/a}, \qquad \frac{a}{2}(e^{q_1/a} + e^{-q_1/a}),$$

resp. Now, these are precisely (II, § **7**, *n*) the forms of q that define the types of surfaces with constant negative curvature that were found in the preceding paragraph. Hence:

Any surface with constant negative curvature can be mapped in various ways onto a pseudo-sphere and other surfaces of rotation that have the same curvature.

However, of the three forms that were found, only the first one will satisfy the conditions (11), and therefore one can only map the surface onto a surface of the corresponding type if one desires that its meridians should stretch all of the geodetics that emanate from a real point. However, if one attributes the second form to Q_2 then one will see that q is annulled only when q_1 is infinite, and that means that the point of concurrence of the geodetics must be supposed to be infinitely far from the surface. Finally, q (or Q_2) is also annulled for any q_2 when one Q_2 assumes the third form, but it is annulled for an imaginary value of $q_1 = \frac{\pi a}{2}\sqrt{-1}$. One will then come to see that:

A surface of constant negative curvature can always be mapped to any surface of rotation with the same curvature in such a way that any system of concurrent geodetics will map to a system of meridians.

Furthermore, the surface of rotation must belong to one of the known types according to whether the point of concurrence of the geodetics is real and situated at a finite distance, real and infinitely distant, or pure imaginary, resp. Naturally, the surfaces of rotation that belong to one type can also be mapped to those of another type, but, contrary to what happened for the surfaces of positive curvature, the meridians will not remain meridians, but will transform into another system of concurrent geodetics. That is due to the possibility (which does not exist for the sphere) that there might exist geodetics that either do not meet or merely meet at infinity. If one desires that the surface should deform while preserving the meridians then it would be necessary that they should also preserve their own type, as one can see easily by means of (9), moreover. The pseudo-sphere is particularly noteworthy, since it cannot be deformed while preserving the meridians unless it slides along itself; that is to say, it is the unique surface of rotation that can be mapped onto the pseudo-sphere in such a way that the meridians stretch into meridians in the pseudo-sphere itself. Indeed, for $f(s) = a\,e^{-s/a}$, equation (9) will become:

$$\rho = \frac{a}{k}\sqrt{e^{2s/a} - k^2},$$

and that will always represent the tractrix of parameter a, as one will see immediately upon changing s into $s + a \log k$. That is easily explained by observing that it is only in the case of the pseudo-sphere that the parallels will have constant geodetic curvature $-1/a$, and that on the other hand, that curvature cannot vary when the surface is deformed by simple flexure.

11. Surfaces with constant mean curvature. – Some interesting questions of physics lead one to consider those surfaces, among which the *elastoids* are particular noteworthy, or the surfaces of zero mean curvature. The equality $H = 0$ says immediately that the Dupin indicatrix is an equilateral hyperbola at any point, and consequently:

The elastoids are characterized by the fact that their asymptotes constitute a doubly-orthogonal system.

Now, if one refers to the preceding result then one can assert that *the only elastoid of rotation is the catenoid,* as one will also see by recalling (II, § **7**, *e*) that the catenary is characterized by the property that its center of curvature at any point M is symmetric with respect to M to the point at which the normal meets the directrix. It is easy to answer the more general question: *Which surfaces of rotation have constant mean curvature?* If $-1/a$ indicates the constant value of H then the curvature of the meridians must satisfy the equality:

$$\frac{1}{\rho}+\frac{1}{a}=\frac{\cos\varphi}{q},$$

from which (V, § **3**, *f*), one will grasp immediately that *the required surfaces are generated by rotating Delaunay curves around their respective directrices.* The surface will be called an *unduloid* or a *nodoid* according to whether the curve belongs to the elliptic or hyperbolic type, resp. By virtue of a known property of Delaunay curves, it is clear that the unduloid and the nodoids are analogues of the two surfaces of the first type that were founs in § **9**. However, that property is true in a more general form, and it reveals an intimate connection between the surfaces with constant *mean curvature* and those of constant *total curvature.* Indeed, if one draws surfaces with principal radii $R_1 \mp a$ and $R_2 \mp a$ that are parallel to a surface upon which one has $R_1 R_2 = a^2$ at the distances a and $-a$, resp., and therefore:

$$\frac{1}{R_1 \mp a}+\frac{1}{R_2 \mp a}=\mp\frac{1}{a},$$

then one will see that the two surfaces have constant mean curvature.

12. – All surfaces of constant mean curvature enjoy the property that: *The lines of curvature constitute an isothermal system.* Indeed, from the first Codazzi formula, one will immediately get the geodetic curvatures in the form:

$$\mathcal{G}_1 = \frac{\partial f}{\partial s_1} - \frac{\partial g}{\partial s_2}, \qquad \mathcal{G}_2 = -\frac{\partial f}{\partial s_2} - \frac{\partial g}{\partial s_1}, \tag{12}$$

in which $f = 0$, $g = \frac{1}{2}\log(\mathcal{N}_1 - \mathcal{N}_2)$, and it is known (XI, § **22**) that the first equality is sufficient ($f = 0$, and even better $\Delta^2 f = 0$), so the theorem is proved. The theorem will also persist for an elastoid when one considers the asymptotes of the lines of curvature instead. Indeed, if one takes the asymptotes to be coordinate lines then the values of the geodetic curvature will result from the aforementioned Codazzi formulas in the form (12), with $f = 0$, $g = \frac{1}{2}\log \mathcal{T}$. Therefore, the *asymptotes of any elastoid constitute an isothermal system.* Furthermore, that theorem is an immediate consequence of the preceding one, since $\omega = \pi / 4$ will certainly satisfy (cf., XI, § **22**) $\Delta^2\omega = 0$. One will arrive at another characteristic property of the elastoid when one demands to know: *Which doubly-orthogonal systems of curves will remain orthogonal in the spherical representation?* That is equivalent (XI, §§ **14**, **15**) to wondering when an orthogonal pair of diameters to a conic is conjugate to another orthogonal pair. Now, we know the only axes that have that property, at least, as long as the conic is not an equilateral hyperboloid, in which case (III, § **4**), the property will belong to any pair of perpendicular diameters. Hence, in general:

The lines of curvature constitute the only system that remains orthogonal in the spherical representation, and it is only on the elastoids that any other double system will remain orthogonal.

Moreover, if one observes that the deviations of the two perpendicular lines are defined by the trigonometric tangents $-\mathcal{T} : \mathcal{N}$ and $\mathcal{T} : \mathcal{N}'$ then an easy calculation will show that the increase in the angle between the two curves in the spherical representation will have the tangent $(R_1 + R_2)\,\mathcal{T}$. Hence, if one wishes that the angle should remain equal to $\pi / 4$, while $\mathcal{T} \neq 0$, then one would need to have $H = 0$.

13. – We now ask: *Are any ruled surfaces elastoids?* One of the two systems of asymptotes will necessarily be composed of the rectilinear generators that are also geodetics, in such a way that one will have $\mathcal{N}_1 = \mathcal{G}_1 = 0$. The other system is composed of the orthogonal trajectories of the generators, and we have already seen (§ **5**) that we have:

$$\mathcal{G}_2 = \frac{t}{t^2 + p^2}, \qquad \mathcal{T} = -\frac{p}{t^2 + p^2} \tag{13}$$

for those curves. Meanwhile, since we must also have $\mathcal{N}_2 = 0$, the first Codazzi formula will show that $\mathcal{T}$ is a function of only the parameter q_1, and therefore p and $t - q_1$, which do not depend upon q_1, will be constants. Having said that, one has $t = 0$ on the line of striction, and consequently, $\mathcal{G}_2 = 0$. Hence, that line, which is asymptotic and geodetic, is a straight line; that is to say, the generators meet a fixed line in space orthogonally. When one varies only $s_2 = s$, t will keep a constant value b, and formulas (13) will give the constant values:

$$\frac{1}{\rho} = -\frac{b}{a^2 + b^2}, \qquad \frac{1}{r} = -\frac{a}{a^2 + b^2}$$

for the measures of the curvatures of any trajectory that is orthogonal to the generators, in which a represents the constant value of p. Therefore (X, § **4**), the curvilinear asymptotes of the surface are circular helices. Finally, if one applies the fundamental formulas to the direction of the generator ($\alpha = 1$, $\beta = \gamma = 0$) under the hypothesis that the origin displaces along an arbitrary curvilinear asymptote then one will find that $\delta\alpha = \delta\beta = 0$, $\delta\gamma = -\mathcal{T}\,ds$. For $t = 0$, one will then see that the angle through which the generator rotates around the axis of striction is $s : a$; i.e., it varies in proportion to the length of the segment that is traversed by that axis from the foot of the generator. One then arrives (IX, § **7**, *h*) at the following theorem by **Catalan**:

The only ruled elastoid is the helicoid with director plane.

Furthermore, that theorem results immediately from the fact that the aforementioned surface is the only one (X, § **9**) that has more than two asymptotes that are orthogonal to the generators.

14. – In order to know on which ruled surfaces the orthogonal trajectories to the generators are lines of constant geodetic curvature (as they are with the helicoid with a director plane), one needs to know when the curvature $\mathcal{G}_1$, which is given by the first formula (13), reduces to a function of the single parameter q_1 . In order for that to be true, it is necessary that the quantities p and $t - q_1$, which are always independent of q_1, should remain constants. Let $p = a$, $t = q_1$. The line of striction ($t = 0$) is therefore a line q_2, and it is a geodetic ($\mathcal{G}_2 = 0$), from which it will follow that the generators admit them as binormals. In addition, $-\mathcal{T}$, which keeps its value of $1 / a$ along the entire line of striction, represents the torsion of that line. Hence, the desired surfaces are the ones that *are composed of the binormals to the lines with constant torsion.* For all of those surfaces, one has:

$$\frac{\partial \log Q_2}{\partial q_1} = \frac{q_1}{q_1^2 + a^2},$$

and consequently, $Q_2 = \sqrt{q_1^2 + a^2}$, so the square of the elementary arc length will take the unique form $dq_1^2 + (q_1^2 + a^2)\,dq_2^2$. Hence, the surfaces that one deals with *can be mapped to the helicoid with director plane.* It is then obvious that they can also be mapped to a surface of rotation whose form one determines by taking $f(s) = \sqrt{s^2 + a^2}$ in (9). For $k = 1$, that equation will become $\rho = a + s^2 / a$, and therefore *the helicoid with director plane* *can be mapped onto the catenoid.* In other words, if one bends an inextensible catenoid that rectifies the minimal parallel then the meridians will be adjusted to constitute a helicoid with director plane.

15. Quadrics. – For the intrinsic study of a surface that is represented by an equation of degree two in the coordinates relative to immobile axis, one begins with the observation (cf., III, § **6**) that if one sets the origin on the surface, directs the z-axis along the normal, and differentiates the equation that expresses the immobility of the point (x, y, z) under the usual conditions [XI, form. (8)] then one will recover the equation from which one started. Having said that, recall that for $x = y = z = 0$, only the quotients $\partial x / \partial s_1$ and $\partial y / \partial s_2$ will take the value – 1, while all of the other ones will be annulled, and one will see immediately that the linear part of the proposed equation must reduce to z. One then sees that for $z = 0$, the surface will be cut by the tangent plane along two lines, which will be real or imaginary: *It will then* (IX, § **7**, *d*) *be a quadric.* We leave it to the reader to prove the converse proposition. If the x and y axes are directed tangentially to the lines of curvature then the terms in xy must also be absent, in such a way that the equation will finally reduce to the form:

$$z = \tfrac{1}{2}(\mathcal{N}_1\, x^2 + \mathcal{N}_2\, y^2 + \gamma z^2) + (\alpha x + \beta y)\, z. \tag{14}$$

After differentiating this once with respect to q_1 and another time with respect to q_2, we need to identify each of the two derived equations by considering the coefficients of y^2 for the first one and those of x^2 for the other. In that way, one will get:

$$\alpha = \frac{1}{\mathcal{N}_2}\frac{\partial \mathcal{N}_2}{\partial s_1}, \qquad \beta = \frac{1}{\mathcal{N}_1}\frac{\partial \mathcal{N}_1}{\partial s_2}. \tag{15}$$

However, if one compares the coefficient of x^2 in the first equation and that of y^2 in the second one with the analogous coefficients in the original equation then one will get the values:

$$\alpha = \frac{1}{3\mathcal{N}_1}\frac{\partial \mathcal{N}_1}{\partial s_2}, \qquad \beta = \frac{1}{3\mathcal{N}_2}\frac{\partial \mathcal{N}_2}{\partial s_2},$$

and if one equates these to the preceding values and integrates then one will recover a characteristic property of the quadric (XI, § **17**), namely, that the independence of the ratios $\mathcal{N}_2^3 : \mathcal{N}_1$ and $\mathcal{N}_1^3 : \mathcal{N}_2$ will give q_1 and q_2, respectively. One is now naturally led to choose the parameters of the lines of curvature by taking:

$$\mathcal{N}_1 = q_1^3\, q_2, \qquad \mathcal{N}_2 = q_1\, q_2^3,$$

after which, the values (15) will become:

$$\alpha = \frac{1}{Q_1\, q_1}, \qquad \beta = \frac{1}{Q_2\, q_2}.$$

Meanwhile, it results from the first two Codazzi formulas that:

$$\frac{\partial \log Q_1}{\partial q_2} = -\frac{q_1^2}{q_2(q_1^2 - q_2^2)}, \qquad \frac{\partial \log Q_2}{\partial q_1} = \frac{q_2^2}{q_1(q_1^2 - q_2^2)}, \tag{16}$$

and upon integrating, one will infer Q_1 and Q_2 ; hence:

$$\alpha = \frac{q_1\, q_2\, \varphi(q_1)}{\sqrt{q_1^2 - q_2^2}}, \qquad \beta = \frac{q_1\, q_2\, \psi(q_2)}{\sqrt{q_1^2 - q_2^2}}.$$

Here, one observes that the ratio of Q_1 to Q_2 proves to be equal to the product of a function of q_1 with a function of q_2. Therefore, *the lines of curvature on a quadric constitute an isothermal system.* In order to determine γ, it is enough to compare the coefficient of z^2 in the original equation with the ones in the two derived ones, and integrate the equations that one gets in that way:

$$\frac{\partial \gamma}{\partial s_1} = \alpha\,(\gamma - 2\mathcal{N}_1), \qquad \frac{\partial \gamma}{\partial s_2} = \beta\,(\gamma - 2\mathcal{N}_2).$$

One will then find that:

$$\gamma = -\, q_1\, q_2\, (q_1^2 + q_2^2 - A),$$

and in order to know all of the coefficients in the equations (14) completely as functions of q, all that remains is to specify the functions φ and ψ. To that end, one compares the coefficient of xz in the first derived equation and that of yz in the second one with the analogous coefficients in the original equation and substitutes the preceding values of α, β, γ, and $\mathcal{N}$ in the equality thus-obtained to get:

$$\frac{\partial \alpha}{\partial s_1} = \alpha^2 - \beta\, \mathcal{G}_1 + \gamma \mathcal{N}_1 - \mathcal{N}_1^2, \qquad \frac{\partial \beta}{\partial s_2} = \beta^2 - \alpha\, \mathcal{G}_2 + \gamma \mathcal{N}_2 - \mathcal{N}_2^2.$$

One sees from some simple considerations that if one sets:

$$f(x) = x^3 - A\, x^3 + Bx - C$$

then one must take:

$$\varphi(q_1) = \sqrt{-f(q_1^2)}, \quad \psi(q_2) = \sqrt{f(q_2^2)},$$

and with that, one has what it takes for the intrinsic study of quadrics.

16. – If one fixes an arbitrary value for z then equation (14) will represent a conic that degenerates into a pair of lines for those values of z that annul the discriminant:

$$\begin{vmatrix} \mathcal{N}_1 & 0 & \alpha z \\ 0 & \mathcal{N}_2 & \beta z \\ \alpha z & \beta z & \gamma z^2 - 2z \end{vmatrix} = q_1^2\, q_2^2\, z\,(Cz - 2q_1\, q_2),$$

i.e., for $z = 0$ and $Cz = 2q_1\, q_2$. Hence, *any plane section of a quadric is a conic*, and for each direction there will exist two degenerate sections whose planes obviously touch the surface. If one skips over the discussion of exceptional cases then one will see that any point M corresponds to a point N whose coordinates with respect to the fundamental trihedron that has its origin at M are defined by the conditions:

$$\mathcal{N}_1 x + \alpha z = 0, \qquad \mathcal{N}_1 y + \beta z = 0, \qquad Cz = 2q_1\, q_2\,,$$

and the correspondence between the two points is such that the tangent planes to the quadric will be parallel under it. With that, it is clear that the surface possesses a *center*, which must divide the chord MN in half, and one can therefore say that the center exists and that its coordinates will be:

$$x_0 = -\frac{q_2}{Cq_1}\sqrt{\frac{f(q_1^2)}{q_2^2 - q_1^2}}\,, \qquad y_0 = -\frac{q_1}{Cq_2}\sqrt{\frac{f(q_2^2)}{q_1^2 - q_2^2}}\,, \qquad z_0 = \frac{q_1\, q_2}{C}, \tag{17}$$

after having verified (as one can easily do) that those coordinates satisfy the immobility conditions. In order for M to be a *vertex*, one needs to have $x_0 = 0$, $y_0 = 0$, and z_0 will then represent the length of the corresponding semi-axis. Hence, if λ, μ, ν are the roots of f, which are assumed to be distinct, then one can define one of the three pairs of vertices, and therefore one of the three axes, by taking (for example) $q_1^2 = \lambda$, $q_2^2 = \mu$, in which case the length c of the semi-axis will be given by $C^2\, c^2 = \lambda\mu$. Hence, if one observes that $C = \lambda\mu\nu$ then one will see that the squares of the semi-axes are:

$$a^2 = \frac{1}{C\lambda}, \qquad b^2 = \frac{1}{C\mu}, \qquad c^2 = \frac{1}{C\nu},$$

and that consequently the value of the constant C will be inverse to $\sqrt{abc}$. The fact that the three axes constitute an orthogonal triple results from the symmetry of the form that equation (14) will take when the origin of the coordinates is transferred to the center of the surface. Indeed, when one changes z into $z + c$ and observes that A is equal to $\lambda + \mu + \nu$, one will obtain:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{18}$$

and it is then easy to appeal to that equation in order to discuss the various forms of real quadrics that correspond to the hypotheses of *three* or *two* pairs of real vertices or just *one* pair. In the first case, one has the *ellipsoid*, in the second, the *hyperboloid with one sheet,* and in the third case, the *hyperboloid of two sheets.* Meanwhile, one knows that the lines of curvature, which are defined by the equation $q_1^2 = \lambda$, will pass through the two pairs of vertices that correspond to the values μ and ν of the other parameter q_2^2, and since one has $\mathcal{G}_2 = 0$, the line considered will be planar (XI, § **2**). Hence, the three principal diametral sections – i.e., the sections that are made by principal planes (viz., planes that pass through two axes) – will be lines of curvature and all of the geodetics of the surface. That is obvious, moreover, if one reflects upon the fact that the normals to the surface along a principal diametral section will all be in the plane of the section, and also if one observes that the principal planes cut the surface at a right angle. However, from what has been said before, it will also result that the three curves are characterized by the fact that, the square of a parameter will be kept constantly equal to one root of f along each of them.

17. – One defines the *umbilics* ($\mathcal{N}_1 = \mathcal{N}_2$) by taking $q_1^2 = q_2^2 = \lambda, \mu, \nu$, in succession, in which case x_0 and y_0 will become indeterminate, but:

$$C^2\ (x_0^2 + y_0^2) = -\lambda^2 + B - \frac{2C}{\lambda} = -f'(\lambda),$$

in such a way that in order to fix the position of the center with respect to the tangent plane and the normal at an umbilic, one will have the formulas:

$$z_0^2 = \pm\, abc\, \lambda^2, \qquad\qquad x_0^2 + y_0^2 = \pm\, abc\, f'(\lambda)$$

then one will find that the distances from the center to the tangent plane to the umbilic (whether real or imaginary) are $bc : a$, $ca : b$, $ab : c$, while the distance to the corresponding normals are:

$$\pm\frac{1}{a}\sqrt{(c^2-a^2)(a^2-b^2)}\,, \qquad \pm\frac{1}{a}\sqrt{(a^2-b^2)(b^2-c^2)}\,, \qquad \pm\frac{1}{a}\sqrt{(b^2-c^2)(c^2-a^2)}\,.$$

Obviously, the umbilics are on three principal diametral sections, but not all real. Hence, for example, in the case of the ellipsoid, if one supposes that $a > b > c$ then the only real umbilics will be the four points that are defined by $q_1^2 = q_2^2 = \mu$. They belong to the section that is determined by the minimum or maximum axis, and they are defined by that situation, along with the fact that their distance to the center is $\sqrt{a^2 + c^2 - b^2}$, which is less than a and greater than c. The other umbilics are imaginary precisely because they belong to spheres that are concentric to the ellipsoid and are described by radii

$\sqrt{b^2+c^2-a^2} < c$, $\sqrt{a^2+b^2-c^2} > a$, and consequently, the former will be too small to cut the ellipsoid, while the latter will be too large. The four real umbilics are associated with two pairs *F*, *G*, and *F′*, *G′* of diametrically-opposite points, and the tangent planes to those points are all at a distance of *ac* : *b* from the ellipsoid. The only (real) circular sections of the surface are obviously in planes that are parallel to those tangent planes, and two of them (viz., the ones that are situated in diametral planes) will have a radius of *b*. The hyperboloid with two sheets also has just two pairs of real umbilics – one for each sheet; however, the umbilics of the other hyperboloid are all imaginary.

18. – *The curvature of a quadric at a point M is proportional to the fourth power of the distance from the center to the tangent plane at M.* Indeed:

$$K = \mathcal{N}_1\,\mathcal{N}_2 = q_1^4\,q_2^4 = \frac{z_0^4}{a^2\,b^2\,c^2}\,.$$

Here, one should note that only the hyperboloids with one sheet ($a^2\ b^2\ c^2 < 0$) have negative curvature, and are consequently the only quadrics with centers that are generated by real lines. We already know that when *M* drifts indefinitely far from the central position along a generator, *K* will tend to zero, and z_0 must also tend to zero; i.e., the tangent plane will conclude by containing the center. Hence, *the asymptotic developable* (§ **5**) *of a quadric is a cone with its vertex at the center.* If the cone is referred to the vertex and the principal planes then its equation will obviously be (18), in which one replaces 1 with 0 in the right-hand side. With that equation, one will see immediately that the asymptotic cone, which is imaginary only in the case of the ellipsoid, is surrounded entirely by the hyperbola with one sheet, although it is surrounded separately by the two sheets of the other hyperboloid. When *M* then traverses a given generator until it occupies the central position, the distance z_0 will become a maximum, and therefore *the planes that touch a quadric along the line of striction are normals to the perpendiculars to the generators that are based at the center.* Hence, if *M* belongs to the line of striction (along the central points of the generators of the *two* systems) then one will have $y_0 = x_0 \tan \omega$, where ω satisfies the condition $\mathcal{N} = 0$ (viz., $q_1^2\cos^2\omega + q_2^2\sin^2\omega = 0$), which defines the inclination of the generators with respect to the lines of curvature. It will then follow that the line of striction is characterized by the relation $q_1^2\,x_0^2 + q_2^2\,y_0^2 = 0$, and therefore, if one takes the values (17) for x_0 and y_0 then one will see that *its equation in curvilinear coordinates is:*

$$q_1^2\,q_2^2(q_1^2+q_2^2-A) + C = 0. \tag{19}$$

That is equivalent to $\gamma z_0 = 1$ and expresses the idea that *the projections of the center onto the normals to the line of striction bisect the segments that are cut out by the surface along those normals.* Indeed, it results from (14) that the length of that segment will be 2 : $\gamma = 2z_0$. By means of the equation (19), one can study the line of striction with no difficulty, which passes through all of the vertices, is inflected, and is symmetric with respect to the principal plane.

19. – A noteworthy example of the curves that are traced on a quadric is that of the *poloids*; i.e., the curves that are defined by the following property: *The tangent planes along a poloid are at equal distances from the center.* From what was said before, it is clear that *the curvature of a quadric is constant along each poloid.* If one observes that the equation of that curve in curvilinear coordinates is $q_1\, q_2$ = constant and that, on the other hand, one has:

$$\frac{\partial}{\partial s}\log q_1\, q_2 = \alpha\cos\omega + \beta\sin\omega,$$

then one will see that the inclination ω of a poloid with respect to a line of curvature is given by the formula:

$$\tan\omega = -\frac{\varphi(q_1)}{\psi(q_2)}.$$

The substitution of that value of ω in formulas (12), (15), and (19) of the preceding chapter will yield all of the elements that are necessary for the intrinsic study of the poloids. One can arrive at another property of those curves if one observes that the section that is made in the quadric by the diametral plane that is parallel to the tangent plane is represented by the equation $\mathcal{N}_1\, x^2 + \mathcal{N}_2\, y^2 = z_0$, in such a way that the squares of the semi-axes of that section are:

$$\frac{x_0}{\mathcal{N}_1} = \frac{1}{C\, q_1^2}, \quad \frac{z_0}{\mathcal{N}_2} = \frac{1}{C\, q_2^2}.$$

It will then follow that the parameters q_1 and q_2 are inversely proportional to the semi-axes of that section. It is a real ellipse in the case of the ellipsoid, and one can then say that *if a point traverses a poloid then the diametral section that is parallel to the tangent plane will preserve a constant area.* Finally, an easy calculation will show that *the normal sections that are made in the ellipsoid tangentially to any poloid have a vertex at the contact point.*

20. – If one multiplies the length of the semi-axis that is parallel to $M\,x$ by z_0 then one will get a value that is independent of q_1 . Hence: *The product of the diameter that is parallel to the tangent with the distance from the center to the tangent plane will be constant along the lines of curvature of a quadric.* We now ask: Is that a characteristic property of the lines of curvature? From what was said (XI, § **13**) about the Dupin indicatrix, one will see immediately that the length λ of the semi-diameter that is parallel to any tangent that is inclined above Mx by ω will be given by the formula:

$$l^2 = \frac{z_0}{\mathcal{N}} = \frac{1}{C\,(q_1^2 \cos^2 \omega + q_1^2 \sin^2 \omega)},$$

and that consequently the stated property will belong to all of the curves along which one has:

$$\frac{\cos^2 \omega}{q_2^2} + \frac{\sin^2 \omega}{q_1^2} = \text{constant}.$$

Meanwhile, if one applies the operation:

$$\frac{\partial}{\partial s} = q_1\ \alpha \cos\ \omega\, \frac{\partial}{\partial q_1} + q_2\, \beta \sin\ \omega\, \frac{\partial}{\partial q_2}$$

to the left-hand side then one will easily get:

$$\frac{\partial}{\partial s}\left(\frac{\cos^2 \omega}{q_2^2} + \frac{\sin^2 \omega}{q_1^2}\right) = -\left(\frac{\alpha \sin \omega}{q_1^2} + \frac{\beta \cos \omega}{q_2^2} + \frac{q_1^2 - q_2^2}{q_1^2\, q_2^2}\frac{\partial \omega}{\partial s}\right) \sin 2\omega,$$

and one can also give the quantity in parentheses in the right-hand side the form:

$$\frac{q_1^2 - q_2^2}{q_1^2\, q_2^2}\left(\mathcal{G}_2 \sin \omega - \mathcal{G}_1 \cos \omega + \frac{\partial \omega}{\partial s}\right),$$

since one has:

$$\mathcal{G}_1 = \frac{\beta\, q_1^2}{q_2^2 - q_1^2}, \qquad \mathcal{G}_2 = \frac{\alpha\, q_2^2}{q_1^2 - q_2^2},$$

by virtue of (16). Hence, if one recalls the first formula in (19) of the preceding chapter:

$$\frac{\partial}{\partial s}\left(\frac{\cos^2 \omega}{q_2^2} + \frac{\sin^2 \omega}{q_1^2}\right) = \frac{q_1^2 - q_2^2}{q_1^2\, q_2^2}\, \mathcal{G} \sin 2\omega \tag{20}$$

then the stated property will not only belong to the *lines of curvature* ($\omega = 0$, $\omega = \pi / 2$), but as **Joachimsthal** observed, also to the *geodetics* ($\mathcal{G} = 0$). In any case, formula (20) will serve to calculate the geodetic curvature of any line that is traced on a quadric.

21. – Some important consequences of Joachimsthal's theorem were pointed out by **Roberts** already. Consider an ellipsoid, and connect any point M to two real umbilics that are not diametrically opposite – for example, F and F'. The product lz_0 keeps a constant value along one or the other geodetic, and then from what was said at the end of § **17**, the value of lz_0 will be ac at F, as well as F', and therefore lz_0, and consequently l, must have a unique value at M, as well. Now, if one ponders the significance of l then one will see immediately that the tangents to the two geodetics at M are inclined equally with respect to the axes of the Dupin indicatrix. Hence, *the lines of curvature of an ellipsoid at any point M will bisect the angles between the geodetics that connect M to two non-diametrically-opposite umbilics.* Consequently, MF and MG are arcs of that geodetic, and since M is a point on the surface that is taken arbitrarily, one will see that *the infinitude of geodetics that emanate from an umbilic will be concurrent at the diametrically-opposite umbilic.* In addition, it results from the essential property of geodetics that the distance between the two umbilics will always be the same when it is computed along any geodetic. With that, it is clear that if a flexible, but inextensible, filament on an ellipsoid tends to become perfectly smooth along the mean principal section then the ellipsoid can rotate freely, like a sphere, around FG or $F'G'$, while deforming the filament, but without it ceasing to remain tense on the surface at any moment. However, in the general case, it can happen that the filament leaves the surface, or even that it is dragged along in such a way that it remains rigidly fixed on it. We would finally like to point out another important property that is limited to the statement:

The lines of curvature of an ellipsoid are the loci of the points whose geodetic distances from two non-diametrically-opposite umbilics have a constant sum or difference.

They will be, so to speak, the ellipses and the hyperbolas of the surface then; the ones that can be considered to be ellipses with respect to the *foci* F and F' are, at the same time, hyperbolas with respect to F and G', and *vice versa.* It will then follow that a pen that moves while keeping a flexible, inextensible filament tense that is fixed at its end points at F and F' will trace out a line of curvature on the surface, and in such a way that when one increases or diminishes the length of the filament, one will succeed in mechanically tracing out all the lines of curvature of one system; for those of the other, it is enough to fix the end points of the filament at F and G', or at F and G.

22. Weingarten surfaces. – The surfaces of rotation, surfaces of constant mean or total curvature, and many other noteworthy surfaces belong to a single class that is characterized by the fact that *a relation exists between the principal curvatures.* All of those surfaces are called **Weingarten** surfaces, from the name of the geometer that discovered their most important properties. We shall now confine ourselves to proving a few theorems that relate to the evolutes of those surfaces, in particular. Recall (XI, § **26**) that the normal curvatures and the geodetic torsion of the first sheet of the evolute of an arbitrary surface are determined by means of the formulas:

$$\frac{\partial R_1}{\partial s_1}\mathcal{N}_1' = \frac{1}{R_1}, \qquad \frac{\partial R_1}{\partial s_1}\mathcal{T}' = \mathcal{G}_1, \qquad \frac{\partial R_1}{\partial s_1}\mathcal{N}_2' = \frac{R_2}{l}\left(\mathcal{G}_2\frac{\partial R_1}{\partial s_1} + \mathcal{G}_1\frac{\partial R_1}{\partial s_2}\right),$$

from which, one deduces that:

$$K = \mathcal{N}_1'\mathcal{N}_2' - \mathcal{T}'^2 = -\frac{1}{l^2}\frac{\partial R_2/\partial s_1}{\partial R_1/\partial s_1},$$

when one takes formulas (30) of the preceding chapter into account. In order to obtain the value K'' of the curvature at the corresponding point of the second sheet, it is enough to switch R_1 and R_2 and s_1 and s_2 in the preceding expression; one will then find that:

$$K'\,K'' = \frac{1}{l^4}\frac{\frac{\partial R_1}{\partial s_2}\frac{\partial R_2}{\partial s_1}}{\frac{\partial R_1}{\partial s_1}\frac{\partial R_2}{\partial s_2}}.$$

On the other hand, in order for a surface to be a Weingarten surface, by virtue of the definition, it is necessary and sufficient that the functional determinant of the R should be zero; i.e., that one should have:

$$\frac{\partial R_1}{\partial s_1}\frac{\partial R_2}{\partial s_2} = \frac{\partial R_1}{\partial s_2}\frac{\partial R_2}{\partial s_1},$$

and consequently $K'\,K'' = 1 : l^4$. One will then find the following theorem of **Halphen**:

In order for a surface to belong to the class of Weingarten surfaces, it is necessary and sufficient that the product of the curvatures of the sheets of the evolute at two corresponding points should be the inverse of the fourth power of the distance between those two points.

Another characteristic property was discovered by **Ribaucour** in his search for the condition under which the asymptotes of one sheet would correspond to those of the other sheet. Let ω be the inclination of the lines of curvature of (M) with respect to the curve that M must traverse in order for the corresponding center C_1 to describe an asymptote of (C_1), and let ω' be the inclination of that asymptotic line with respect to the tangent Mz. A relation exists between ω and ω' that one easily deduces from formulas (27) of the preceding chapter by dividing the third one by the second one:

$$\frac{l}{R_2}\cos\omega' = \frac{\partial R_1}{\partial s_1}\cot\omega + \frac{\partial R_1}{\partial s_2}.$$

Now, if one would like to determine ω in such a way that C_1 describes an asymptote of (C_1) then one would need to have:

$$\mathcal{N}_1'\cot^2\omega' - 2\mathcal{T}'\cot\omega' + \mathcal{N}_2' = 0\;;$$

i.e., if one substitutes the values that were obtained before for the $\mathcal{N}'$, $\mathcal{T}'$, and ω' then:

$$\cot\,\omega = \pm\frac{l\,R_1}{R_2}\sqrt{-K'}\,.$$

In order for the asymptotes on two sheets to correspond – that is to say, since displacing M in the direction that is defined by the angle ω with Mx (or $\pi/2 - \omega$ with My) will make the centers C_1 and C_2 tend to displace along the asymptotes of the respective sheet – one will need to have:

$$\cot\left(\frac{\pi}{2}-\omega\right) = \mp\frac{l\,R_2}{R_1}\sqrt{-K''}$$

for that value of ω, so $K'K'' = 1 : l^4$. Hence:

In order for the asymptotes of a surface to correspond on the two sheets of its evolute, it is necessary and sufficient that it should belong to the Weingarten class.

23. – One can look for the conditions under which the lines of curvature of the two sheets might correspond in an analogous way, but that and other research can be accomplished with greater facility by studying the correspondence between the two sheets directly, that is to say, between an arbitrary surface (M) and one of its complements (M'). Consider a system of curvilinear geodetics ($\mathcal{N}_1 \neq 0$) on the first surface and take them to be lines q_1, in such a way that one has:

$$\mathcal{G}_1 = 0, \qquad \mathcal{G}_2 = -\frac{1}{l},$$

if one lets l denote the distance MM', and one meanwhile observes that from the third Gauss formula, the function l will be coupled with the curvature of (M) by means of the differential equation:

$$\frac{\partial l}{\partial s_1} + 1 = -K\,l^2. \tag{21}$$

Having said that, when the fundamental formulas are applied to the point M whose coordinates are $x = l$, $y = z = 0$, that will give:

$$\frac{\delta x}{\partial s_1} = \frac{\partial l}{\partial s_1} + 1, \qquad \frac{\delta y}{\partial s_1} = 0, \qquad \frac{\delta z}{\partial s_1} = \mathcal{N}_1\,l,$$

$$\frac{\delta x}{\partial s_2} = \frac{\partial l}{\partial s_2}, \qquad \frac{\delta y}{\partial s_2} = 0, \qquad \frac{\delta z}{\partial s_2} = -\mathcal{T}\,l\,.$$

Hence, the z'-axis, which is normal to the surface (M'), will be parallel to the y-axis at M', and if the y'-axis is directed parallel to Mz and the x'-axis is directed in the opposite sense to Mx then if one proceeds as in § **26** of the preceding chapter, one will easily arrive at the relations:

$$-k\frac{\partial}{\partial s_1'} = \mathcal{T}\frac{\partial}{\partial s_1} + \mathcal{N}_1\frac{\partial}{\partial s_2}, \qquad kl\frac{\partial}{\partial s_2'} = \frac{\partial l}{\partial s_2}\frac{\partial}{\partial s_1} - \left(\frac{\partial l}{\partial s_1}+1\right)\frac{\partial}{\partial s_2},$$

in which one sets:

$$k = \mathcal{T}\left(\frac{\partial l}{\partial s_1}+1\right) + \mathcal{N}_1\frac{\partial l}{\partial s_2},$$

for brevity. It is then enough to write out the immobility condition for the point $(-x+l, z, y)$ with respect to the trihedron (M'), while taking into account those conditions for the point (x, y, z) with respect to the trihedron (M), in order to arrive at the anticipated results:

$$\mathcal{G}_1' = 0, \qquad \mathcal{G}_2' = -\frac{1}{l},$$

as in the aforementioned § **26**, along with the other ones:

$$-\frac{\mathcal{N}_1'}{\mathcal{N}_1} = -\frac{\mathcal{T}'}{Kl} = \frac{\mathcal{N}_2'}{\mathcal{N}_2\left(\frac{\partial l}{\partial s_1}+1\right)+\mathcal{T}\frac{\partial l}{\partial s_2}} = \frac{1}{kl}.$$

Here, one should note that one has:

$$K = \mathcal{N}_1'\mathcal{N}_2' - \mathcal{T}'^2 = -\frac{\mathcal{T}'}{Kl^2}, \tag{22}$$

and consequently, when one uses (21):

$$KK' = \frac{1}{l^4} - \frac{\mathcal{N}_1}{kl^4}\frac{\partial l}{\partial s_2},$$

from which, one will deduce that the condition that **Halphen** found for the evolute of the Weingarten surface is equivalent to $\partial l : \partial s_2 = 0$, and that will say that $\mathcal{G}_2$ is a function of only the parameter q_1. Hence:

In order for a surface to be one sheet of the evolute of a Weingarten surface, it is necessary that it should admit a system of parallel geodetic lines with constant geodetic curvature.

The condition is not sufficient. Indeed, we have always excluded the systems of rectilinear geodetics, since it is indispensible that the tangents to the geodetics considered should form a congruence, and on the other hand, in § **14**, we had occasion to consider the existence of ruled surfaces whose generators moved orthogonally to the lines of constant geodetic curvature.

24. – The observation that was made in § **8** will now permit us to assert that *each sheet of the evolute of a Weingarten surface can be mapped onto a surface of rotation.* If one is given a relation $R_2 = \varphi(R_1)$ then if one observes that $dq_1 = dR_2$, one can infer from $\mathcal{G}_2 = -1 : l$, in succession, that:

$$\frac{\partial \log Q_2}{\partial q_1} = \frac{1}{R_1 - R_2}, \qquad Q_2 = e^{\int \frac{dR_1}{R_1 - R_2}},$$

in such a way that it is enough to set:

$$f(s) = e^{\int \frac{ds}{s - \varphi(s)}},$$

since (9) represents the meridian of a surface of rotation onto which the first sheet of the evolute of the Weingarten surface that is defined by $R_2 = \varphi(R_1)$ will be mapped. As one sees, the form of that surface of rotation will depend uniquely upon the nature of the constraint that intercedes between R_1 and R_2. Conversely, *any surface that can be mapped onto a surface of rotation is one sheet of the evolute of a Weingarten surface*, as long as it is not composed of the normals to a line of constant torsion. That theorem is due to **Weingarten.**

25. – To conclude, we return to the question that was posed at the beginning of § **23**. If we apply the fundamental formulas to the cosines $\alpha = 0$, $\beta = 1$, $\gamma = 0$ then we will get:

$$\frac{\delta\alpha}{ds} = \frac{\sin\omega}{l}, \qquad \frac{\delta\beta}{ds} = 0, \qquad \frac{\delta\gamma}{ds} = -\mathcal{T}\cos\omega + \mathcal{N}_2 \sin\omega,$$

and if we desire that $M'z'$ should generate a developable then we must [cf., IX, form. (19)] determine ω in such a way that the condition:

$$\begin{vmatrix} \alpha & \delta\alpha & \delta x \\ \beta & \delta\beta & \delta y \\ \gamma & \delta\gamma & \delta z \end{vmatrix} = 0$$

is satisfied; i.e., (from the calculations that were made) we must have:

$$\mathcal{T}_\omega + (\mathcal{T}\cos\omega - \mathcal{N}_2 \sin\omega)\,\frac{dl}{ds} = 0.$$

That is the equivalence that defines the inclination ω of Mx with respect to the lines of (M) that correspond to the lines of curvature of (M'). In order for them to also be lines of curvature on (M), one will need, in addition, that ω should annul $\mathcal{T}_\omega$, and therefore the conditions:

$$\mathcal{T} \cos 2\omega + \tfrac{1}{2}(\mathcal{N}_1 - \mathcal{N}_2) \sin 2\omega = 0,$$

$$(\mathcal{T} \cos \omega - \mathcal{N}_2 \sin \omega)\left(\cos\omega \frac{\partial l}{\partial s_1} + \sin\omega \frac{\partial l}{\partial s_2}\right) = 0$$

must reduce to just one condition. Setting aside the case of developable surfaces, it will result from identification that:

$$\frac{\partial l}{\partial s_1} = 0, \qquad \frac{\partial l}{\partial s_2} = 0 ;$$

i.e., l must have a constant value a. One can then state another theorem of **Ribaucour**'s:

In order for the lines of curvature of one sheet of the evolute of a surface to correspond to the lines of curvature on the other sheet, it is necessary and sufficient that the distance between the principal centers of curvature of the surface in question should be constant.

In other words, it must be a special Weingarten surface that is defined by the relation $R_1 - R_2$ = constant. Under that hypothesis, formulas (21) and (22) will give:

$$K = K' = -\frac{1}{a^2} ;$$

i.e., *the two sheets are surfaces with constant negative curvature.* Conversely, if one is given a surface with curvature $-1 / a^2$ and one constructs, tangentially to a simple infinitude of geodetics that are concurrent at an infinitely-distant point, rectilinear segments that start from the contact point and have length a in the direction in which the geodetics point to the common point then the end points of those segments will be on another surface of curvature $-1 / a^2$ that constitutes an evolute of a Weingarten surface, along with the first one, and:

The lines of curvature, as well as the asymptotes, of the given surface will correspond to the analogous lines of the other one.

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CHAPTER XIII

INFINITESIMAL DEFORMATION OF SURFACES

1. Imagine that the points of a surface (M) are displaced infinitely little in such a way that they constitute another surface (M'). We propose to study the alteration that is produced in the fundamental curvatures of (M). Let u, v, w be the projections of the displacement MM' onto two orthogonal tangents and the normal to (M), resp., at the point M. When M traverses a segment ds in the tangent plane that is inclined from the x-axis by ω, the coordinates ($x = u$, $y = v$, $z = w$) of the point M' will experience variations that are given by the usual fundamental formulas (XI, § **9**):

$$\left.\begin{aligned}\frac{\delta x}{ds} &= (1+u_1)\cos\omega + u_2 \sin\omega,\\ \frac{\delta y}{ds} &= v_1\cos\omega + (1+v_2)\sin\omega,\\ \frac{\delta z}{ds} &= w_1\cos\omega + w_2\sin\omega,\end{aligned}\right\} \tag{1}$$

in which we have set:

$$\left.\begin{aligned} u_1 &= \frac{\partial u}{\partial s_1} + \mathcal{G}_1 v - \mathcal{N}_1 w, & u_2 &= \frac{\partial u}{\partial s_2} + \mathcal{T} w - \mathcal{G}_1 v,\\ v_1 &= \frac{\partial v}{\partial s_1} + \mathcal{T} w - \mathcal{G}_1 u, & v_2 &= \frac{\partial v}{\partial s_2} + \mathcal{G}_2 u - \mathcal{N}_2 w,\\ w_1 &= \frac{\partial w}{\partial s_1} + \mathcal{N}_1 u - \mathcal{T} v, & w_2 &= \frac{\partial w}{\partial s_2} + \mathcal{N}_2 v - \mathcal{T} u,\end{aligned}\right\} \tag{2}$$

for brevity. Squaring and summing (1), one will get $ds' = (1 + \Phi)\, ds$, in which:

$$\Phi = u_1 \cos^2\omega + (v_1 + u_2)\cos\omega\sin\omega + v_2\sin^2\omega,$$

up to higher-order infinitesimals. Obviously, Φ, which represents the *linear dilatation* per unit length, reduces to u_1 in the direction Mx and to v_2 in the direction My. The surface element $ds_1\, ds_2$ then transforms into $(1 + u_1)(1 + v_2)\, ds_1\, ds_2$, so if one represents the area of the deformed element by $(1 + \Theta)\, ds_1\, ds_2$ – i.e., if Θ is the unit *surface dilatation* – then one will have $\Theta = u_1 + v_2$, if one neglects higher-order infinitesimals, or, by virtue of (2):

$$\Theta = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right) u + \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right) v - Hw. \tag{3}$$

That formula shows that *it is only for the elastoids* ($H = 0$) *that the normal displacement has no influence upon the dilatations.* When the surface is inextensible, the function Φ

must reduce to zero identically, that is, one must have $u_1 = 0$, $v_2 = 0$, $v_1 + u_2 = 0$, and consequently, one will also have $\Theta = 0$.

2. Formula (3) gives us the right to exhibit an important property of the elastoids. A closed planar line cuts out an area on any curved surface that is certainly greater than the one that it bounds in the plane that contains it. However, if the line is skew then another surface must take the place of the plane if the given line is to determine a minimal area inside it. Such a surface is called one *of minimal area*, or simply, *minimal*. If one deforms it infinitely little in such a way that it does not cease to contain the given curve then it will be necessary that the first variation of the area that is bounded by that curve should be zero; i.e., that one should have $\iint \Theta\, ds_1\, ds_2 = 0$. Meanwhile, it is clear *a priori* that the tangential displacements cannot vary the area that is enclosed by the fixed contour, and we can always suppose, moreover, that the one surface passes to the other by means of arbitrary normal displacements w. Hence, by virtue of (3), the preceding condition will become $\iint Hw\, ds_1\, ds_2 = 0$, and in order for that to be satisfied for any w, it is necessary that we must have $H = 0$ at each point. Therefore, *any minimal surface is an elastoid*, but since the stated condition is not sufficient, because the area is minimal, it is clear that, exceptionally, an elastoid cannot have minimal area. In order to better see why the tangential displacements have no influence on the first variation of area, observe that:

$$\iint \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right) f\, ds_1\, ds_2 = \iint \frac{\partial \mathcal{G}_2 f}{\partial q_1}\, dq_1\, dq_2 = \int f\, ds_2\,,$$

in which the last integral is meant to be taken over the entire contour. Hence, since u and v are zero on the contour:

$$\iint \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right) u\, ds_1\, ds_2 = 0, \qquad \iint \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right) v\, ds_1\, ds_2 = 0.$$

3. The direction cosines of the tangents to the trajectory of M' obviously have the values given in (1), either divided by $1 + \Phi$ or multiplied by $1 - \Phi$ (up to higher-order infinitesimals). Such a multiplication will lead to the results:

$$\cos\omega - \varphi\sin\omega, \qquad \sin\omega + \varphi\cos\omega, \qquad w_1\cos\omega - w_2\sin\omega$$

if one sets:

$$\varphi = v_1\cos^2\omega - (u_1 - v_2)\cos\omega\sin\omega - u_2\sin^2\omega,$$

and because the first two expressions are equivalent to $\cos(\omega + \varphi)$ and $\sin(\omega + \varphi)$, resp., one will see clearly that φ is the angle through which the tangent plane to the line considered has rotated. In particular, the rotated axes of v_1 and $-u_2$, and therefore, $v_1 + u_2$, which is the coefficient of the rectangular term in Φ, will represent the mutual angular displacement of the tangent axes. Now, since Φ is generally reducible to a canonical form in just one way, one can state that *only one orthogonal pair of tangents will remain orthogonal under the deformation*, and it will be rotated rigidly in the tangent plane through an angle of $v_1 = - u_2 = \frac{1}{2}(v_1 - u_2)$. Since $v_1 - u_2$ is then an orthogonal invariant of the form φ, one will see that no matter how one orients the tangent axes, the *geodetic rotation* of the surface particle is always expressed by means of $\vartheta = v_1 - u_2$. Hence: *Any orthogonal pair of tangents, after having participated in a common rotation that is measured by* $\frac{1}{2}(v_1 - u_2)$, *will generally become oblique by an angular displacement* $\frac{1}{2}(v_1 + u_2)$ *that each of the two tangents exhibits with respect to the other.* The expression for ϑ results from (2) in the simple form:

$$\vartheta = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right) v - \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right) u ,$$

which does not depend upon the normal displacement.

4. Before we go any further, we must observe that the six functions u_1, u_2, v_1, v_2, w_1, w_2 are not arbitrary. Indeed, if we apply the known integrability condition:

$$\left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\frac{\partial}{\partial s_2} = \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\frac{\partial}{\partial s_1}$$

to the derivatives of the displacements when we are given (2) then we will find the relations:

$$\left.\begin{aligned}
\frac{\partial u_2}{\partial s_1} - \frac{\partial u_1}{\partial s_2} &= \mathcal{G}_1(u_1 - v_2) - \mathcal{G}_2(v_1 + u_2) + \mathcal{T} w_1 + \mathcal{N}_1 w_2, \\
\frac{\partial v_2}{\partial s_1} - \frac{\partial v_1}{\partial s_2} &= \mathcal{G}_1(v_1 + u_2) + \mathcal{G}_2(u_1 - v_2) - \mathcal{T} w_2 - \mathcal{N}_2 w_2, \\
\frac{\partial w_2}{\partial s_1} - \frac{\partial w_1}{\partial s_2} &= \mathcal{G}_1 w_1 - \mathcal{G}_2 w_2 - \mathcal{N}_1 u_1 + \mathcal{N}_2 v_1 - \mathcal{T}(u_1 - v_2).
\end{aligned}\right\} \tag{4}$$

Now, we can observe that if the values of w_1 and w_2 are given the forms:

$$w_1 = U\,\mathcal{N}_1 - V\,\mathcal{T}, \qquad w_2 = V\,\mathcal{N}_2 - U\,\mathcal{T} \tag{5}$$

then the first two relations will give:

$$\left.\begin{aligned} KU &= \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right) v_1 + \left(\frac{\partial}{\partial s_1} - \mathcal{G}_2\right) v_2 + \mathcal{G}_1 u_2 + \mathcal{G}_2 u_1, \\ KV &= \left(\frac{\partial}{\partial s_2} + \mathcal{G}_2\right) u_2 - \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right) u_1 + \mathcal{G}_1 v_2 + \mathcal{G}_2 v_1; \end{aligned}\right\} \tag{6}$$

i.e., by virtue of (2) and the Codazzi formulas:

$$U = u + \left(\frac{\mathcal{N}_2}{K}\frac{\partial}{\partial s_1} + \frac{\mathcal{T}}{K}\frac{\partial}{\partial s_2}\right) w, \qquad V = v + \left(\frac{\mathcal{N}_1}{K}\frac{\partial}{\partial s_2} + \frac{\mathcal{T}}{K}\frac{\partial}{\partial s_1}\right) w.$$

If one then substitutes the values (5) in the third equation (4) then one will see not only that w_1 and w_2 depend upon the other four functions, but that the differential relation:

$$\left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)(V\mathcal{N}_2 - U\mathcal{T}) - \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)(U\mathcal{N}_1 - V\mathcal{T}) + \mathcal{N}_1 u_2 - \mathcal{N}_2 v_1 + \mathcal{T}(u_2 - v_2) = 0 \tag{7}$$

will intercede among them.

5. In the case of the inextensible surface, if one sets:

$$u_1 = v_2 = 0, \qquad v_1 = -u_2 = \varphi, \quad \vartheta = v_1 - u_2 = 2\varphi$$

then the formulas (6) will give:

$$KU = \frac{\partial \varphi}{\partial s_2}, \qquad KV = -\frac{\partial \varphi}{\partial s_1};$$

hence, (7) will become $\mathbf{D}\varphi = 0$ after one sets:

$$\mathbf{D} = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\left(\frac{\mathcal{N}_2}{K}\frac{\partial}{\partial s_1} + \frac{\mathcal{T}}{K}\frac{\partial}{\partial s_2}\right) + \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\left(\frac{\mathcal{N}_1}{K}\frac{\partial}{\partial s_2} + \frac{\mathcal{T}}{K}\frac{\partial}{\partial s_1}\right) + H.$$

Therefore, the angles φ and ϑ will satisfy a second-order differential equation that is called the *characteristic equation*. It is clear that conversely any solution φ of the characteristic equation will correspond to a possible deformation, since if (7) is satisfied then the integrability conditions (4) will also be satisfied, and then the functions u, v, w that are defined by (2) will exist. In particular, if α, β, γ are the cosines that define an invariable direction then one will easily verify that the characteristic equation is satisfied by the function γ, and on the other hand, one can assert that this function does not correspond to a true deformation, but only to a change of position of the whole surface in space. Indeed, any infinitesimal rigid motion of the surface can be considered to be the result of a rotation ε around a certain line (α, β, γ, ξ, η, ζ) and a translation ε' that is parallel to that line in such a way that the displacements will take on the form:

$$u = \varepsilon\,\xi + \varepsilon'\alpha, \qquad v = \varepsilon\,\eta + \varepsilon'\beta, \qquad w = \varepsilon\,\zeta + \varepsilon'\gamma;$$

(2) will then give:

$$u_1 = v_2 = 0, \quad v_1 = -u_2 = \varphi = \varepsilon\,\gamma, \quad w_1 = -\varepsilon\,\gamma, \quad w_2 = \varepsilon\,\alpha\,.$$

It then follows that for the study of a deformation, properly speaking, it will always be legitimate to neglect terms in u, v, w that have the form that was indicated in the final place.

6. Assume that the new axes in the plane that touches the deformed surface at M are the lines that correspond to the values $-v_1$ and $\pi/2 + u_2$ for ω. Hence, by virtue of formulas (1), the direction cosines of the new axes with respect to the old ones will be:

for the x'-axis:	1,	0,	w_1,
″ y' ″ :	1,	1,	w_2,
″ z' ″ :	$-w_1$,	$-w_2$,	0 .

One will then have the following relations between the old and new coordinates:

$$\left.\begin{array}{lll} x' = x - (u - w_1 z), & y' = y - (u - w_2 z), & z' = z - (w + w_1 x + w_2 y), \\ x = x' + (u - w_1 z'), & y = y' + (u - w_2 z'), & z = z' + (w + w_1 x' + w_2 y'). \end{array}\right\} \tag{8}$$

In addition, the differential coefficients that relate to the arc-length ds' that M' can traverse are expressed in terms of the old ones by means of the formula:

$$\frac{\partial}{\partial s'} = (1 - \Phi)\left(\cos\omega\frac{\partial}{\partial s_1} + \sin\omega\frac{\partial}{\partial s_2}\right),$$

from which, when one takes ω to be $-v_1$ and $\pi/2 + u_2$, successively, one will deduce the differential quotients relative to the new axes:

$$(1 - u_1)\left(\frac{\partial}{\partial s_1} - v_1\frac{\partial}{\partial s_2}\right), \qquad (1 - v_2)\left(\frac{\partial}{\partial s_2} - u_2\frac{\partial}{\partial s_1}\right).$$

Thus:

$$\frac{\partial}{\partial s_1'} = \frac{\partial}{\partial s_1} - \left(u_1\frac{\partial}{\partial s_1} + v_1\frac{\partial}{\partial s_2}\right), \qquad \frac{\partial}{\partial s_2'} = \frac{\partial}{\partial s_2} - \left(u_2\frac{\partial}{\partial s_1} + v_2\frac{\partial}{\partial s_2}\right). \tag{9}$$

7. Given that, we propose to calculate the variations that the various curvatures experience as a result of the deformation. The first condition of the immobility of the point (x', y', z') with respect to the deformed surface is:

$$\frac{\partial x'}{\partial s_1'} = (\mathcal{N}_1 + \mathbf{D}\mathcal{N}_1)\, z' - (\mathcal{G}_1 + \mathbf{D}\mathcal{G}_1)\, y' - 1,$$

in which one adopts the symbol $\mathcal{D}$ in order to indicate the variations that are produced by the deformation, and one writes $\mathbf{D}$, instead of $\mathcal{D}$, when one passes from one surface to the other by displacing along the coordinates lines. Meanwhile, if one applies (9) to (8) then one will have:

$$\frac{\partial x'}{\partial s_1'} = \frac{\partial}{\partial s_1}(x - u + w_1 z) - \left(u_1 \frac{\partial x}{\partial s_1} + v_1 \frac{\partial x}{\partial s_2}\right),$$

which will also make the new and old coordinates coincide when one finds that they have been multiplied by infinitesimals. Similarly, the left-hand side of the formula in question is equivalent to:

$$\frac{\partial x}{\partial s_1} - \mathcal{N}_1\,(w + w_1\, x + w_1\, y) + \mathcal{G}_1\,(v - w_2\, z) + z\,\mathbf{D}\mathcal{N}_1 - y\,\mathbf{D}\mathcal{G}_1 \;.$$

Hence, if one observes (2) then one will see that $z\,\mathbf{D}\mathcal{N}_1 - y\,\mathbf{D}\mathcal{G}_1$ must be identically equal to:

$$\frac{\partial w_1 z}{\partial s_1} + \mathcal{G}_1\, w_2\, z + \mathcal{N}_1\,(w_1\, x + w_2\, y) - u_1\left(\frac{\partial z}{\partial s_1} + 1\right) - v_1 \frac{\partial x}{\partial s_2}\,.$$

One will get the values of $\mathbf{D}\mathcal{N}_1$ and $\mathbf{D}\mathcal{G}_1$ by developing this and taking into account the immobility condition that relates to the original surface by identification. One calculates $\mathbf{D}\mathcal{N}_2$ and $\mathbf{D}\mathcal{G}_2$ by an analogous procedure when one starts with the second immobility condition for the second set of three. One will then arrive at the following results:

$$\left.\begin{aligned}
\mathbf{D}\mathcal{N}_1 &= \frac{\partial w_1}{\partial s_1} - \mathcal{N}_1 u_1 + \mathcal{T} v_1 + \mathcal{G}_1 w_2, \qquad \mathbf{D}\mathcal{G}_1 = \mathcal{G}_2 v_1 - \mathcal{G}_1 u_1 - \mathcal{T} w_1 - \mathcal{N}_1 w_2, \\
\mathbf{D}\mathcal{N}_2 &= \frac{\partial w_2}{\partial s_2} - \mathcal{N}_2 v_2 + \mathcal{T} u_2 + \mathcal{G}_2 w_1, \quad \mathbf{D}\mathcal{G}_2 = \mathcal{G}_1 u_2 - \mathcal{G}_2 u_2 - \mathcal{T} w_2 - \mathcal{N}_2 w_1.
\end{aligned}\right\} \tag{10}$$

One should note that, by virtue of (4), the formulas on the right can also be written in the following way:

$$\mathbf{D}\,\mathcal{G}_1 = \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right) u_1 - \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right) u_2 - \Theta\,\mathcal{G}_1\,,$$

$$\mathbf{D}\mathcal{G}_2 = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)v_2 - \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)v_1 - \Theta\,\mathcal{G}_2\,.$$

As for the geodetic torsion, it is enough to operate analogously on the second immobility condition for the first set of three, or on the first such condition for the second set of three in order to obtain:

$$\mathbf{D}\mathcal{T} = -\frac{\partial w_2}{\partial s_1} - \mathcal{T}\,u_1 + \mathcal{N}_2\,v_1 + \mathcal{G}_1\,w_1$$

and also

$$\mathbf{D}\mathcal{T} = -\frac{\partial w_1}{\partial s_2} - \mathcal{T}\,v_2 + \mathcal{N}_1\,u_2 + \mathcal{G}_2\,w_2\,.$$

Those two expressions are equivalent by virtue of the third formula in (4).

8. We can now calculate the variations that the deformations produce in the *mean curvature* and the *total curvature*. The formulas (10) on the left immediately give:

$$\mathcal{D}H = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)w_1 + \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)w_2 - \mathcal{N}_1\,u_1 - \mathcal{N}_2\,w_2 + \mathcal{T}(v_1 + u_2), \tag{11}$$

or also, upon substituting the values (2):

$$\mathcal{D}H = u\frac{\partial H}{\partial s_1} + v\frac{\partial H}{\partial s_2} + \left(\mathcal{N}_1^2 + \mathcal{N}_2^2 + 2\mathcal{T}^2\right)w + \Delta^2 w\,.$$

If one applies (10) and the last formula that was obtained to $K = \mathcal{N}_1\,\mathcal{N}_2 - \mathcal{T}^2$ then one will easily arrive at the result:

$$\mathcal{D}K$$

$$= -K\Theta + \mathcal{N}_2\frac{\partial w_1}{\partial s_1} + \mathcal{N}_1\frac{\partial w_2}{\partial s_2} + \mathcal{T}\left(\frac{\partial w_1}{\partial s_2} + \frac{\partial w_2}{\partial s_1}\right) + (\mathcal{N}_1\,\mathcal{G}_2 - \mathcal{T}\mathcal{G}_1)\,w_1 + (\mathcal{N}_2\,\mathcal{G}_1 - \mathcal{T}\mathcal{G}_2)\,w_2\,,$$

which can be put into various forms. For example, if one substitutes the values (5), while keeping the Codazzi formulas in mind, then:

$$\mathcal{D}K = -K\Theta + \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)KU + \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)KV\,, \tag{12}$$

and one will see immediately (§ **5**) that under the hypotheses of inextensibility, one will have $\mathcal{D}K = 0$; i.e. (cf., XI, § **25**): *The curvature of an inextensible surface will remain* *invariant at each point when the surface is flexed.* If one substitutes given in (2) for w_1 and w_2, instead of (11), then one will find the formula:

$$\mathcal{D}\ln K = \left(U\frac{\partial}{\partial s_1} + V\frac{\partial}{\partial s_2}\right)\ln K + \mathbf{D}\, w,$$

from which, it will result, for example, that if one would like to deform a surface with constant curvature in such a way that the curvature remains invariant then one will need to give an infinitesimal normal displacement that satisfies the characteristic equation to any point.

9. Among the infinitude of possible deformations, it is natural that the attention should be focused on the ones for which the representative conic of the form Φ reduces to a circle. The same thing will be true for φ, and one will have:

$$u_1 = v_2 = \Phi = \tfrac{1}{2}\Theta, \qquad v_1 = -u_2 = \varphi = \tfrac{1}{2}\vartheta,$$

independently of ω. (6) will then give:

$$-KU = \frac{\partial\Phi}{\partial s_1} - \frac{\partial\varphi}{\partial s_2}, \qquad -KV = \frac{\partial\Phi}{\partial s_2} + \frac{\partial\varphi}{\partial s_1},$$

and (12) will reduce to the simple form:

$$\mathcal{D}K = -(K + \tfrac{1}{2}\Delta^2)\,\Theta, \tag{13}$$

independently of the rotation. One will then have the differential relation (7) between the dilatations and rotations, which will assume the remarkable form:

$$\mathbf{d}\Theta + \mathbf{D}\vartheta = 0, \tag{14}$$

if one sets:

$$\mathbf{d} = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_2\right)\left(\frac{\mathcal{N}_2}{K}\frac{\partial}{\partial s_2} - \frac{\mathcal{T}}{K}\frac{\partial}{\partial s_1}\right) - \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\left(\frac{\mathcal{N}_1}{K}\frac{\partial}{\partial s_1} - \frac{\mathcal{T}}{K}\frac{\partial}{\partial s_2}\right),$$

for brevity. As we saw in § **5**, (14) will reduce to the characteristic equation for the inextensible surfaces. However, that will hold for only those values of Θ that satisfy the equation $\mathbf{d} = 0$, and in particular, that will persist when Θ has a constant value, which by virtue of (13) will measure the unit decrease in the total curvature. Furthermore, no matter what the function Θ is that satisfies the equation $\mathbf{d} = 0$, it will always correspond to a possible deformation that is characterized by the absence of geodetic rotation – i.e., one for which $\vartheta = 0$ – and consequently:

$$u = \frac{\partial \psi}{\partial s_1}, \qquad v = \frac{\partial \psi}{\partial s_2}, \qquad \Delta^2 \psi = \Theta + Hw, \qquad \text{etc.}$$

10. Turning to the general case, we propose to determine the alternating products of the deformations of the curvature of the various lines. When M is displaced in a definite direction in the plane tangent to (M) that is at an angle ω, M' will traverse a curve that is inclined by $\omega' = \omega + \varphi$ with respect to the x'-axis, which would be the effect of the deformation if one had $\mathbf{D}\omega = 0$. Thus, if one recalls (XI, §§ **10**, **18**) that:

$$\mathcal{N}_\omega = \mathcal{N}_1 \cos^2 \omega - 2\mathcal{T} \cos \omega \sin \omega + \mathcal{N}_2 \sin^2 \omega, \qquad \frac{\partial \mathcal{N}_\omega}{\partial \omega} = -2\mathcal{T}\omega$$

then one will obtain:

$$\mathcal{DN}_\omega = \cos^2 \omega \cdot \mathbf{D}\mathcal{N}_1 - 2 \cos \omega \sin \omega \cdot \mathbf{D}\mathcal{T} + \sin^2 \omega \cdot \mathbf{D}\mathcal{N}_2 - 2\varphi\, \mathcal{T}_\omega,$$

and, in particular:

$$\mathcal{DN}_1 = \frac{\partial w_1}{\partial s_1} - \mathcal{N}_1\, u_1 - \mathcal{T} v_1 + \mathcal{G}_1\, w_2, \qquad \mathcal{DN}_2 = \frac{\partial w_2}{\partial s_2} - \mathcal{N}_2\, v_2 - \mathcal{T} u_2 + \mathcal{G}_2\, w_1 .$$

Similarly, from the formulas:

$$\mathcal{T}_\omega = \mathcal{T} \cos 2\omega + (\mathcal{N}_1 - \mathcal{N}_2) \sin 2\omega, \qquad \frac{\partial \mathcal{T}_\omega}{\partial \omega} = \mathcal{N}_\omega - \mathcal{N}_{\omega+\pi/2}\ ,$$

one will deduce that:

$$\mathcal{DT}_\omega = \cos 2\omega \cdot \mathbf{D}\mathcal{T} + \sin 2\omega \cdot (\mathbf{D}\mathcal{N}_1 - \mathbf{D}\mathcal{N}_2) + (\mathcal{N}_\omega - \mathcal{N}_{\omega+\pi/2})\ \varphi.$$

In particular, the alternating product of the geodetic torsion of the line q_1 is given by the formula:

$$\mathcal{DT} = -\frac{\partial w_2}{\partial s_1} - \mathcal{T} u_1 + \mathcal{N}_1\, v_1 + \mathcal{G}_1\, w_1\, ,$$

and it is easy to verify that when the displacements have the form that was indicated at the end of § **5**, $\mathcal{DT}$, as well as $\mathcal{DN}$, will be zero. Conversely, if the surface is inextensible then the given formula will follow from the hypotheses that $\mathcal{DN}_1 = 0$, $\mathcal{DN}_2 = 0$, $\mathcal{DT} = 0$. Indeed, one can deduce from that and (4) that:

$$\left.\begin{aligned}
&\frac{\partial w_2}{\partial s_1} = \mathcal{N}_1\varphi + \mathcal{G}_1 w_1, &&\frac{\partial w_2}{\partial s_2} = -\mathcal{G}_2 w_1 - \mathcal{T}\varphi,\\
&\frac{\partial w_1}{\partial s_1} = -\mathcal{G}_1 w_2 + \mathcal{T}\varphi, &&\frac{\partial w_1}{\partial s_2} = -\mathcal{N}_2\varphi + \mathcal{G}_2 w_2,\\
&\frac{\partial \varphi}{\partial s_1} = -\mathcal{T}_1 w - \mathcal{N}_1 w_2, &&\frac{\partial \varphi}{\partial s_2} = \mathcal{T} w_2 + \mathcal{N}_2 w_1;
\end{aligned}\right\} \tag{15}$$

hence:

$$\frac{\partial}{\partial s_1}(w_2^2 + w_1^2 + \varphi^2) = 0, \qquad \frac{\partial}{\partial s_2}(w_2^2 + w_1^2 + \varphi^2) = 0.$$

Therefore, $w_2^2 + w_1^2 + \varphi^2$ will have a *constant* value ε^2. Now, if we set $w_2 = \varepsilon\alpha$, $w_1 = -\varepsilon\beta$, $\varphi = \varepsilon\gamma$ then the relations (15) will be precisely the ones that express the *invariability* of the direction (α, β, γ).

11. Finally, in order to know how the geodetic curvature of any line will vary, observe that the formula:

$$\mathcal{G}_\omega + \frac{\partial \omega}{\partial s} = \mathcal{G}_1 \cos\omega - \mathcal{G}_2 \sin\omega$$

will become:

$$\mathcal{G}_\omega + \mathcal{D}\mathcal{G}_\omega + (1 - \Phi)\frac{\partial}{\partial s}(\omega + \varphi) = (\mathcal{G}_1 + \mathbf{D}\mathcal{G}_1)\cos(\omega + \varphi) - (\mathcal{G}_2 + \mathbf{D}\mathcal{G}_2)\sin(\omega + \varphi)$$

on the surface (M'). It then follows that:

$$\mathcal{D}\mathcal{G}_\omega = \Phi\frac{\partial \omega}{\partial s} + \left[\mathbf{D}\mathcal{G}_1 - \left(\frac{\partial}{\partial s_1} + \mathcal{G}_1\right)\varphi\right]\cos\omega - \left[\mathbf{D}\mathcal{G}_2 + \left(\frac{\partial}{\partial s_2} + \mathcal{G}_1\right)\varphi\right]\sin\omega.$$

If one distinguishes the derivative with respect to the direction $\omega + \pi/2$ by a prime then one will come to the formula:

$$\mathcal{D}\mathcal{G}_\omega = \left(\frac{\partial}{\partial s'} + \mathcal{G}_\omega\right)\Phi - \left(\frac{\partial}{\partial s} + \mathcal{G}'_\omega\right)(2\varphi - \vartheta) - \Theta\mathcal{G}_\omega,$$

after a simple calculation, which we shall omit, for brevity. The right-hand side will obviously reduce to zero when the surface is inextensible, and one will then recover the fact, which is obvious moreover, of the *invariability of the geodetic curvature* of a line that is traced on a flexible, inextensible surface.

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CHAPTER XIV

CONGRUENCES

1. – The intrinsic geometry of systems of lines can be based upon considerations that are analogous to the ones in the preceding chapter that permitted us to undertake the infinitesimal study of systems of points. In a congruence, or doubly-infinite system of lines, one considers two of them, g and g' that are infinitely close. One takes g to be the z-axis, and the x-axis also meets g' orthogonally. Let ds and $r\,ds$ be the angle and the distance, resp., between the lines considered, in such a way that p will represent (cf., IX, § **7**, *c*; § **8**) the *distributor parameter* of the tangent planes to the element gg' of the ruled surface. When the axes go to the positions that are specified by another surface element $g'g''$, the variations δx, δy, δz that the coordinates x, y, z of a point suffer with respect to the initial position will be equal to the possible variations dx, dy, dz relative to the moving axes, augmented by the variations that are due solely to the motion of the reference trihedron – i.e., to the translation ($p\,d\sigma$; $h\,d\sigma$) and the rotation ($d\sigma$; 0, $k\,d\sigma$). One will then have the formulas:

$$\frac{\delta x}{\partial\sigma} = \frac{\partial x}{\partial\sigma} - k\,y + p, \qquad \frac{\delta y}{\partial\sigma} = \frac{\partial y}{\partial\sigma} + k\,x - z, \qquad \frac{\delta z}{\partial\sigma} = \frac{\partial z}{\partial\sigma} + y + h,$$

which are also true in the case where x, y, z are direction cosines, as long as one ignores p and h, which are characteristic of translations. When the y-plane rotates around g, the origin (viz., *central point*) will be displaced along the generator. If one imagines that it is transported to a fixed point of g that is at a distance q from the central point then the preceding formulas will become:

$$\frac{\delta x}{\partial\sigma} = \frac{\partial x}{\partial\sigma} - k\,y + p, \qquad \frac{\delta y}{\partial\sigma} = \frac{\partial y}{\partial\sigma} + k\,x - z + q, \qquad \frac{\delta z}{\partial\sigma} = \frac{\partial z}{\partial\sigma} + y + r, \tag{1}$$

in which, we have set:

$$r = h - \frac{dq}{d\sigma}, \tag{2}$$

for brevity. Having done that, let the index 1 distinguish the quantities that relate to a given position (which is arbitrary, moreover) of the y-plane, and demand that the ones that do not include that index should refer to the position that is occupied by the plane itself after a rotation ω, in such a way:

$$x = x_1 \cos\omega + y_1 \sin\omega, \qquad y = -x_1 \sin\omega + y_1 \cos\omega, \qquad z = z_1 .$$

However, if one lets the index 2 distinguish everything that refers to the position $\omega = \pi / 2$ then one will see that $x_2 = y_1$, $y_2 = -x_1$, $z_2 = z_1$, and therefore formulas (1) will become:

$$\frac{\delta x}{\partial\sigma} = \frac{\partial x}{\partial\sigma} - k\,y + p - q', \qquad \frac{\delta y}{\partial\sigma} = \frac{\partial y}{\partial\sigma} + k'\,x + p, \quad \frac{\delta z}{\partial\sigma} = \frac{\partial z}{\partial\sigma} - x + r' \tag{1'}$$

for a position of the y-plane that is perpendicular to the original one. Hence, if the y-plane is fixed in the initial position ($\omega = 0$) then the necessary and sufficient condition for the immobility of the point (x, y, z) will result from (1) and (1′) in the following form:

$$\left.\begin{aligned} &\frac{\partial x}{\partial\sigma_1} = k_1 y - p_1, && \frac{\partial y}{\partial\sigma_1} = z - k_1 x - q_1, && \frac{\partial z}{\partial\sigma_1} = -y - r_1, \\ &\frac{\partial x}{\partial\sigma_2} = -z + k_2 y + q_2, && \frac{\partial y}{\partial\sigma_2} = -k_2 x - p_2, && \frac{\partial z}{\partial\sigma_2} = x - r_1. \end{aligned}\right\} \tag{3}$$

These are the fundamental formulas for the intrinsic analysis of the congruence.

2. – The differential quotients $\partial / \partial\sigma$ that relate to an arbitrary angle ω are composed very simply in terms of the quotients $\partial / \partial\sigma_1$ and $\partial / \partial\sigma_2$. Indeed, it is enough to observe that the direction cosines of g, which are obviously equal to:

$$\sin\omega\, d\sigma, \qquad -\cos\omega\, d\sigma, \qquad 1,$$

prove to be equal to $d\sigma_2$, $-d\sigma_1$, 1, resp., from formulas (10). Therefore:

$$\frac{\partial}{\partial\sigma} = \cos\omega\,\frac{\partial}{\partial\sigma_1} + \sin\omega\,\frac{\partial}{\partial\sigma_2}\,. \tag{4}$$

Having said that, the first of (1) will show that $-p$ is the value on g of the differential quotient of x with respect to σ, and in this one must understand x to mean $x \cos\omega + y \sin\omega$, in such a way that if one takes formulas (3) and (4) into account:

$$p = (p_2 \cos\omega - q_1 \sin\omega) \cos\omega + (q_2 \cos\omega + p_1 \sin\omega) \sin\omega;$$

i.e.:

$$p = p_1 \cos^2\omega + (q_1 - q_2) \cos\omega \sin\omega + p_2 \sin^2\omega. \tag{5}$$

Operating analogously on the second formula in (1) will yield:

$$q = q_1 \cos^2\omega - (p_1 - p_2) \cos\omega \sin\omega + q_2 \sin^2\omega. \tag{6}$$

These two relations (*viz.*, *Hamilton's formulas*) comprise the fundamental properties of a congruence. When ω is eliminated, they will allow one will arrive at the very simple rule that exists between p and q:

$$(p - p_1)(p - p_2) + (q - q_1)(q - q_2) = 0.$$

3. – When one adds $\pi / 2$ to ω, one will get values p' and q' for p and q, resp., such that:

$$p + p' = p_1 + p_2 , \qquad q + q' = q_1 + q_2 .$$

The first equality leads (cf., XI, § **23**) to the notion of the *mean distributor parameter* $p_0 = \frac{1}{2}(p_1 + p_2)$ of the congruence around g. The second one shows that there exists a point (viz., the *midpoint*) on g at a distance $q_0 = \frac{1}{2}(q_1 + q_2)$ from the origin, with respect to which the central points are symmetric relative to two arbitrary orthogonal planes. One also deduces from the same formulas (5) and (6) that:

$$p - p' = (p_1 - p_2) \cos 2\omega + (q_1 - q_2) \sin 2\omega, \quad q - q' = (q_1 - q_2) \cos 2\omega - (p_1 - p_2) \sin 2\omega,$$

and therefore, if one sets:

$$p - p' = 2l \cos 2\omega_0 , \quad q - q' = 2l \sin 2\omega_0$$

then one will see that one has, more generally:

$$p - p' = 2l \cos 2(\omega_0 - \omega), \qquad q - q' = 2l \sin 2(\omega_0 - \omega) . \tag{7}$$

Another invariant is therefore:

$$4l^2 = (p - p')^2 + (q - q')^2 = (p_1 - p_2)^2 + (q_1 - q_2)^2 ;$$

however, one can substitute:

$$\kappa = pp' + qq' = p_1 p_2 + q_1 q_2$$

for this, since $\kappa = p_0^2 + q_0^2 - l^2$. The three orthogonal invariants that were mentioned above then make up the discriminants $\kappa - p_0^2$ and $\kappa - q_0^2$ in the forms (5) and (6).

4. – The discussion of (7) leads one to distinguish the pairs of orthogonal planes that pass through g that correspond to the values ω_0 and $\omega_0 + \pi / 4$ of ω from the infinitude of all of such pairs. For the first one, the central points coincide with the midpoint, and the distributor parameters attain the extreme values $p_0 \pm l$. For the second one, which is composed of *principal planes*, one will have $p = p'$, and the central points will become the *limit points*; i.e., they will be the end points of the segment of length $2l$ that includes all of the central points. When the figure refers to the first pair, for example, formula (5) will give $p = p_0 - l \cos 2\omega$, and one will have $p = 0$ in the two directions (real or imaginary) that are defined by $\cos 2\omega = p_0 : l$; i.e., g' will meet g. The planes that correspond to those directions are the *focal planes*, and the relative central points (i.e., *foci*), which are also symmetric with respect to the midpoint, and at a distance of $\sqrt{l^2 - p_0^2}$ from it. That situation will emerge even more clearly from the geometric interpretation of the relations that were found between p and q, which can be considered to be the equations of a circle of radius l, center (p_0, q_0), and power κ with respect to the origin. We have already encountered the preceding property for $p_0 = 0$ in the congruence that is composed of the normals to an arbitrary surface, and moreover, one will see that it is only for those congruences that one calls *normal congruences* that one can have a mean distributor parameter that is zero. For them, one has that the foci fall upon the limit points, and the *principal surface*, which is the locus of all the limit points, will become the evolute of the surface that one considers.

5. – We now turn to (3) in order to apply the integrability condition to the normal congruences, and that condition will always have the form:

$$\frac{\partial^2}{\partial\sigma_1\,\partial\sigma_2} - \frac{\partial^2}{\partial\sigma_2\,\partial\sigma_1} = \lambda_2\frac{\partial}{\partial\sigma_2} - \lambda_1\frac{\partial}{\partial\sigma_1}, \tag{8}$$

as long as one determines the λ conveniently. In order for z to exist, it is necessary and sufficient that one should have:

$$(\lambda_2 - k_2)\,x + (\lambda_1 + k_1)\,y + (p_1 + p_2) = \left(\frac{\partial}{\partial\sigma_1} + \lambda_2\right)r_2 - \left(\frac{\partial}{\partial\sigma_2} + \lambda_1\right)r_1, \tag{9}$$

no matter what x and y are. Therefore, $\lambda_1 = - k_1$, $\lambda_2 = k_2$, and the condition (8) will become:

$$\frac{\partial^2}{\partial\sigma_1\,\partial\sigma_2} - \frac{\partial^2}{\partial\sigma_2\,\partial\sigma_1} = k_1\frac{\partial}{\partial\sigma_1} + k_2\frac{\partial}{\partial\sigma_2}, \tag{10}$$

while the relations (9) will reduce to:

$$\frac{\partial r_1}{\partial\sigma_2} - \frac{\partial r_2}{\partial\sigma_1} - (p_1 + p_2) = k_1\,r_1 + k_2\,r_2\,. \tag{0}$$

Similarly, if one applies the condition (10) to the functions x and y on the generator then one will get:

$$\frac{\partial p_1}{\partial\sigma_2} + \frac{\partial q_2}{\partial\sigma_1} + r_1 = - k_1\,(p_1 - p_2) + k_2\,(q_1 - q_2), \tag{1}$$

$$\frac{\partial q_1}{\partial\sigma_2} + \frac{\partial q_2}{\partial\sigma_1} + r_2 = k_1\,(q_1 - q_2) - k_2\,(p_1 - p_2), \tag{2}$$

and it is easy to insure by means of the equality (2) that these formulas are not linked to a choice of origin. If one then considers the coefficients of x or y, respectively, in the conditions (10) that apply to y or x then one will find that:

$$\frac{\partial k_1}{\partial \sigma_2} - \frac{\partial k_2}{\partial \sigma_1} = 1 + k_1^2 + k_2^2 . \tag{3}$$

The last four relations are, so to speak, *the Cozazzi formulas* for the theory of congruences, and they will essentially reduce to the three known Codazzi formulas (XI, § **9**) when the congruence is composed of normals to a surface. Moreover, if you carry out the *spherical representation* of the congruence – i.e., if (cf., XI, § **15**) you consider the radii in a sphere of radius 1 that are parallel to the lines of the congruence – then it will be easy to see that $\mathcal{G}_1 = k_2$, $\mathcal{G}_2 = k_1$, either by writing the integrability condition and comparing it to (10) or by transforming (3) into the immobility conditions that relate to a spherical surface (and observing that $\mathcal{N}_1 = \mathcal{N}_2 = 1$, $\mathcal{T} = 0$). If one interprets the k by observing that, in addition:

$$\frac{\partial}{\partial s_1} = - \frac{\partial}{\partial \sigma_1} , \qquad \frac{\partial}{\partial s_2} = - \frac{\partial}{\partial \sigma_1} ,$$

then one will see directly that **(3)** is precisely the Gauss formula that relates to the sphere, while the other two Codazzi formulas are satisfied identically.

6. – In order to better exhibit the aforementioned reduction that we would like to carry out, we replace the differential quotients $\partial / \partial \sigma$ with other quotients that are determined by the relations:

$$\frac{\partial}{\partial \sigma_1} = p_1 \frac{\partial}{\partial s_1} + q_1 \frac{\partial}{\partial s_2} , \qquad \frac{\partial}{\partial \sigma_2} = - q_2 \frac{\partial}{\partial s_1} + p_2 \frac{\partial}{\partial s_2} ,$$

and replace the k with other quantities that are defined in the following way:

$$k_1 = - p_1 \, \mathcal{G}_1 + q_1 \, \mathcal{G}_2 , \quad k_2 = q_2 \, \mathcal{G}_1 + p_2 \, \mathcal{G}_2 .$$

The transformation of formulas **(1)** and **(2)** presents no difficulties. If one combines the transformed relations conveniently then one will arrive at the formulas:

$$\frac{\partial}{\partial s_1} \frac{q_2}{\kappa} - \frac{\partial}{\partial s_2} \frac{p_2}{\kappa} + \frac{1}{\kappa} [(p_1 - p_2) \, \mathcal{G}_1 - (q_1 - q_2) \, \mathcal{G}_2] + \frac{1}{\kappa^2} (p_2 \, r_1 + q_2 \, r_2) = 0,$$

$$\frac{\partial}{\partial s_2} \frac{q_1}{\kappa} + \frac{\partial}{\partial s_1} \frac{p_1}{\kappa} + \frac{1}{\kappa} [(q_1 - q_2) \, \mathcal{G}_1 + (p_1 - p_2) \, \mathcal{G}_2] + \frac{1}{\kappa^2} (p_1 \, r_2 - q_1 \, r_1) = 0;$$

when one takes **(1)** and **(2)** into account, **(3)** will be easily transformed into:

$$\frac{\partial \mathcal{G}_1}{\partial s_2} + \frac{\partial \mathcal{G}_2}{\partial s_1} + \mathcal{G}_1^2 + \mathcal{G}_2^2 = \frac{1}{\kappa}(\mathcal{G}_1\, r_1 - \mathcal{G}_2\, r_2 - 1).$$

Having said that, if one sets:

$$p_1 = -\,\kappa \mathcal{T}_2, \quad p_2 = -\,\kappa \mathcal{T}_1, \quad q_1 = \kappa \mathcal{N}_1, \qquad q_2 = \kappa \mathcal{N}_2$$

and observes that:

$$\frac{1}{\kappa} = \mathcal{N}_1\, \mathcal{N}_2 + \mathcal{T}_1\, \mathcal{T}_2$$

then the preceding relations will take on the definitive form:

$$\left.\begin{aligned}
&\frac{\partial \mathcal{N}_2}{\partial s_1} + \frac{\partial \mathcal{T}_1}{\partial s_2} + (\mathcal{T}_1 - \mathcal{T}_2)\,\mathcal{G}_1 - (\mathcal{N}_1 - \mathcal{N}_2)\,\mathcal{G}_2 = (\mathcal{N}_1\mathcal{N}_2 + \mathcal{T}_1\,\mathcal{T}_2)(\mathcal{T}_1\, r_2 - \mathcal{N}_1\, r_2),\\
&\frac{\partial \mathcal{N}_1}{\partial s_2} - \frac{\partial \mathcal{T}_2}{\partial s_1} + (\mathcal{N}_1 - \mathcal{N}_2)\,\mathcal{G}_1 - (\mathcal{T}_1 - \mathcal{T}_2)\,\mathcal{G}_2 = (\mathcal{N}_1\mathcal{N}_2 + \mathcal{T}_1\,\mathcal{T}_2)(\mathcal{T}_2\, r_2 + \mathcal{N}_1\, r_1),\\
&\frac{\partial \mathcal{G}_1}{\partial s_2} + \frac{\partial \mathcal{G}_2}{\partial s_1} + \mathcal{G}_1^2 + \mathcal{G}_2^2 = (\mathcal{N}_1\mathcal{N}_2 + \mathcal{T}_1\,\mathcal{T}_2)(\mathcal{G}_1\, r_1 - \mathcal{G}_2\, r_2 - 1).
\end{aligned}\right\} \tag{11}$$

If one also introduces the new notation in formulas (3) then they will become:

$$\left.\begin{aligned}
&\frac{\partial x}{\partial s_1} = \mathcal{N}_1\, x - \mathcal{G}_1 y - 1, && \frac{\partial x}{\partial s_2} = \mathcal{G}_2\, y + \mathcal{T}_2\, z,\\
&\frac{\partial y}{\partial s_1} = \mathcal{G}_1\, x - \mathcal{T}_1\, z, && \frac{\partial y}{\partial s_2} = \mathcal{N}_2\, z - \mathcal{G}_2\, x - 1,\\
&\frac{\partial z}{\partial s_1} = \mathcal{T}_1(y + r_1) - \mathcal{N}_1(x - r_2), && \frac{\partial z}{\partial s_2} = -\mathcal{T}_2(x - r_1) + \mathcal{N}_2(y + r_1).
\end{aligned}\right\} \tag{12}$$

7. – How does one characterize the normal congruences? In order for the lines of a congruence to all be normal to a surface, it is necessary and sufficient that one must have $\delta z = 0$ for (at least) one point of the generator ($x = 0$, $y = 0$); i.e., if one recalls (4):

$$\left(\frac{\partial z}{\partial \sigma_1} + r_1\right)\cos\omega + \left(\frac{\partial z}{\partial \sigma_2} + r_2\right)\sin\omega = 0,$$

for any ω. It is therefore necessary and sufficient that there should exist a function z that admits the differential quotients $-\,r_1$ and $-\,r_2$, and by virtue of (10), for that to happen, it is enough that one should have:

$$\frac{\partial r_1}{\partial \sigma_2} - \frac{\partial r_2}{\partial \sigma_1} = k_1\, r_1 + k_2\, r_2\,,$$

i.e., from **(0)**, that *the mean distributor parameter should be zero,* or what amounts to the same thing, that *the focal planes should be perpendicular*, or simply that *the foci must fall upon the limit points.* That will permit one to take $\mathcal{T}_1 = -\mathcal{T}_2 = \mathcal{T}$, and therefore, when one also chooses the origin to be on the surface, formulas (12) will become the known fundamental formulas for the intrinsic analysis of surfaces, while (11) will finally reduce to the Codazzi formulas.

8. – We conclude with just one application of formulas (3) to the determination of the surface that is enveloped by the planes perpendicular to the lines of the congruence. If we differentiate the equation $z = 0$ then we will immediately get, by means of the (3), the coordinates $x = r_2$, $y = -r_1$ of the contact point of the z-plane with its own envelope. When the same formulas are applied to that point, that will tell us, for any ω, the direction of a tangent that is defined in the z-plane by direction cosines α and β that are proportional to:

$$P_1 \cos \omega - Q_2 \sin \omega, \qquad Q_1 \cos \omega + P_2 \sin \omega,$$

resp., in which we set:

$$P_1 = p_1 + k_1\, r_1 + \frac{\partial r_2}{\partial \sigma_1}, \qquad P_2 = p_2 + k_2\, r_2 - \frac{\partial r_1}{\partial \sigma_2}\,,$$

$$Q_1 = q_1 + k_1\, r_2 - \frac{\partial r_1}{\partial \sigma_1}, \qquad Q_2 = q_2 - k_2\, r_1 - \frac{\partial r_2}{\partial \sigma_2}\,,$$

for brevity. We then need to note that by virtue of (0), we can take $P_1 = -P_2 = P$. Having done that, let:

$$\alpha = \lambda\,(P \cos \omega - Q_2 \sin \omega), \qquad \beta = \lambda\,(Q_1 \cos \omega - P \sin \omega),$$

and consequently:

$$\lambda \cos \omega = \frac{Q_2 \beta - P\alpha}{Q_1\, Q_2 - P^2}\,, \qquad \lambda \sin \omega = \frac{P\beta - Q_1\, \alpha}{Q_1\, Q_2 - P^2}\,.$$

The curvature of the normal section is determined by the angles ω and $\lambda\, d\gamma : d\sigma$, and therefore, by virtue of (3), it can be expressed in the following way:

$$\mathcal{N} = \lambda\,(\beta \cos \omega - \alpha \sin \omega) = \frac{Q_1 \alpha^2 - 2P\alpha\beta + Q_2 \beta^2}{Q_1\, Q_2 - P^2}\,.$$

Hence, if one adopts the usual notation:

$$\mathcal{N}_1 = \frac{Q_1}{Q_1\,Q_2 - P^2}, \qquad \mathcal{N}_2 = \frac{Q_2}{Q_1\,Q_2 - P^2}, \qquad \mathcal{T} = \frac{P}{Q_1\,Q_2 - P^2}.$$

The *total curvature* ($\mathcal{N}_1\ \mathcal{N}_2 - \mathcal{T}^2$) is then inverse to $Q_1\ Q_2 - P^2$, and the ratio of the *mean curvature* to the total curvature is one-half of $Q_1 + Q_2$; i.e.:

$$2q_0 - \left(\frac{\partial}{\partial\sigma_1} + k_2\right) r_1 - \left(\frac{\partial}{\partial\sigma_2} - k_1\right) r_2 . \tag{13}$$

The case of *isotropic congruences* is noteworthy, which are the ones for which p is constant on each generator. By virtue of (5), q will also have a constant value at each point of g then, and formulas **(1)** and **(2)** will give:

$$\frac{\partial p}{\partial\sigma_2} + \frac{\partial q}{\partial\sigma_1} = -\,r_1, \qquad \frac{\partial p}{\partial\sigma_1} - \frac{\partial q}{\partial\sigma_2} = r_2 .$$

Therefore, if one takes (10) into account then the expression (13) will change into the left-hand side of the equation:

$$\left[\left(\frac{\partial}{\partial\sigma_1} + k_2\right)\frac{\partial}{\partial\sigma_1} + \left(\frac{\partial}{\partial\sigma_2} - k_1\right)\frac{\partial}{\partial\sigma_2} + 2\right] q = 0,$$

which will then characterize the envelopes with zero mean curvature. On the other hand, if the preceding values of r_1 and r_2 are substituted in **(0)** then that will become:

$$\left[\left(\frac{\partial}{\partial\sigma_1} + k_2\right)\frac{\partial}{\partial\sigma_1} + \left(\frac{\partial}{\partial\sigma_2} - k_1\right)\frac{\partial}{\partial\sigma_2} + 2\right] p = 0.$$

Hence, one will get an infinitude of elastoids by taking q to be proportional to p, and in particular, for $q = 0$, one will arrive at the following theorem by **Ribaucour:**

The mean envelope of an isotropic congruence is an elastoid.

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