"Sulle formole del Volterra, fondamentali nella teoria della distorsioni elastiche," Rend. Accad. delle sci. fis. Napoli **45** (1906), 311-321.

On VOLTERRA's formulas, which are fundamental in the theory of elastic distortions.

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While teaching my students about the recent research of prof. VOLTERRA on the distortions of elastic bodies, I was led to a simple proof of the formula that was at the basis (*) of that study, and now I believe that it would be useful to make that formula (slightly modified) known, either for the somewhat simpler and more symmetric aspect that I succeeded in giving to it or because it offered me a way of examining some results that were obtained by Volterra that persist independently of the Euclidian hypothesis.

Let u, v, w be the components of the displacement of an arbitrary point (x, y, z), and let:

$$a = \frac{\partial u}{\partial x}, \qquad b = \frac{\partial v}{\partial y}, \qquad c = \frac{\partial w}{\partial z}, \tag{1}$$

$$f = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \qquad g = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \qquad h = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{2}$$

be the components of a *regular* deformation. These six functions are therefore assumed to be finite, continuous, and monodromic, as well as all of their first and second derivatives; however, one does not necessarily assume this for u, v, w, and the components of the rotations:

$$p = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad q = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad r = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{3}$$

In order to express u at an arbitrary point M, one can always write:

$$u = u_0 + \int\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right),$$

(*) Rendiconti dei Lincei, 1905, pp. 129.

in which u_0 denotes the value of u at an arbitrary point M, and the integration is extended along an arc that goes from M_0 to M_1. Therefore:

$$u = u_0 + \int (a\,dx + h\,dy + g\,dz) + \int (q\,dz - r\,dy)\,.$$

In order to make only the components of the deformation appear in it, the second integral is then given the form:

$$\int [r\,d(y_1 - y) - r\,(z_1 - z)] = q_0\,(z_1 - z_0) - r_0\,(y_1 - y_0) + \int [(z_1 - z)\,dq - (y_1 - y)\,dr].$$

and one notes that one has:

$$\frac{\partial q}{\partial x} = \frac{\partial a}{\partial z} - \frac{\partial g}{\partial x}, \qquad \frac{\partial r}{\partial x} = \frac{\partial h}{\partial x} - \frac{\partial a}{\partial y}, \qquad \frac{\partial q}{\partial y} = \frac{\partial h}{\partial z} - \frac{\partial f}{\partial x}, \qquad \text{etc.} \tag{4}$$

It will then follow that:

$$u = u_0 + q_0\,(z_1 - z_0) - r_0\,(y_1 - y_0) + \int (\xi\,dx + \eta\,dy + \zeta\,dz), \tag{5}$$

in which:

$$\left\{\begin{aligned} \xi &= a + (y_1 - y)\left(\frac{\partial a}{\partial y} - \frac{\partial h}{\partial x}\right) + (z_1 - z)\left(\frac{\partial a}{\partial z} - \frac{\partial g}{\partial x}\right), \\ \eta &= h + (y_1 - y)\left(\frac{\partial h}{\partial y} - \frac{\partial b}{\partial x}\right) + (z_1 - z)\left(\frac{\partial h}{\partial z} - \frac{\partial f}{\partial x}\right), \\ \zeta &= g + (y_1 - y)\left(\frac{\partial g}{\partial y} - \frac{\partial f}{\partial x}\right) + (z_1 - z)\left(\frac{\partial g}{\partial z} - \frac{\partial c}{\partial x}\right). \end{aligned}\right. \tag{6}$$

Formula (5), and the analogous ones for v and w, are (with slight modifications) precisely Volterra's formulas, which permit one to calculate the displacements (u, v, w) from the deformations (a, b, c, f, g, h). In them, p_0, q_0, r_0 are considered to be arbitrary constants that must then be equal to the values of p, q, r at M_0. In fact, if one differentiates (5) with respect to y_1 and z_1 then one will find that:

$$\frac{\partial u}{\partial y_1} = h_1 - r_0 + \int\left[\left(\frac{\partial a}{\partial y} - \frac{\partial h}{\partial x}\right)dx + \left(\frac{\partial h}{\partial y} - \frac{\partial b}{\partial x}\right)dy + \left(\frac{\partial g}{\partial y} - \frac{\partial f}{\partial x}\right)dz\right],$$

$$\frac{\partial u}{\partial z_1} = g_1 + q_0 + \int\left[\left(\frac{\partial a}{\partial z} - \frac{\partial g}{\partial x}\right)dx + \left(\frac{\partial h}{\partial z} - \frac{\partial f}{\partial x}\right)dy + \left(\frac{\partial g}{\partial z} - \frac{\partial c}{\partial x}\right)dz\right],$$

from which, it results by circular permutation and suppressing the index 1, in turn, that:

$$\left\{\begin{aligned} p &= p_0 + \int\left[\left(\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}\right)dx+\left(\frac{\partial f}{\partial y}-\frac{\partial b}{\partial z}\right)dy+\left(\frac{\partial c}{\partial y}-\frac{\partial f}{\partial z}\right)dz\right],\\ q &= q_0 + \int\left[\left(\frac{\partial a}{\partial z}-\frac{\partial g}{\partial x}\right)dx+\left(\frac{\partial h}{\partial z}-\frac{\partial f}{\partial x}\right)dy+\left(\frac{\partial g}{\partial z}-\frac{\partial c}{\partial x}\right)dz\right],\\ r &= r_0 + \int\left[\left(\frac{\partial h}{\partial x}-\frac{\partial a}{\partial y}\right)dx+\left(\frac{\partial b}{\partial x}-\frac{\partial h}{\partial y}\right)dy+\left(\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}\right)dz\right], \end{aligned}\right. \tag{7}$$

which indeed leads one to expect (4). In order for the function u that is defined by (5) to be monodromic in an acyclic space, it is necessary and sufficient that one have:

$$\frac{\partial\eta}{\partial z}=\frac{\partial\zeta}{\partial y},\qquad \frac{\partial\zeta}{\partial x}=\frac{\partial\xi}{\partial z},\qquad \frac{\partial\xi}{\partial y}=\frac{\partial\eta}{\partial x}, \tag{8}$$

and from (6), it will result that these conditions, and the analogous ones for v and w, are just the noted conditions of *Saint-Venant*, which are necessary and sufficient for the existence of the u, v, w. When the space that the body occupies is cyclic instead (for simplicity, suppose that just one cut ς is sufficient to render it acyclic), the integral (5) will lead to different values u' and u'' of u on the faces of the cut. One thus has a discontinuity $u' - u'' = \bar{u}$ in u when one crosses ς, and the same thing will happen for the components of the rotation that were given by (7). Therefore, when one takes this into account, one can give (5) the form:

$$u + ry - qz = u_0 + r_0\, y_0 - q_0\, z_0 + \int\left\{\left[a+\left(\frac{\partial h}{\partial x}-\frac{\partial a}{\partial y}\right)y+\left(\frac{\partial g}{\partial x}-\frac{\partial a}{\partial z}\right)z\right]dx+\cdots\right\},$$

and one will see that one has $\bar{u} = \bar{l} + \bar{q}z - \bar{r}y$, etc., in which $\bar{l}$ denotes the value of the last integral when taken along an arbitrary closed line that crosses ς just once inside of the space considered. When one physically performs the cut ς, the last formula will define a *rigid* motion of one face of ς with respect to the other one that is equal and opposite to the distortion that is capable of constraining the body in the state of tension that determined by the sextuple a, b, c, f, g, h.

Now, if one takes stereographic coordinates x, y, z in a non-Euclidian space then the linear element will be represented by:

$$Q\sqrt{dx^2+dy^2+dz^2}, \quad \text{where} \quad 1/Q = 1 + \tfrac{1}{4}K(x^2+y^2+z^2),$$

and one lets u, v, w be the variations in x, y, z that are produced by the deformation, namely, one lets Qu, Qv, Qw be the components of the infinitesimal displacement of each point (x, y, z). The components f, g, h of the deformation are always given by (2), but (1) no longer applies to a, b, c. Rather, one has:

$$\frac{\partial u}{\partial x} = a + \varepsilon, \quad \frac{\partial u}{\partial y} = h - \gamma, \quad \frac{\partial u}{\partial z} = g + \beta,$$

where:

$$\varepsilon = \tfrac{1}{2} KQ\,(u\alpha + vy + wz),$$

and consequently:

$$u = u_0 + \int (\alpha\, dx + h\, dy + g\, dz) + \int (\varepsilon\, dx - \gamma dy + \beta\, dz). \tag{9}$$

Here, one should note that the components of the rotation are not given by (3) – namely, α, β, γ – but rather by:

$$p = \alpha + \tfrac{1}{2} KQ\,(vz - wy),$$
$$q = \beta + \tfrac{1}{2} KQ\,(wx - uz),$$
$$r = \gamma + \tfrac{1}{2} KQ\,(uy - vx),$$

and that in place of (4), one has:

$$\frac{\partial \beta}{\partial x} = \frac{\partial a}{\partial z} - \frac{\partial g}{\partial x} + \frac{\partial \varepsilon}{\partial z}, \qquad \frac{\partial \gamma}{\partial x} = \frac{\partial h}{\partial x} - \frac{\partial a}{\partial y} - \frac{\partial \varepsilon}{\partial y}, \quad \text{etc.}$$

Meanwhile:

$$\int (\varepsilon\, dx - \gamma dy + \beta\, dz) = \varepsilon_0\,(x_1 - x_0) - \gamma_0\,(y_1 - y_0) + \beta_0\,(z_1 - z_0) - \int [(x_1 - x)\, d\alpha - (y_1 - y)\, d\gamma + (z_1 - z)\, d\beta],$$

and the last integral splits into one part:

$$\begin{aligned}\int \Bigg\{ &\left[(y_1 - y)\left(\frac{\partial a}{\partial y} - \frac{\partial h}{\partial x}\right) + (z_1 - z)\left(\frac{\partial a}{\partial y} - \frac{\partial h}{\partial x}\right)\right] dx \\ &+ \left[(y_1 - y)\left(\frac{\partial h}{\partial y} - \frac{\partial b}{\partial x}\right) + (z_1 - z)\left(\frac{\partial h}{\partial z} - \frac{\partial f}{\partial x}\right)\right] dy \\ &+ \left[(y_1 - y)\left(\frac{\partial g}{\partial y} - \frac{\partial f}{\partial x}\right) + (z_1 - z)\left(\frac{\partial g}{\partial y} - \frac{\partial c}{\partial x}\right)\right] dz \Bigg\}\end{aligned} \tag{10}$$

and another one that depends upon just ε:

$$\begin{aligned}\int \Bigg\{ &\left[(x_1 - x)\frac{\partial \varepsilon}{\partial x} + (y_1 - y)\frac{\partial \varepsilon}{\partial y} + (z_1 - z)\frac{\partial \varepsilon}{\partial z}\right] dx \\ &+ \left[(x_1 - x)\frac{\partial \varepsilon}{\partial y} + (y_1 - y)\frac{\partial \varepsilon}{\partial x}\right] dy \\ &+ \left[(x_1 - x)\frac{\partial \varepsilon}{\partial z} + (z_1 - z)\frac{\partial \varepsilon}{\partial x}\right] dz \Bigg\}\end{aligned} \tag{11}$$

It remains for us to express the latter in terms of a, b, c, f, g, h. Now, one has:

$$\frac{\partial \varepsilon}{\partial x} = \tfrac{1}{2} KQ(u + gy - bz + ax + hy + gz), \qquad \text{etc.,}$$

and consequently the expression (11), in turn, will split into one part:

$$\begin{aligned}\tfrac{1}{2} K \int Q\{&[(x_1 - x)(ax + hy + gz) + (y_1 - y)(hx + by + fz) + (z_1 - z)(gx + fy + cz)]\,dx \\ &+ [(x_1 - x)(hx + by + fz) + (y_1 - y)(hx + by + fz) + (z_1 - z)(ax + by + gz)]\,dy \\ &+ [(x_1 - x)(gx + fy + cz) + (y_1 - y)(hx + by + fz) + (z_1 - z)(ax + hy + gz)]\,dz\end{aligned} \tag{12}$$

and another one:

$$\begin{aligned}-\tfrac{1}{2} K \int Q\{&[(x_1 - x)(u + \gamma y - \beta z) + (y_1 - y)(v + \alpha z - \gamma x) + (z_1 - z)(w + \beta x - \alpha y)]\,d(x_1 - x) \\ &+ [(x_1 - x)(v + \alpha y - \gamma z) - (y_1 - y)(u + \gamma y - \beta z)]\,d(y_1 - y) \\ &+ [(x_1 - x)(w + \beta y - \alpha z) - (z_1 - z)(u + \gamma y - \beta z)]\,d(z_1 - z),\end{aligned}$$

which integration by parts will transform into:

$$\begin{aligned}&-\tfrac{1}{4} KQ_0[(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2](u_0 + \gamma_0 y_0 - \beta_0 z_0) \\ &+ \tfrac{1}{2} KQ_0 (x_1 - x_0)\{(x_1 - x_0)(u_0 + \gamma_0 y_0 - \beta_0 z_0) \\ &\qquad + (y_1 - y_0)(v_0 + \alpha_0 z_0 - \gamma_0 x_0) \\ &\qquad + (z_1 - z_0)(w_0 + \beta_0 x_0 - \alpha_0 y_0)\} \\ &- \tfrac{1}{4} K \int [(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2]\,d[Q(u + \gamma y - \beta z)] \\ &+ \tfrac{1}{2} K \int (x_1 - x_0)\{(x_1 - x_0)\,d[Q(u + \gamma y - \beta z)] \\ &\qquad + (y_1 - y_0)\,d[Q(v_0 + \alpha_0 z_0 - \gamma_0 x_0)] \\ &\qquad + (z_1 - z_0)\,d[Q(w_0 + \beta_0 x_0 - \alpha_0 y_0)]\}.\end{aligned} \tag{13}$$

Meanwhile, an easy calculation gives:

$$\left\{\begin{aligned}
&\frac{1}{Q}\frac{\partial}{\partial x}[Q(u+\gamma y-\beta z)]\\
&=a+\left(\frac{\partial h}{\partial x}-\frac{\partial a}{\partial y}\right)y+\left(\frac{\partial g}{\partial x}-\frac{\partial a}{\partial z}\right)z+\tfrac{1}{2}KQx(ax+hy+gz)-KQ\omega,\\
&\frac{1}{Q}\frac{\partial}{\partial y}[Q(u+\gamma y-\beta z)]\\
&=h+\left(\frac{\partial b}{\partial x}-\frac{\partial h}{\partial y}\right)y+\left(\frac{\partial f}{\partial x}-\frac{\partial h}{\partial z}\right)z+\tfrac{1}{2}KQy(ax+hy+gz),\\
&\frac{1}{Q}\frac{\partial}{\partial z}[Q(u+\gamma y-\beta z)]\\
&=g+\left(\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}\right)y+\left(\frac{\partial c}{\partial x}-\frac{\partial g}{\partial z}\right)z+\tfrac{1}{2}KQz(ax+hy+gz),
\end{aligned}\right. \tag{14}$$

where

$$\omega=\tfrac{1}{2}(ax^2+by^2+cz^2+2fyz+2gzx+2hxy),$$

although the expression (13) can be considered to be known when one is given just the functions $a, b, \dots, h$. Now, (9) becomes:

$$\begin{gathered}
u=u_0+\beta_0(z_1-z_0)-\gamma_0(y_1-y_0)\\
-\tfrac{1}{4}KQ_0[(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2](u_0+\gamma_0 y_0-\beta_0 z_0)\\
+\tfrac{1}{2}KQ_0(x_1-x_0)[x_1(u_0+\gamma_0 y_0-\beta_0 z_0)+y_1(v_0+\alpha_0 z_0-\gamma_0 x_0)+z_1(w_0+\gamma_0 x_0-\alpha_0 y_0)]\\
+\int(\xi\,dx+\eta\,dy+\zeta\,dz),
\end{gathered} \tag{15}$$

where ξ, η, ζ are obtained by collecting the terms that are multiplied by dx, dy, dz, respectively, in the first integral (9), the integrals (10) and (12), and in the two integral (13).

The forms that the displacements assume in the case of a *rigid* motion result immediately from (15). In fact, it suffices to suppose that all of the components of the deformation are zero (which implies that ξ, η, ζ will also reduce to zero) in order to find that:

$$\left\{\begin{aligned}
u&=l+\mu z-\nu y+\tfrac{1}{2}Kx(lx+my+nz)-\tfrac{1}{4}Kl(x^2+y^2+z^2),\\
v&=m+\nu x-\lambda z+\tfrac{1}{2}Ky(lx+my+nz)-\tfrac{1}{4}Km(x^2+y^2+z^2),\\
w&=n+\lambda y-\mu x+\tfrac{1}{2}Kz(lx+my+nz)-\tfrac{1}{4}Kn(x^2+y^2+z^2),
\end{aligned}\right. \tag{16}$$

when one lets l, m, n, λ, μ, ν denote the values of the functions:

$$\begin{aligned}
l&=Q\,(u+\gamma y-\beta z), &\qquad \lambda&=\alpha+\tfrac{1}{2}K\,(ny-mz),\\
m&=Q\,(v+\alpha z-\gamma x), &\qquad \mu&=\beta+\tfrac{1}{2}K\,(lz-nx),\\
n&=Q\,(w+\beta x-\alpha y), &\qquad \nu&=\gamma+\tfrac{1}{2}K\,(mx-ly)
\end{aligned}$$

at M_0 . One notes that if $l, m, \ldots, \nu$ take on their expressions at the point M, instead, then (16) will be satisfied identically, because that would be equivalent to taking M_0 to M. The rigid motion can thus be characterized by saying that the functions $l, m, \ldots, \nu$ keep their values $l_0, m_0, \ldots, \nu_0$ throughout. One should also put (16) into the form:

$$Qu = l + Q\,(mz - ny) + \tfrac{1}{2}KQ\,[x(lx + my + nz) - l\,(x^2 + y^2 + z^2)], \qquad \text{etc.,} \tag{17}$$

in order to exhibit the duality that exists between the displacements and the rotations:

$$p = \lambda + KQ\,(mz - ny) + \tfrac{1}{2}KQ\,[x\,(\lambda x + \mu y + \nu z) - \lambda\,(x^2 + y^2 + z^2)], \quad \text{etc.,}$$

and it is easy to convince oneself that simultaneously annulling $a, b, \ldots, h$ will insure the rigidity of the motion in all of space; that is, the distance r between two arbitrary points M and M_0 will remain invariant. In fact, starting with the formula:

$$\sin^2\left(\tfrac{1}{2}\rho\sqrt{K}\right) = \tfrac{1}{4}\,KQQ_0\,[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2],$$

and observing that the deformation will produce the variation $-\varepsilon Q$ in Q, one easily finds that for rigidity, one must have that:

$$\begin{aligned}&(x - x_0)\,(u - u_0) + (y - y_0)\,(v - v_0) + (z - z_0)\,(w - w_0)\\ &\quad = \tfrac{1}{2}(\varepsilon + \varepsilon_0)\,[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2],\end{aligned}$$

and this relation, which makes the conditions $a = b = \ldots = h = 0$ necessary for an infinitesimal ρ, is verified for an arbitrary ρ by (16). Indeed, one arrives at this by integrating:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = \varepsilon, \qquad \frac{\partial w}{\partial y} = -\frac{\partial v}{\partial z} = \alpha, \qquad \text{etc.,}$$

and taking into account the relations:

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y} = \frac{\partial \gamma}{\partial z} = 0, \qquad \frac{\partial \gamma}{\partial y} = -\frac{\partial \beta}{\partial z} = \frac{\partial \varepsilon}{\partial x}, \qquad \text{etc.,}$$

from which, it is easy to deduce that all of the second derivatives of ε must be zero. Now, if one takes $\varepsilon = \tfrac{1}{2}K\,(lx + my + nz)$ then one will have:

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial z^2} = \tfrac{1}{2}Kl,$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \tfrac{1}{2}Km, \qquad \frac{\partial^2 u}{\partial x\,\partial z} = \tfrac{1}{2}Kn, \qquad \frac{\partial^2 u}{\partial y\,\partial z} = 0,$$

and if one develops $u - u_0$ in a Taylor series then (after some transformations) one will recover the first of (16).

In order to give (15) a more compact form, one agrees to introduce the functions l, m, n, and their values at M_0, and writes:

$$u = \frac{l_0}{Q_0} + \beta_0\, z_1 - \gamma_0\, y_1 - \tfrac{1}{4} K l_0\, [(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]$$
$$+ \tfrac{1}{2}\, K(x_1 - x_0)\,(l_0\, x_1 + m_0\, y_1 + n_0\, z_1) + \int (\xi\, dx + \eta\, dy + \zeta\, dz), \qquad \text{etc.} \qquad (18)$$

where

$$\begin{aligned}
\xi &= \frac{1}{Q}\frac{\partial l}{\partial x} + \left(\frac{\partial a}{\partial y} - \frac{\partial h}{\partial x}\right) y_1 + \left(\frac{\partial a}{\partial z} - \frac{\partial g}{\partial x}\right) z_1 \\
&\quad + \tfrac{1}{2} KQ\, [(x_1 - x)\, \omega_x + (y_1 - y)\, \omega_y + (z_1 - z)\, \omega_z] \\
&\quad - \tfrac{1}{4} K\, [(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]\, \frac{\partial l}{\partial x} \\
&\quad + \tfrac{1}{2} K\, (x_1 - x) \left[(x_1 - x)\frac{\partial l}{\partial x} + (y_1 - y)\frac{\partial m}{\partial x} + (z_1 - z)\frac{\partial n}{\partial x}\right],
\end{aligned}$$

$$\begin{aligned}
\eta &= \frac{1}{Q}\frac{\partial l}{\partial y} + \left(\frac{\partial h}{\partial y} - \frac{\partial b}{\partial x}\right) y_1 + \left(\frac{\partial h}{\partial z} - \frac{\partial f}{\partial x}\right) z_1 \\
&\quad + \tfrac{1}{2} KQ\, [(x_1 - x)\, \omega_y + (y_1 - y)\, \omega_x] \\
&\quad - \tfrac{1}{4} K\, [(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]\, \frac{\partial l}{\partial y} \\
&\quad + \tfrac{1}{2} K\, (x_1 - x) \left[(x_1 - x)\frac{\partial l}{\partial y} + (y_1 - y)\frac{\partial m}{\partial y} + (z_1 - z)\frac{\partial n}{\partial y}\right],
\end{aligned}$$

$$\begin{aligned}
\zeta &= \frac{1}{Q}\frac{\partial l}{\partial z} + \left(\frac{\partial g}{\partial y} - \frac{\partial f}{\partial x}\right) y_1 + \left(\frac{\partial g}{\partial z} - \frac{\partial c}{\partial x}\right) z_1 \\
&\quad + \tfrac{1}{2} KQ\, [(x_1 - x)\, \omega_z - (z_1 - z)\, \omega_x] \\
&\quad - \tfrac{1}{4} K\, [(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]\, \frac{\partial l}{\partial z} \\
&\quad + \tfrac{1}{2} K\, (x_1 - x) \left[(x_1 - x)\frac{\partial l}{\partial z} + (y_1 - y)\frac{\partial m}{\partial z} + (z_1 - z)\frac{\partial n}{\partial z}\right].
\end{aligned}$$

The symbols ω_x, ω_y, ω_z briefly represent the partial derivatives of ω, while leaving a, b, …, h constant. Assuming the existence of the functions l, m, n, it is enough to apply (8) to the preceding expressions for ξ, η, ζ in order to find the conditions:

$$K\left[f+\frac{1}{2}\left(y\frac{\partial b}{\partial z}+z\frac{\partial c}{\partial y}\right)+\frac{1}{2}x\left(\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}-\frac{\partial f}{\partial x}\right)\right]+\frac{1}{Q}\left[\frac{\partial^2 a}{\partial y\,\partial z}-\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}-\frac{\partial f}{\partial x}\right)\right]=0,$$

$$K\left[g+\frac{1}{2}\left(z\frac{\partial c}{\partial x}+x\frac{\partial a}{\partial z}\right)+\frac{1}{2}y\left(\frac{\partial h}{\partial z}+\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}\right)\right]+\frac{1}{Q}\left[\frac{\partial^2 b}{\partial z\,\partial x}-\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial z}+\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}\right)\right]=0,$$

$$K\left[h+\frac{1}{2}\left(x\frac{\partial a}{\partial y}+y\frac{\partial b}{\partial x}\right)+\frac{1}{2}z\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}\right)\right]+\frac{1}{Q}\left[\frac{\partial^2 c}{\partial x\,\partial y}-\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}\right)\right]=0$$

that are necessary for the monodromy of the functions that are defined by the integrals (18). In addition, for the existence of l, m, n – that is, for the integrability of (14) and the analogous relations that correspond to m and n – one must also satisfy the following conditions:

$$K\left[b+c+x\left(\frac{\partial g}{\partial z}+\frac{\partial h}{\partial y}-\frac{1}{2}\frac{\partial}{\partial z}(b+c)\right)+y\left(\frac{\partial f}{\partial z}+\frac{1}{2}\frac{\partial}{\partial y}(b-c)\right)+z\left(\frac{\partial f}{\partial y}-\frac{1}{2}\frac{\partial}{\partial z}(b-c)\right)\right]$$
$$+\frac{1}{Q}\left(\frac{\partial^2 b}{\partial z^2}+\frac{\partial^2 c}{\partial y^2}-2\frac{\partial^2 f}{\partial y\,\partial z}\right)=K^2\,Q\omega,$$

$$K\left[c+a+y\left(\frac{\partial h}{\partial x}+\frac{\partial f}{\partial z}-\frac{1}{2}\frac{\partial}{\partial y}(c+a)\right)+z\left(\frac{\partial g}{\partial x}+\frac{1}{2}\frac{\partial}{\partial z}(c-a)\right)+x\left(\frac{\partial g}{\partial z}-\frac{1}{2}\frac{\partial}{\partial x}(c-a)\right)\right]$$
$$+\frac{1}{Q}\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 a}{\partial z^2}-2\frac{\partial^2 g}{\partial z\,\partial x}\right)=K^2\,Q\omega,$$

$$K\left[a+b+z\left(\frac{\partial f}{\partial y}+\frac{\partial g}{\partial x}-\frac{1}{2}\frac{\partial}{\partial z}(a+b)\right)+x\left(\frac{\partial h}{\partial y}+\frac{1}{2}\frac{\partial}{\partial x}(a-b)\right)+y\left(\frac{\partial h}{\partial x}-\frac{1}{2}\frac{\partial}{\partial y}(a-b)\right)\right]$$
$$+\frac{1}{Q}\left(\frac{\partial^2 a}{\partial y^2}+\frac{\partial^2 b}{\partial x^2}-2\frac{\partial^2 h}{\partial x\,\partial y}\right)=K^2\,Q\omega.$$

These six conditions thus insure the existence of l, m, n, and then that of the displacements u, v, w, which are given by (18) in monodromic form in any acyclic part of space that one considers. The predicted conditions reduce to those of *Saint-Venant* for $K = 0$, and for arbitrary K they will be included in the general conditions that were given by *Padova*. If one assumes that they are satisfied in a cyclic region of space in which (18) continue to give the displacements then they will generally be expressed in a polydromic form. Proceeding as for $K = 0$, one can immediately see by means of (14) that the discontinuities of the displacements upon crossing the surface of a cut ς are given by the formulas:

$$Q\bar{u}=\bar{l}+Q(\bar{\beta}z-\bar{\gamma}y)\ \text{, etc.}\tag{19}$$

in which $\bar{l}$ denotes the integral:

$$\int Q\left\{\left[a+\left(\frac{\partial h}{\partial x}-\frac{\partial a}{\partial y}\right)y+\left(\frac{\partial h}{\partial x}-\frac{\partial a}{\partial y}\right)y-\tfrac{1}{2}KQ(y\omega_y+z\omega_y)\right]dx\right.$$
$$+\left[h+\left(\frac{\partial b}{\partial x}-\frac{\partial h}{\partial y}\right)y+\left(\frac{\partial f}{\partial x}-\frac{\partial h}{\partial z}\right)z-\tfrac{1}{2}KQy\omega_x\right]dy$$
$$\left.+\left[g+\left(\frac{\partial f}{\partial x}-\frac{\partial g}{\partial y}\right)y+\left(\frac{\partial c}{\partial x}-\frac{\partial g}{\partial z}\right)z-\tfrac{1}{2}KQz\omega_x\right]dz\right\},$$

which is extended over a closed line that can be continuously deformed inside the region considered without varying the integral. If $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ also remain invariant then for $K \neq 0$, (19) will define a deformation of ς, but it is easy to see that the quantity above will depend upon the position of the point at which the line of integration crosses ς, and it is important to specify the form of that dependency. It is easy to deduce α, β, γ from (18) by differentiating with respect to x_1, y_1, z_1. The value of α at M_1 is:

$$\alpha_1 = \alpha_0 + \tfrac{1}{2}K\,[m_0\,(z_1 - z_0) - n_0\,(y_1 - y_0)]$$
$$+\tfrac{1}{2}\int\left\{\left[\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}-\tfrac{1}{2}K(y_1-y)\frac{\partial n}{\partial x}+\tfrac{1}{2}K(z_1-z)\frac{\partial m}{\partial x}\right]dx\right.$$
$$+\left[\frac{\partial f}{\partial y}-\frac{\partial b}{\partial z}-\tfrac{1}{2}K(y_1-y)\frac{\partial n}{\partial y}+\tfrac{1}{2}K(z_1-z)\frac{\partial m}{\partial y}-\tfrac{1}{2}KQ\omega_z\right]dy$$
$$\left.+\left[\frac{\partial c}{\partial y}-\frac{\partial f}{\partial z}-\tfrac{1}{2}K(y_1-y)\frac{\partial n}{\partial z}+\tfrac{1}{2}K(z_1-z)\frac{\partial m}{\partial z}+\tfrac{1}{2}KQ\omega_y\right]dz\right\},$$

namely, if one extends the integration to ends that include the coordinates of M_1 then:

$$\alpha_1 = \alpha_0 + \tfrac{1}{2}K\,[(m_1\,z_1 - m_0\,z_0) - (n_1\,y_1 - n_0\,y_0)]$$
$$+\tfrac{1}{2}\int\left\{\left[\frac{\partial g}{\partial y}-\frac{\partial h}{\partial z}-\tfrac{1}{2}K\left(y\frac{\partial n}{\partial x}-z\frac{\partial m}{\partial x}\right)\right]dx\right.$$
$$+\left[\frac{\partial f}{\partial y}-\frac{\partial b}{\partial z}+\tfrac{1}{2}K\left(y\frac{\partial n}{\partial y}-z\frac{\partial m}{\partial y}\right)-\tfrac{1}{2}KQ\omega_z\right]dy$$
$$\left.+\left[\frac{\partial c}{\partial y}-\frac{\partial f}{\partial z}+\tfrac{1}{2}K\left(y\frac{\partial n}{\partial z}-z\frac{\partial m}{\partial z}\right)+\tfrac{1}{2}KQ\omega_y\right]dz\right\}.$$

Therefore, if one denotes the last part of this expression by $\bar{\lambda}$ (i.e., the one that has the form of an integral) then if one assumes that the integration is extended along one of the usual closed lines then one will get:

$$\bar{\alpha} = \tfrac{1}{2} K(\bar{m}z - \bar{n}y) + \bar{\lambda}, \quad \text{etc.},$$

at an arbitrary point (x, y, z) of ς, with $\bar{\lambda}$, $\bar{\mu}$, $\bar{\nu}$ constants. It is now sufficient to substitute these results into (19) in order to find that:

$$Q\bar{u} = \bar{l} + Q(\bar{\mu}z - \bar{\nu}\,y) + \tfrac{1}{2} KQ[x(\bar{l}x + \bar{m}y + \bar{n}z) - \bar{l}\,(x^2 + y^2 + z^2)], \quad \text{etc.},$$

and to convince oneself by comparing this with (17) that the elastic distortions will imply only *rigid* motions even in non-Euclidian spaces. Granted, this result can be considered to be an immediate consequence of the hypothesis that the deformation is *regular*, which is, in fact, expressed by setting $\bar{a}$, $\bar{b}$, …, $\bar{h}$ simultaneously equal to 0, but it is still useful to see how the calculations that are developed by starting with the fundamental formulas lead to the same result, as well as to know how the constants of the distortion depend upon the components of the given deformation.