CHAPTER XV

THREE-DIMENSIONAL SPACES

1. – Consider a continuous function of the points of a three-dimensional space; i.e., a variable u that takes a prescribed value at each location M in a triply-infinite system of points and varies infinitely-little when M passes to an infinitely-close position M'. Imposing a constant value upon M is equivalent to singling out a double infinitude of points from the triple infinitude – i.e., a surface – and changing the value of u will mean passing from one surface to another. It is then clear that any real function of the points of a space will be the analytical representation of a simply-infinite system of surfaces. The ratio of the variation that the function suffers when the point goes from M to M' to MM' is the *differential quotient* relative to the direction MM', and it is easy (cf., VIII, § **2**) to see that when one knows the differential quotients in three mutually-perpendicular directions, the quotient relative to the direction that is defined by the cosines α, β, γ will result from the operation:

$$\frac{d}{ds} = \alpha\frac{\partial}{\partial s_1} + \beta\frac{\partial}{\partial s_2} + \gamma\frac{\partial}{\partial s_3},$$

and yet, if one sets:

$$\Delta = \left(\frac{\partial}{\partial s_1}\right)^2 + \left(\frac{\partial}{\partial s_2}\right)^2 + \left(\frac{\partial}{\partial s_3}\right)^2$$

then one will have:

$$\frac{du}{ds} = \sqrt{\Delta u} \cdot \cos\theta,$$

in which θ is the angle that the variable direction in question makes with the fixed direction that is defined by the cosines:

$$\frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_1}, \quad \frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_2}, \quad \frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_3}. \tag{1}$$

That will then always be the direction along which *u tends to vary most rapidly*, and the value of the differential quotient in that direction will be precisely $\sqrt{\Delta u}$. However, if one sets $\theta = 0$ then one will see that *u tends to remain constant* along the infinite perpendicular that goes through M in the direction (1). In other words, a plane passes through any point M that is characterized by the property that the variation of u when M is displaced infinitely little in that plane is an infinitesimal of an order that is higher than one with respect to the magnitude of the displacement of M. On the other hand, if M traverses any line on the surface that belongs to the system that is defined by the function u then it will remain constant, and it will also remain constant, up to higher infinitesimals, when M is, on the contrary, displaced infinitely little along the tangent to that line, since, if one neglects the quantities that are higher order infinitesimals with respect to the arc length that is traversed then the line can coincide with its tangent. Therefore, the tangent to the line necessarily belongs to the preceding plane that was found, i.e., *the tangents to all of the lines that pass through M on the surface are in a plane,* even though not all of the differential quotients (1) are zero. One can then add that the cosines (1) are precisely the ones that define the direction of the *normal* to the surface at the point *M*.

2. – One can base a system of curvilinear coordinates on three families of surfaces that are defined by the functions q_1, q_2, q_3, from what was said in § **7** of Chapter Eight. A *surface* q_i will then pass through any point M in space that is the locus of points at which the parameter q_i keeps the value that it has at M. The surfaces q_1, q_2, q_3 that pass through M intersect each other along three lines (viz., the *coordinate lines*), and one says the *line* q_i to mean the line along which only the parameter q_i varies. We shall always suppose that the coordinate surfaces (and consequently the lines) are pair-wise mutually-perpendicular at any point, and assume that the axes are the tangents Mx_1, Mx_2, Mx_3 to those lines. Now, if x_1, x_2, x_3 represent the coordinates of a fixed point in space then we would like to write down the condition of immobility with respect to the trihedron that is, in turn, considered to be the fundamental trihedron of the three coordinate surfaces. First, suppose that the functions Q that enter into the expression for the square of the elementary arc-length:

$$ds^2 = Q_1^2\, dq_1^2 + Q_2^2\, dq_2^2 + Q_3^2\, dq_3^2 \tag{2}$$

are defined and set:

$$\mathcal{G}_{ij} = \frac{1}{Q_i\, Q_j} \frac{\partial Q_i}{\partial q_j}. \tag{3}$$

Let *ijk* denote an even permutation of the indices 123 and consider one of the three surfaces – q_k, for example. On it, the lines q_1 and q_2 will be denoted by q_i and q_j, resp., and the curvatures that are denoted by $\mathcal{G}_1$ and $\mathcal{G}_2$ in the theory of surfaces (XI, § **6**) will be denoted by $\mathcal{G}_{ij}$ and $\mathcal{G}_{ji}$, resp. Once more, let $\mathcal{N}_{ik}$, $\mathcal{N}_{jk}$, and $\mathcal{T}_k$ represent the quantities $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{T}$ that relate to the surface in question and write down the conditions of immobility (XI, § **9**) of the point $(x = x_i, y = x_j, z = x_k)$:

$$\frac{\partial x_i}{\partial s_i} = \mathcal{N}_{ik}\, x_k - \mathcal{G}_{ij}\, x_j - 1, \qquad \frac{\partial x_j}{\partial s_i} = \mathcal{G}_{ij}\, x_i - \mathcal{T}_k\, x_k, \qquad \frac{\partial x_k}{\partial s_i} = \mathcal{T}_k\, x_j - \mathcal{N}_{ik}\, x_i,$$

$$\frac{\partial x_i}{\partial s_j} = \mathcal{G}_{ji}\, x_j - \mathcal{T}_k\, x_k, \qquad \frac{\partial x_j}{\partial s_j} = \mathcal{N}_{jk}\, x_k - \mathcal{G}_{ji}\, x_i - 1, \qquad \frac{\partial x_k}{\partial s_j} = \mathcal{T}_k\, x_i - \mathcal{N}_{jk}\, x_j.$$

Having done that, if one writes the first pair of formulas for the surface q_j and the second pair for q_i, while leaving the third pair intact, then one will get:

$$\frac{\partial x_k}{\partial s_i} = \mathcal{G}_{ik}\, x_i - \mathcal{T}_j\, x_j = \mathcal{T}_k\, x_j - \mathcal{N}_{ik}\, x_i,$$

$$\frac{\partial x_k}{\partial s_j} = \mathcal{G}_{jk}\, x_j - \mathcal{T}_i\, x_i = \mathcal{T}_k\, x_i - \mathcal{N}_{jk}\, x_j\,,$$

$$\frac{\partial x_k}{\partial s_k} = \mathcal{N}_{kj}\, x_j - \mathcal{G}_{ki}\, x_i - 1 = \mathcal{N}_{ki}\, x_i - \mathcal{G}_{kj}\, x_j - 1;$$

hence, $\mathcal{T}_i = \mathcal{T}_j = -\mathcal{T}_k$ for any i, j, k, and consequently:

$$\mathcal{T}_1 = 0, \qquad \mathcal{T}_2 = 0, \qquad \mathcal{T}_3 = 0. \tag{4}$$

In addition, for any pair of values of i and j:

$$\mathcal{N}_{ij} = -\mathcal{G}_{ij}\,. \tag{5}$$

(4) says immediately that *in any triply-orthogonal system, any surface will be cut by the surfaces of the other families along the lines of curvature.* That is the very important *Dupin theorem.* As for (5), it expresses an obvious fact, namely, the equivalence (up to sign) of the normal curvature to the line q_i, when it is considered to belong to the surface q_j, and the geodetic curvature of that line on the surface q_k. It will then follow that the six functions $\mathcal{G}$, which are the only ones that we shall continue to consider in our calculations, represent precisely the principal curvatures of the three coordinate surfaces, with a change of sign. Finally, the preceding formulas assume the definitive form:

$$\left\{\begin{aligned} &\frac{\partial x_1}{\partial s_1} = -\mathcal{G}_{12}x_2 - \mathcal{G}_{13}x_3 - 1, \quad \frac{\partial x_2}{\partial s_3} = \mathcal{G}_{32}x_3, \quad \frac{\partial x_3}{\partial s_2} = \mathcal{G}_{23}x_2,\\ &\frac{\partial x_2}{\partial s_2} = -\mathcal{G}_{23}x_3 - \mathcal{G}_{21}x_1 - 1, \quad \frac{\partial x_3}{\partial s_1} = \mathcal{G}_{13}x_1, \quad \frac{\partial x_1}{\partial s_3} = \mathcal{G}_{31}x_3,\\ &\frac{\partial x_3}{\partial s_3} = -\mathcal{G}_{31}x_1 - \mathcal{G}_{32}x_2 - 1, \quad \frac{\partial x_1}{\partial s_2} = \mathcal{G}_{21}x_2, \quad \frac{\partial x_2}{\partial s_1} = \mathcal{G}_{12}x_1. \end{aligned}\right. \tag{6}$$

These are the *necessary and sufficient conditions for the immobility of the point that is defined by the coordinates* x_1, x_2, x_3.

3. – If one applies the integrability conditions that relate to the surfaces q_k to the functions x, namely:

$$\left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji}\right)\frac{\partial}{\partial s_j} = \left(\frac{\partial}{\partial s_j} + \mathcal{G}_{ij}\right)\frac{\partial}{\partial s_i}\,, \tag{7}$$

then one will find six differential relations between the $\mathcal{G}$ that are known by the name of *Lamé formulas*; however, those relations can also be obtained without calculation by writing down the Codazzi formulas that relate to the three coordinate surfaces. For example, when one sets $\mathcal{T} = 0$ and changes $\mathcal{G}_1, \mathcal{G}_2, \mathcal{N}_1, \mathcal{N}_2$ into $\mathcal{G}_{ij}$, $\mathcal{G}_{ji}$, $-\mathcal{G}_{ik}$, $-\mathcal{G}_{jk}$, resp., the Gauss formula:

$$\frac{\partial \mathcal{G}_1}{\partial s_2} + \frac{\partial \mathcal{G}_2}{\partial s_1} + \mathcal{G}_1^2 + \mathcal{G}_2^2 = \mathcal{T}^2 - \mathcal{N}_1 \mathcal{N}_2$$

will then become:

$$\frac{\partial \mathcal{G}_1}{\partial s_2} + \frac{\partial \mathcal{G}_2}{\partial s_1} + \mathcal{G}_1^2 + \mathcal{G}_2^2 + \mathcal{G}_{ik}\,\mathcal{G}_{jk} = 0,$$

and will give rise to a first triple of relations. Similarly, the formulas:

$$\frac{\partial \mathcal{N}_2}{\partial s_1} = (\mathcal{N}_1 - \mathcal{N}_2)\,\mathcal{G}_2, \qquad \frac{\partial \mathcal{N}_1}{\partial s_2} = (\mathcal{N}_2 - \mathcal{N}_1)\,\mathcal{G}_1$$

will become:

$$\frac{\partial \mathcal{G}_{jk}}{\partial s_i} + (\mathcal{G}_{jk} - \mathcal{G}_{ik})\,\mathcal{G}_{ji} = 0, \qquad \frac{\partial \mathcal{G}_{ik}}{\partial s_j} + (\mathcal{G}_{ik} - \mathcal{G}_{jk})\,\mathcal{G}_{ij} = 0,$$

but they will reduce to just one triple, because one has, by virtue of (3):

$$\frac{\partial \mathcal{G}_{kj}}{\partial s_i} = \frac{1}{Q_i\,Q_j}\left(\frac{\partial^2 \log Q_k}{\partial q_i\,\partial q_j} - \frac{\partial \log Q_k}{\partial q_j}\frac{\partial \log Q_j}{\partial q_i}\right),$$

$$\frac{\partial \mathcal{G}_{ki}}{\partial s_j} = \frac{1}{Q_i\,Q_j}\left(\frac{\partial^2 \log Q_k}{\partial q_i\,\partial q_j} - \frac{\partial \log Q_k}{\partial q_i}\frac{\partial \log Q_i}{\partial q_j}\right),$$

namely:

$$\frac{1}{Q_i\,Q_j}\frac{\partial^2 \log Q_k}{\partial q_i\,\partial q_j} = \frac{\partial \mathcal{G}_{kj}}{\partial s_i} + \mathcal{G}_{kj}\,\mathcal{G}_{ji} = \frac{\partial \mathcal{G}_{ki}}{\partial s_j} + \mathcal{G}_{ki}\,\mathcal{G}_{ij},$$

and consequently:

$$\frac{\partial \mathcal{G}_{kj}}{\partial s_i} + (\mathcal{G}_{kj} - \mathcal{G}_{ij})\,\mathcal{G}_{ji} = \frac{\partial \mathcal{G}_{ki}}{\partial s_j} + (\mathcal{G}_{ki} - \mathcal{G}_{ji})\,\mathcal{G}_{ij}. \tag{8}$$

Hence, the Lamé formulas will become:

$$\left\{\begin{aligned}
&\frac{\partial \mathcal{G}_{32}}{\partial s_2} + \frac{\partial \mathcal{G}_{22}}{\partial s_3} + \mathcal{G}_{32}^2 + \mathcal{G}_{23}^2 + \mathcal{G}_{21}\,\mathcal{G}_{31} = 0,\\
&\frac{\partial \mathcal{G}_{12}}{\partial s_3} + \frac{\partial \mathcal{G}_{31}}{\partial s_1} + \mathcal{G}_{13}^2 + \mathcal{G}_{31}^2 + \mathcal{G}_{32}\,\mathcal{G}_{12} = 0,\\
&\frac{\partial \mathcal{G}_{21}}{\partial s_1} + \frac{\partial \mathcal{G}_{12}}{\partial s_2} + \mathcal{G}_{21}^2 + \mathcal{G}_{12}^2 + \mathcal{G}_{12}\,\mathcal{G}_{23} = 0,
\end{aligned}\right.$$

$$\left\{\begin{aligned} &\frac{\partial \mathcal{G}_{13}}{\partial s_2} + (\mathcal{G}_{12} - \mathcal{G}_{23})\,\mathcal{G}_{12} = 0, \\ &\frac{\partial \mathcal{G}_{21}}{\partial s_3} + (\mathcal{G}_{21} - \mathcal{G}_{31})\,\mathcal{G}_{23} = 0, \\ &\frac{\partial \mathcal{G}_{32}}{\partial s_1} + (\mathcal{G}_{32} - \mathcal{G}_{12})\,\mathcal{G}_{31} = 0, \end{aligned}\right. \qquad \text{or} \qquad \left\{\begin{aligned} &\frac{\partial \mathcal{G}_{12}}{\partial s_3} + (\mathcal{G}_{12} - \mathcal{G}_{32})\,\mathcal{G}_{13} = 0, \\ &\frac{\partial \mathcal{G}_{23}}{\partial s_1} + (\mathcal{G}_{23} - \mathcal{G}_{13})\,\mathcal{G}_{21} = 0, \\ &\frac{\partial \mathcal{G}_{31}}{\partial s_2} + (\mathcal{G}_{31} - \mathcal{G}_{21})\,\mathcal{G}_{32} = 0, \end{aligned}\right.$$

and, as one sees, they establish a necessary coupling between the principal curvatures of the coordinate surfaces and their variations. Since they are easily transformed into:

$$\left\{\begin{aligned} &\frac{\partial}{\partial q_2}\left(\frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{1}{Q_3}\frac{\partial Q_2}{\partial q_3}\right) + \frac{1}{Q_1^2}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_3}{\partial q_1} = 0, \\ &\frac{\partial}{\partial q_3}\left(\frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\right) + \frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_3}{\partial q_1}\right) + \frac{1}{Q_2^2}\frac{\partial Q_3}{\partial q_2}\frac{\partial Q_1}{\partial q_2} = 0, \\ &\frac{\partial}{\partial q_1}\left(\frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\right) + \frac{1}{Q_3^2}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_2}{\partial q_3} = 0, \\ &\frac{\partial^2 Q_1}{\partial q_2\,\partial q_3} = \frac{1}{Q_2}\frac{\partial Q_1}{\partial q_2}\frac{\partial Q_2}{\partial q_3} + \frac{1}{Q_3}\frac{\partial Q_1}{\partial q_3}\frac{\partial Q_3}{\partial q_2}, \\ &\frac{\partial^2 Q_2}{\partial q_3\,\partial q_1} = \frac{1}{Q_3}\frac{\partial Q_2}{\partial q_3}\frac{\partial Q_3}{\partial q_1} + \frac{1}{Q_1}\frac{\partial Q_2}{\partial q_1}\frac{\partial Q_1}{\partial q_3}, \\ &\frac{\partial^2 Q_3}{\partial q_1\,\partial q_2} = \frac{1}{Q_1}\frac{\partial Q_3}{\partial q_1}\frac{\partial Q_1}{\partial q_2} + \frac{1}{Q_2}\frac{\partial Q_3}{\partial q_2}\frac{\partial Q_2}{\partial q_1} \end{aligned}\right. \tag{9}$$

by means of formula (3), we can also consider these to be the second-order partial differential equations that the functions Q must satisfy.

4. – Whereas a pair of orthogonal directions that are defined for any point in the plane can always be considered to be the directions of the tangents at that point to two lines in a doubly-orthogonal system, by contrast, *the analogous property for an orthogonal triple in space is not true*, even if it is defined at every point. That noteworthy fact results from the restrictive condition that we will find ourselves having to deal with when the triad that is defined by the elements of the orthogonal determinant:

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 1$$

that relates to the variable triad of tangents to the lines of one triply-orthogonal system can be the triad of tangents to the lines of another triply-orthogonal system. When space is referred to the new system, one can express the idea that the immobility conditions for the point (x_1, x_2, x_3) can be satisfied by the new coordinates:

$$x_i' = \alpha_i x_1 + \beta_i x_2 + \gamma_i x_3 ,$$

while being careful that the formulas:

$$\frac{\partial}{\partial s_i'} = \alpha_i \frac{\partial}{\partial s_1} + \beta_i \frac{\partial}{\partial s_2} + \gamma_i \frac{\partial}{\partial s_3} \tag{10}$$

for $i = 1, 2, 3$ are the ones that pertain to the differential quotients that relate to the new axes. Meanwhile, in order for the conditions (6) to be satisfied, it is necessary that one must have:

$$\frac{\partial x_i'}{\partial s_j'} = \mathcal{G}_{ji}' x_j' , \quad \frac{\partial x_i'}{\partial s_k'} = \mathcal{G}_{ki}' x_k' \tag{11}$$

identically; i.e., if one considers only the first equality, for now:

$$\left(\alpha_j \frac{\partial}{\partial s_1} + \beta_j \frac{\partial}{\partial s_2} + \gamma_j \frac{\partial}{\partial s_3}\right)(\alpha_i x_1 + \beta_i x_2 + \gamma_i x_3) = \mathcal{G}_{ji}' (\alpha_j x_1 + \beta_j x_2 + \gamma_j x_3).$$

This splits into three:

$$\begin{cases} \alpha_j \mathcal{G}_{ji}' = \dfrac{\partial \alpha_i}{\partial s_j'} + \beta_i (\alpha_j \mathcal{G}_{12} - \beta_j \mathcal{G}_{21}) - \gamma_i (\gamma_j \mathcal{G}_{31} - \alpha_j \mathcal{G}_{13}), \\ \beta_j \mathcal{G}_{ji}' = \dfrac{\partial \beta_i}{\partial s_j'} + \gamma_i (\beta_j \mathcal{G}_{23} - \beta_j \mathcal{G}_{32}) - \alpha_i (\alpha_j \mathcal{G}_{12} - \beta_j \mathcal{G}_{21}), \\ \gamma_j \mathcal{G}_{ji}' = \dfrac{\partial \gamma_i}{\partial s_j'} + \alpha_i (\gamma_j \mathcal{G}_{31} - \beta_j \mathcal{G}_{13}) - \beta_i (\beta_j \mathcal{G}_{23} - \gamma_j \mathcal{G}_{32}), \end{cases} \tag{12}$$

which will give:

$$\begin{aligned} \mathcal{G}_{ji}' = &- \alpha_j (\varepsilon_{k1} + \gamma_k \mathcal{G}_{12} - \beta_k \mathcal{G}_{13}) \\ &- \beta_j (\varepsilon_{k2} + \alpha_k \mathcal{G}_{23} - \gamma_k \mathcal{G}_{21}) \\ &- \gamma_j (\varepsilon_{k3} + \beta_k \mathcal{G}_{31} - \alpha_k \mathcal{G}_{32}) \end{aligned} \tag{13}$$

when one multiplies them by α_j, β_j, γ_j and sums, while setting:

$$\alpha_i \frac{\partial \alpha_j}{\partial s_\nu} + \beta_i \frac{\partial \beta_j}{\partial s_\nu} + \gamma_i \frac{\partial \gamma_j}{\partial s_\nu} = -\left(\alpha_j \frac{\partial \alpha_i}{\partial s_\nu} + \beta_j \frac{\partial \beta_i}{\partial s_\nu} + \gamma_j \frac{\partial \gamma_i}{\partial s_\nu}\right) = \varepsilon_{k\nu} ,$$

and observing that each element of the orthogonal determinant:

$$\begin{vmatrix} \alpha_i & \beta_i & \gamma_i \\ \alpha_j & \beta_j & \gamma_j \\ \alpha_k & \beta_k & \gamma_k \end{vmatrix} = 1$$

is equal to its own algebraic complement. Similarly, one can deduce:

$$\begin{aligned} \mathcal{G}'_{ki} = \; & \alpha_k\,(\varepsilon_{j1} + \gamma_j\,\mathcal{G}_{12} - \beta_j\,\mathcal{G}_{13}) \\ & - \beta_k\,(\varepsilon_{j2} + \alpha_j\,\mathcal{G}_{23} - \gamma_j\,\mathcal{G}_{21}) \\ & - \gamma_k\,(\varepsilon_{j3} + \beta_j\,\mathcal{G}_{31} - \alpha_j\,\mathcal{G}_{32}) \end{aligned} \tag{13}$$

from the three relations that are analogous to (12) when one splits the second equality in (11). It is then easy to verify that if (11) is satisfied then one will also have:

$$\frac{\partial x'_i}{\partial s'_i} = -\mathcal{G}'_{ji}\,x'_j - \mathcal{G}'_{ik}\,x'_k - 1$$

identically; i.e., all of the conditions (6) will also be satisfied in the new system. Having said that, the elimination of the $\mathcal{G}'$ from (12) or from the analogous relations will lead, in any case, to the triple of conditions:

$$\alpha_i\,\varepsilon_{i1} + \beta_i\,\varepsilon_{i2} + \gamma_i\,\varepsilon_{i3} = (\mathcal{G}_{21} - \mathcal{G}_{31})\,\beta_i\,\gamma_i + (\mathcal{G}_{32} - \mathcal{G}_{12})\,\gamma_i\,\alpha_i + (\mathcal{G}_{13} - \mathcal{G}_{23})\,\alpha_i\,\beta_i\,. \tag{15}$$

5. – Those conditions come from the necessity of *orienting the new triad of axes in such a way that Dupin's theorem will still be valid* in the new system. Indeed, if one applies the fundamental formulas to the direction (α_i, β_i, γ_i) then one will find from (12) that one can give it the form:

$$\alpha_j\,\mathcal{G}'_{ji} = \frac{\delta\alpha_i}{\partial s'_j}, \qquad \beta_j\,\mathcal{G}'_{ji} = \frac{\delta\beta_i}{\partial s'_j}, \qquad \gamma_j\,\mathcal{G}'_{ji} = \frac{\delta\gamma_i}{\partial s'_j},$$

and it is clear that any relation that is obtained by eliminating $\mathcal{G}'_{ji}$ from that equality will necessarily be contained in (15). Now, if one sums the predicted equality, after having multiplied it by α_k, β_k, γ_k, respectively, then one will get the relation:

$$\alpha_k\,\frac{\delta\alpha_i}{\partial s'_j} + \beta_k\,\frac{\delta\beta_i}{\partial s'_j} + \gamma_k\,\frac{\delta\gamma_i}{\partial s'_j} = 0,$$

which one can then consider to be a new form of (15). Meanwhile, that relation, when written in the following way:

$$\begin{vmatrix} \alpha_i & \alpha_j & \delta\alpha_i \\ \beta_i & \beta_j & \delta\beta_i \\ \gamma_i & \gamma_j & \delta\gamma_i \end{vmatrix} = 0,$$

expresses precisely [cf., IX, form. (19)] the idea that the axis x_i' will generate a developable when the origin displaces along the axis x_j'. It will therefore be true that *the condition* (15) *in itself shows how much it is necessary and sufficient for the conservation of Dupin's theorem.*

6. – Let us turn to (15) in order to find out how we could deduce a single relation that each triple of cosines must satisfy by itself. First, observe that since:

$$\varepsilon_{iv} = \sum \alpha_j \frac{\partial \alpha_k}{\partial s_v} = -\begin{vmatrix} \alpha_i & \alpha_k & \dfrac{\partial \alpha_k}{\partial s_v} \\ \beta_i & \beta_k & \dfrac{\partial \beta_k}{\partial s_v} \\ \gamma_i & \gamma_k & \dfrac{\partial \gamma_k}{\partial s_v} \end{vmatrix},$$

the left-hand side of (15) can be given the form:

$$-\begin{vmatrix} \alpha_i & \alpha_k & \alpha_i \dfrac{\partial \alpha_k}{\partial s_1} + \beta_i \dfrac{\partial \alpha_k}{\partial s_2} + \gamma_i \dfrac{\partial \alpha_k}{\partial s_3} \\ \beta_i & \beta_k & \alpha_i \dfrac{\partial \beta_k}{\partial s_1} + \beta_i \dfrac{\partial \beta_k}{\partial s_2} + \gamma_i \dfrac{\partial \beta_k}{\partial s_3} \\ \gamma_i & \gamma_k & \alpha_i \dfrac{\partial \gamma_k}{\partial s_1} + \beta_i \dfrac{\partial \gamma_k}{\partial s_2} + \gamma_i \dfrac{\partial \gamma_k}{\partial s_3} \end{vmatrix}.$$

It will then follow that if one lets $F_k(\alpha, \beta, \gamma)$ represent the quadratic form:

$$\begin{vmatrix} \alpha & \alpha_k & \alpha \dfrac{\partial \alpha_k}{\partial s_1} + \beta \dfrac{\partial \alpha_k}{\partial s_2} + \gamma \dfrac{\partial \alpha_k}{\partial s_3} \\ \beta & \beta_k & \alpha \dfrac{\partial \beta_k}{\partial s_1} + \beta \dfrac{\partial \beta_k}{\partial s_2} + \gamma \dfrac{\partial \beta_k}{\partial s_3} \\ \gamma & \gamma_k & \alpha \dfrac{\partial \gamma_k}{\partial s_1} + \beta \dfrac{\partial \gamma_k}{\partial s_2} + \gamma \dfrac{\partial \gamma_k}{\partial s_3} \end{vmatrix} + \sum (\mathcal{G}_{21} - \mathcal{G}_{31}) \beta\gamma$$

then *the triples* α_i, β_i, γ_i *and* α_j, β_j, γ_j *will be the two solutions of the system:*

$$F_k(\alpha, \beta, \gamma) = 0, \qquad \alpha\,\alpha_k + \beta\,\beta_k + \beta\,\beta_k = 0, \qquad \alpha^2 + \beta^2 + \gamma^2 = 1,$$

which is to say, those triples will define the directions of those generators of the quadric cone $F_k = 0$ that are in the plane perpendicular to the direction (α_j, β_j, γ_j). Note that the two directions must prove to be mutually-perpendicular, and the form F_k will not be arbitrary then, but must (as is easy to see) reduce to the sum of the coefficients of the quadratic terms, which is to say that one must have:

$$\sum\left(\beta_k\frac{\partial\gamma_k}{\partial s_1} - \gamma_k\frac{\partial\beta_k}{\partial s_1}\right) = \sum(\mathcal{G}_{21} - \mathcal{G}_{31})\beta_k\gamma_k\,;$$

i.e.:

$$\sum\alpha_k\left[\left(\frac{\partial}{\partial s_3} + \mathcal{G}_{23}\right)\beta_k - \left(\frac{\partial}{\partial s_2} + \mathcal{G}_{32}\right)\gamma_k\right] = 0.$$

However, that condition is satisfied identically, as one will see immediately when one observes that if u is the function that defines the system of surfaces that are normal to the direction (α_k, β_k, γ_k) then one will have:

$$\alpha_k = \frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_1}, \qquad \beta_k = \frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_2}, \qquad \gamma_k = \frac{1}{\sqrt{\Delta u}}\frac{\partial u}{\partial s_3}.$$

So far, we have not taken the third condition in (15) into account, which one can give in one or the other of the following forms:

$$F_i(\alpha_k, \beta_k, \gamma_k) = 0, \qquad F_j(\alpha_k, \beta_k, \gamma_k) = 0.$$

If one substitutes the values of α_i, β_i, γ_i or α_j, β_j, γ_j in one of these relations for the ones that the preceding system provides as functions of the cosines α_k, β_k, γ_k and their first derivatives then one must add a relation that is not an identity in those cosines and their first and second derivatives, which is a relation that must obviously be satisfied by the other triples of cosines, as well. If one then sets α_k, β_k, γ_k equal to the aforementioned values, when written as functions of u, then one will get *Bonnet's* relation; viz., *an equation in the third partial derivatives of u that is necessary and sufficient for u to define a system of surfaces that belong to a triply-orthogonal system.* Therefore, while any simple infinitude of plane curves, along with its orthogonal trajectories, constitutes a double system of curves, it happens very rarely that a system of surfaces in three-dimensional space will belong to a triply-orthogonal system. That is always explained by Dupin's theorem, since when one is given a system of surfaces, it is very difficult for its lines of curvature to be associated in such a way that they would constitute two other systems of surfaces that are orthogonal to the given system. On the contrary, any surface belongs to a triply-orthogonal system, because it is enough, for example, to associate it with the infinitude of parallel surfaces (XI, § **27**) and to construct the other two systems from the developables of the common normals along the lines of curvature. In other words, *any system of parallel surfaces belongs to a triply-orthogonal system.* It is also easy to see that *any system of planes or spheres belongs to a triply-orthogonal system*, and that is due precisely (XI, § **2**) to the complete freedom that one enjoys in the choice of the lines of curvature on the plane and the sphere.

7. – In order to calculate *the second differential parameter*, we need to consider the sum $\mathcal{G}_i$ of the $\mathcal{G}$ that have their second index equal to *i*, namely:

$$\mathcal{G}_i = \mathcal{G}_{ji} + \mathcal{G}_{ki} = \frac{\partial}{\partial s_i} \log Q_j\, Q_k \,. \tag{16}$$

If one observes that the system:

$$\alpha_i \frac{\partial \alpha_i}{\partial s_\nu} + \beta_i \frac{\partial \beta_i}{\partial s_\nu} + \gamma_i \frac{\partial \gamma_i}{\partial s_\nu} = 0,$$

$$\alpha_j \frac{\partial \alpha_j}{\partial s_\nu} + \beta_j \frac{\partial \beta_j}{\partial s_\nu} + \gamma_k \frac{\partial \gamma_k}{\partial s_\nu} = -\,\varepsilon_{k\nu} \,,$$

$$\alpha_k \frac{\partial \alpha_k}{\partial s_\nu} + \beta_k \frac{\partial \beta_k}{\partial s_\nu} + \gamma_k \frac{\partial \gamma_k}{\partial s_\nu} = \varepsilon_{j\nu}$$

implies that:

$$\frac{\partial \alpha_i}{\partial s_\nu} = \alpha_k\, \varepsilon_{j\nu} - \alpha_j\, \varepsilon_{k\nu} \,, \quad \frac{\partial \beta_i}{\partial s_\nu} = \beta_k\, \varepsilon_{j\nu} - \beta_j\, \varepsilon_{k\nu} \,, \quad \frac{\partial \gamma_i}{\partial s_\nu} = \gamma_k\, \varepsilon_{j\nu} - \gamma_j\, \varepsilon_{k\nu}$$

then when the formulas (13) and (14) are summed, they will give:

$$\mathcal{G}_i' = \left(\frac{\partial}{\partial s_1} + \mathcal{G}_1\right)\alpha_i + \left(\frac{\partial}{\partial s_2} + \mathcal{G}_2\right)\beta_i + \left(\frac{\partial}{\partial s_3} + \mathcal{G}_3\right)\gamma_i \,.$$

Having done that, when one repeats the operation (10), one will get:

$$\frac{\partial^2}{\partial s_i'^2} = \alpha_i^2 \frac{\partial^2}{\partial s_1^2} + \beta_i^2 \frac{\partial^2}{\partial s_2^2} + \ldots + \beta_i \gamma_i \left(\frac{\partial^2}{\partial s_2\, \partial s_3} + \frac{\partial^2}{\partial s_3\, \partial s_2}\right) + \ldots$$
$$+ \left(\alpha_i \frac{\partial \alpha_i}{\partial s_1} + \beta_i \frac{\partial \alpha_i}{\partial s_2} + \gamma_i \frac{\partial \alpha_i}{\partial s_1}\right)\frac{\partial}{\partial s_1} + \left(\alpha_i \frac{\partial \beta_i}{\partial s_1} + \beta_i \frac{\partial \beta_i}{\partial s_2} + \gamma_i \frac{\partial \beta_i}{\partial s_1}\right)\frac{\partial}{\partial s_2} + \ldots,$$

so

$$\left(\frac{\partial}{\partial s_i'} + \mathcal{G}_i'\right)\frac{\partial}{\partial s_i'} = \alpha_i^2 \left(\frac{\partial}{\partial s_1} + \mathcal{G}_1\right)\frac{\partial}{\partial s_1} + \beta_i^2 \left(\frac{\partial}{\partial s_2} + \mathcal{G}_2\right)\frac{\partial}{\partial s_2} + \ldots$$

$$+\beta_i\gamma_i\left[\left(\frac{\partial}{\partial s_3}+\mathcal{G}_3\right)\frac{\partial}{\partial s_2}+\left(\frac{\partial}{\partial s_2}+\mathcal{G}_2\right)\frac{\partial}{\partial s_3}\right]+\ldots$$

$$+\left[\frac{\partial}{\partial s_1}(\alpha_i^2)+\frac{\partial}{\partial s_2}(\alpha_i\beta_i)+\frac{\partial}{\partial s_3}(\alpha_i\gamma_i)\right]\frac{\partial}{\partial s_1}$$

$$+\left[\frac{\partial}{\partial s_1}(\beta_i\,\alpha_i)+\frac{\partial}{\partial s_2}(\beta_i^2)+\frac{\partial}{\partial s_3}(\beta_i\gamma_i)\right]\frac{\partial}{\partial s_2}$$

$$+\left[\frac{\partial}{\partial s_1}(\gamma_i\,\alpha_i)+\frac{\partial}{\partial s_2}(\gamma_i\,\beta_i)+\frac{\partial}{\partial s_3}(\gamma_i^2)\right]\frac{\partial}{\partial s_3},$$

and finally:

$$\sum\left(\frac{\partial}{\partial s_i'}+\mathcal{G}_i'\right)\frac{\partial}{\partial s_i'}=\sum\left(\frac{\partial}{\partial s_i}+\mathcal{G}_i\right)\frac{\partial}{\partial s_i}.$$

That exhibits the invariant property of the operation:

$$\Delta^2=\left(\frac{\partial}{\partial s_1}+\mathcal{G}_1\right)\frac{\partial}{\partial s_1}+\left(\frac{\partial}{\partial s_2}+\mathcal{G}_2\right)\frac{\partial}{\partial s_2}+\left(\frac{\partial}{\partial s_3}+\mathcal{G}_3\right)\frac{\partial}{\partial s_3},$$

which one can also give the form that **Lamé** indicated:

$$\Delta^2=\frac{1}{Q_1Q_2Q_3}\left[\frac{\partial}{\partial q_1}\left(\frac{Q_2\,Q_3}{Q_1}\frac{\partial}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{Q_3\,Q_1}{Q_2}\frac{\partial}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{Q_1\,Q_2}{Q_3}\frac{\partial}{\partial q_3}\right)\right],$$

from (16).

8. – The relations (9) warn us that the space that has been considered so far, and in which we have previously studied curves and surfaces, is not the most general three-dimensional space that we can imagine. Indeed, we envision space to be a triply-infinite system of points, each of which is individualized by a triple of values that are attributed to the parameters q_1, q_2, q_3 and are connected to the infinitely-close points by the condition that their distances are expressed by means of formula (2) for an *arbitrary* triple of given functions Q. The relations that were found between the Q are therefore an obvious clue to a particularization of space, and that is, in fact, true as a consequence of the hypothesis that we tacitly introduced that it is legitimate to implant a Cartesian coordinate system in the space considered, and consequently, there will exist three functions x_1, x_2, x_3 of q_1, q_2, q_3 such that the expression (2) will reduce to the form:

$$ds^2=dx_1^2+dx_2^2+dx_3^2. \tag{17}$$

It is, moreover, easy to recover the Lamé formula as the necessary and sufficient condition for the possibility of making that reduction by a direct process. Here, it is worth recalling that we already encountered (VIII, § **10**) a relation in the plane between the curvatures of the lines of any doubly-orthogonal system, which was a relation that expressed precisely the possibility of reducing the square of the elementary arc-length to the form $dx_1^2 + dx_2^2$, while on a curved surface, no link will necessarily intercede between the functions Q. The space that we study, and that will be distinguished from now on from the other ones by calling it *linear*, will then be represented amongst all the possible three-dimensional spaces in the same way that the plane is distinguished amongst the curved surfaces, and we then call the nonlinear spaces *curved*, while deferring the definition of the concept of curvature to later.

9. – Just as in the study of curved lines and surfaces we referred the space that contained them to a Cartesian system, it will be useful to suppose that the triple infinitude of points that are defined by the triple (q_1, q_2, q_3) and arranged according to the law (2) belongs to a linear quadruple infinitude of points, or as one says, it is immersed in a *four-dimensional linear space.* Therefore, imagine a four-times-infinite system of points, each of which is distinguished by a quadruple of values that are attributed to the parameters q_1, q_2, q_3, q_4, and is connected to the infinitely-close points by the condition that its distances to them should have a square that is expressed by the form $Q_1^2\,dq_1^2 + \ldots + Q_4^2\,dq_4^2$, which is assumed to transform linearly into $dx_1^2 + \ldots + dx_4^2$. That will reduce to (17) when one of the x is held constant, and since any linear orthogonal transformation that is applied to the x will leave that form unaltered, one can generally affirm that in a four-dimensional linear space *any linear equation between the Cartesian coordinates relative to the immobile axes represents a three-dimensional linear space.* That will therefore justify the term *linear* space, and at the same time, one will find a way to rapidly adapt the geometric terminology and fundamental principles of the ordinary analytic geometry of lines and planes to those spaces. We shall leave to the reader the task of familiarizing himself with that extension and of repeating the considerations of § **1** in order to see how obligating a function u of the coordinates of the points in a four-dimensional linear space to keep a given value amounts to singling out a (generally curved) three-dimensional space from that space that admits a *normal* line at every point M in the direction of the most rapid variation of u and a three-dimensional *tangent* linear space that is determined by the line along which the variation of u is infinitesimal of a higher order than the displacement of M.

10 – Consider a point that displaces in a four-dimensional linear space along a curve, and let M', M'', … be the positions that it successively occupies at infinitesimal intervals when it starts from an arbitrary position M. The *tangent* to the curve at M is always defined as the limit of the line MM' when M' tends to M, and we shall briefly say that the linear element MM' determines the tangent when we intend to mean passing to the limit any time that we employ analogous locutions. That immediately says that *two* successive elements MM' and $M'M''$ determine the *osculating plane*, and *three* elements determine the *osculating linear space*, which generally varies from one point to the other along the curve. The perpendicular to the osculating space that goes through M can indeed be called the *trinormal,* which is perpendicular to the three infinitely-close tangents. It belongs to the *binormal plane,* which is the locus of the infinitude of *binormals,* or perpendiculars to two successive elements through M, just as the binormal plane belongs to the *normal space* in which one finds all of the *normals* to the tangents through M in a double infinitude. The perpendicular that is raised at M to the trinormal in the binormal plane is the *principal binormal* and the perpendicular to the binormal plane that is raised at M in the normal space is the *principal normal.* That then constitutes the fundamental quadruple of the curve, namely, the tangent trinormal, principal binormal, and principal normal. If those lines, which are pair-wise mutually-perpendicular, are taken to be the axes, and if the Cartesian coordinates of a fixed point with respect to them are represented by x_1', x_2', x_3', x', as usual, then one will see in the next chapter that the necessary and sufficient conditions for the immobility of the point are:

$$\frac{dx_1'}{ds} = \frac{x'}{\rho} - 1, \qquad \frac{dx_2'}{ds} = \frac{x_3'}{\tau}, \qquad \frac{dx_3'}{ds} = \frac{x'}{\rho} - \frac{x_2'}{\tau}, \qquad \frac{dx'}{ds} = -\frac{x_1'}{\rho} - \frac{x_3'}{r}, \tag{18}$$

in which τ, like ρ and r, is a radius of curvature, which will lead us to consider how to measure the tendency that the point M has to leave the osculating space more or less rapidly as it traverses the curve.

11. – Take a line in a curved space, whose normal at M is determined in the normal space by its cosines α_1, β_1, γ_1 with respect to the axes x_2', x_3', x', resp. The other two lines, which are perpendicular to each other and to the first one, are determined in the normal space by the cosines α_2, β_2, γ_2 and α_3, β_3, γ_3, such that one will have:

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 1. \tag{19}$$

The coordinates of the fixed point with respect to the three normals that were defined just now are $x_1 = x_1'$, and:

$$x = \alpha_1\, x_1' + \beta_1\, x_2' + \gamma_1 x', \qquad x_2 = \alpha_3\, x_2' + \beta_3\, x_3' + \gamma_3 x', \qquad x_3 = \alpha_2\, x_2' + \beta_2\, x_3' + \gamma_2 x'.$$

Having said that, the conditions (18) easily transform into the following ones:

$$\left.\begin{aligned} \frac{dx}{ds} &= -\mathcal{T}_2\, x_2 + \mathcal{T}_3\, x_3 - \mathcal{N}\, x_1, & \frac{dx_2}{ds} &= \mathcal{G}_2\, x_1 + \mathcal{T}_2\, x - \mathcal{T}_1\, x_3, \\ \frac{dx_1}{ds} &= -\mathcal{G}_2\, x_2 - \mathcal{G}_3\, x_3 + \mathcal{N}\, x - 1, & \frac{dx_3}{ds} &= \mathcal{G}_3\, x_1 - \mathcal{T}_3\, x + \mathcal{T}_1\, x_2, \end{aligned}\right\} \tag{20}$$

in which one sets:

$$\mathcal{N} = \frac{\gamma_1}{\rho}, \qquad \mathcal{G}_2 = -\frac{\gamma_3}{\rho}, \qquad \mathcal{G}_3 = -\frac{\gamma_2}{\rho},$$

and also:

$$\mathcal{T}_1 = \frac{\alpha_1}{r} + \frac{\gamma_1}{\tau} - \varepsilon_1, \qquad \mathcal{T}_2 = \frac{\alpha_2}{r} + \frac{\gamma_2}{\tau} - \varepsilon_2, \qquad \mathcal{T}_3 = \frac{\alpha_3}{r} + \frac{\gamma_3}{\tau} - \varepsilon_3 .$$

One will return to the known (XI, § **3**) immobility conditions that relate to the surface from formulas (20) when one supposes that the third curvature is zero, in which case the determinant (19) will become:

$$\begin{vmatrix} 0 & \sin\psi & \cos\psi \\ 0 & \cos\psi & -\sin\psi \\ -1 & 0 & 0 \end{vmatrix},$$

and $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{G}_2$ will be constantly equal to zero.

12. – Now imagine other curves that are tangent to the axes x_2 and x_3 at M, and distinguish everything that pertains to the tangent curve to Mx_i by an index i. Formulas (20) will then give rise to the following relations:

$$(A) \quad \left\{ \begin{aligned} \frac{\partial x}{\partial s_1} &= -\mathcal{N}_1 x_1 - \mathcal{T}_{12}\, x_2 + \mathcal{T}_{13}\, x_3, & \frac{\partial x_1}{\partial s_1} &= \mathcal{N}_1 x - \mathcal{G}_{12}\, x_2 - \mathcal{G}_{13}\, x_3 - 1, \\ \frac{\partial x}{\partial s_2} &= \mathcal{T}_{21}\, x_1 - \mathcal{N}_2\, x_2 - \mathcal{T}_{23}\, x_3, & \frac{\partial x_2}{\partial s_2} &= -\mathcal{G}_{21}\, x_1 + \mathcal{N}_2\, x - \mathcal{G}_{23}\, x_3 - 1, \\ \frac{\partial x}{\partial s_3} &= -\mathcal{T}_{31}\, x_1 + \mathcal{T}_{32}\, x_2 - \mathcal{N}_3\, x_3, & \frac{\partial x_3}{\partial s_3} &= -\mathcal{G}_{31}\, x_1 - \mathcal{G}_{32}\, x_2 + \mathcal{N}_3\, x_3 - 1, \end{aligned} \right\} \quad (B)$$

$$(C) \quad \left\{ \begin{aligned} \frac{\partial x_3}{\partial s_2} &= \mathcal{G}_{23}\, x_2 + \mathcal{T}_{23}\, x - \mathcal{T}_{22}\, x_1, & \frac{\partial x_2}{\partial s_3} &= \mathcal{G}_{32}\, x_3 - \mathcal{T}_{32}\, x + \mathcal{T}_{33}\, x_1, \\ \frac{\partial x_1}{\partial s_3} &= \mathcal{G}_{31}\, x_3 + \mathcal{T}_{31}\, x - \mathcal{T}_{33}\, x_2, & \frac{\partial x_2}{\partial s_1} &= \mathcal{G}_{13}\, x_1 - \mathcal{T}_{13}\, x + \mathcal{T}_{11}\, x_2, \\ \frac{\partial x_2}{\partial s_1} &= \mathcal{G}_{12}\, x_1 + \mathcal{T}_{12}\, x - \mathcal{T}_{11}\, x_3, & \frac{\partial x_1}{\partial s_2} &= \mathcal{G}_{21}\, x_2 - \mathcal{T}_{21}\, x + \mathcal{T}_{22}\, x_3. \end{aligned} \right\} \quad (C')$$

It will follow directly from this that for $x = x_1 = x_2 = x_3 = 0$, the differential quotients:

$$\frac{\partial^2 x}{\partial s_1^2}, \frac{\partial^2 x}{\partial s_2^2}, \frac{\partial^2 x}{\partial s_3^2}; \qquad \frac{\partial^2 x}{\partial s_2\,\partial s_3}, \frac{\partial^2 x}{\partial s_3\,\partial s_1}, \frac{\partial^2 x}{\partial s_1\,\partial s_2}; \qquad \frac{\partial^2 x}{\partial s_3\,\partial s_2}, \frac{\partial^2 x}{\partial s_1\,\partial s_3}, \frac{\partial^2 x}{\partial s_2\,\partial s_1}$$

will take the values:

$$\mathcal{N}_1\,,\mathcal{N}_2\,,\mathcal{N}_3\,;\qquad \mathcal{T}_{23}\,,\mathcal{T}_{31}\,,\mathcal{T}_{12}\,;\qquad -\mathcal{T}_{32}\,,-\mathcal{T}_{13}\,,-\mathcal{T}_{21}\,,$$

and that the analogous values for the functions $x_1 = x_2 = x_3$ will be:

$$\begin{matrix} 0, & -\mathcal{G}_{21} & -\mathcal{G}_{31}\,; & -\mathcal{T}_{22} & 0 & \mathcal{G}_{12}\,; & \mathcal{T}_{33} & \mathcal{G}_{13} & 0 \\ -\mathcal{G}_{12} & 0 & -\mathcal{G}_{32}\,; & \mathcal{G}_{23} & -\mathcal{T}_{33} & 0\,; & 0 & \mathcal{T}_{11} & \mathcal{G}_{21} \\ -\mathcal{G}_{13} & -\mathcal{G}_{23} & 0\,; & 0 & \mathcal{G}_{31} & -\mathcal{T}_{11}\,; & -\mathcal{T}_{33} & 0 & \mathcal{T}_{22}\,. \end{matrix}$$

Now, when the known condition:

$$\frac{\partial^2}{\partial s_i\,\partial s_j} - \frac{\partial^2}{\partial s_j\,\partial s_i} = \frac{\partial \log Q_j}{\partial s_i}\frac{\partial}{\partial s_j} - \frac{\partial \log Q_i}{\partial s_j}\frac{\partial}{\partial s_i} \tag{21}$$

is applied to the functions x, it will immediately give $\mathcal{T}_{ij} + \mathcal{T}_{ji} = 0$, and one can then set:

$$T_1 = \mathcal{T}_{31} = -\mathcal{T}_{23}\,,\qquad T_2 = \mathcal{T}_{13} = -\mathcal{T}_{31}\,,\qquad T_3 = \mathcal{T}_{21} = -\mathcal{T}_{12}\,.$$

However, when that condition is applied to the functions x_1, x_2, x_3 , one will say that the $\mathcal{G}_{ij}$ are once more expressed by the formulas (3), and in addition, one has the equalities $\mathcal{T}_{ii} = 0$, in which one finds *Dupin's theorem*. With that, the immobility conditions can be put into their ultimate form. (*B*) will remain unchanged, but (*A*) will become:

$$\left\{\begin{aligned} \frac{\partial x}{\partial s_1} &= -\mathcal{N}_1 x_1 + T_3\,x_2 + T_2\,x_3, \\ \frac{\partial x}{\partial s_2} &= \quad T_3 x_1 - \mathcal{N}_2\,x_2 + T_1\,x_3, \\ \frac{\partial x}{\partial s_3} &= \quad T_2 x_1 + T_1\,x_2 - \mathcal{N}_3\,x_3, \end{aligned}\right.$$

while (*C*) and (*C′*) will reduce to the simple form:

$$\left\{\begin{aligned} &\frac{\partial x_2}{\partial s_1} = \mathcal{G}_{11}\,x_1 - T_2\,x, \quad \frac{\partial x_3}{\partial s_2} = \mathcal{G}_{23}\,x_2 + T_1\,x, \quad \frac{\partial x_1}{\partial s_3} = \mathcal{G}_{31}\,x_3 - T_2\,x, \\ &\frac{\partial x_3}{\partial s_1} = \mathcal{G}_{13}\,x_1 - T_2\,x, \quad \frac{\partial x_1}{\partial s_2} = \mathcal{G}_{31}\,x_2 - T_3\,x, \quad \frac{\partial x_2}{\partial s_3} = \mathcal{G}_{32}\,x_2 - T_1\,x. \end{aligned}\right.$$

One notes that one can also give (*A*) the form:

$$\frac{\partial x}{\partial s_1} = -\frac{1}{2}\frac{\partial \Phi}{\partial x_1}, \qquad \frac{\partial x}{\partial s_2} = -\frac{1}{2}\frac{\partial \Phi}{\partial x_2}, \qquad \frac{\partial x}{\partial s_3} = -\frac{1}{2}\frac{\partial \Phi}{\partial x_3}, \tag{22}$$

in which Φ represents the quadratic form that is defined by the discriminant:

$$K = \begin{vmatrix} \mathcal{N}_1 & -T_3 & -T_2 \\ -T_3 & \mathcal{N}_2 & -T_1 \\ -T_2 & -T_1 & \mathcal{N}_3 \end{vmatrix},$$

which has great importance in the study of curvature. We shall soon see that K can be expressed in terms of only the curvatures $\mathcal{G}$. To that end, we agree to make use of the reciprocal determinant, whose elements are represented in the following way:

$$\begin{aligned} K_{11} &= \mathcal{N}_2\,\mathcal{N}_3 - T_1^2, & K_{23} &= K_{32} = \mathcal{N}_1\,T_1 + T_2\,T_3, \\ K_{32} &= \mathcal{N}_3\,\mathcal{N}_1 - T_2^2, & K_{31} &= K_{13} = \mathcal{N}_2\,T_2 + T_3\,T_1, \\ K_{33} &= \mathcal{N}_1\,\mathcal{N}_2 - T_3^2, & K_{12} &= K_{21} = \mathcal{N}_3\,T_3 + T_1\,T_2. \end{aligned}$$

13. – Before we go further, we shall take advantage of the preceding results in order to show how one extends *Euler's theorem* (XI, § **10**) to three-dimensional spaces. The curvature of the planar normal section whose tangent is determined in the tangent linear space by the direction cosines α, β, γ is always measured by d^2x / ds^2 for $x = x_1 = x_2 = x_3 = 0$, in which:

$$\frac{d}{ds} = \alpha\frac{\partial}{\partial s_1} + \beta\frac{\partial}{\partial s_2} + \gamma\frac{\partial}{\partial s_3}.$$

It is then given by:

$$\alpha^2\frac{\partial^2 x}{\partial s_1^2} + \beta^2\frac{\partial^2 x}{\partial s_2^2} + \ldots + \beta\gamma\left(\frac{\partial^2 x}{\partial s_2\,\partial s_3} + \frac{\partial^2 x}{\partial s_3\,\partial s_2}\right) + \ldots$$

for $x = x_1 = x_2 = x_3 = 0$, namely:

$$\frac{1}{\rho} = \Phi(\alpha, \beta, \gamma).$$

The discussion of that formula will prove to be completely analogous to the discussion of the one the theory of surfaces, and in particular, it will lead one to consider three *principal curvatures*, which correspond to the axes of the quadric cone $\Phi = 0$, which is the locus of the tangents to the infinitude of asymptotics (real or imaginary) that pass through any point. The product of the principal curvatures is precisely K and can serve to measure the *total curvature*, while the orthogonal invariants:

$$\tfrac{1}{3}(\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3), \qquad \tfrac{1}{3}(K_{11} + K_{22} + K_{33})$$

measure two *mean curvatures* of the space around the point considered. If one would wish that the normal to the space should generate a developable then that would lead one to express the idea that the major determinants of the matrix:

$$\begin{vmatrix} 0 & \dfrac{\partial\Phi}{\partial\alpha} & \dfrac{\partial\Phi}{\partial\beta} & \dfrac{\partial\Phi}{\partial\gamma} \\ 0 & \alpha & \beta & \gamma \\ 1 & 0 & 0 & 0 \\ & & & \end{vmatrix}$$

should all be zero, which is to say that one must have

$$\frac{\partial\Phi}{\partial\alpha} : \alpha = \frac{\partial\Phi}{\partial\beta} : \beta = \frac{\partial\Phi}{\partial\gamma} : \gamma,$$

and one would then find the axes of Φ. The systems of curvature are then characterized by the constant vanishing of T_1, T_2, T_3. The relations (A), (C), and (C') reduce to the exceedingly simple forms for them:

$$\frac{\partial x}{\partial s_i} = -\mathcal{N}_i\, x_i\,, \qquad \frac{\partial x_j}{\partial s_i} = \mathcal{G}_{ij}\, x_i\,.$$

14. – We now return to the condition (21), which can now be put into the form (7), and apply it to the function x:

$$\left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji}\right)(-\mathcal{N}_j\, x_j + T_i\, x_k + T_k\, x_i) = \left(\frac{\partial}{\partial s_j} + \mathcal{G}_{ij}\right)(-\mathcal{N}_i\, x_i + T_k\, x_j + T_j\, x_k).$$

That equality will reduce to a linear relation between the x by means of the immobility conditions, and by the arbitrariness in those variables, that will give rise to the following groups of formulas:

$$(\alpha) \qquad \begin{cases} \dfrac{\partial T_2}{\partial s_2} - \dfrac{\partial T_3}{\partial s_3} + T_2\,\mathcal{G}_{32} - T_3\,\mathcal{G}_{32} = T_1(\mathcal{G}_{21} - \mathcal{G}_{31}), \\ \dfrac{\partial T_3}{\partial s_3} - \dfrac{\partial T_1}{\partial s_1} + T_3\,\mathcal{G}_{13} - T_1\,\mathcal{G}_{31} = T_2(\mathcal{G}_{32} - \mathcal{G}_{12}), \\ \dfrac{\partial T_1}{\partial s_1} - \dfrac{\partial T_2}{\partial s_2} + T_1\,\mathcal{G}_{21} - T_2\,\mathcal{G}_{12} = T_3(\mathcal{G}_{13} - \mathcal{G}_{23}), \end{cases}$$

$$(\beta)\qquad \begin{cases} \dfrac{\partial \mathcal{N}_3}{\partial s_2}+\dfrac{\partial T_1}{\partial s_3}+2T_1\,\mathcal{G}_{23}+T_3\,\mathcal{G}_{31}=(\mathcal{N}_2-\mathcal{N}_3)\,\mathcal{G}_{32}, \\ \dfrac{\partial \mathcal{N}_1}{\partial s_3}+\dfrac{\partial T_2}{\partial s_1}+2T_2\,\mathcal{G}_{31}+T_1\,\mathcal{G}_{12}=(\mathcal{N}_3-\mathcal{N}_1)\,\mathcal{G}_{13}, \\ \dfrac{\partial \mathcal{N}_2}{\partial s_1}+\dfrac{\partial T_3}{\partial s_2}+2T_3\,\mathcal{G}_{12}+T_2\,\mathcal{G}_{23}=(\mathcal{N}_1-\mathcal{N}_2)\,\mathcal{G}_{21}, \end{cases}$$

$$(\beta')\qquad \begin{cases} \dfrac{\partial \mathcal{N}_2}{\partial s_3}+\dfrac{\partial T_1}{\partial s_2}+2T_1\,\mathcal{G}_{32}+T_2\,\mathcal{G}_{21}=(\mathcal{N}_3-\mathcal{N}_2)\,\mathcal{G}_{23}, \\ \dfrac{\partial \mathcal{N}_3}{\partial s_1}+\dfrac{\partial T_2}{\partial s_3}+2T_2\,\mathcal{G}_{13}+T_3\,\mathcal{G}_{32}=(\mathcal{N}_1-\mathcal{N}_3)\,\mathcal{G}_{31}, \\ \dfrac{\partial \mathcal{N}_1}{\partial s_2}+\dfrac{\partial T_3}{\partial s_1}+2T_3\,\mathcal{G}_{21}+T_1\,\mathcal{G}_{13}=(\mathcal{N}_2-\mathcal{N}_1)\,\mathcal{G}_{12}. \end{cases}$$

Similarly, if one applies the condition (7) to the variables x_i then one will get the following formulas:

$$(\gamma)\qquad \begin{cases} \dfrac{\partial \mathcal{G}_{32}}{\partial s_2}+\dfrac{\partial \mathcal{G}_{23}}{\partial s_3}+\mathcal{G}_{22}^2+\mathcal{G}_{23}^2+\mathcal{G}_{21}\,\mathcal{G}_{31}=-K_{11}, \\ \dfrac{\partial \mathcal{G}_{13}}{\partial s_3}+\dfrac{\partial \mathcal{G}_{31}}{\partial s_1}+\mathcal{G}_{13}^2+\mathcal{G}_{31}^2+\mathcal{G}_{32}\,\mathcal{G}_{12}=-K_{22}, \\ \dfrac{\partial \mathcal{G}_{21}}{\partial s_1}+\dfrac{\partial \mathcal{G}_{12}}{\partial s_2}+\mathcal{G}_{21}^2+\mathcal{G}_{12}^2+\mathcal{G}_{13}\,\mathcal{G}_{23}=-K_{33}, \end{cases}$$

$$(\delta)\qquad \begin{cases} \dfrac{\partial \mathcal{G}_{13}}{\partial s_3}+(\mathcal{G}_{13}-\mathcal{G}_{23})\,\mathcal{G}_{12}=\dfrac{\partial \mathcal{G}_{12}}{\partial s_3}+(\mathcal{G}_{12}-\mathcal{G}_{32})\,\mathcal{G}_{13}=K_{23}, \\ \dfrac{\partial \mathcal{G}_{31}}{\partial s_3}+(\mathcal{G}_{21}-\mathcal{G}_{31})\,\mathcal{G}_{23}=\dfrac{\partial \mathcal{G}_{23}}{\partial s_1}+(\mathcal{G}_{23}-\mathcal{G}_{13})\,\mathcal{G}_{21}=K_{31}, \\ \dfrac{\partial \mathcal{G}_{32}}{\partial s_1}+(\mathcal{G}_{32}-\mathcal{G}_{12})\,\mathcal{G}_{31}=\dfrac{\partial \mathcal{G}_{31}}{\partial s_2}+(\mathcal{G}_{31}-\mathcal{G}_{21})\,\mathcal{G}_{32}=K_{12}. \end{cases}$$

Here, one should note that, thanks to (γ) and (δ), *the total curvature can be expressed in terms of only $\mathcal{G}$*, since one has:

$$K^2=\begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix},$$

namely, if one recalls the observation that was made at the end of § **3**, K is a function of only Q and its first and second partial derivatives. When one then applies the condition

(7) to any other x, one will get back to the formulas that were obtained already. Meanwhile, the formulas (α) reduce to just two distinct ones, and (δ) will constitute just one triple, in substance, by virtue of the identity (8). One therefore has *fourteen* formulas, in all, which will take the place of the *three* Codazzi formulas in the study of surfaces for the study of the three-dimensional curved spaces that are immersed in a four-dimensional linear space.

15. – Consider, for example, a *spherical space* – i.e., the locus of points in a four-dimensional linear space that are equidistant from a fixed point. The coordinates of that point must constantly satisfy the relation:

$$x_1^2 + x_2^2 + x_3^2 + x^2 = R^2$$

and the immobility conditions. If one differentiates that relation with respect to the three coordinate arcs, in turn, then one will get $x_1 = x_2 = x_3 = 0$, and consequently $x = R$. If one substitutes those results into the immobility conditions then one will find that one must have:

$$\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \frac{1}{R}, \qquad T_1 = T_2 = T_3 = 0.$$

One will then see that the formulas (α), (β), and (β') are satisfied identically, in such a way that only the conditions (γ) and (δ) will remain, in which:

$$K_{11} = K_{22} = K_{33} = \frac{1}{R^2}, \qquad K_{23} = K_{31} = K_{12} = 0. \tag{23}$$

The six relations thus-obtained then characterize the spherical spaces. They are due to **Beltrami**, and when R increases to infinity, they will become the six **Lamé** [form. (9)] characteristics of the linear space. Naturally, it is not possible to define a Cartesian coordinate system in a spherical space, but one can always establish a curvilinear coordinate system in which the triple of functions Q reduces to a single function Q, as it does in the Cartesian system. In order for that to happen then, it is necessary and sufficient that the conditions (23) should be satisfied when one sets the K equal to the values that they get from formulas (γ) and (δ), namely:

$$K_{ii} = -\frac{1}{Q}\frac{\partial^2(1/Q)}{\partial q_i^2} + \frac{1}{Q^2}\sum \frac{\partial}{\partial q_i}\left(Q\frac{\partial(1/Q)}{\partial q_i}\right), \qquad K_{ij} = -\frac{\partial^2(1/Q)}{\partial q_i\,\partial q_j}.$$

An easy integration will lead one to take:

$$\frac{1}{Q} = 1 + \frac{q_1^2 + q_2^2 + q_3^2}{4R^2},$$

and one will arrive at a coordinate system in which the elementary arc-length is given by the formula:

$$ds^2 = \frac{dq_1^2 + dq_2^2 + dq_3^2}{\left(1 + \frac{1}{4R^2}(q_1^2 + q_2^2 + q_3^2)\right)^2}.$$

It is the system of *stereographic coordinates* that **Riemann** pointed out and that **Beltrami** utilized for the study of spaces of constant curvature.

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CHAPTER XVI

CURVES IN HYPERSPACE

1. – Consider the successive positions M', M'', … of a point along a curve (viz., a continuous simple infinitude of points in an n-dimensional linear space) that are infinitely close to the arbitrary initial position M. As in § **10** of the preceding chapter, say that the element MM' determines the *tangent*, which will always be assumed to the x_1-axis. Take the $(n – 1)$-*normal* to be the x_2-axis; i.e., the line through M that is perpendicular to the $n – 1$ consecutive elements MM', $M'M''$, … Obviously, that line is in the plane that is determined by all of the perpendiculars through M to the $n – 2$ consecutive elements, among which one selects the x_3-axis to be the one that one can very well call the $(n – 2)$-*principal normal*, since it is perpendicular to the $(n – 1)$-*normal*. The x_2 and x_3 axes, along with all of the $(n – 3)$-*normals* are in a three-dimensional linear space, in which one agrees to take the x_4-axis to be the perpendicular to the $x_2\,x_3$-plane. If one always proceeds in the same way then one will eventually choose the x_{n-1}-axis to be the *principal binormal*, which is determined in the (n – 2-dimensional) binormal linear space by the demand that it must be perpendicular to the preceding axes. Finally, in the (n – 1-dimensional) normal linear space that includes all of the normals, one distinguishes the *principal normal* from among them and chooses it to be the x_n-axis, which is perpendicular to the binormal space. Let $\alpha_{i1}, \dots, \alpha_{i2}, \dots, \alpha_{in}$ be the direction cosines of the x_i-axis with respect to any system of n pair-wise mutually-perpendicular axes, and note that the definition that was given of that axis translates into the relations:

$$\sum_{\nu=1}^{n} \alpha_{i\nu}\, d^j \alpha_{i\nu} = 0 \qquad \text{for} \qquad 1 \le j \le n - 1, \tag{1}$$

$$\sum_{\nu=1}^{n} \alpha_{i\nu}\, \alpha_{j\nu} = 0 \qquad \text{for} \qquad 1 \le j \le i - 1. \tag{2}$$

In particular, if one differentiates the equality:

$$\alpha_{i1}^2 + \alpha_{i2}^2 + \alpha_{i3}^2 + \dots + \alpha_{in}^2 = 1 \tag{3}$$

then one will get:

$$\sum_{\nu=1}^{n} \alpha_{i\nu}\, d\alpha_{j\nu} = 0 \qquad \text{for} \qquad 1 \le i \le i - 1 \tag{4}$$

from (1) with $j = 1$. Meanwhile, the relations (2) and (3) say that the determinant that is defined by the general element α_{ij} is orthogonal: If one so desires, its value can be equal to unity, and each element is equal to its own algebraic complement. Having done that, one will get from (4) that:

$$d\alpha_{1\nu} = \varepsilon_1\, \alpha_{n\nu} \tag{5}$$

for all values of v, in which ε_1 represents the angle between two infinitely-close tangents. More generally, if one sets:

$$\sum_{v=1}^{n} \alpha_{jv}\, d\alpha_{iv} = \varepsilon_{ij} ,$$

in such a way that:

$$\varepsilon_{ij} = 0, \qquad \varepsilon_{ij} = - \varepsilon_{ji} , \tag{6}$$

then one will have:

$$d\alpha_{ij} = \sum_{v=1}^{n} \varepsilon_{iv}\, \alpha_{vi} . \tag{7}$$

One gets from (5) for $i = 1$ and:

$$\varepsilon_{11} = \varepsilon_{12} = \varepsilon_{13} = \ldots = \varepsilon_{1,n-1} = 0, \qquad \varepsilon_{1n} = \varepsilon_1 , \tag{8}$$

from which, it will result that:

$$\varepsilon_{11} = \varepsilon_{21} = \varepsilon_{31} = \ldots = \varepsilon_{n-1,1} = 0, \qquad \varepsilon_{n1} = - \varepsilon_1 , \tag{9}$$

by virtue of (6).

2. – Thanks to (7), one succeeds in expressing the successive differentials of the direction cosines as linear functions of those cosines. If one starts with the angle ε_{ij} and temporarily represents it by $\varepsilon_{ij}^{(1)}$, and one then calculates a succession of quantities $\varepsilon_{ij}^{(2)}$, $\varepsilon_{ij}^{(3)}$, … according to the rule:

$$\varepsilon_{ij}^{(k+1)} = d\varepsilon_{ij}^{(k)} + \sum_{v=1}^{n} \varepsilon_{iv}^{(k)} \varepsilon_{vj} \tag{10}$$

then one will find upon successively differentiating (7) and repeatedly employing that formula that:

$$d^{\,k}\, \alpha_{ij} = \sum_{v=1}^{n} \varepsilon_{iv}^{(k)} \alpha_{vj} .$$

Now, (1) will become:

$$\sum_{i,j=1}^{n} \varepsilon_{1i}^{(k)} \alpha_{vj} \alpha_{ij} = 0,$$

which is to say, if one observes (3) and (4), that:

$$\varepsilon_{iv}^{(k)} = 0 \qquad \text{for} \qquad 2 \le v \le n - k.$$

If one successively sets $k = 1, 2, 3, \ldots$ and substitutes the ultimate result in (10) then one will get:

$$\sum_{i_1, i_2, \cdots, i_k}^{n} \varepsilon_{ii_1} \varepsilon_{i_1 i_2} \varepsilon_{i_2 i_3} \cdots \varepsilon_{i_k v} = 0 \qquad 2 \le v \le n - k - 1. \tag{11}$$

If one sets $k = 1$, for example, then one will find that:

$$\sum_{i=1}^{n} \varepsilon_{ii}\, \varepsilon_{iv} = 0;$$

i.e., by virtue of (8):

$$\varepsilon_{nv} = 0 \qquad \text{for} \qquad 2 \le v \le n - 2.$$

Similarly, for $k = 2$, the relation (11) will become:

$$\sum_{i=1}^{n} \varepsilon_{1i}\, \varepsilon_{ij}\, \varepsilon_{jv} = 0,$$

and from that, if one takes the preceding result into account then one will get:

$$\varepsilon_{n-1,v} = 0 \qquad \text{for} \qquad 2 \le v \le n - 3.$$

If one proceeds in that way then one will expect that in general one will have:

$$\varepsilon_{n-k+1,v} = 0 \qquad \text{for} \qquad 2 \le v \le n - k - 1. \tag{12}$$

Suppose that this equality is true, along with the ones that preceded it, and prove that it will still be true when one changes k into $k + 1$. Set $i_{k+1} = j$, so the relation (11) will give:

$$\sum_{j=1}^{n} (\varepsilon_{jv} \sum_{i_1, i_2, \cdots, i_k}^{n} \varepsilon_{ii_1} \varepsilon_{i_1 i_2} \varepsilon_{i_2 i_3} \cdots \varepsilon_{i_k j}) = 0 \qquad \text{for} \qquad 2 \le v \le n - k - 2.$$

The sum over k is zero for $j = 2, 3, 4, \ldots, n - k - 1$. On the other hand, by virtue of (12) and the preceding equality, ε_{jv} will be zero for $j = n - k + 1, n - k + 2, \ldots, n - 1, n$. All that remain then will be the terms that correspond to the values $j = 1$ and $j = n - k$: The first one is zero by virtue of (8), and one will then have:

$$\varepsilon_{n-k,v} = 0 \qquad \text{for} \qquad 2 \le v \le n - k - 2\,;$$

i.e., one will come back to (12), in which one will find that k has changed into $k + 1$.

3. – Now, take the n principal lines to be the axes and consider the positions that they will occupy when the origin M moves to M'. It is clear that ε_{ij} represents the cosine of the angle that the new axis x'_i makes with x_i when $i \ne j$. With that, formulas (12) will take on a geometric interpretation that one can easily utilize for the direct proof of those formulas. Meanwhile, if one sets:

$$\varepsilon_{2,n} = \varepsilon_2\,, \qquad \varepsilon_{3,n-1} = \varepsilon_3\,, \qquad \varepsilon_{4,n-1} = \varepsilon_4\,, \ldots, \varepsilon_{n-1,3} = \varepsilon_{n-1}$$

then formulas (12), (8), and (9) will say that the direction cosines of the principal lines that have their origins at M' with respect to the ones that go through M are given by the following table:

	x_1	x_2	x_3	$\cdots$	x_{n-2}	x_{n-1}	x_n
x'_1	1	0	0	$\cdots$	0	0	ε_1
x'_2	0	1	ε_{n-1}	$\cdots$	0	0	0
x'_3	0	$-\varepsilon_{n-1}$	1	$\cdots$	0	0	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x'_{n-2}	0	0	0	$\cdots$	1	ε_2	0
x'_{n-1}	0	0	0	$\cdots$	$-\varepsilon_3$	1	ε_2
x'_n	$-\varepsilon_1$	0	0	$\cdots$	0	$-\varepsilon_2$	1

Having done that, let $x_1, x_2, \ldots, x_n$ be the coordinates of a point P with respect to the moving axes. Let δx be the absolute variation of an x coordinate in space when M passes to M' and P simultaneously passes to P'. Let dx be the variation that the coordinate experiences with respect to the moving axes. If one projects $M'P'$ onto the axes through M then one will get:

$$\delta x_1 = dx_1 - \varepsilon_1 (x_n + dx_n) + ds, \qquad \delta x_2 = dx_2 - \varepsilon_{n-1} (x_n + dx_n), \qquad \text{etc.},$$

or

$$\left\{\begin{aligned} &\frac{\delta x_1}{ds} = \frac{dx_1}{ds} - \frac{x_n}{\rho_1} + 1, \quad \frac{\delta x_2}{ds} = \frac{dx_2}{ds} - \frac{x_3}{\rho_{n-1}}, \quad \frac{\delta x_n}{ds} = \frac{dx_n}{ds} + \frac{x_n}{\rho_1} + \frac{x_{n-1}}{\rho_2}, \\ &\frac{\delta x_i}{ds} = \frac{dx_i}{ds} + \frac{x_{i-1}}{\rho_{n-i+2}} - \frac{x_{i+1}}{\rho_{n-i+1}} \quad (i = 3, 4, \ldots, n-1), \end{aligned}\right. \tag{13}$$

after having set:

$$ds = \varepsilon_1\, \rho_1 = \varepsilon_2\, \rho_2 = \varepsilon_3\, \rho_3 = \ldots = \varepsilon_{n-1}\, \rho_{n-1}\,. \tag{14}$$

4. – (13) are the *fundamental formulas for the intrinsic analysis of the curves* that are contained in an n-dimensional linear space. Those curves can then be associated with n – 1 *curvatures* that one can measure by means of the *radii* ρ. When the point M deviates from the tangent as it traverses the curve, it will produce the first curvature. One has a second curvature for the greater or lesser tendency of the point M to deviate from the osculating plane, and then a third one that is due to the tendency that M has to leave the osculating space that is determined by the three consecutive elements MM', $M'M''$, $M''M'''$, and one continues in that way until the deviation of M from the (n – 1-dimensional linear) osculating space that is perpendicular to the (n – 1)-normal implies an (n – 1)$^{\text{th}}$, and final, curvature for the line. Formulas (13) are also valid when one replaces the x with the cosines that define an arbitrary direction, as long as one removes the 1 from the first one. In particular, when the given formulas are applied to the x_{n-i+1}-axis, one will easily find that the *angle* e_i *between two infinitely-close i-normals* is given by the formula:

$$e_i^2 = \varepsilon_i^2 + \varepsilon_{i+1}^2 ,$$

which will also be true for $i = n - 1$ if one agrees to let $\varepsilon_n = 0$, as one must suppose if one briefly thinks of the space considered as being immersed in a linear space of higher dimension. If, in addition, one calls the angle ($= \varepsilon_1$) between two infinitely-close tangents e_0 then it will be interesting to note the following *theorem of Lancret:*

$$e_0^2 - e_1^2 + e_2^2 - \ldots \pm e_{n-1}^2 = 0.$$

5. – The proof of formula (13) can be facilitated by some very simple mechanical considerations that have the advantage of showing what the path is that one must follow in order to obtain more general analogous formulas that refer to nonlinear spaces. When one attributes arbitrary variations δx to the coordinates x in an n-dimensional space in which one has:

$$ds^2 = dx_1^2 + dx_2^2 + \cdots + dx_n^2 ,$$

the last relations will give:

$$ds\ \delta ds = \tfrac{1}{2} \sum_{i,j} \left(\frac{\partial \delta x_i}{\partial x_j} + \frac{\partial \delta x_j}{\partial x_i} \right) dx_i\, dx_j ,$$

and from this, one will deduce that the conditions:

$$\frac{\partial \delta x_i}{\partial x_j} + \frac{\partial \delta x_j}{\partial x_i} = 0$$

will be *necessary*, and when taken altogether, *sufficient for rigidity*. If one integrates them then one will get:

$$\delta x_i = a_i + \omega_{i1}\, x_1 + \omega_{i2}\, x_2 + \ldots + \omega_{in}\, x_n ,$$

in which $\omega_{ij} + \omega_{ji} = 0$. Therefore, any infinitesimal rigid motion will result from a *translation* ($a_1, a_2, \ldots, a_n$) and a *rotation* that decomposes into n (n – 1) / 2 rotations parallel to the coordinate planes, in such a way that for each component rotation, any point of the system will move in a plane that is parallel to a coordinate plane, and will submit to a rotation ω_{ij} in it that is computed from x_j with respect to x_i . Having said that, the system of n principal lines will be considered to be rigid, and one will then study the passage from the position that it occupies at a point M to the one that it takes at an infinitely-close point M'. The ($n - i + 1$)-principal normal x_i will remain perpendicular to $n - i$ consecutive elements, and it must therefore move in the space that is normal to the i dimensions $x_2\, x_3 , \ldots, x_i\, x_{i+1}$, which are perpendicular to the remaining axes x_{i+2} , $x_{i+3}, \ldots, x_n$. It will then follow that $\omega_{ij} = 0$ for:

$$i > 1, \quad j = i + 2, i + 3, \ldots, n.$$

If one then observes that $\omega_{ij} = -\omega_{ji}$ then one can add that $\omega_{ij} = 0$ for:

$$j > 1, \quad i = j + 2, j + 3, \ldots, n.$$

Therefore, in summary, one will have $\omega_{ij} = 0$ for:

$$i > 1, \quad j = 2, 3, \ldots, i - 3, i - 2, i, i + 2, i + 3, \ldots, n - 1, n.$$

As for x_1, it is clear that since it must remain perpendicular to all of the multinormals, it cannot leave the osculating space $x_1\ x_n$; let ε_1 be the angle through which it is rotated towards x_n. One will have:

$$\begin{aligned} \omega_{n1} &= \ \ \varepsilon_1, \qquad \omega_{n1} = \omega_{31} = \ldots = \omega_{n-1,1} = 0, \\ \omega_{1n} &= -\varepsilon_1, \qquad \omega_{12} = \omega_{13} = \ldots = \omega_{1,n-1} = 0. \end{aligned}$$

One then lets ε_{n-1}, ε_{n-2}, …, ε_2 be the angles through which x_2, x_3, …, x_{n-1} are rotated towards x_3, x_4, …, x_n, respectively, in such a way that:

$$\omega_{i+1,i} = -\omega_{i,i+1} = \varepsilon_{n-i+1}.$$

The rigid system that is individuated by the n principal lines is then subjected to the rotations that are defined by the angles ω that were just determined, along with the translation ds along x_1, in its passage from M to M'. As for the point (x_1, x_2, …, x_n), instead of being invariably coupled to the n lines, it suffers the displacement (dx_1, dx_2, …, dx_n) with respect to them, while the components of the absolute displacement in space will be:

$$\begin{cases} \delta x_1 = dx_1 - \varepsilon_1 x_n + ds, \quad \delta x_2 = dx_2 - \varepsilon_{n-1} x_3, \quad \delta x_n = dx_n + \varepsilon_1 x_1 + \varepsilon_2 x_{n-1}, \\ \delta x_i = dx_i + \varepsilon_{n-1} x_{i-1} - \varepsilon_{n-i+1} x_i \quad (i = 3, 4, \ldots, n-1), \end{cases}$$

by virtue of the formula that was proved to begin with and the last results that were obtained. If this is divided by (14) then that will give the fundamental formulas precisely.

6. – Let us pass on to see how we might easily extend the principles of barycentric analysis (VII, § **1**) to linear spaces of more than two dimensions. Fix the points A_1, A_2, …, A_{n+1} in an n-dimensional linear space, which one can consider to be the vertices of the simplest n-dimensional polyhedral entity, to which, following **Stringham**, we give the name of *n-tuple* (*n* + 1)*-hedron*. Let x_{i1}, x_{i2}, …, x_{in} be the coordinates of A_i with respect to the moving axes. An arbitrary point M can always be assumed to be defined in space as the barycenter of a certain system of $n + 1$ masses (viz., *barycentric coordinates*) that are applied to the vertices of the fundamental (n + 1)-hedroid and satisfy the relation:

$$\mu_1 + \mu_2 + \mu_3 + \ldots + \mu_{n+1} = 1,$$

in such a way that its Cartesian coordinates relative to an arbitrary system of orthogonal axes will be given by the formula:

$$x_h = \mu_1\, x_{1h} + \mu_2\, x_{2h} + \mu_3\, x_{2h} + \ldots + \mu_{n+1}\, x_{n+1,h}\, .$$

Meanwhile, one has:

$$\sum_{i,j}^{n+1} (x_{ik} - x_{jk})^2\, d\mu_i\, d\mu_j = 2\sum_{i=1}^{n+1} d\mu_i \cdot \sum_{i=1}^{n+1} x_{ik}^2\, d\mu_i - 2\left(\sum_{i=1}^{n+1} x_{ik}\, d\mu_i\right)^2$$

identically. By virtue of the preceding equality, the right-hand side reduces to $-\,2dx_k^2$, and therefore if one sets $k = 1, 2, 3, \ldots, n$ then when one sums, one will get:

$$ds^2 = -\tfrac{1}{2}\sum_{i,j}^{n+1} a_{ij}^2\, d\mu_i\, d\mu_j\, ,$$

in which a_{ij} represents the length of the edge $A_i\, A_j$, which is to say that one has set:

$$a_{ij}^2 = (x_{i1} - x_{j1})^2 + (x_{i2} - x_{j2})^2 + \ldots + (x_{in} - x_{jn})^2.$$

If the coordinates μ are given, for example, as functions of one parameter t, which might represent time, if one so desires, then the preceding formula will immediately give rise to the square of the velocity:

$$\kappa^2 = -\tfrac{1}{2}\sum_{i,j}^{n+1} a_{ij}^2\, \frac{d\mu_i}{dt}\frac{d\mu_j}{dt}\, . \tag{15}$$

7. – Consider the Wronskian determinant:

$$W = \begin{vmatrix} \mu_1 & \dfrac{d\mu_1}{ds} & \dfrac{d^2\mu_1}{ds^2} & \cdots & \dfrac{d^n\mu_1}{ds^n} \\ \mu_2 & \dfrac{d\mu_2}{ds} & \dfrac{d^2\mu_2}{ds^2} & \cdots & \dfrac{d^n\mu_2}{ds^n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{n+1} & \dfrac{d\mu_{n+1}}{ds} & \dfrac{d^2\mu_{n+1}}{ds^2} & \cdots & \dfrac{d^n\mu_{n+1}}{ds^n} \end{vmatrix}$$

and multiply it by the constant:

$$a^n = \begin{vmatrix} 1 & x_{11} & x_{11} & \cdots & x_{1n} \\ 1 & x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{n+1,1} & x_{n+1,2} & \cdots & x_{n+1,n} \end{vmatrix},$$

which can be made to depend upon only a_{ij}, since one finds that:

$$(-2a^2)^n = -\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & a_{11}^2 & a_{12}^2 & \cdots & a_{1,n+1}^2 \\ 1 & a_{21}^2 & a_{22}^2 & \cdots & a_{2,n+1}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n+1,1}^2 & a_{n+1,2}^2 & \cdots & a_{n+1,n+1}^2 \end{vmatrix}$$

by using simple transformations that are not necessary to reproduce, since it is pointless to prove that $a^n : n!$ measures the volume of the fundamental $(n + 1)$-hedroid. Meanwhile, the aforementioned multiplication will yield the value of a^nW in the form of a determinant of order n whose general element is:

$$\sigma_{ij} = \sum_{k=1}^{n+1} x_{kj} \frac{d^i \mu_k}{ds^i}.$$

Having done that, if one applies the fundamental formulas to the points A then one will get:

$$\begin{cases} \dfrac{dx_{i1}}{ds} = \dfrac{x_{in}}{\rho_1} - 1, \quad \dfrac{dx_{i2}}{ds} = \dfrac{x_{i3}}{\rho_{n-1}}, \quad \dfrac{dx_{in}}{ds} = -\left(\dfrac{x_{i1}}{\rho_1} + \dfrac{x_{i,n-1}}{\rho_2}\right), \\ \dfrac{dx_{ij}}{ds} = \dfrac{x_{i,j+1}}{\rho_{n-j+1}} - \dfrac{x_{i,j-1}}{\rho_{n-j+2}} \quad (j = 3, 4, \ldots, n-1), \end{cases}$$

which will yield some other equations:

$$\begin{cases} \sigma_{i+1,1} = \dfrac{d\sigma_{i1}}{ds} - \dfrac{\sigma_{in}}{\rho_1}, \quad \sigma_{i+1,2} = \dfrac{d\sigma_{i2}}{ds} - \dfrac{\sigma_{i3}}{\rho_{n-1}}, \quad \sigma_{i+1,n} = \dfrac{d\sigma_{in}}{ds} + \dfrac{\sigma_{i,n-1}}{\rho_2} - \dfrac{\sigma_{i1}}{\rho_1}, \\ \sigma_{i+1,j} = \dfrac{d\sigma_{ij}}{ds} + \dfrac{\sigma_{i,j-1}}{\rho_{n-j+2}} - \dfrac{\sigma_{i,j+1}}{\rho_{n-j+1}} \quad (j = 3, 4, \ldots, n-1), \end{cases} \tag{16}$$

by means of which, if one knows the first column of the determinant a^nW then one can calculate all of the other ones. Moreover, a first derivation of the defining equality:

$$\sum_{i=1}^{n+1} \mu_i \, x_{ij} = 0 \qquad (j = 1, 2, 3\ldots, n)$$

will show that:

$$\sigma_{11} = 1, \qquad \sigma_{12} = \sigma_{13} = \ldots = \sigma_{1n} = 0 .$$

(16) will then give:

$$\sigma_{21} = \sigma_{22} = \sigma_{23} = \ldots = \sigma_{2,n-1} = 0, \qquad \sigma_{2,n} = \frac{1}{\rho_1},$$

so

$$\sigma_{31} = -\frac{1}{\rho_1^2}, \quad \sigma_{32} = \sigma_{33} = \ldots = \sigma_{3,n-2} = 0, \quad \sigma_{3,n-1} = -\frac{1}{\rho_1\,\rho_2}, \qquad \sigma_{3,n} = -\frac{d}{ds}\frac{1}{\rho_1},$$

etc. One expects that one must have:

$$\sigma_{ij} = 0 \qquad \text{for} \qquad 2 \leq j \leq n - i + 1, \tag{18}$$

such that if one observes (17) then one will have:

$$\sigma_{i+1,j} = 0 \qquad \text{for} \qquad 2 \leq j \leq n - 1,$$

and that is precisely (17) when one substitutes $i + 1$ for i. Therefore:

$$a^{\,n}W = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \sigma_{2,n} \\ \sigma_{31} & 0 & 0 & \cdots & 0 & \sigma_{3,n-1} & \sigma_{3,n} \\ \sigma_{41} & 0 & 0 & \cdots & \sigma_{4,n-2} & \sigma_{4,n-1} & \sigma_{4,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{n-1,1} & 0 & \sigma_{n-1,3} & \cdots & \sigma_{n-1,n-2} & \sigma_{n-1,n-1} & \sigma_{n-1,n} \\ \sigma_{n,1} & \sigma_{n,2} & \sigma_{n,3} & \cdots & \sigma_{n,n-2} & \sigma_{n,n-1} & \sigma_{n,n} \end{vmatrix},$$

i.e.:

$$a^nW = (-1)^{(n-1)(n-2)/2} \cdot \sigma_{2,n}\,\sigma_{3,n-1}\,\sigma_{4,n-2}\ldots\sigma_{n,2} . \tag{19}$$

On the other hand, formula (18) will give:

$$\sigma_{i+1,n-i+1} = -\,\frac{\sigma_{i,n-i+2}}{\rho_i}$$

for $j = n - i + 1$, and one will deduce from this that:

$$\sigma_{i+1,n-i+1} = \frac{(-1)^{i-1}}{\rho_1\,\rho_2\,\rho_3\cdots\rho_i}.$$

Finally, if one substitutes that result in (19) then one will arrive at the following noteworthy formula:

$$a^n \rho_1^{n-1} \rho_2^{n-2} \rho_3^{n-3} \ldots \rho_{n-1}\, W = 1. \tag{20}$$

8. – The last formula always gives one relation between the $n - 1$ curvatures of a given curve, and one will need $n - 2$ more in order to be able to establish the intrinsic equations of that curve. It is clear that the relation (20) will suffice in the case of a plane curve: The required intrinsic equation will then be obtained by eliminating t from the equalities:

$$a^2 \rho\, W = 1, \qquad s = \int \kappa\, dt\,,$$

in which one first thinks of replacing κ and W as functions of t by means of formula (15), as well as:

$$\kappa^{n(n+1)/2}\, W = \begin{vmatrix} \mu_1 & \dfrac{d\mu_1}{dt} & \dfrac{d^2\mu_1}{dt^2} & \cdots & \dfrac{d^n\mu_1}{dt^n} \\ \mu_2 & \dfrac{d\mu_2}{dt} & \dfrac{d^2\mu_2}{dt^2} & \cdots & \dfrac{d^n\mu_2}{dt^n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_{n+1} & \dfrac{d\mu_{n+1}}{dt} & \dfrac{d^2\mu_{n+1}}{dt^2} & \cdots & \dfrac{d^n\mu_{n+1}}{dt^n} \end{vmatrix},$$

which is a simple consequence of the definition of W. One will get the relation:

$$a^3 \rho^2\, r\, W = 1, \tag{21}$$

and one needs one more. In order to find it, consider the Wronskian matrix:

$$\begin{vmatrix} \mu_1 & \dfrac{d\mu_1}{ds} & \dfrac{d^2\mu_1}{ds^2} & \dfrac{d^3\mu_1}{ds^3} & \dfrac{d^4\mu_1}{ds^4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_4 & \dfrac{d\mu_4}{ds} & \dfrac{d^2\mu_4}{ds^2} & \dfrac{d^4\mu_4}{ds^3} & \dfrac{d^4\mu_4}{ds^4} \end{vmatrix}, \tag{22}$$

which will become W when one suppresses the last column, and dW / ds when one suppresses the penultimate one. Furthermore, let W' be the determinant that is obtained by suppressing the second column instead, which is a determinant that can be calculated easily as a function of t, since one will have:

$$\kappa^{10}\, W' = \begin{vmatrix} 0 & \dfrac{ds}{dt} & \dfrac{d^2 s}{dt^2} & \dfrac{d^3 s}{dt^3} & \dfrac{d^4 s}{dt^4} \\ \mu_1 & \dfrac{d\mu_1}{dt} & \dfrac{d^2\mu_1}{dt^2} & \dfrac{d^3\mu_1}{dt^3} & \dfrac{d^4\mu_1}{dt^4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu_4 & \dfrac{d\mu_4}{dt} & \dfrac{d^2\mu_4}{dt^2} & \dfrac{d^4\mu_4}{dt^3} & \dfrac{d^4\mu_4}{dt^4} \end{vmatrix},$$

by virtue of a general property of the Wronskian. In order for the equalities:

$$\sum \mu_i\, x_{i1} = 0, \quad \sum x_{i1} \frac{d\mu_i}{ds} = 1, \quad \sum x_{i1} \frac{d^2\mu_i}{ds^2} = 0, \quad \sum x_{i1} \frac{d^3\mu_i}{ds^3} = -\frac{1}{\rho^2}, \quad \sum x_{i1} \frac{d^4\mu_i}{ds^4} = \frac{3}{\rho^3}\frac{d\rho}{ds},$$

in which i goes from 1 to 4, to coexist, it is necessary that the determinant that is formed by adding the row:

$$0, \quad 1, \quad 0, \quad -\frac{1}{\rho^2}, \quad \frac{3}{\rho^3}\frac{d\rho}{ds}$$

to the matrix (22) should be zero – i.e., that one should have:

$$\rho^3\, W' = \frac{d}{ds}(\rho^3\, W), \tag{23}$$

from which, one infers that:

$$\frac{1}{\rho^2} = -\tfrac{2}{3} W^{2/3} \int W' W^{-3/2} \kappa\, dt, \tag{24}$$

as long as W and W' are non-zero. When one replaces κ; W, W' with their expressions as functions of t, the last formula will make ρ known, and one will then get r from (21). Moreover, one will also arrive at the equation that gave us (21) from (24) by an analogous procedure. Indeed, if one multiplies the determinants a^3 and W' together then one will get:

$$a^3\, \rho^5\, W' = \frac{d}{ds}\frac{\rho}{r},$$

and one will get back to (23) from this formula by eliminating r by means of (21). In addition, one will see that $W' = 0$ is the necessary and sufficient condition for the curve to be a *helix*, while $W = 0$ is the condition for the curve to be *planar*. The reader can apply the preceding formulas to the study of the tetrahedral potential (cf., VII, § **20**) as an exercise.

CHAPTER XVII

HYPERSPACES

1. – Consider a line in an *n*-dimensional curved space that is immersed in a linear space and recall that by virtue of the fundamental formulas that were established in the preceding chapter, the coordinates of a fixed point with respect to the $n + 1$ principal lines of the curve are functions of the arc length whose derivatives can be expressed linearly in those coordinates. It is clear that this property will be preserved when the n normal lines, which are regarded as concurrent, rotate in their space until one of them becomes normal to the curved space in question. If one imagines $n - 1$ curves in it that are tangents to the other $n - 1$ lines then if one calls the coordinates of the fixed point $x_0, x_1, \ldots, x_n$, one can write:

$$\frac{\partial x_i}{\partial s_j} = \sum_{k=0}^{n} A_{ik}^{(j)} x_k - e_{ij} , \tag{1}$$

in which e_{ij} is equal to 1 or 0 according to whether $i = j$ or $i \neq j$, resp. We shall soon see that the $n\ (n + 1)^2$ coefficients A reduce to just $n\ (3n - 1) / 2$ linearly-independent ones and that $n\ (n - 1)(5n - 1) / 4$ substantially-diverse relations exist among them and their derivatives that are analogous to the formulas that **Codazzi** established for the surface.

2. Dupin's theorem. – First of all, note that formulas (1) will still be true when e_{ij} is always zero if the x have the significance of the direction cosines, in which case, one must have:

$$x_0 \frac{\partial x_0}{\partial s_k} + x_1 \frac{\partial x_1}{\partial s_k} + x_2 \frac{\partial x_2}{\partial s_k} + \ldots + x_n \frac{\partial x_n}{\partial s_k} = 0$$

identically; hence:

$$A_{ij}^{(k)} + A_{ji}^{(k)} = 0, \tag{2}$$

and in particular, $A_{ii}^{(k)} = 0$. On the other hand, if one differentiates (1) then one will get:

$$\frac{\partial^2 x_k}{\partial s_i\, \partial s_j} = \sum_{l=0}^{n} \frac{\partial A_{kl}^{(i)}}{\partial s_j} x_l + \sum_{l,m} A_{kl}^{(i)} A_{lm}^{(j)} x_m - A_{kj}^{(i)} , \tag{3}$$

and then, for $x_0 = x_1 = \ldots = x_n = 0$:

$$\frac{\partial x_i}{\partial s_j} = - e_{ij} , \qquad \frac{\partial^2 x_k}{\partial s_i\, \partial s_j} = - A_{kj}^{(i)} .$$

Having done that, the integrability conditions:

$$\frac{\partial^2 x_k}{\partial s_i\,\partial s_j} - \frac{\partial^2 x_k}{\partial s_j\,\partial s_i} = \frac{\partial \log Q_j}{\partial s_i}\frac{\partial x_k}{\partial s_j} - \frac{\partial \log Q_i}{\partial s_j}\frac{\partial x_k}{\partial s_i} \tag{4}$$

will become:

$$A_{ki}^{(j)} - A_{kj}^{(i)} = e_{ki}\frac{\partial \log Q_j}{\partial s_i} - e_{kj}\frac{\partial \log Q_j}{\partial s_i},$$

and in particular, for $k = i$:

$$A_{ij}^{(i)} = -\frac{\partial \log Q_i}{\partial s_j}. \tag{5}$$

However, if one supposes that k is different from i and j then one will get:

$$A_{ki}^{(j)} = A_{kj}^{(i)}. \tag{6}$$

Finally, when formulas (2) and (6) are adopted as an alternative, that will give:

$$A_{ij}^{(k)} = -A_{ji}^{(k)} = -A_{k}^{(i)} = A_{kj}^{(i)} = A_{ki}^{(j)} = -A_{ik}^{(j)} = -A_{ij}^{(k)}.$$

It will then result that:

$$A_{ij}^{(k)} = 0 \tag{7}$$

whenever i, j, k are all different and non-zero. That equality is the generalized expression for *Dupin's theorem* in hyperspace, because we saw before (XV, § **12**) that the aforementioned theorem is due precisely to the independence of the quotients $\partial x_i / \partial s_j$ from the x with non-zero indices that are different from i and j, and we will soon see that the geometrical significance of the equality (7) is also the natural extension of the one that is already found in three-dimensional space.

3. – If one sets $k = 0$ and $A_{0j}^{(i)} = \mathcal{T}_{ij}$ then one can only assert that $\mathcal{T}_{ij} = \mathcal{T}_{ji}$, by virtue of (6). In addition, we represent $\mathcal{T}_{ii}$ by $-\mathcal{N}_i$ and $A_{ij}^{(i)}$ by $-\mathcal{G}_{ij}$, in such a way that we will have:

$$\mathcal{G}_{ij} = \frac{\partial \log Q_i}{\partial s_j}, \tag{8}$$

according to (5). The only coefficients A that remain will then be the ones that are denoted by $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, …, $\mathcal{G}_{12}$, $\mathcal{G}_{21}$, $\mathcal{G}_{13}$, $\mathcal{G}_{31}$, $\mathcal{G}_{23}$, $\mathcal{G}_{32}$, … or $\mathcal{T}_{12}$, $\mathcal{T}_{13}$, $\mathcal{T}_{23}$, …, and which we call *normal curvatures, geodetic curvatures, and geodetic torsions,* respectively. Therefore, in summary, we do not cease to keep in mind that any coefficient A will change sign when one transposes the lower indices and that the normal curvatures and geodetic torsions and curvatures are expressed in the following ways:

$$\mathcal{N}_i = A_{i0}^{(i)}, \qquad \mathcal{T}_{ij} = \mathcal{T}_{ji} = A_{0j}^{(i)}, \qquad \mathcal{G}_{ij} = A_{ji}^{(i)}.$$

In any case, $A = 0$. In addition, we feel that allows us to write $-\mathcal{T}_{ii}$, instead of $\mathcal{N}_i$, and $\mathcal{G}_{ii}$, instead of 0, for ease of writing.

4. Codazzi formulas. – Now, if one observes (8) then the conditions (4) will change into:

$$\left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji}\right)\frac{\partial x_k}{\partial s_j} = \left(\frac{\partial}{\partial s_j} + \mathcal{G}_{ij}\right)\frac{\partial x_k}{\partial s_i},$$

and thanks to (3) this can be transformed into a linear relation in the x that splits into the following conditions:

$$\left(\frac{\partial}{\partial s_j} + \mathcal{G}_{ij}\right)A_{kl}^{(i)} - \left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji}\right)A_{kl}^{(j)} = \sum_{m=0}^{n}(A_{km}^{(i)}A_{lm}^{(j)} - A_{km}^{(j)}A_{lm}^{(i)}), \tag{9}$$

which are necessary and sufficient for the existence of the functions x. When each of the numbers i, j, k, l is assumed to be positive and different from the other three, the left-hand side will be zero, by virtue of (7), and one will get:

$$\mathcal{T}_{ik}\,\mathcal{T}_{jl} - \mathcal{T}_{il}\,\mathcal{T}_{jk} = 0. \tag{10}$$

That equality permits one to express $n(n-3)/2$ of the coefficients $\mathcal{T}$ in terms of the other n, and the number of coefficients A to which one can arbitrarily assign a value at a point will then be found to reduce to $n(n+1)$ for $n > 2$. (9) is the *universal Codazzi formula*, so to speak, from which one deduces other groups of formulas according to the various meanings of the coefficients A. When one of the indices k, l is supposed to be zero and the other one is different from i and j, formula (9) will give:

$$\frac{\partial \mathcal{T}_{ik}}{\partial s_j} - \frac{\partial \mathcal{T}_{jk}}{\partial s_i} + \mathcal{T}_{ik}\,\mathcal{G}_{ij} - \mathcal{T}_{jk}\,\mathcal{G}_{ji} + \mathcal{T}_{ij}\,(\mathcal{G}_{ik} - \mathcal{G}_{jk}) = 0. \tag{α}$$

However, if the non-zero indices are set equal to i or j then one will get:

$$\frac{\partial \mathcal{N}_j}{\partial s_i} + \frac{\partial \mathcal{T}_{ij}}{\partial s_j} + 2\mathcal{T}_{ij}\,\mathcal{G}_{ij} + \sum_{k=1}^{i-1}\mathcal{T}_{ik}\,\mathcal{G}_{jk} + \sum_{k=i+1}^{i-1}\mathcal{T}_{ik}\,\mathcal{G}_{jk} = (\mathcal{N}_i - \mathcal{N}_j)\,\mathcal{G}_{ji}. \tag{β}$$

Now set $k = i$, $l = j$. Under that hypothesis, formula (9) will become:

$$\frac{\partial \mathcal{G}_{ij}}{\partial s_j}+\frac{\partial \mathcal{G}_{ji}}{\partial s_i}+\mathcal{G}_{ij}^2+\mathcal{G}_{ji}^2+\sum_{k=1}^{i-1}\mathcal{G}_{ik}\,\mathcal{G}_{jk} = \mathcal{T}_{ij}^2-\mathcal{N}_i\,\mathcal{N}_j\,. \tag{γ}$$

If one finally supposes that just one of the positive numbers k, l is equal to i or j then one will find that:

$$\frac{\partial \mathcal{G}_{jk}}{\partial s_j}+(\mathcal{G}_{ik}-\mathcal{G}_{jk})\,\mathcal{G}_{ij} = \mathcal{N}_i\,\mathcal{T}_{jk}+\mathcal{T}_{ik}\,\mathcal{T}_{ji}\,, \tag{δ}$$

as well as:

$$\frac{\partial \mathcal{G}_{ij}}{\partial s_k}+(\mathcal{G}_{ij}-\mathcal{G}_{kj})\,\mathcal{G}_{ik} = \mathcal{N}_i\,\mathcal{T}_{jk}+\mathcal{T}_{ik}\,\mathcal{T}_{ji}\,.$$

However, the last formula is no different from the preceding one, since (cf., XV, § **3**) one sees the identity between their left-hand sides, by virtue of (8). The formulas of the group (δ), like those of the group (α), are distributed into n (n – 1)(n – 2) / 4 triples; however, any triple in (α) contains only two substantially different formulas. The groups (β) and (γ) obviously contain n (n – 1) and n (n – 1) / 2 formulas, in such a way that one will have:

$$\tfrac{5}{6}\,n(n-1)(n-2)+\tfrac{3}{2}\,n(n-1) = \tfrac{1}{6}\,n(n-1)(5n-1)$$

relations, in all, which are the analogues of the Codazzi formulas for curved n-dimensional spaces.

5. – The quadratic form:

$$\Phi = \mathcal{N}_1\,x_1^2+\mathcal{N}_2\,x_2^2+\ldots-2\,\mathcal{T}_{12}\,x_1\,x_2\,\ldots$$

is fundamental to the intrinsic study of these spaces, and its first partial derivatives are proportional to the derivatives of x_0, by virtue of (1). It is useful to observe the simple form that these relations will assume as a consequence of the determination of the coefficients A that was carried out in § **3**. One will have:

$$\frac{\partial x_i}{\partial s_i} = \mathcal{N}_i\,x_0-\sum_{j=1}^{n}\mathcal{G}_{ij}\,x_j-1, \qquad \frac{\partial x_j}{\partial s_i} = -\,\mathcal{T}_{ij}\,x_0+\mathcal{G}_{ij}\,x_j\,.$$

The discussion of Φ leads to *Euler's theorem* and the notion of *systems of curvature*, which are characterized by the conditions $\mathcal{T} = 0$. If one then supposes that the immobility conditions have been written down in an n – 1-dimensional space that belongs to the system that is defined in the given curved space by a function q_i then one will immediately realize that the $\mathcal{T}_{jk}$ that relate to the aforementioned space q_i are no different from the coefficients $A_{ik}^{(j)}$, in such a way that when the n spaces q_i are associated in such a manner as to constitute the curved space in question, the equality (7) will say that all of the $\mathcal{T}$ are zero – i.e., *those spaces will necessarily intersect along their systems of curvature.* Finally, the discussion of Φ also leads us one to consider n principal curvatures, whose product K, which is equal to the discriminant of Φ, can serve to measure the *total curvature*. Formulas (γ) and (δ), along with (10), then provide values to all of the quadratic minors in K, and one can then say that *the total curvature depends uniquely upon the geodetic curvature and its variations.* One should also notice that in the case of a linear space, the function Φ will vanish identically, and the Codazzi formulas will reduce (cf., XV, § **15**) to $n (n - 1)^2 / 2$ *necessary and sufficient conditions for the linearity of that n-dimensional space.*

6. – Let us now apply the preceding formulas to the study of the *infinitesimal deformations* of hypersurfaces. A point M in a curved n-dimensional space that is immersed in a linear space of one higher dimension is displaced infinitely little in that space. Let $u_0, u_1, u_2, \ldots, u_n$ be its coordinates at the new position M' with respect to the moving axes whose origin is M and which are chosen in the previously-described way, and set:

$$u_{ij} = \frac{\partial u_i}{\partial s_j} - \sum_{k=0}^{n} A_{ik}^{(j)} u_k \,. \tag{11}$$

The fundamental formulas show immediately that when M traverses the infinitesimal segment ds_i along the i-axis, the coordinates of the point M' will vary by:

$$u_{0i}\, ds_i\,, \quad u_{1i}\, ds_i\,, \quad u_{2i}\, ds_i\,, \;\ldots, \quad (u_{ii} + 1)\, ds_i\,, \quad \ldots, u_{ni}\, ds_i\,.$$

Therefore, more generally, if the point M moves in the direction that is defined by the cosines $\alpha_1, \alpha_2, \ldots, \alpha_n$ in the tangent linear space and describes the segment ds then the coordinates of M' will submit to the variations:

$$(\alpha_i + \sum_{j=1}^{n} \alpha_j u_{ij})\, ds\,, \tag{12}$$

and therefore if one squares and sums this then one will find that the segment that is traversed by M is $ds' = (1 + \Omega)\, ds$, and when one omits higher-order infinitesimals, one will have:

$$\Omega = \sum_{i,j} \alpha_i\, \alpha_j\, u_{ij}\,.$$

In particular, the u_{ii} represent the unit elongations along the axes, and the consideration of the solid element that is constructed from the edges $ds_1, ds_2, \ldots, ds_n$ will show that $u_{ij} + u_{ji}$ is the mutual angular displacement between the i and j axes, and that the unit solid dilatation is:

$$\Theta = u_{11} + u_{22} + u_{33} + \ldots + u_{nn}\,;$$

i.e., by virtue of (11):

$$\Theta = \sum_{i=1}^{n}\left(\frac{\partial}{\partial s_i} + \mathcal{G}_i\right)u_i - u_0 \sum_{i=1}^{n} \mathcal{N}_i\,, \tag{13}$$

in which $\mathcal{G}_i$ represents the sum of all the $\mathcal{G}$ that have their second index equal to i.

7. – The direction cosines of the tangents to the trajectory of M' are obviously obtained by multiplying the quantity (12) by $1 - \Omega$ and dividing by ds. They will then have the values:

$$\alpha_i - \Omega\ \alpha_i + \sum_{j=1}^{n} \alpha_j\, u_{ij}\ . \tag{14}$$

At the same time, one can give the form:

$$\sum_{j,k} \alpha_j \alpha_k (\alpha_k u_{ij} - \alpha_i u_{kj}) = \alpha_1\ \omega_1 + \alpha_2\ \omega_2 + \ldots + \alpha_n\ \omega_n$$

to the increment that α_i gives directly by setting:

$$\omega_{ij} = \sum_{k=1}^{n} \alpha_k (\alpha_j u_{ik} - \alpha_i u_{jk})\,,$$

and since $\omega_{ji} = -\ \omega_{ij}$, one will see that the direction $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ will submit to the rotation in the tangent linear space that is defined precisely (XVI, § **5**) by the semi-symmetric matrix:

$$\begin{vmatrix} 0 & \omega_{12} & \omega_{13} & \cdots & \omega_{1n} \\ \omega_{21} & 0 & \omega_{23} & \cdots & \omega_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_{n1} & \omega_{n2} & \omega_{n3} & \cdots & 0 \end{vmatrix}.$$

In particular, the i and j axes rotate in their plane through $-\ u_{ji}$ and u_{ij}, and one will then see, once again, that $u_{ij} + u_{ji}$ represents the mutual angular displacement of those axes. Meanwhile, since Ω is generally reducible to its canonical form in just one way, one will generally have that just one orthogonal system of axes will remain orthogonal under the deformation, in such a way that any pair (i, j) of those axes will rotate rigidly in its plane through an angle of $u_{ij} = -\ u_{ji} = (u_{ij} - u_{ji})\ /\ 2$. Since the quantity $\vartheta_{ij} = u_{ij} - u_{ji}$ is an orthogonal invariant of the form ω_{ij}, one will easily see that the ϑ_{ij} then represent twice the components of the geodetic rotation, in any case. From (11), one now has:

$$\vartheta_{ij} = \left(\frac{\partial}{\partial s_j} + \mathcal{G}_{ij}\right)u_i - \left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji}\right)u_j\ . \tag{15}$$

Here, one should note that by virtue of the integrability condition:

$$\left(\frac{\partial}{\partial s_j} + \mathcal{G}_{ij}\right)\frac{\partial}{\partial s_i} = \left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji}\right)\frac{\partial}{\partial s_j}, \tag{16}$$

the ϑ are all annulled in the potential deformations of space intrinsically – i.e., when the displacement is tangential and has the differential quotients of a function u with respect to the tangent axes for its components. One will then also have $\Theta = \Delta^2 u$ from (13), since with our symbols, the *Lamé formula*, which serves to express the second differential parameter, can be written (cf., XV, § **7**) in the following way:

$$\Delta^2 = \sum_{i=1}^{n}\left(\frac{\partial}{\partial s_i} + \mathcal{G}_i\right)\frac{\partial}{\partial s_j}.$$

8. – We shall now go on to the choice of those axes in the deformed space. We always take the 0 axis to be the normal to that space, and choose the other ones to have positions that differ infinitely little from the one that the original tangent axes will occupy as a result of the deformation. Therefore, assume, for the moment, that the i axis is allowed to assume the line that is defined in the original tangent linear space by the cosines:

$$\alpha_j = \begin{cases} 1 & \text{for} \quad j = i, \\ -u_{ji} & \text{for} \quad j \neq i, \end{cases} \tag{17}$$

when considered in the position it occupies after the deformation. If one observes the expressions (13) then one will find that the direction cosines of the new tangent axes will be given by the table:

$$\begin{array}{ccccc} u_{01} & 1 & 0 & \dots & 0 \\ u_{02} & 0 & 1 & \dots & 0 \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots} \\ u_{0n} & 0 & 0 & \dots & 1 \end{array}$$

and the direction cosines of the new normal axes will then be:

$$1 \quad -u_{01} \qquad -u_{02} \quad \dots \qquad -u_{0n}\,.$$

It will then follow that under the passage from the old system to the new one, the x coordinates will submit to the variations:

$$Dx_i = -u_i + u_{0i}\,x_0 \tag{18}$$

for $i > 0$, and:

$$Dx_0 = -u_0 - \sum_{i=1}^{n} u_{0i}\,x_i\,. \tag{19}$$

It is now easy to express the new differential quotients in terms of the old ones, since one obviously has:

$$\frac{\partial}{\partial s'} = (1 - \Omega) \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial s_i},$$

and in particular, if one makes the hypothesis (17) then the left-hand side will become the symbol for the differential quotient with respect to the new i axis, while the right-hand side will reduce to:

$$(1 - u_{ii}) \left(\frac{\partial}{\partial s_i} + u_{ii} \frac{\partial}{\partial s_i} - \sum_{j=1}^{n} u_{ji} \frac{\partial}{\partial s_j} \right).$$

Hence:

$$\frac{\partial}{\partial s_i'} = \frac{\partial}{\partial s_i} - \sum_{j=1}^{n} u_{ji} \frac{\partial}{\partial s_j},$$

and therefore the variations of the original differential quotients that are produced by the deformation will be given in the form:

$$D \frac{\partial}{\partial s_i'} = \frac{\partial}{\partial s_i} D - \sum_{j=1}^{n} u_{ji} \frac{\partial}{\partial s_j}. \tag{20}$$

9. – Having said that, in order to calculate the variations that the curvatures experience as a result of the deformation, it is enough to apply the last formula to the relations (1). One will immediately get:

$$\sum_{k=0}^{n} (A_{ik}^{(j)} Dx_k + x_k DA_{ik}^{(j)}) = \frac{\partial}{\partial s_i} Dx_i - \sum_{k=1}^{n} u_{ki} \frac{\partial x_i}{\partial s_k},$$

so if one supposes that $i > 0$ and invokes formulas (18) and (19) then:

$$\sum_{k=0}^{n} x_k DA_{ik}^{(j)} = \left[\frac{\partial u_{0i}}{\partial s_j} + \sum_{k=0}^{n} (A_{0i}^{(k)} u_{kj} - A_{ik}^{(j)} u_{0k}) \right] x_0 + \sum_{k=0}^{n} (A_{0k}^{(j)} u_{0i} - A_{0i}^{(j)} u_{0k} - \sum_{l=1}^{n} A_{ik}^{(l)} u_{lj}) x_k. \tag{21}$$

Upon equating the coefficients of x_0 and supposing that $i = j$, in addition, it will follow that:

$$D\mathcal{N}_i = \frac{\partial u_{0i}}{\partial s_i} + \sum_{j=1}^{n} (\mathcal{G}_{ij} u_{0j} + \mathcal{T}_{ij} u_{ji}).$$

However, for $i \neq j$, one will get:

$$D\mathcal{T}_{ij} = -\frac{\partial u_{0i}}{\partial s_j} + \mathcal{G}_{ji} u_{0j} - \sum_{k=1}^{n} \mathcal{T}_{ki} u_{kj}.$$

However, it is clear that one can also write:

$$D\mathcal{T}_{ij} = -\frac{\partial u_{0j}}{\partial s_i} + \mathcal{G}_{ij}\, u_{0i} - \sum_{k=1}^{n} \mathcal{T}_{kj}\, u_{ki}\,.$$

If one equates the coefficients of x_j between them then one will find that:

$$D\mathcal{G}_{ij} = \mathcal{G}_{ji}\; u_{ji} - \mathcal{G}_{ij}\; u_{ii} - \mathcal{T}_{ji}\; u_{0i} - \mathcal{N}_i\; u_{0j}\,.$$

10. – The identity between the two expressions that were found for $D\mathcal{G}_{ij}$ can be established directly by applying the condition (16) to u_0 . When it is applied, more generally, to u_k , that will give:

$$\left(\frac{\partial}{\partial s_j} + \mathcal{G}_{ij}\right) u_{ki} - \left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji}\right) u_{kj} = \sum_{l=0}^{n} (A_{kl}^{(j)}\, u_{li} - A_{kl}^{(i)}\, u_{lj})\,, \tag{22}$$

if one takes the relations (9) into account. For $k = 0$, one will get:

$$\left(\frac{\partial}{\partial s_j} + \mathcal{G}_{ij}\right) u_{0i} - \left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji}\right) u_{0j} = \sum_{k=0}^{n} (\mathcal{T}_{kj}\, u_{ki} - \mathcal{T}_{ki}\, u_{kj})\,.$$

However, for $k = i$:

$$\left(\frac{\partial}{\partial s_j} + \mathcal{G}_{ij}\right) u_{ii} - \left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji}\right) u_{ij} = \mathcal{G}_{ji}\; u_{ji} - \mathcal{T}_{ij}\; u_{0i} - \mathcal{N}_i\; u_{0j} + \sum_{k=0}^{n} \mathcal{G}_{ik}\, u_{kj}\,.$$

Finally, when i, j, k are supposed to be different from each other and positive, one will find the relations:

$$\left.\begin{aligned}
\left(\frac{\partial}{\partial s_k} + \mathcal{G}_{jk}\right) u_{ij} - \left(\frac{\partial}{\partial s_j} + \mathcal{G}_{kj}\right) u_{ik} &= \mathcal{G}_{ki}\, u_{kj} - \mathcal{G}_{ji}\, u_{jk} + \mathcal{T}_{ij}\, u_{0k} - \mathcal{T}_{ik}\, u_{0j}\,, \\
\left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ki}\right) u_{jk} - \left(\frac{\partial}{\partial s_k} + \mathcal{G}_{ik}\right) u_{ji} &= \mathcal{G}_{ij}\, u_{ik} - \mathcal{G}_{kj}\, u_{ki} + \mathcal{T}_{jk}\, u_{0i} - \mathcal{T}_{ji}\, u_{0k}\,, \\
\left(\frac{\partial}{\partial s_j} + \mathcal{G}_{ij}\right) u_{ki} - \left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji}\right) u_{kj} &= \mathcal{G}_{jk}\, u_{ji} - \mathcal{G}_{ik}\, u_{ij} + \mathcal{T}_{ki}\, u_{0j} - \mathcal{T}_{kj}\, u_{0i}\,.
\end{aligned}\right\} \tag{23}$$

11. – Meanwhile, the consequences of the identity (21) have not been exhausted, since we still need to express the idea that the zero coefficients must remain zero, and we will then arrive at the triple of relations:

$$\left.\begin{aligned} \mathcal{G}_{kj}\,u_{ki} - \mathcal{G}_{jk}\,u_{ji} + \mathcal{T}_{ij}\,u_{0k} - \mathcal{T}_{ik}\,u_{0j} &= 0, \\ \mathcal{G}_{ik}\,u_{ij} - \mathcal{G}_{ki}\,u_{kj} + \mathcal{T}_{jk}\,u_{0i} - \mathcal{T}_{ji}\,u_{0k} &= 0, \\ \mathcal{G}_{ji}\,u_{jk} - \mathcal{G}_{ij}\,u_{ik} + \mathcal{T}_{ki}\,u_{0j} - \mathcal{T}_{kj}\,u_{0i} &= 0 \end{aligned}\right\} \tag{24}$$

for any triple i, j, k of distinct positive numbers. These relations, which we can consider to be the conditions for the *permanence of Dupin's theorem* in the deformed space, constrain the displacements by means of first-order partial differential equations, and with them, one will discover that *the deformation that has been studied so far is not the most general one possible* for $n > 2$. The previously-established formulas are then valid in full generality *only for surfaces*, and it is easy to verify that for $n = 2$ they will effectively reduce to the ones that we proved in Chapter XIII. The specialization that was found for $n > 2$ came about as a consequence of the choice of axes, since (cf., XV, § **4**) *the totality of all axes that are tangent to the space cannot always be considered to constitute the tangents to a n-fold orthogonal system of curves*, even though the orientation of the aforementioned axes varies in a continuous manner with the position of the origin.

12. – Other restrictive conditions can give rise to the *permanence of the universal Codazzi formula*, but since that formula is obtained by applying the condition (16) to the coordinates of a fixed point, it is enough to investigate whether there is any constraint that one can subject the displacements to in order for the stated condition to persist in the deformed space. Now, one easily deduces from (20) that:

$$D\frac{\partial^2}{\partial s_i\,\partial s_j} = \frac{\partial^2}{\partial s_i\,\partial s_j}D - \sum_{k=1}^{n}\frac{\partial u_{ki}}{\partial s_j}\frac{\partial}{\partial s_k} - \sum_{k=1}^{n}\left(u_{ki}\frac{\partial^2}{\partial s_k\,\partial s_j} + u_{kj}\frac{\partial^2}{\partial s_i\,\partial s_k}\right).$$

Hence, if one switches i with j and then subtracts one equality from the other one, while keeping relations (16), (22), (24) in mind, along with:

$$\mathcal{G}_{ij}\,u_{ik} + \mathcal{G}_{jk}\,u_{ji} + \mathcal{G}_{ki}\,u_{kj} = \mathcal{G}_{ik}\,u_{ij} + \mathcal{G}_{ji}\,u_{jk} + \mathcal{G}_{kj}\,u_{ki}\,,$$

which is a consequence of any triple (24), then one will arrive at an identity. Therefore, other than (24), there exist no other restrictions that one must impose upon the displacements. One would arrive at the same conclusion less rapidly by the direct route; i.e., by calculating the variations that the deformation brings to the Codazzi formulas, and in order to exhibit the final identity, one must opportunely employ integration by parts and some other artifice, in addition.

13. – We shall turn to the study of the general deformation. The pseudo-symmetric matrix:

$$\begin{vmatrix} 1 & v_{10} & v_{20} & \cdots & v_{n0} \\ v_{01} & 1 & v_{21} & \cdots & v_{n1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_{0n} & v_{1n} & v_{2n} & \cdots & 1 \end{vmatrix}$$

is the one that defines the orientation of the axes in the deformed space with respect to the original axes. If one continues to assume that the 0 axis is normal to the space then one will have $v_{0i} = u_{0i}$, and the other n $(n - 1) / 2$ infinitesimal quantities v must satisfy conditions that would validate and insure the existence of an n-fold orthogonal system of curves in the deformed space that are tangent to the axes 1, 2, 3, …, n at any of their points. We shall leave the v arbitrary, for now, since the desired conditions will arise spontaneously from the calculations that one must do, and one will note that they will be precisely the ones that insure the permanence of Dupin's theorem in the deformed space. First, observe that the direction that is defined by the cosines:

$$\alpha_j = \begin{cases} 1 & \text{for} \quad j = i \\ -u_{ji} + v_{ji} & \text{for} \quad j \neq i \end{cases}$$

is carried by the deformation in such a way that it will coincide with that of the new i axis. Indeed, one will see from the expressions (14) that in order to effect the deformation, the aforementioned cosines must acquire the values:

$$(1 - u_{ii})\ \alpha_j + u_{ji} = \begin{cases} 1 & \text{for} \quad j = i, \\ v_{ji} & \text{for} \quad j \neq i. \end{cases}$$

Therefore, the new differential quotients relative to the i axis will be expressed in the following way:

$$(1 - u_{ii}) \sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial s_j} = (1 - u_{ii}) \left(\frac{\partial}{\partial s_i} + u_{ii} \frac{\partial}{\partial s_i} - \sum_{j=1}^{n} (u_{ji} - v_{ji}) \frac{\partial}{\partial s_j} \right) .$$

Therefore, one needs to replace (20) with:

$$D \frac{\partial}{\partial s_i} = \frac{\partial}{\partial s_i} D - \sum_{j=1}^{n} (u_{ji} - v_{ji}) \frac{\partial}{\partial s_j} . \tag{25}$$

Similarly, instead of (18) and (19), one will have:

$$Dx_i = -u_i - \sum_{j=0}^{n} v_{ij}\, x_j .$$

14. – If one takes advantage of the preceding formulas and varies all of immobility conditions then one will get a linear relation between the coordinates x that will split into:

$$DA_{ik}^{(j)} = -\frac{\partial v_{ik}}{\partial s_j} - \sum_{l=1}^{n} A_{ik}^{(l)}(u_{lj} - v_{lj}) - \sum_{l=0}^{n}(A_{li}^{(j)}\,u_{lk} - A_{lk}^{(j)}\,v_{li}), \tag{26}$$

given their arbitrariness. If one recalls that for any triple i, j, k of distinct positive numbers, one must have:

$$A_{ik}^{(j)} = 0, \qquad DA_{ik}^{(j)} = 0$$

then one will find the conditions:

$$\left.\begin{aligned}
&\frac{\partial v_{jk}}{\partial s_i} + (\mathcal{G}_{ij} - \mathcal{G}_{kj})\,v_{ki} - (\mathcal{G}_{jk} - \mathcal{G}_{ik})\,v_{ij} + \mathcal{G}_{kj}\,u_{ki} - \mathcal{G}_{jk}\,u_{ji} + \mathcal{T}_{ij}\,u_{0k} - \mathcal{T}_{ik}\,u_{0j} = 0, \\
&\frac{\partial v_{ki}}{\partial s_j} + (\mathcal{G}_{jk} - \mathcal{G}_{ik})\,v_{ij} - (\mathcal{G}_{ki} - \mathcal{G}_{ji})\,v_{jk} + \mathcal{G}_{ik}\,u_{ij} - \mathcal{G}_{ki}\,u_{kj} + \mathcal{T}_{jk}\,u_{0i} - \mathcal{T}_{ji}\,u_{0k} = 0, \\
&\frac{\partial v_{ij}}{\partial s_k} + (\mathcal{G}_{ki} - \mathcal{G}_{ji})\,v_{jk} - (\mathcal{G}_{ij} - \mathcal{G}_{kj})\,v_{ki} + \mathcal{G}_{ji}\,u_{jk} - \mathcal{G}_{ij}\,u_{ik} + \mathcal{T}_{ki}\,u_{0j} - \mathcal{T}_{kj}\,u_{0i} = 0,
\end{aligned}\right\} \tag{27}$$

which can also be given the remarkably simple form:

$$\left.\begin{aligned}
\left(\frac{\partial}{\partial s_k} + \mathcal{G}_{jk} - \mathcal{G}_{ik}\right)(u_{ij} - v_{ij}) &= \left(\frac{\partial}{\partial s_j} + \mathcal{G}_{kj} - \mathcal{G}_{ij}\right)(u_{ik} - v_{ik}), \\
\left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ki} - \mathcal{G}_{ji}\right)(u_{jk} - v_{jk}) &= \left(\frac{\partial}{\partial s_k} + \mathcal{G}_{ik} - \mathcal{G}_{jk}\right)(u_{ji} - v_{ji}), \\
\left(\frac{\partial}{\partial s_j} + \mathcal{G}_{ij} - \mathcal{G}_{kj}\right)(u_{ki} - v_{ki}) &= \left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji} - \mathcal{G}_{ki}\right)(u_{kj} - v_{kj}),
\end{aligned}\right\} \tag{28}$$

by virtue of (23). The latter equations are also noteworthy since, thanks to the identity:

$$\left(\frac{\partial}{\partial s_k} + \mathcal{G}_{jk}\right)\mathcal{G}_{ij} = \left(\frac{\partial}{\partial s_j} + \mathcal{G}_{kj}\right)\mathcal{G}_{ik},$$

they will be satisfied when one substitutes $\mathcal{G}_{ij}$ for any $u_{ij} - v_{ij}$. By some straightforward, but tedious, calculations, one can then verify that (27) are integrable, and on the other hand, if one takes advantage of the formula:

$$D\frac{\partial^2}{\partial s_i\,\partial s_j} = \frac{\partial^2}{\partial s_i\,\partial s_j}D - \sum_{k=1}^{n}\frac{\partial(u_{ki} - v_{ki})}{\partial s_j}\frac{\partial}{\partial s_k} - \sum_{k=1}^{n}\left[(u_{ki} - v_{ki})\frac{\partial^2}{\partial s_k\,\partial s_j} + (u_{kj} - v_{kj})\frac{\partial^2}{\partial s_i\,\partial s_k}\right],$$

which is an easy consequence of (25), one will see that the integrability conditions (16), and consequently, the universal Codazzi formula, will persist in deformed space when (27) are satisfied: *The choice of the v is therefore subordinate to only those conditions.* If one finds an arbitrary system of functions v that satisfy (27) then the formula (26) will lead rapidly to the knowledge of the alternating products of the deformations of the varied curvatures:

$$D\mathcal{N}_i = \frac{\partial u_{0i}}{\partial s_i} + \sum_{j=1}^{n} (\mathcal{G}_{ij}\, u_{0j} + \mathcal{T}_{ij}\, u_{ji}) + 2\sum_{j=1}^{n} \mathcal{T}_{ij}\, v_{ij}\,,$$

$$D\mathcal{T}_{ij} = -\frac{\partial u_{0i}}{\partial s_j} + \mathcal{G}_{ji}\, u_{0j} - \sum_{k=1}^{n} \mathcal{T}_{ki}\, u_{kj} - \sum_{k=1}^{n} (\mathcal{T}_{ik}\, v_{jk} + \mathcal{T}_{jk}\, v_{ik})\,,$$

$$D\mathcal{G}_{ij} = \mathcal{G}_{ji}\, u_{ji} - \mathcal{G}_{ij}\, u_{ii} - \mathcal{T}_{ij}\, u_{0i} - \mathcal{N}_i\, u_{0j} + \left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji}\right) v_{ij} - \sum_{k=1}^{n} \mathcal{G}_{ik}\, v_{jk}\,.$$

These formulas can be transformed in various ways, thanks to (22), and the following transformation of the last one is particular noteworthy:

$$D\mathcal{G}_{ij} = \frac{\partial u_{ii}}{\partial s_j} - \left(\frac{\partial}{\partial s_i} + \mathcal{G}_{ji}\right)(u_{ij} - v_{ij}) - \sum_{k=1}^{n} \mathcal{G}_{ik}\,(u_{jk} - v_{kj})\,. \tag{29}$$

15. Beez's theorem. – When space is assumed to be *inextensible*, the form Ω will then become identically zero, the functions u, like the v, will enjoy the property $u_{ji} = -u_{ij}$, and consequently, if $n > 2$ then one can take $v = u$, since when (28) is satisfied, the conditions (27) will necessarily reduce to (23). If one chooses the v in that way then one will see directly for the situation that was discussed in § **13** that the new axes are found in the positions that the (arbitrary) old ones occupy as a result of the deformation, and one can then substitute $\mathcal{D}$ for the sign D (cf., XIII, § **6**). Therefore, as in the deformations of inextensible surfaces, any orthogonal system of axes will remain orthogonal; however, for $n > 2$, that is nothing peculiar, since in reality space *is not deformed.* Indeed, from (29), one will have immediately that $\mathcal{D}\mathcal{G}_{ij} = 0$, which is to say that the geodetic curvature does not vary, and then, by virtue of the groups (γ) and (δ) of Codazzi formulas, one will see that if one observes (25) then the normal curvatures and geodetic torsions will also remain unaltered, since for any triple of values that are attributed to i, j, k, six functions, which are (generally) independent, will remain invariant, and they will depend upon the six curvatures $\mathcal{N}_i$, $\mathcal{N}_j$, …, $\mathcal{T}_{ij}$, $\mathcal{N}_i\,\mathcal{T}_{jk} + \mathcal{T}_{ij}\,\mathcal{T}_{ik}$, $\mathcal{N}_j\,\mathcal{N}_k - \mathcal{T}_{jk}^2$, and the other analogous ones. Therefore, *all* of the curvatures will remain invariant, and one will then discover an important fact that was pointed out by **Beez**, and then exhibited mostly by **Ricci**, i.e., *the impossibility of deforming an inextensible space* of more than two dimensions. Whereas an inextensible filament can be flexed until it is given an arbitrary form, one has already seen (XI, § **25**) that an inextensible surface cannot assume an arbitrary form when it is flexed, and Beez's theorem will then say that it is enough to make a space of dimension three or higher inextensible in order to determine its complete rigidity, that is to say, to impede any change of form, and everything will then happen as if the increasing number of dimensions tends to destroy the flexibility of the space. In addition, the preceding analysis reveals that the impossibility that was discovered by **Beez** is due, for the most part, to the demand (viz., $\mathcal{D}\mathcal{A}_{ik}^{(j)} = 0$) that space has to deform in such a way that the lower-dimensional spaces that constitute it do not cease to intersect in the manner that is prescribed by Dupin's theorem, as one that would lead to such a rigidity in the geometric structure of the space that it would not be possible to provoke a deformation without the spaces that constitute it extending or contracting. Exceptionally, that rigidity will cease when all of the third-order principal minors in the determinant K are zero when they include one or two given principal elements, so one or more normal curvatures and geodetic torsions can vary, and that restitution of flexibility is properly due to the greater liberty by which the space can satisfy Dupin's theorem, by reason of the partial or total indeterminacy in its system of curvatures.

16. – In order for no doubt to remain about the preceding proof of Beez's theorem, we would also like to show that *space can only move rigidly when the curvatures do not vary*. Assume that we have an orthogonal system of immobile axes: Let $\alpha_{i0}, \alpha_{i1}, \ldots, \alpha_{in}$ be the direction cosines of the i axis, and let $x_0, x_1, \ldots, x_n$ be the coordinates of the origin. We notice from § **7** that under the hypothesis of inextensibility, the u_{ij} will be precisely the components of the *rotation*, just as the u_i are the components of the *translation*. Let υ_{ij} and υ_i be the analogous quantities with respect to the moving axes, and in order to calculate them, observe that the variations of the coordinates:

$$\xi_i = -\sum \alpha_{ij}\, x_j$$

of M with respect to the immobile axes obviously have the values $\sum \alpha_{ij}\, u_j$, and on the other hand, we must be able to express them in terms of the υ and ξ in the following way:

$$\upsilon_i + \sum \upsilon_{ij}\, \xi_j = \sum \alpha_{ij}\, u_j\,. \tag{30}$$

It is enough to the switch the two systems of axes with each other in order to see that one also has:

$$u_i + \sum u_{ij}\, x_j = \sum \alpha_{ji}\, \upsilon_j\,,$$

and to deduce the first of the formulas:

$$\upsilon_i = \sum \alpha_{ij}\, u_j + \sum \alpha_{ij}\, u_{jk}\, x_k\,, \qquad \upsilon_{ij} = \sum \alpha_{ik}\, \alpha_{jl}\, u_{kl}\,. \tag{31}$$

As for the second one, one substitutes υ_i in (30) and compares the coefficients of ξ_j, after observing that:

$$x_i = -\sum \alpha_{ji}\,\xi_j\,.$$

Having said that, we differentiate the second formula in (31), while taking the immobility condition:

$$\frac{\partial \upsilon_{ij}}{\partial s_v} = \sum \alpha_{ik}\,\alpha_{jl}\,\frac{\partial u_{kl}}{\partial s_v} + \sum (A_{km}^{(v)}\,\alpha_{im}\,\alpha_{jl} + A_{lm}^{(v)}\,\alpha_{jm}\,\alpha_{ik})\,u_{kl}$$

into account. If one switches k with m in the first part of the second sum and l with m in the second part then one can also write:

$$\frac{\partial \upsilon_{ij}}{\partial s_v} = \sum_{k,l} \alpha_{ik}\,\alpha_{jl}\left[\frac{\partial u_{kl}}{\partial s_v} + \sum_{m=0}^{n} (A_{km}^{(v)}\,u_{lm} - A_{lm}^{(v)}\,u_{km})\right],$$

and since (20) implies that one can express the invariability of all the curvatures by:

$$\frac{\partial u_{kl}}{\partial s_v} = \sum_{m=0}^{n} (A_{lm}^{(v)}\,u_{km} - A_{km}^{(v)}\,u_{lm})\,,$$

one will see that:

$$\frac{\partial \upsilon_{ij}}{\partial s_1} = 0, \qquad \frac{\partial \upsilon_{ij}}{\partial s_2} = 0, \;\ldots,\; \frac{\partial \upsilon_{ij}}{\partial s_n} = 0;$$

i.e.: *all of the* υ_{ij} *are constants.* Similarly, if one keeps (11) in mind then one will find that:

$$\frac{\partial}{\partial s_v}\sum_j \alpha_{ij}\,u_j = \sum_j \alpha_{ij}\,u_{jv} + \sum_{j,k} A_{jk}^{(v)}(\alpha_{ik}\,u_j + \alpha_{ij}\,u_k)\,,$$

and it is clear that the last sum is zero, since the elements that correspond to the permutations jk and kj in the indices are equal and have opposite sign. Meanwhile, one has:

$$\frac{\partial}{\partial s_v}\sum_j \upsilon_{ij}\,\xi_j = \sum_j \alpha_{iv}\,\upsilon_{ij} + \sum_j \alpha_{ij}\,u_{jv}\,.$$

Therefore, the derivation of (30) will give:

$$\frac{\partial \upsilon_i}{\partial s_1} = 0, \qquad \frac{\partial \upsilon_i}{\partial s_2} = 0, \qquad \ldots, \qquad \frac{\partial \upsilon_i}{\partial s_n} = 0,$$

and one will also have that *the* υ_i *are all constants* then. It is therefore true that space can displace only rigidly, in such a way that all of its points will submit to the translation (υ_0, υ_1, …, υ_n) and the rotation that was defined by the constants υ_{ij}.

NOTE I

ON THE USE OF GRASSMANN NUMBERS

The use of *alternating numbers* confers a very precise and elegant form to the results of the intrinsic analysis of surfaces, and in particular, permits one to combine the three *Codazzi formulas* into just one. From known conditions (XI, § **3**), it is necessary and sufficient for the immobility of the point (x, y, z) that one can given them the form:

$$\left.\begin{aligned}\mathbf{i}\frac{dx}{ds} &= \mathbf{i}\mathcal{T}\cdot\mathbf{i}\,x+\mathbf{k}\mathcal{G}\cdot\mathbf{j}\,y+\mathbf{j}\mathcal{N}\cdot\mathbf{k}\,z-\mathbf{i},\\ \mathbf{j}\frac{dy}{ds} &= \mathbf{k}\mathcal{G}\cdot\mathbf{i}\,x+\mathbf{j}\mathcal{N}\cdot\mathbf{j}\,y+\mathbf{i}\mathcal{T}\cdot\mathbf{k}\,z,\\ \mathbf{k}\frac{dz}{ds} &= \mathbf{j}\mathcal{N}\cdot\mathbf{i}\,x+\mathbf{k}\mathcal{T}\cdot\mathbf{j}\,y+\mathbf{k}\mathcal{G}\cdot\mathbf{k}\,z.\end{aligned}\right\}\tag{1}$$

We agree than the *units* **i**, **j**, **k** should have zero squares, and that in addition, they should satisfy the conditions:

$$\mathbf{i} = \mathbf{jk} = -\,\mathbf{kj},\quad \mathbf{j} = \mathbf{ki} = -\,\mathbf{ik},\quad \mathbf{k} = \mathbf{ij} = -\,\mathbf{ji}\,.\tag{2}$$

Now, (1), when summed, can be combined into the single formula:

$$\frac{d\mathbf{\Omega}}{ds} = \boldsymbol{\omega}\,\mathbf{\Omega} - \mathbf{i},\tag{3}$$

in which only the vectors:

$$\mathbf{\Omega} = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z,\qquad \boldsymbol{\omega} = \mathbf{i}\,\mathcal{T} + \mathbf{j}\,\mathcal{N} + \mathbf{k}\,\mathcal{G}$$

appear. The derivative with respect to the arc length will then reduce to the simple vectorial operation that is represented by the symbol $\boldsymbol{\omega}$. It will then be important to know the effect of the operations $\boldsymbol{\omega}^2$, $\boldsymbol{\omega}^3$, … on the fundamental units.

We agree to observe, first of all, that from the conventions (2), the products of the three fundamental units is generally zero, except when the second or third factor along is equal to the first one, in which two cases, the product will reduce to the remaining factors, when taken with a changed sign or the original sign, respectively. In other words:

$$\mathbf{iij} = -\,\mathbf{j},\qquad \mathbf{iji} = \mathbf{j},\qquad \ldots\tag{4}$$

More generally, if one considers the vectorial operations:

$$\boldsymbol{\omega}_1 = \mathbf{i}\,a_1 + \mathbf{j}\,b_1 + \mathbf{k}\,c_1\,,\qquad \boldsymbol{\omega}_2 = \mathbf{i}\,a_2 + \mathbf{j}\,b_2 + \mathbf{k}\,c_2$$

then it will follow that the operation:

$$\begin{aligned}\boldsymbol{\omega}_1\,\boldsymbol{\omega}_2 = {} & \mathbf{ii}\,a_1\,a_2 + \mathbf{ij}\,a_1\,b_2 + \mathbf{ik}\,a_1\,c_2 \\ & + \mathbf{ji}\,b_1\,a_2 + \mathbf{jj}\,b_1\,b_2 + \mathbf{jk}\,b_1\,c_2 \\ & + \mathbf{ki}\,c_1\,a_2 + \mathbf{kj}\,c_1\,b_2 + \mathbf{kk}\,c_1\,c_2\,,\end{aligned}$$

when applied to the unit **i**, for example, will produce the result:

$$\mathbf{iji}\,a_1\,b_2 + \mathbf{iki}\,a_1\,c_2 + \mathbf{jji}\,b_1\,b_2 + \mathbf{kki}\,c_1\,c_2\,,$$

i.e.:

$$\boldsymbol{\omega}_1\,\boldsymbol{\omega}_2\,\mathbf{i} = -\,\mathbf{i}\,(a_1\,a_2 + b_1\,b_2 + c_1\,c_2) + \boldsymbol{\omega}_2\,a_1\,. \tag{5}$$

In particular, $\boldsymbol{\omega}^2 = 0$, and:

$$\boldsymbol{\omega}^2\,\mathbf{i} = -\,\mathbf{i}\,\kappa^2 + \boldsymbol{\omega}\,\mathcal{T}, \quad \boldsymbol{\omega}^2\,\mathbf{j} = -\,\mathbf{j}\,\kappa^2 + \boldsymbol{\omega}\,\mathcal{N}, \quad \boldsymbol{\omega}^2\,\mathbf{k} = -\,\mathbf{i}\,\kappa^2 + \boldsymbol{\omega}\,\mathcal{G}, \tag{7}$$

in which, κ represents the modulus of $\boldsymbol{\omega}$.

Now, it is easy to find the formulas by which the successive differential quotients of x, y, z are expressed linearly in terms of x, y, z. Indeed, when one omits the variations of the curvatures, (3) will give:

$$\frac{d^n\boldsymbol{\Omega}}{ds^n} = \boldsymbol{\omega}^n\,\boldsymbol{\Omega} - \boldsymbol{\omega}^{n-1}\,\mathbf{i},$$

and everything will reduce to the calculation of the results of the operations $\boldsymbol{\omega}^n$ on the fundamental units, and one will arrive at those results easily by means of the formulas (7) and the other obvious ones:

$$\boldsymbol{\omega}\,\mathbf{i} = \mathbf{j}\,\mathcal{G} - \mathbf{k}\,\mathcal{N}, \qquad \boldsymbol{\omega}\,\mathbf{j} = \mathbf{k}\,\mathcal{T} - \mathbf{i}\,\mathcal{G}, \qquad \boldsymbol{\omega}\,\mathbf{k} = \mathbf{i}\,\mathcal{N} - \mathbf{j}\,\mathcal{T}. \tag{8}$$

(That is why one observes that if the result of more vectorial operations is identical to any scalar then it will be zero.) One will obtain:

$$\boldsymbol{\omega}^{2n+1}\cdot\mathbf{i} = (-1)^n\,\boldsymbol{\omega}\,\mathbf{i}\,\kappa^{2n}, \qquad \boldsymbol{\omega}^{2n+2}\cdot\mathbf{i} = (-1)^n\,\boldsymbol{\omega}^2\,\mathbf{i}\,\kappa^{2n}.$$

In particular, if one wishes to have the formulas that exhibit the second derivatives then one will have:

$$\frac{d^2\boldsymbol{\Omega}}{ds^2} = \boldsymbol{\omega}^2\,\mathbf{i}\,x + \boldsymbol{\omega}^2\,\mathbf{j}\,y + \boldsymbol{\omega}^2\,\mathbf{k}\,z - \boldsymbol{\omega}^2\,\mathbf{i}\,;$$

i.e., by virtue of (7) and (8):

$$\frac{d^2\boldsymbol{\Omega}}{ds^2} = -\,\boldsymbol{\Omega}^2\,\kappa^2 + \mathbf{k}\,\mathcal{N} - \mathbf{j}\,\mathcal{G} + \boldsymbol{\omega}\,(\mathcal{T}x + \mathcal{N}y + \mathcal{G}z).$$

That equality obviously splits into:

$$\frac{d^2x}{ds^2} = -\kappa^2 x + \mathcal{T}(\mathcal{T}x + \mathcal{N}y + \mathcal{G}z),$$
$$\frac{d^2y}{ds^2} = -\kappa^2 y + \mathcal{N}(\mathcal{T}x + \mathcal{N}y + \mathcal{G}z),$$
$$\frac{d^2z}{ds^2} = -\kappa^2 z + \mathcal{G}(\mathcal{T}x + \mathcal{N}y + \mathcal{G}z).$$

One needs to add terms to the right-hand sides that are provided by the variation of the curvatures, i.e.:

$$z\frac{d\mathcal{N}}{ds} - y\frac{d\mathcal{G}}{ds}, \qquad x\frac{d\mathcal{G}}{ds} - z\frac{d\mathcal{T}}{ds}, \qquad y\frac{d\mathcal{T}}{ds} - x\frac{d\mathcal{N}}{ds}.$$

Another noteworthy consequence can be derived from formula (5) if one observes that in the first place:

$$(\boldsymbol{\omega}_1\ \boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\ \boldsymbol{\omega}_1)\ \mathbf{i} = \boldsymbol{\omega}_2\ a_1 - \boldsymbol{\omega}_1\ a_2\ . \tag{9}$$

Let $\mathbf{i}\ a + \mathbf{j}\ b + \mathbf{k}\ c$ be the vectorial operation that is equivalent to $\boldsymbol{\omega}_1\ \boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\ \boldsymbol{\omega}_1$, i.e., let:

$$(\mathbf{i}\ a + \mathbf{j}\ b + \mathbf{k}\ c)\ \boldsymbol{\Omega} = (\boldsymbol{\omega}_1\ \boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\ \boldsymbol{\omega}_1)\ \boldsymbol{\Omega}.$$

By virtue of (4), one will have:

$$\mathbf{i}\ (\boldsymbol{\omega}_1\ \boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\ \boldsymbol{\omega}_1)\ \mathbf{i} = \mathbf{i}\ (\mathbf{i}\ a + \mathbf{j}\ b + \mathbf{k}\ c)\ \mathbf{i} = \mathbf{j}\ b + \mathbf{k}\ c.$$

Hence, if one observes (9) then:

$$\begin{cases} \mathbf{j}b + \mathbf{k}\,c = \mathbf{i}\ \boldsymbol{\omega}_2 a_1 - \mathbf{i}\,\boldsymbol{\omega}_1 a_2, \\ \mathbf{k}\,c + \mathbf{i}\,a = \mathbf{j}\ \boldsymbol{\omega}_2 b_1 - \mathbf{j}\boldsymbol{\omega}_1 b_2, \\ \mathbf{i}\,a + \mathbf{j}b = \mathbf{k}\,\boldsymbol{\omega}_2 c_1 - \mathbf{k}\,\boldsymbol{\omega}_1 c_2; \end{cases}$$

hence, summing these equations will give:

$$\mathbf{i}\ a + \mathbf{j}\ b + \mathbf{k}\ c = \tfrac{1}{2}(\boldsymbol{\omega}_1\ \boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\ \boldsymbol{\omega}_1) = \boldsymbol{\omega}_1\ \boldsymbol{\omega}_2\ . \tag{10}$$

Therefore:

The operation $\boldsymbol{\omega}_1\ \boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\ \boldsymbol{\omega}_1$*, which is equivalent to* $2\ \boldsymbol{\omega}_1\ \boldsymbol{\omega}_2$ *when it is applied to scalar quantities, will, however, reduce to* $\boldsymbol{\omega}_1\ \boldsymbol{\omega}_2$ *when it is applied to a vector.*

Assuming that, consider another curve on the surface that is tangent to the origin of the y-axis and distinguish everything that refers to the first or second curve with the indices 1 or 2, respectively. Let q_1 and q_2 be parameters that define the two curves in a double orthogonal system that is traced on the surface and set:

$$\boldsymbol{\omega}_1 = \mathbf{i}\,\mathcal{T}_1 + \mathbf{j}\,\mathcal{N}_1 + \mathbf{k}\,\mathcal{G}_1\,, \qquad \boldsymbol{\omega}_2 = \mathbf{i}\,\mathcal{N}_2 - \mathbf{j}\,\mathcal{T}_2 + \mathbf{k}\,\mathcal{G}_2\,.$$

The conditions (3), when written for the curve in one or the other system, become:

$$\frac{\partial \boldsymbol{\Omega}}{\partial s_1} = \boldsymbol{\omega}_1\,\boldsymbol{\Omega} - \mathbf{i}\,, \qquad \frac{\partial \boldsymbol{\Omega}}{\partial s_2} = -\,\boldsymbol{\omega}_2\,\boldsymbol{\Omega} - \mathbf{j}\,, \tag{11}$$

and in order for $\boldsymbol{\Omega}$ to exist, it is enough that one should have:

$$\frac{\partial^2 \boldsymbol{\Omega}}{\partial s_1\,\partial s_2} + \frac{\partial \log Q_1}{\partial s_2}\frac{\partial \boldsymbol{\Omega}}{\partial s_1} = \frac{\partial^2 \boldsymbol{\Omega}}{\partial s_2\,\partial s_1} + \frac{\partial \log Q_2}{\partial s_1}\frac{\partial \boldsymbol{\Omega}}{\partial s_2}\,. \tag{12}$$

Meanwhile, one deduces from (11) that:

$$\frac{\partial^2 \boldsymbol{\Omega}}{\partial s_1\,\partial s_2} = \left(\frac{\partial \boldsymbol{\omega}_1}{\partial s_2} - \boldsymbol{\omega}_1\boldsymbol{\omega}_2\right)\boldsymbol{\Omega} + \mathbf{j}\,\boldsymbol{\omega}_1\,, \quad \frac{\partial^2 \boldsymbol{\Omega}}{\partial s_2\,\partial s_1} = -\left(\frac{\partial \boldsymbol{\omega}_2}{\partial s_1} + \boldsymbol{\omega}_2\boldsymbol{\omega}_1\right)\boldsymbol{\Omega} - \mathbf{j}\,\boldsymbol{\omega}_2\,,$$

and in particular, for $\Omega = 0$, (12) will become:

$$\mathbf{j}\,\boldsymbol{\omega}_1 + \mathbf{i}\,\boldsymbol{\omega}_2 = \mathbf{i}\,\frac{\partial \log Q_1}{\partial s_2} - \mathbf{j}\,\frac{\partial \log Q_2}{\partial s_1}\,.$$

The left-hand side has the value $\mathbf{i}\,\mathcal{G}_1 - \mathbf{j}\,\mathcal{G}_2 - \mathbf{k}(\mathcal{T}_1 + \mathcal{T}_2)$; hence:

$$\mathcal{G}_1 = \frac{\partial \log Q_1}{\partial s_2}\,, \qquad \mathcal{G}_2 = \frac{\partial \log Q_2}{\partial s_1}\,, \qquad \mathcal{T}_1 + \mathcal{T}_2 = 0,$$

and one can then set $\mathcal{T}_1 = -\,\mathcal{T}_2 = \mathcal{T}$. After that, the formula (12) will reduce immediately to:

$$\left(\frac{\partial \boldsymbol{\omega}_1}{\partial s_2} + \frac{\partial \boldsymbol{\omega}_2}{\partial s_1} + \boldsymbol{\omega}_1\mathcal{T}_1 + \boldsymbol{\omega}_2\mathcal{G}_2\right)\boldsymbol{\Omega} = (\boldsymbol{\omega}_1\,\boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\,\boldsymbol{\omega}_1)\,\boldsymbol{\Omega}\,.$$

Hence, if one takes the theorem (10) into account:

$$\frac{\partial \boldsymbol{\omega}_1}{\partial s_2} + \frac{\partial \boldsymbol{\omega}_2}{\partial s_1} + \boldsymbol{\omega}_1\mathcal{T}_1 + \boldsymbol{\omega}_2\mathcal{G}_2 = \boldsymbol{\omega}_1\,\boldsymbol{\omega}_2\,.$$

That is the equality that is contained in the three Codazzi formulas, to which one can arrive (cf., XI, § **9**) if one observes that by virtue of (6), the right-hand side will have the value:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathcal{T} & \mathcal{N}_1 & \mathcal{G}_1 \\ \mathcal{N}_2 & \mathcal{T} & \mathcal{G}_2 \end{vmatrix}.$$

We invite the reader to try performing an analogous calculation in hyperspace, while imagining a system of units (ij) with two indices that are endowed, first of all, with the property that that they will change sign when one transposes the indices. One then needs to suppose that one has $(ij)(kl) = 0$ when i, j, k, l are all different from each other, and that one has $(ij)(jk) = (ik)$, in such a way that one has $(ij)^2 = -(ji)(ij) = -(jj) = 0$, in particular, while the unit (ij) will not change when it is multiplied on the left by (ii) or on the right by (jj); etc. When the calculations are based upon these conventions, one will get a clear geometric image by supposing that after having enumerated the vertices of an $(n-1)$-tuple n-hedron from 1 to n one represents the operation that consists of traversing the edge that goes from the vertex i to the vertex j by (ij). The numbers that were previously adopted refer to the case of $n = 3$, in which the units are:

$$\mathbf{i} = (32), \qquad \mathbf{j} = (13), \qquad \mathbf{k} = (21).$$

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NOTE II

ON THE EQUILIBRIUM OF FLEXIBLE, INEXTENSIBLE FILAMENTS

Given a filament that is completely deformable in an n-dimensional linear space, take the axes to be the tangent, $(n - 1)$-normal, …, principal normal at a moving point of the filament. It is assumed to be infinitely thin, but in such a way that each element ds nevertheless has a certain *mass* $q\, ds$. Let X_i be the components along the i axis of the force per unit mass that acts upon $q\, ds$, and let u_i be the projection of the displacement onto that axis. The direction cosines of the element of the filament after deformation will obviously be proportional to $ds + \delta u_1$, δu_2, δu_3, …, δu_n , and therefore if one calls the tension per unit length T then one will have:

$$q\, X_i\, ds + \delta\left(T\,\frac{\delta u_i}{ds}\right) = 0$$

for the equilibrium of the external force, if one takes care to append $T\,\delta s$ and $T\,\delta s_i$ when $i = 1$. At the same time, it is important to note that the fundamental formulas (XVI, § **4**) that relate to the direction $(\alpha_1, \alpha_2, …, \alpha_n)$, when written in the form:

$$\frac{\delta\alpha_i}{ds} = \frac{d\alpha_i}{ds} + \frac{\alpha_{i-1}}{\rho_{n-i+2}} - \frac{\alpha_{i+1}}{\rho_{n-i+1}},$$

and if one agrees to set:

$$\alpha_{i+n} = -\,\alpha_i, \qquad \rho_{i+n} = \rho_i, \qquad \frac{1}{\rho_0} = 0,$$

can, as always, still persist when one considers the projections of an arbitrary variable segment onto the axes, instead of α. Indeed:

$$\delta\, p\alpha_i = \alpha_i\, dp + p\, d\alpha_i = d\, p\alpha_i + \left(\frac{p\alpha_{i-1}}{\rho_{n-i+2}} - \frac{p\alpha_{i+1}}{\rho_{n-i+1}}\right) ds\,.$$

One can then write:

$$\delta\left(T\,\frac{\delta u_i}{ds}\right) = d\left(T\,\frac{\delta u_i}{ds}\right) + \frac{T\,\delta u_{i-1}}{\rho_{n-i+2}} - \frac{T\,\delta u_{i+1}}{\rho_{n-i+1}},$$

and the equations of equilibrium will become, in general:

$$q\, X_i + \frac{d}{ds}\left(T\,\frac{\delta u_i}{ds}\right) + \frac{T}{\rho_{n-i+2}}\frac{\delta u_{i-1}}{ds} - \frac{T}{\rho_{n-i+1}}\frac{\delta u_{i+1}}{ds} = 0.$$

Finally, after totally eliminating the δ sign:

$$q\,X_i + \frac{d}{ds}\left[T\left(\frac{du_i}{ds} + \frac{u_{i-1}}{\rho_{n-i+2}} - \frac{u_{i+1}}{\rho_{n-i+1}}\right)\right] + \frac{T}{\rho_{n-i+2}}\frac{du_{i-1}}{ds} - \frac{T}{\rho_{n-i+1}}\frac{du_{i+1}}{ds}$$

$$+ \frac{T\,u_{i-2}}{\rho_{n-i+3}\,\rho_{n-i+2}} + \frac{T\,u_{i+2}}{\rho_{n-i+1}\,\rho_{n-i}} - Tu_i\left(\frac{1}{\rho_{n-i+2}^2} + \frac{1}{\rho_{n-i+1}^2}\right) = 0.$$

Hence, for $i = 1, 2, 3, \ldots, n$, if one takes into account all of the conventions that were made, one will arrive at the *intrinsic fundamental equations for the equilibrium of a filament* in an n-dimensional linear space:

$$q\,X_1 + \frac{d}{ds}\left[T\left(\frac{du_1}{ds} - \frac{u_n}{\rho_1} + 1\right)\right] - \frac{T}{\rho_1}\left(\frac{du_n}{ds} + \frac{u_1}{\rho_1} + \frac{u_{n-2}}{\rho_2}\right) = 0,$$

$$q\,X_2 + \frac{d}{ds}\left[T\left(\frac{du_2}{ds} - \frac{u_3}{\rho_{n-1}}\right)\right] - \frac{T}{\rho_{n-1}}\left(\frac{du_3}{ds} + \frac{u_2}{\rho_{n-1}} - \frac{u_4}{\rho_{n-2}}\right) = 0,$$

$$q\,X_3 + \frac{d}{ds}\left[T\left(\frac{du_3}{ds} + \frac{u_2}{\rho_{n-1}} - \frac{u_1}{\rho_{n-2}}\right)\right] + \frac{T}{\rho_{n-1}}\left(\frac{du_2}{ds} - \frac{u_3}{\rho_{n-1}}\right) - \frac{T}{\rho_{n-2}}\left(\frac{du_4}{ds} + \frac{u_3}{\rho_{n-2}} - \frac{u_5}{\rho_{n-3}}\right) = 0,$$

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$$q\,X_{n-1} + \frac{d}{ds}\left[T\left(\frac{du_{n-1}}{ds} + \frac{u_{n-2}}{\rho_3} - \frac{u_n}{\rho_2}\right)\right] - \frac{T}{\rho_2}\left(\frac{du_n}{ds} + \frac{u_1}{\rho_1} + \frac{u_{n-1}}{\rho_2}\right) + \frac{T}{\rho_3}\left(\frac{du_{n-2}}{ds} + \frac{u_{n-3}}{\rho_4} - \frac{u_{n-1}}{\rho_3}\right) = 0,$$

$$q\,X_n + \frac{d}{ds}\left[T\left(\frac{du_n}{ds} + \frac{u_1}{\rho_1} + \frac{u_{n-1}}{\rho_2}\right)\right] + \frac{T}{\rho_1}\left(\frac{du_1}{ds} - \frac{u_n}{\rho_1} + 1\right) + \frac{T}{\rho_2}\left(\frac{du_{n-1}}{ds} + \frac{u_{n-2}}{\rho_3} - \frac{u_n}{\rho_2}\right) = 0\,.$$

In particular, for $n = 3$, if one lets X, Y, Z, u, v, w denote the components of the accelerating force and displacement, resp., and lets ρ and r be the radii of flexion and torsion, resp., then one will get the equations:

$$\begin{cases} q\,X + \dfrac{d}{ds}\left[T\left(\dfrac{du}{ds} - \dfrac{w}{\rho} + 1\right)\right] - \dfrac{T}{\rho}\left(\dfrac{dw}{ds} + \dfrac{u}{\rho} + \dfrac{v}{r}\right) = 0, \\ qY + \dfrac{d}{ds}\left[T\left(\dfrac{dv}{ds} - \dfrac{w}{r}\right)\right] - \dfrac{T}{r}\left(\dfrac{dw}{ds} + \dfrac{u}{\rho} + \dfrac{v}{r}\right) = 0, \\ qZ + \dfrac{d}{ds}\left[T\left(\dfrac{dw}{ds} + \dfrac{u}{\rho} + \dfrac{v}{r}\right)\right] + \dfrac{T}{\rho}\left(\dfrac{du}{ds} - \dfrac{w}{\rho} + 1\right) + \dfrac{T}{r}\left(\dfrac{dv}{ds} - \dfrac{w}{r}\right) = 0, \end{cases}$$

which were pointed out by **Maggi**. If the filament is inextensible then it will be enough to observe that the variation of the element *ds* results from the relation:

$$(ds + \delta\, ds)^2 = (ds + \delta u_1)^2 + (\delta u_2)^2 + \ldots + (\delta u_n)^2,$$

from which it will follow that $\delta\, ds = \delta u_1$ in the case of infinitesimal displacements, in order for one to see that *inextensibility* is always expressed by the equality:

$$\frac{du_1}{ds} = \frac{u_n}{\rho_1}.$$

When one omits the displacements, the equations that were found will reduce to the simpler form:

$$q\, X_1 + \frac{dT}{ds} = 0, \qquad q\, X_n + \frac{T}{\rho_1} = 0, \qquad X_2 = X_3 = \ldots = X_{n-1} = 0,$$

and one will see that the filament is always arranged in such a way that the osculating plane contains the accelerating force at any point. The equilibrium curve will then be planar in the case of forces that emanate from a center. If the accelerating force X has an invariable direction then what was expressed in (II, § **1**) can be written:

$$\frac{d\varphi}{ds} = \frac{1}{\rho},$$

in which φ is the inclination of the tangent to the filament with respect to the direction of X. The first two equations of equilibrium, which are the only ones that we agree to take into account, become:

$$q\, X \cos\varphi + \frac{dT}{ds} = 0, \qquad q\, X \sin\varphi = \frac{T}{\rho},$$

and when one eliminates X and integrates, it will be easy to deduce that $T \sin\varphi$ keeps a *constant* value T_0 all along the filament, in such a way that one has:

$$T = \frac{T_0}{\sin\varphi}, \qquad X = \frac{T_0}{q\rho \sin^2\varphi}.$$

That presents two noteworthy special cases: If the filament is *homogeneous* (i.e., q is constant) then the last equation will give:

$$X = \frac{a}{\rho \sin^2\varphi}, \qquad \int X\, ds = -\, a \cot\varphi,$$

after one sets $T_0 = aq$. It will then follow that the intrinsic equation of the equilibrium curve will be:

$$\rho = \frac{1}{X}\left[a + \frac{1}{a}\left(\int X\,ds\right)^2\right].$$

Hence, if the filament is inhomogeneous (but one can still vary the density from one extreme to the other in such a way that one has *equal resistance* to the action of deformation everywhere) then one will need to set $T = aq$, with a constant, in which case, one can deduce from the second equation of equilibrium that:

$$X = \frac{a}{\rho \sin \varphi}, \qquad \int X\,ds = a \log \tan \frac{\varphi}{2};$$

hence:

$$\rho = \frac{a}{2X}\left(e^{\frac{1}{a}\int X\,ds} + e^{-\frac{1}{a}\int X\,ds}\right).$$

For example, when X is constant (and one can always suppose that $X = 1$ then), as one has for a ponderous filament that is fixed at two points and is in equilibrium under the action of gravity, the two preceding intrinsic equations that were obtained will become:

$$\rho = a + \frac{s^2}{a}, \qquad \rho = \frac{a}{2}\left(e^{s/a} + e^{-s/a}\right),$$

which represent the ordinary catenary and the catenary of equal resistance, resp. That explains the reason for the names that are given to those curves (I, § **5**, *b*, *c*).

One can treat other known questions of mechanics with equal speed and simplicity of means, and we encourage the reader to attempt to apply the method that was discussed to the study of the deformations of fibers or material lines that run through an elastic body and consider, in place of tension, the internal forces that act on each element of the fiber in all directions. The formulas that one obtains in that way can offer advantages in the treatment of special problems that are analogous to those of curvilinear coordinates.

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Additional note:

The theorem that was stated above (viz., the filament is always arranged in such a way that the osculating plane contains the accelerating force at any point) is another way of explaining (XI, § **8**) why a filament that is stretched on a surface will take the form of a geodetic. Indeed, the surface tends to oppose the tendency of the filament to rectify with a *normal* reaction F, which must also lie in the osculating plane of the equilibrium curve. It is then such that the osculating plane at each point will be normal to the surface, and therefore a geodetic. In addition, one will see that:

$$\rho\, q\, F = T = \text{constant};$$

i.e., the *reaction*, when computed per unit length, *is proportional to the curvature of the filament*, and that will also explain why the reaction is missing from the points of contact between the filament and the asymptotes of the surface.

NOTE III

ON THE EQUATIONS OF ELASTICITY IN HYPERSPACES

The calculations that **Beltrami** carried out in the paper "Sulle equazioni generali dell'elasticità" can also be done with a certain expediency, and without loss of elegance, for a curved space of as many dimensions as one desires by making use of the notations that we adopted in the first chapter. First, recall (XVII, §§ **6**, **7**) that if $u_0 = 1$ then the coefficients of elongation and unitary solid dilatation will be given by the formulas:

$$\theta_i = u_{ii} = \frac{\partial u_i}{\partial s_i} + \sum \mathcal{G}_{ij} u_j , \qquad \Theta = \sum \theta_{ii} = \sum \left(\frac{\partial}{\partial s_i} + \mathcal{G}_i \right) u_j .$$

In addition, one has to consider the mutual sliding θ_{ij} of the linear coordinate elements, and twice the components ϑ_{ij} of the rotation of the medium. Their expressions can be obtained from the formulas:

$$\tfrac{1}{2}(\theta_{ij} + \vartheta_{ij}) = u_{ij} = \frac{\partial u_i}{\partial s_j} - \mathcal{G}_{ji} u_j , \qquad \tfrac{1}{2}(\theta_{ij} - \vartheta_{ij}) = u_{ji} = \frac{\partial u_j}{\partial s_i} - \mathcal{G}_{ij} u_j , \tag{1}$$

which will reduce to just one [XVII, form. (15)], in substance, if one observes that:

$$\theta_{ij} = \theta_{ji} , \qquad \vartheta_{ij} = -\vartheta_{ji} .$$

Given that, when one assumes that:

$$-\tfrac{1}{2}\left(A\Theta^2 + B \sum \vartheta_{ij}^2 \right) \tag{2}$$

is the only effective part of the potential in the formation of the indefinite equations, one will arrive at the equations

$$X_i + A \frac{\partial \Theta}{\partial s_i} + B \sum \left(\frac{\partial}{\partial s_j} + \mathcal{G}_j - \mathcal{G}_{ij} \right) \vartheta_{ij} + 2B\, a_i = 0 \tag{3}$$

by the usual process, which are free of the last term on the left-hand side. That term is the one that one needs to calculate in order for (3) to be the general equations of elasticity for isotropic media in any curved space or hyperspace if one omits the variations of the isotropy constants. Meanwhile, if one follows the process that **Beltrami** used to find formula (4) in his paper then one will obtain the equations:

$$X_i = \left(\frac{\partial}{\partial s_i} + \mathcal{G}_i\right) T_i - \sum \mathcal{G}_{ij} T_j + \sum_{(i)} \left(\frac{\partial}{\partial s_j} + \mathcal{G}_i + \mathcal{G}_{ij}\right) T_{ij} \,, \tag{4}$$

in place of our (3), in which T_i and T_{ij} are the tensions in the (linear and surface, resp.) coordinate elements. The index i that is placed in the final summation sign serves to remind one that one needs to exclude terms with distinct values of i and j from that corresponding sum. Formulas (4) are independent of the geometric nature of the space, as well as the physical constitution of the medium. When that peculiarity is introduced with the isotropy hypothesis, one will have:

$$T_i = -(A - 2B)\,\Theta - 2B\,\theta_i, \qquad T_{ij} = -B\,\theta_{ij}\,,$$

and equations (4) will become:

$$X_i + A\frac{\partial \Theta}{\partial s_i} - 2B\frac{\partial}{\partial s_i}\sum_{(i)} \theta_j + 2B\sum \mathcal{G}_{ij}(\theta_j - \vartheta_j) + B\sum_{(i)}\left(\frac{\partial}{\partial s_j} + \mathcal{G}_j + \mathcal{G}_{ij}\right)\theta_{ij} = 0.$$

Now, a comparison with (3) will give immediately, upon observing (1):

$$a_i = -\frac{\partial}{\partial s_i}\sum_{(i)} \theta_j + \sum \mathcal{G}_{ij}(\theta_j - \vartheta_j) + \sum \mathcal{G}_{ij}\left(\frac{\partial u_i}{\partial s_j} - \mathcal{G}_{ij} u_j\right) + \sum_{(i)}\left(\frac{\partial}{\partial s_j} + \mathcal{G}_j\right)\left(\frac{\partial u_j}{\partial s_i} - \mathcal{G}_{ij} u_i\right). \tag{5}$$

Meanwhile:

$$\frac{\partial}{\partial s_i}\sum_{(i)} \theta_j = \sum_{(i)} \frac{\partial^2 u_j}{\partial s_j\,\partial s_i} + \frac{\partial}{\partial s_i}\sum (\mathcal{G}_j - \mathcal{G}_{ij})\,u_j\,.$$

On the other hand, by virtue of the integrability conditions [XVII, form. (16)], one also has:

$$\sum_{(i)} \frac{\partial^2 u_j}{\partial s_j\,\partial s_i} = \sum_{(i)}\left(\frac{\partial}{\partial s_j} + \mathcal{G}_j\right)\frac{\partial u_j}{\partial s_i} + \mathcal{G}_i\,\frac{\partial u_i}{\partial s_i} - \sum \mathcal{G}_{ji}\,\frac{\partial u_j}{\partial s_j} - \sum (\mathcal{G}_i - \mathcal{G}_{ij})\,\frac{\partial u_j}{\partial s_i}\,;$$

hence:

$$\frac{\partial}{\partial s_i}\sum_{(i)} \theta_j = \sum_{(i)}\left(\frac{\partial}{\partial s_j} + \mathcal{G}_j\right)\frac{\partial u_j}{\partial s_i} + \mathcal{G}_i\,\frac{\partial u_i}{\partial s_i} - \sum \mathcal{G}_{ji}\,\frac{\partial u_j}{\partial s_j} + \sum u_j\,\frac{\partial}{\partial s_i}(\mathcal{G}_j - \mathcal{G}_{ij})\,.$$

If one substitutes this in (5) then one will get:

$$a_i = \mathcal{G}_j\left(\theta_i - \frac{\partial u_j}{\partial s_j}\right) - \sum \mathcal{G}_{ji}\left(\theta_j - \frac{\partial u_j}{\partial s_j}\right) - u_i\sum\left(\frac{\partial \mathcal{G}_{ij}}{\partial s_j} + \mathcal{G}_j\mathcal{G}_{ij}\right)$$

$$-\sum\left[\frac{\partial}{\partial s_i}(\mathcal{G}_j-\mathcal{G}_{ij})+\mathcal{G}_{ij}\mathcal{G}_{ji}\right]u_j\,.$$

That proves that a_i is a linear form in the u:

$$a_i = \sum a_{ij}\,u_j\ .$$

If one collects the terms that are multiplied by u_j then one will get:

$$a_{ij} = (\mathcal{G}_j - \mathcal{G}_{ij})\ \mathcal{G}_{ij}\ -\frac{\partial}{\partial s_i}(\mathcal{G}_j - \mathcal{G}_{ij}) - \sum \mathcal{G}_{hi}\mathcal{G}_{hj} \tag{6}$$

for $i \neq j$. Moreover:

$$a_{ii} = -\ \frac{\partial \mathcal{G}_i}{\partial s_i} - \sum\left(\frac{\partial \mathcal{G}_i}{\partial s_j}+\mathcal{G}_j\mathcal{G}_{ij}+\mathcal{G}_{ij}^2\right). \tag{7}$$

Now, we can express the coefficients a by means of the functions Q. However, it is more convenient to introduce the *normal curvature* $\mathcal{N}$ and *geodetic torsion* $\mathcal{T}$, while bearing in mind the groups (γ) and (δ) of general Codazzi formulas (XVII, § **4**). Formula (7) can be written in the following way:

$$a_{ii} = -\ \sum\left(\frac{\partial \mathcal{G}_{ij}}{\partial s_j}+\frac{\partial \mathcal{G}_{ji}}{\partial s_i}+\mathcal{G}_{ij}^2+\mathcal{G}_{ji}^2\right) - \sum(\mathcal{G}_j - \mathcal{G}_{ij})\mathcal{G}_{ij}\,.$$

The second sum is equal to:

$$\sum_j\sum_{h,(i)}\mathcal{G}_{hj}\mathcal{G}_{ij}\ =\ \sum_{h,(i)}\sum_j\mathcal{G}_{hj}\mathcal{G}_{ij}\ =\ \sum_{j,(i)}\sum_h\mathcal{G}_{ih}\mathcal{G}_{jk}\ .$$

Hence:

$$a_{ii} = -\ \sum_{(i)}\left(\frac{\partial \mathcal{G}_{ij}}{\partial s_j}+\frac{\partial \mathcal{G}_{ji}}{\partial s_i}+\mathcal{G}_{ij}^2+\mathcal{G}_{ji}^2+\sum\mathcal{G}_{ih}\mathcal{G}_{jh}\right),$$

or, from (γ):

$$a_{ii} = \sum(\mathcal{N}_i\mathcal{N}_j - \mathcal{T}_{ij}^2)\,. \tag{8}$$

Similarly, one can give (6) the form:

$$a_{ij} = \mathcal{G}_{ij}\sum_{(j)}\mathcal{G}_{hi} - \frac{\partial}{\partial s_i}\sum_{(i)}\mathcal{G}_{hj} - \sum\mathcal{G}_{hi}\mathcal{G}_{hj}\ = -\sum_{(i,j)}\left[\frac{\partial \mathcal{G}_{hj}}{\partial s_i}+(\mathcal{G}_{hj}-\mathcal{G}_{ij})\,\mathcal{G}_{hj}\right],$$

i.e., by virtue of (δ):

$$a_{ij} = -\ \sum(\mathcal{N}_h\mathcal{T}_{ij}+\mathcal{T}_{ih}\mathcal{T}_{jh})\ . \tag{9}$$

This formula shows that $a_{ij} = a_{ji}$. One is then led to consider the quadratic form:

$$U = \tfrac{1}{2}\sum a_{ij}u_i u_j \,, \tag{10}$$

whose first partial derivatives are precisely the a_i. In order to understand the significance of U, note that one can also arrive at equations (3) by assuming that the effective part of the potential is the expression (2), augmented by $2BU$. That can be expressed by saying that the curvature of space produces a *loss of elastic energy*, as if one part of that energy were expended by the body to overcome the difficulty that it encountered by deforming in a *nonlinear* space. However, it can happen that $U < 0$, and then the elastic energy will be, by contrast, more intense than what one has in a linear space, as if the form of the space is such that it tends to facilitate the elastic deformation rather than oppose it. In other words, if one imagines the space to be non-rigid in its geometric essence, and on the other hand, one supposes that the matter is endowed with a type of *inertia*, by virtue of which it will always tend to deform *as if it were found in a linear space*, then one can say that the space reacts to that tendency with a force that admits the potential $2BU$.

For example, in the case of a two-dimensional space, one has $a_{11} = a_{22} = K$, $a_{12} = 0$. Hence, $U = \tfrac{1}{2}K(u_1^2 + u_2^2)$, and equations (3) will become:

$$X_1 + A\frac{\partial\Theta}{\partial s_1} - B\frac{\partial\vartheta}{\partial s_2} + 2BKu_1 = 0, \qquad X_2 + A\frac{\partial\Theta}{\partial s_2} + B\frac{\partial\vartheta}{\partial s_1} + 2BKu_2 = 0,$$

and will remain unchanged under deformations of the surface, which one assumes to be flexible, but inextensible. Hence, for a surface, the loss of elastic energy is proportional to the square of the displacement and to the curvature of the surface at the point that one considers. One will have an analogous state of affairs for an arbitrary space. Indeed, imagine that the space is referred to its *system of curvatures*. All of the torsions $\mathcal{T}$ will then be zero, and from (9), one will have that $a_{ij} = 0$, while from (8), one will see that a_{ii} is the sum of the total curvatures of all coordinate surfaces that contain the line q_i . Now, if one represents the projections of the displacements q_i q_j onto the surface by u_{ij} , and represents its total curvature by K_{ij} then the equality (10) will become:

$$U = \tfrac{1}{2}\sum K_{ij}\,u_{ij}^2 \,.$$

The loss of elastic energy in an n-dimensional curved space is then equal to the sum of the losses that are due to the $\tfrac{1}{2}n\,(n-1)$ surfaces of curvature.