Excerpted from E. Cesàro, *Lezioni di geometria intrinseca*, from the author-publisher, Naples, 1896. "Sull'uso dei numeri di Grassmann," pp. 250-253.

On the use of Grassmann numbers

The use of *alternating numbers* confers a very precise and elegant form to the results of the intrinsic analysis of surfaces, and in particular, permits one to combine the three *Codazzi formulas* into just one. From the known conditions (XI, 3), it is necessary and sufficient for the immobility of the point (x, y, z) that one can given them the form:

$$\left.\begin{aligned}\mathbf{i}\frac{dx}{ds} &= \mathbf{i}\mathcal{T}\cdot\mathbf{i}\,x + \mathbf{k}\mathcal{G}\cdot\mathbf{j}\,y + \mathbf{j}\mathcal{N}\cdot\mathbf{k}\,z - \mathbf{i},\\ \mathbf{j}\frac{dy}{ds} &= \mathbf{k}\mathcal{G}\cdot\mathbf{i}\,x + \mathbf{j}\mathcal{N}\cdot\mathbf{j}\,y + \mathbf{i}\mathcal{T}\cdot\mathbf{k}\,z,\\ \mathbf{k}\frac{dz}{ds} &= \mathbf{j}\mathcal{N}\cdot\mathbf{i}\,x + \mathbf{k}\mathcal{T}\cdot\mathbf{j}\,y + \mathbf{k}\mathcal{G}\cdot\mathbf{k}\,z.\end{aligned}\right\} \tag{1}$$

We agree than the *units* **i**, **j**, **k** should have zero squares, and that in addition, they should satisfy the conditions:

$$\mathbf{i} = \mathbf{jk} = -\,\mathbf{kj}, \quad \mathbf{j} = \mathbf{ki} = -\,\mathbf{ik}, \quad \mathbf{k} = \mathbf{ij} = -\,\mathbf{ji}\,. \tag{2}$$

Now, (1), when summed, can be combined into the single formula:

$$\frac{d\mathbf{\Omega}}{ds} = \boldsymbol{\omega}\,\mathbf{\Omega} - \mathbf{i}, \tag{3}$$

in which only the vectors:

$$\mathbf{\Omega} = \mathbf{i}\,x + \mathbf{j}\,y + \mathbf{k}\,z, \qquad \boldsymbol{\omega} = \mathbf{i}\,\mathcal{T} + \mathbf{j}\,\mathcal{N} + \mathbf{k}\,\mathcal{G}$$

appear. The derivative with respect to the arc length will then reduce to the simple vectorial operation that is represented by the symbol $\boldsymbol{\omega}$. If is then important to know the effect of the operations $\boldsymbol{\omega}^2$, $\boldsymbol{\omega}^3$, … on the fundamental units.

We agree to observe, first of all, that from the conventions (2), the products of the three fundamental units is generally zero, except when the second or third factor along is equal to the first one, in which two cases, the product will reduce to the remaining factors, when taken with a changed sign or the original sign, respectively. In other words:

$$\mathbf{iij} = -\,\mathbf{j}, \qquad \mathbf{iji} = \mathbf{j}, \qquad \ldots \tag{4}$$

More generally, if one considers the vectorial operations:

$$\boldsymbol{\omega}_1 = \mathbf{i}\,a_1 + \mathbf{j}\,b_1 + \mathbf{k}\,c_1\,, \qquad \boldsymbol{\omega}_2 = \mathbf{i}\,a_2 + \mathbf{j}\,b_2 + \mathbf{k}\,c_2\,,$$

it will follow that the operations:

$$\begin{aligned}\boldsymbol{\omega}_1\ \boldsymbol{\omega}_2 = &\ \mathbf{ii}\, a_1\, a_2 + \mathbf{ij}\, a_1\, b_2 + \mathbf{ik}\, a_1\, c_2 \\ &+ \mathbf{ji}\, b_1\, a_2 + \mathbf{jj}\, b_1\, b_2 + \mathbf{jk}\, b_1\, c_2 \\ &+ \mathbf{ki}\, c_1\, a_2 + \mathbf{kj}\, c_1\, b_2 + \mathbf{kk}\, c_1\, c_2\,,\end{aligned}$$

when applied to the unit **i**, for example, will produce the result:

$$\mathbf{iji}\, a_1\, b_2 + \mathbf{iki}\, a_1\, c_2 + \mathbf{jji}\, b_1\, b_2 + \mathbf{kki}\, c_1\, c_2\,,$$

i.e.:

$$\boldsymbol{\omega}_1\ \boldsymbol{\omega}_2\ \mathbf{i} = -\,\mathbf{i}\,(a_1\, a_2 + b_1\, b_2 + c_1\, c_2) + \boldsymbol{\omega}_2\, a_1\,. \tag{5}$$

In particular, $\boldsymbol{\omega}^2 = 0$, and:

$$\boldsymbol{\omega}^2\ \mathbf{i} = -\,\mathbf{i}\ \kappa^2 + \boldsymbol{\omega}\,\mathcal{T}, \qquad \boldsymbol{\omega}^2\ \mathbf{j} = -\,\mathbf{j}\ \kappa^2 + \boldsymbol{\omega}\,\mathcal{N}, \qquad \boldsymbol{\omega}^2\ \mathbf{k} = -\,\mathbf{i}\ \kappa^2 + \boldsymbol{\omega}\,\mathcal{G}, \tag{7}$$

in which, κ represents the modulus of $\boldsymbol{\omega}$.

Now, it is easy to find the formulas by which the successive differential quotients of x, y, z are expressed linearly in terms of x, y, z. Indeed, when one omits the variations of the curvatures, (3) will give:

$$\frac{d^n\boldsymbol{\Omega}}{ds^n} = \boldsymbol{\omega}^n\,\boldsymbol{\Omega} - \boldsymbol{\omega}^{n-1}\,\mathbf{i},$$

and everything will reduce to the calculation of the results of the operations $\boldsymbol{\omega}^n$ on the fundamental units, and one will arrive at those results easily by means of the formulas (7), and the other obvious ones:

$$\boldsymbol{\omega}\,\mathbf{i} = \mathbf{j}\,\mathcal{G} - \mathbf{k}\,\mathcal{N}, \qquad \boldsymbol{\omega}\,\mathbf{j} = \mathbf{k}\,\mathcal{T} - \mathbf{i}\,\mathcal{G}, \qquad \boldsymbol{\omega}\,\mathbf{k} = \mathbf{i}\,\mathcal{N} - \mathbf{j}\,\mathcal{T}. \tag{8}$$

(That is why one observes that if the result of more vectorial operations is identical to any scalar then it will be zero.) One will obtain:

$$\boldsymbol{\omega}^{2n+1}\cdot\mathbf{i} = (-1)^n\ \boldsymbol{\omega}\,\mathbf{i}\ \kappa^{2n}, \qquad \boldsymbol{\omega}^{2n+2}\cdot\mathbf{i} = (-1)^n\ \boldsymbol{\omega}^2\ \mathbf{i}\ \kappa^{2n}.$$

In particular, if one wishes to have the formulas that exhibit the second derivatives then one will have:

$$\frac{d^2\boldsymbol{\Omega}}{ds^2} = \boldsymbol{\omega}^2\ \mathbf{i}\, x + \boldsymbol{\omega}^2\ \mathbf{j}\, y + \boldsymbol{\omega}^2\ \mathbf{k}\, z - \boldsymbol{\omega}^2\ \mathbf{i}\,;$$

i.e., by virtue of (7) and (8):

$$\frac{d^2\boldsymbol{\Omega}}{ds^2} = -\,\boldsymbol{\Omega}^2\ \kappa^2 + \mathbf{k}\,\mathcal{N} - \mathbf{j}\,\mathcal{G} + \boldsymbol{\omega}\,(\mathcal{T}x + \mathcal{N}y + \mathcal{G}z).$$

That equality obviously splits into:

$$\frac{d^2x}{ds^2} = -\,\kappa^2\,x + \mathcal{T}(\mathcal{T}x + \mathcal{N}y + \mathcal{G}z),$$
$$\frac{d^2y}{ds^2} = -\,\kappa^2\,y + \mathcal{N}(\mathcal{T}x + \mathcal{N}y + \mathcal{G}z),$$
$$\frac{d^2z}{ds^2} = -\,\kappa^2\,z + \mathcal{G}(\mathcal{T}x + \mathcal{N}y + \mathcal{G}z).$$

One needs to add terms to the right-hand sides that are provided by the variation of the curvatures, i.e.:

$$z\frac{d\mathcal{N}}{ds} - y\frac{d\mathcal{G}}{ds}, \qquad x\frac{d\mathcal{G}}{ds} - z\frac{d\mathcal{T}}{ds}, \qquad y\frac{d\mathcal{T}}{ds} - x\frac{d\mathcal{N}}{ds}.$$

Another noteworthy consequence can be derived from formula (5) if one observes that in the first place:

$$(\boldsymbol{\omega}_1\,\boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\,\boldsymbol{\omega}_1)\,\mathbf{i} = \boldsymbol{\omega}_2\,a_1 - \boldsymbol{\omega}_1\,a_2\,. \tag{9}$$

Let $\mathbf{i}\,a + \mathbf{j}\,b + \mathbf{k}\,c$ be the vectorial operation that is equivalent to $\boldsymbol{\omega}_1\,\boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\,\boldsymbol{\omega}_1$, i.e., let:

$$(\mathbf{i}\,a + \mathbf{j}\,b + \mathbf{k}\,c)\,\boldsymbol{\Omega} = (\boldsymbol{\omega}_1\,\boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\,\boldsymbol{\omega}_1)\,\boldsymbol{\Omega}.$$

By virtue of (4), one will have:

$$\mathbf{i}\,(\boldsymbol{\omega}_1\,\boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\,\boldsymbol{\omega}_1)\,\mathbf{i} = \mathbf{i}\,(\mathbf{i}\,a + \mathbf{j}\,b + \mathbf{k}\,c)\,\mathbf{i} = \mathbf{j}\,b + \mathbf{k}\,c.$$

Hence, if one observes (9) then:

$$\begin{cases} \mathbf{j}\,b + \mathbf{k}\,c = \mathbf{i}\;\boldsymbol{\omega}_2 a_1 - \mathbf{i}\,\boldsymbol{\omega}_1 a_2, \\ \mathbf{k}\,c + \mathbf{i}\,a = \mathbf{j}\;\boldsymbol{\omega}_2 b_1 - \mathbf{j}\,\boldsymbol{\omega}_1 b_2, \\ \mathbf{i}\,a + \mathbf{j}\,b = \mathbf{k}\,\boldsymbol{\omega}_2 c_1 - \mathbf{k}\,\boldsymbol{\omega}_1 c_2; \end{cases}$$

hence, summing these equations will give:

$$\mathbf{i}\,a + \mathbf{j}\,b + \mathbf{k}\,c = \tfrac{1}{2}(\boldsymbol{\omega}_1\,\boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\,\boldsymbol{\omega}_1) = \boldsymbol{\omega}_1\,\boldsymbol{\omega}_2\,. \tag{10}$$

Therefore:

The operation $\boldsymbol{\omega}_1\,\boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\,\boldsymbol{\omega}_1$*, which is equivalent to* $2\,\boldsymbol{\omega}_1\,\boldsymbol{\omega}_2$ *when it is applied to scalar quantities, will, however, reduce to* $\boldsymbol{\omega}_1\,\boldsymbol{\omega}_2$ *when it is applied to a vector.*

Assuming that, consider another curve on the surface that is tangent to the origin of the *y*-axis and distinguish everything that refers to the first or second curve with the indices 1 or 2, respectively. Let q_1 and q_2 be parameters that define the two curves in a double orthogonal system that is traced on the surface and set:

$$\boldsymbol{\omega}_1 = \mathbf{i}\,\mathcal{T}_1 + \mathbf{j}\,\mathcal{N}_1 + \mathbf{k}\,\mathcal{G}_1\,, \qquad \boldsymbol{\omega}_2 = \mathbf{i}\,\mathcal{N}_2 - \mathbf{j}\,\mathcal{T}_2 + \mathbf{k}\,\mathcal{G}_2\,.$$

The conditions (3), when written for the curve in one or the other system, become:

$$\frac{\partial\boldsymbol{\Omega}}{\partial s_1} = \boldsymbol{\omega}_1\boldsymbol{\Omega} - \mathbf{i}\,, \qquad \frac{\partial\boldsymbol{\Omega}}{\partial s_2} = -\,\boldsymbol{\omega}_2\boldsymbol{\Omega} - \mathbf{j}\,, \tag{11}$$

and in order for $\boldsymbol{\Omega}$ to exist, it is enough that one should have:

$$\frac{\partial^2\boldsymbol{\Omega}}{\partial s_1\,\partial s_2} + \frac{\partial\log Q_1}{\partial s_2}\frac{\partial\boldsymbol{\Omega}}{\partial s_1} = \frac{\partial^2\boldsymbol{\Omega}}{\partial s_2\,\partial s_1} + \frac{\partial\log Q_2}{\partial s_1}\frac{\partial\boldsymbol{\Omega}}{\partial s_2}\,. \tag{12}$$

Meanwhile, one deduces from (11) that:

$$\frac{\partial^2\boldsymbol{\Omega}}{\partial s_1\,\partial s_2} = \left(\frac{\partial\boldsymbol{\omega}_1}{\partial s_2} - \boldsymbol{\omega}_1\boldsymbol{\omega}_2\right)\boldsymbol{\Omega} + \mathbf{j}\,\boldsymbol{\omega}_1\,, \quad \frac{\partial^2\boldsymbol{\Omega}}{\partial s_2\,\partial s_1} = -\left(\frac{\partial\boldsymbol{\omega}_2}{\partial s_1} + \boldsymbol{\omega}_2\boldsymbol{\omega}_1\right)\boldsymbol{\Omega} - \mathbf{j}\,\boldsymbol{\omega}_2\,,$$

and in particular, for $\Omega = 0$, (12) will become:

$$\mathbf{j}\,\boldsymbol{\omega}_1 + \mathbf{i}\,\boldsymbol{\omega}_2 = \mathbf{i}\,\frac{\partial\log Q_1}{\partial s_2} - \mathbf{j}\,\frac{\partial\log Q_2}{\partial s_1}\,.$$

The left-hand side has the value $\mathbf{i}\,\mathcal{G}_1 - \mathbf{j}\,\mathcal{G}_2 - \mathbf{k}(\mathcal{T}_1 + \mathcal{T}_2)$; hence:

$$\mathcal{G}_1 = \frac{\partial\log Q_1}{\partial s_2}\,, \qquad \mathcal{G}_2 = \frac{\partial\log Q_2}{\partial s_1}\,, \qquad \mathcal{T}_1 + \mathcal{T}_2 = 0,$$

and one can then set $\mathcal{T}_1 = -\,\mathcal{T}_2 = \mathcal{T}$. After that, the formula (12) will reduce immediately to:

$$\left(\frac{\partial\boldsymbol{\omega}_1}{\partial s_2} + \frac{\partial\boldsymbol{\omega}_2}{\partial s_1} + \boldsymbol{\omega}_1\mathcal{T}_1 + \boldsymbol{\omega}_2\mathcal{G}_2\right)\boldsymbol{\Omega} = (\boldsymbol{\omega}_1\,\boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\,\boldsymbol{\omega}_1)\,\boldsymbol{\Omega}\,.$$

Hence, if one takes the theorem (10) into account:

$$\frac{\partial\boldsymbol{\omega}_1}{\partial s_2} + \frac{\partial\boldsymbol{\omega}_2}{\partial s_1} + \boldsymbol{\omega}_1\mathcal{T}_1 + \boldsymbol{\omega}_2\mathcal{G}_2 = \boldsymbol{\omega}_1\,\boldsymbol{\omega}_2\,.$$

That is the equality that is contained in the three Codazzi formulas, to which one can arrive (cf., XI, 9) if one observes that by virtue of (6), the right-hand side will have the value:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathcal{T} & \mathcal{N}_1 & \mathcal{G}_1 \\ \mathcal{N}_2 & \mathcal{T} & \mathcal{G}_2 \end{vmatrix}.$$

We invite the reader to try performing an analogous calculation in hyperspace, while imagining a system of units (*ij*) with two indices that are endowed, first of all, with the property that that they will change sign when one transposes the indices. One then needs to suppose that one has (*ij*)(*kl*) = 0 when *i*, *j*, *k*, *l* are all different from each other, and that one has (*ij*)(*jk*) = (*ik*), in such a way that one has $(ij)^2 = -(ji)(ij) = -(jj) = 0$, in particular, while the unit (*ij*) will not change when it is multiplied on the left by (*ii*) or on the right by (*jj*); etc. When the calculations are based upon these conventions, one will get a clear geometric image by supposing that, after having enumerated the vertices of an (*n* – 1)-tuple *n*-hedron from 1 to *n*, one represents the operation that consists of traversing the edge that goes from the vertex *i* to the vertex *j* by (*ij*). The numbers that were previously adopted refer to the case of *n* = 3, in which the units are:

$$\mathbf{i} = (32), \qquad \mathbf{j} = (13), \qquad \mathbf{k} = (21).$$

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