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On the Bäcklund transformations

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INTRODUCTION

The celebrated Laplace method for the integration of second-order, linear, partial differential equations is founded upon the properties of a very simple transformation that replaces each integral of the given equation with an integral of another second-order, linear equation, and conversely. Upon studying the linear equations, and in particular, upon seeking to obtain equations of that type for which it is possible to find a general integral, geometers were led to consider transformations that differ from the Laplace transformations in some of their properties, but which nonetheless enjoy the characteristic property of that transformation: If one is given an arbitrary surface then there exists a transformed surface, while the transformation makes each integral of a certain second-order, linear equation correspond to either one integral or an infinitude of integrals of an analogous equation. The theory of linear equations was presented by Darboux ([1]), to whom a large number of the important results are due. I will simply cite the names Moutard, Lucien Lévy, and Roger Liouville, who have studied certain modes of transformation for linear equations.

The transformations of nonlinear, second-order, partial differential equations have not been the object of as great a number of works, although some very general results have been proved by Bäcklund in his papers ([2]), which are, unfortunately, a bit difficult to read. Upon seeking to analytically study the transformation of surfaces with constant total curvature that were discovered by Bianchi, Lie remarked ([3]) that this transformation does not apply to an arbitrary surface, but only to surfaces of constant total curvature. Bäcklund showed that in certain cases of systems of partial differential equations that are analogous to the system that was the object of Lie's work, one is allowed to deduce from any integral of a Monge-Ampére equation, an infinitude of integrals of an equation of the same type by the integration of ordinary differential equations. At the same time, Bäcklund defined a transformation of surfaces of constant total curvature that includes

([1]) *Leçons sur la théorie générale des surfaces*, t. II.

([2]) BAECKLUND, "Zur Theorie der partiellen Differentialgleichungen erster Ordnung" Mathematische Annalen **XVII** (1880), 285, and "Zur Theorie der Flächentransformationen, *ibid*., **XIX** (1882), 387.

([3]) *Archiv for Mathematik og Naturvidenskab,* **V** (1880), 282.

the Bianchi transformation as a special case. A little later, Darboux ([1]) generalized the last result and indicated a very elegant method for proving the theorems that were obtained by Bäcklund.

Some particular transformations of second-order partial differential equations were studied by Gomes Teixeira ([2]). More recently, Goursat ([3]) showed that in an extended case of the surfaces that the transformations that were considered by Bäcklund were applied to – i.e., the transformations that are defined, in ordinary notation, by a system of four equations in $x, y, z, p, q, x', y', z', p', q'$ – are the integrals of a second-order equation.

The present paper, which is divided into three parts, has for its object of study the transformations that are defined as we just explained in the case where each of two families of surfaces that correspond to them are composed of integrals of a second-order equation. These transformations are called *Bäcklund transformations.*

In the first part, I point out the general properties of these transformations. In particular, after having briefly reviewed the results that were obtained by Bäcklund, I examine a case that has not been further considered where four equations between $x, y, z, p, q, x', y', z', p', q'$ define a transformation that is applied to the integrals of a second-order equation.

In the second part, I study the Bäcklund transformations that make the integrals of the two equations correspond in a one-to-one manner. It is easy to obtain some very general and very precise propositions.

When the transformation makes an integral of at least one of the two equations correspond to an infinitude of integrals of the other equation, the theory seems very complicated. Although they are very incomplete, the results that I prove in the third part are possibly capable of providing some useful indications of a more profound study.

The principal results that are contained in the first two parts were presented to the Académie des Sciences (sessions on 5 February and 9 April of 1900 and 11 February 1901).

([1]) *Leçons sur la théorie générale des surfaces*, t. III, pp. 438.

([2]) *Comptes rendus*, **XCIII** (1881), 702. – *Bulletin de l'Académie de Belgique* (3) **III** (1882), 486.

([3]) GOURSAT, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre* **II**, pp. 284.

PART ONE

1. If we are given two systems of first-order elements in space then, as is customary, we let x, y, z, p, q denote the coordinates of an arbitrary element of the first system, which we call the system (E); we employ the same letters with primes for the second system. We say that a surface whose elements all belong to one of the two systems belongs to that system. We have to consider contact transformations that either the elements of (E) or those of (E') are subjected to. Often, to abbreviate, we shall call the former transformations (T) and the latter ones (T').

One knows that if the point x, y, z describes a surface $z = f(x, y)$ then the coordinates of the elements of that surface have the values:

$$p = \frac{\partial f}{\partial x}, \qquad q = \frac{\partial f}{\partial y}$$

so we further write:

$$r = \frac{\partial^2 f}{\partial x^2}, \qquad s = \frac{\partial^2 f}{\partial x \, \partial y}, \qquad t = \frac{\partial^2 f}{\partial y^2},$$

and for the derivatives of higher order, we write:

$$p_{i,k} = \frac{\partial^{i+k} f}{\partial x^i \, \partial y^k} \qquad (i + k > 2).$$

Let V be a function of x, y, z, and the derivatives of z up to order n, and let $\frac{d^{i+k} f}{dx^i \, dy^k}$ be the derivative taken i times with respect to x and k times with respect to y. If one subtracts the terms that contain the derivatives of z of order $n + i + k$ then what remains is an expression that denote by:

$$\left(\frac{d^{i+k} f}{dx^i \, dy^k}\right).$$

The problem that Bäcklund proposed can be stated in the following manner:

Being given four equations:

$$F_i(x, y, z, p, q; x', y', z', p', q') = 0 \qquad (i = 1, 2, 3, 4) \tag{1}$$

between the coordinates of the elements of the two systems (E), (E'), *determine the surfaces of the systems* (E) *that correspond to the surfaces of* (E').

We shall show that equations (1) always define a transformation that makes certain surfaces of (E) correspond to some surfaces of (E'). We do not consider two transformations that are defined by two systems of equations to be distinct when they can be converted into each other by the transformations (T) and (T'), and we likewise regard two partial differential equations as identical when one can deduce one from the other by a contact transformation.

At first, we shall not occupy ourselves with the case in which one can deduce an equation that depends upon only x, y, z, p, q from system (1). In addition, we suppose that equations (1) are solved with respect to the variables x', y', p', q'. If that solution is impossible then it will suffice to first perform a change of variables or a conveniently chosen transformation (T) – for example, the Ampère transformation. The system (1) then becomes:

$$\left\{\begin{array}{ll} x' = f_1(x, y, z, p, q; z'), & y' = f_2(x, y, z, p, q; z'), \\ p' = f_3(x, y, z, p, q; z'), & q' = f_4(x, y, z, p, q; z'). \end{array}\right. \tag{2}$$

The elements of (E'), which generate a surface, satisfy the equation:

$$dz' - p'\,dx' - q'\,dy' = 0,$$

and if one replaces x', y', p', q' with their values then this becomes:

$$A\,dz' + B\,dx + C\,dy + \alpha\,dp + \beta\,dq = 0, \tag{3}$$

or further:

$$A\,dz' + (B + \alpha\,r + \beta\,s)\,dx + (C + \alpha\,s + \beta\,t)\,dy = 0 \tag{3$'$}$$

on the condition that one must set:

$$A = f_3\frac{\partial f_1}{\partial z'} + f_4\frac{\partial f_2}{\partial z'} - 1, \qquad B = f_3\left(\frac{\partial f_1}{\partial x}\right) + f_4\left(\frac{\partial f_2}{\partial x}\right), \qquad C = f_3\left(\frac{\partial f_1}{\partial y}\right) + f_4\left(\frac{\partial f_2}{\partial y}\right),$$

$$\alpha = f_3\frac{\partial f_1}{\partial p} + f_4\frac{\partial f_2}{\partial p}, \quad \beta = f_3\frac{\partial f_1}{\partial q} + f_4\frac{\partial f_2}{\partial q}.$$

The integrability condition for (3′) can be written:

$$H\,r + 2K\,s + L\,t + M + N(rt - s^2) = 0, \tag{4}$$

where H, K, L, M, N are functions of x, y, z, p, q, z' that are easy to calculate.

Two cases can then present themselves: If equation (4) depends upon z' then one solves for z' as a function of x, y, z, p, q, r, s, t, and after having inserted it into (3)′, what remains are two third-order, partial differential equations that determine z; these equations are compatible. In general, there exist ∞^1 integrals that pass through a given curve and admit a given tangent plane at each point of that curve ([1]). Only one surface in (E') corresponds an integral of these equations.

It might likewise happen that equation (4) does not depend upon z'; it is then a *Monge-Ampère equation*. If one replaces z with an integral of (4) then equation (3)′ is completely integrable and determines an infinitude of surfaces in (E') that depend upon an arbitrary parameter. One immediately sees that the same thing will be true, in particular, if z' does not figure in equations (2). If these transformations admit an infinitesimal transformation (T) then one can, with the aid of a transformation (T'), replace the system (2) with a system that is analogous to four equations that no longer contain z', and the integrability condition (4) again reduces to a Monge-Ampère equation.

2. These results are due to Bäcklund, who proved them in the papers that were cited above. I will add the following remark: Here, I will consider only the case where one is led to a Monge-Ampère equation (4). It is always possible, by a contact transformation, to make the term in $rt - s^2$ disappear from that equation, which then admits a family of integrals that are each composed of a point and all of the planes that pass through it.

From this, the equation:

$$A\,dz' + \alpha\,dp + \beta\,dq = 0,$$

in which one regards x, y, z as constants, is completely integrable. We thus have, upon letting ρ and θ denote two functions of x, y, z, p, q, z':

$$A\,dz' + \alpha\,dp + \beta\,dq = \rho\,d\theta,$$

on the condition that one differentiate θ with respect to only the variables p, q, z'.

If one now takes the ordinary differential then one can write:

$$A\,dz' + \alpha\,dp + \beta\,dq = \rho\left[d\theta - \left(\frac{d\theta}{dx}\right)dx - \left(\frac{d\theta}{dy}\right)dy\right].$$

Substitute this value for $A\,dz' + \alpha\,dp + \beta\,dq$ in (3) and, at the same time, replace z' with its expression as a function of x, y, z, p, q, and a new variable that is defined by:

$$\zeta = \theta(x, y, z, p, q, z'),$$

and equation (3) becomes:

(3)″ $$dz - \lambda\,dx - \mu\,dy = 0,$$

in which λ and μ denote functions of x, y, z, p, q, z, and that equation will be completely integrable at the same time as (3)′. A linear equation in r, s, t is not, in general, identical to the condition of integrability for an expression such as (3)″. For example, if one supposes that the equation considered contains neither s not t then it is necessary that the equations must be bilinear in r and q. Consequently, a Monge-Ampère equation is not generally derived from a transformation (2) by the method that was indicated in the

([1]) BAECKLUND, Math. Ann. **XVII**, pp. 291 and **XIX**, pp. 389.

preceding paragraph. It seems difficult to determine all of the equations that enjoy this property, but it suffices to suppose that λ and μ do not depend upon ζ in order to have an infinitude of them.

Upon writing that H, K, L, M, N are proportional to the coefficients of a given Monge-Ampère equation, one obtains four partial differential equations to determine four functions f_1, f_2, f_3, f_4, but this system of equations is a singular system to which the theorems of Cauchy do not apply, and which does not, in general, admit a solution.

3. If (4) is a Monge-Ampère equation then as soon as one knows an integral of that equation z' will be determined by an integration. x', y', p', q' are given by equations (2) as functions of x, y, z, p, q, z'. In a general manner, an arbitrary derivative $p'_{i,k}$ of z' is expressed as a function of z', x, y, z, and the derivatives of z up to order $i + k$. One sees this by proceeding step-by-step. For example, in order to obtain s' and t', one differentiates $q' = f_4(x, y, z, p, q; z')$, upon replacing dx', dy', dz' with their values that one infers from (2) and (3)′, and one then annuls the coefficients of dx and dy. The determinant of the system of linear equations that one thus obtains in order to calculate two consecutive derivatives $p'_{i,k}$, $p'_{i-1,k+1}$ is always $\dfrac{D(f_1, f_2)}{D(x, y)}$. It is zero only if the multiplicities of (E') that correspond to multiplicities of (E) satisfy the equation:

$$y' = \text{arbitrary function of } x'.$$

It suffices to perform a transformation (T') in order for this situation to not be present. Moreover, one easily comprehends why this case is distinguished from the others. If one is given an integral form which y' is a function of x' then z' is no longer a function of two independent variables, and there is no reason to consider the partial derivatives of that quantity.

4. The arguments of Bäcklund suppose essentially that the coefficient A in equation (3) is non-zero. If this coefficient is identically zero then things will happen differently. The geometric interpretation of the condition:

$$S = f_3 \frac{\partial f_1}{\partial z'} + f_4 \frac{\partial f_2}{\partial z'} - 1 = 0$$

is simple. That equality expresses the idea that an element x, y, z, p, q corresponds to the union of ∞^1 elements of (E').

The equation:

$$dz' - p'\,dx' - q'\,dy' = 0,$$

which is satisfied by the elements of (E') that generate a surface, reduces here, after x', y', p', q' have been replaced by their values, to:

$$(B + \alpha r + \beta s)\,dx + (C + \alpha s + \beta t)\,dy = 0,$$

and can be verified only if one has:

(5) $$\begin{cases} B + \alpha r + \beta s = 0, \\ C + \alpha s + \beta t = 0 \end{cases}$$

at that time

Upon eliminating z' from these equations, one is led to one second-order, partial differential equation:

(6) $$F(x, y, z, p, q, r, s, t) = 0,$$

which defines the surfaces of (E) that the surfaces of (E') correspond to. For the moment, regard x, y, z, p, q as arbitrary parameters and r, s, t as Cartesian coordinates in a three-dimensional space. The two equations (5) represent a line that belongs to the complex (G) of lines that are parallel to the generators of the cone $rt - s^2 = 0$. Equation (6) represents a ruled surface whose generators belong to (G), so that equation possesses a system of first-order characteristics. Each integral of (6) corresponds to only one surface of (E'), so equations (2) and (5) give x', y', z', p', q'.

Conversely, if one is given a second-order, partial differential equation (6) that possesses a system of first-order characteristics then there exists an infinitude of transformations (2) such that the proposed equation is derived from any of these transformations in the manner that we just spoke of.

Equation (6) can be replaced by a system of two equations:

$$\lambda r + \mu s + \nu = 0,$$
$$\lambda s + \mu t + \rho = 0,$$

in which λ, μ, ν, ρ are coupled by two homogeneous relations:

$$\psi_1(x, y, z, p, q, \lambda, \mu, \nu, \rho) = 0, \qquad \psi_2(x, y, z, p, q, \lambda, \mu, \nu, \rho) = 0.$$

On the other hand, the second-order, partial differential equation that is derived from a transformation (2) for which one has:

(7) $$f_3 \frac{\partial f_1}{\partial z'} + f_4 \frac{\partial f_2}{\partial z'} = 1$$

is, as we have just seen, equivalent to the system of two equations (5), where B, C, α, β have the values that were indicated above (no. 1). That equation will be identical to (6) if one has:

(8) $$\psi_1(x, y, z, p, q, \alpha, \beta, B, C) = 0, \qquad \psi_2(x, y, z, p, q, \alpha, \beta, B, C) = 0.$$

These two equations are homogeneous with respect to f_3 and f_4. One can thus eliminate f_3 and f_4, and what remains is just one first-order, partial differential equation that f_1 and f_2 must satisfy. One can take one of these functions arbitrarily – f_2, for example, – and as soon as one knows a solution f_1 of the partial differential equations thus obtained, one will have f_3 and f_4 without any new integration upon solving the system that is formed by equation (7) and one of equations (8).

When one has determined a transformation from which one derives the proposed equation, one will know an infinitude of them, since for this, it will suffice to perform an arbitrary transformation (T'), but, from our conventions, we do not consider two transformations are deduced from each other as distinct. Moreover, the problem that we just treated admits an infinitude of distinct solutions, in the sense that we gave to that word, and later on we shall point out a remarkable relation between these solutions.

If we let (C) denote the system of first-order characteristics of the proposed equation that is defined by the equations:

$$dz = p\,dx + q\,dy,$$

$$\psi_1(x, y, z, p, q, dx, dy, -dp, -dq) = 0,$$
$$\psi_2(x, y, z, p, q, dx, dy, -dp, -dq) = 0$$

then we say that the transformations that we just studied are deduced from the system of characteristics (C).

5. Recall the four equations:

$$x' = f_1(x, y, z, p, q; z'), \qquad y' = f_2(x, y, z, p, q; z'),$$
$$p' = f_3(x, y, z, p, q; z'), \qquad q' = f_4(x, y, z, p, q; z'),$$

in which f_1, f_2, f_3, f_4 satisfy the equality (7), and suppose that f_1 and f_2 depend upon variables x, y, z, p, q by the intermediary of three functions φ_1, φ_2, φ_3 of these variables, i.e., that the point-like multiplicities that support the multiplicities that are generated by the elements of (E') that correspond to an element (x, y, z, p, q) form a three-parameter family. A contact transformation will permit one to replace them with the set of points in space ([1]). The equations of the transformation that we just studied will then be:

$$(9) \qquad \begin{cases} x' = \varphi_1(x, y, z, p, q), \\ y' = \varphi_2(x, y, z, p, q), \\ z' = \varphi_3(x, y, z, p, q), \\ \Phi(x, y, z, p, q; p', q') = 0. \end{cases}$$

In practice, it is frequently interesting, when such a thing is possible, to consider the equations of a transformation in this form rather than in the form (2). It results from the general theorem of the preceding paragraph that (9) will lead to a second-order equation for $z(x, y)$, as was already shown by Goursat. We propose to characterize these equations.

The condition:

$$dz' - p'\,dx' - q'\,dy' = 0$$

([1]) Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, t. 1, pp. 12.

is equivalent to the two equations:

$$(10)\qquad \begin{cases} \dfrac{d\varphi_3}{dx} - p'\dfrac{d\varphi_1}{dx} - q'\dfrac{d\varphi_2}{dx} = 0, \\ \dfrac{d\varphi_3}{dy} - p'\dfrac{d\varphi_1}{dy} - q'\dfrac{d\varphi_2}{dy} = 0. \end{cases}$$

Upon eliminating p', q' from (9) and (10), one obtains the desired second-order equation:

$$F(x, y, z, p, q, r, s, t) = 0.$$

Write the system (10) in the form:

$$(10)'\qquad \begin{cases} \lambda r + \mu s + \nu = 0, \\ \lambda s + \mu t + \nu = 0, \end{cases}$$

where λ, μ, ν, ρ have values that are easy to calculate and satisfy the relation:

$$(11)\qquad h\nu + l\rho + m\lambda + n\mu = 0,$$

h, l, m, n being quantities that are defined up to a factor by the equations:

$$h\left(\frac{d\varphi_i}{dx}\right) + l\left(\frac{d\varphi_i}{dy}\right) + m\frac{\partial\varphi_i}{\partial p} + n\frac{\partial\varphi_i}{\partial q} = 0 \qquad (i = 1, 2, 3).$$

The relation (11) has a very simple significance: It expresses the idea that all of the lines (10)′ that generate the surface that is represented by the second-order equation meet a fixed line:

$$(12)\qquad \begin{cases} hr + ls + m = 0, \\ hs + lt + n = 0, \end{cases}$$

which then belong to a complex (G).

Conversely, if a second-order, partial differential equation $F = 0$ represents a ruled surface whose generators belong to the complex (G) and meet a fixed line of that complex then there exists an infinitude of transformations (9) that the proposed equation can be derived from. The parameters λ, μ, ν, ρ of the generators of the surface satisfy the relation (11) upon supposing that the fixed line is represented by the equations (12). If φ_1, φ_2, φ_3 are three distinct integrals of the linear equation:

$$h\left(\frac{d\varphi}{dx}\right) + l\left(\frac{d\varphi}{dy}\right) + m\frac{\partial\varphi}{\partial p} + n\frac{\partial\varphi}{\partial q} = 0$$

then the values of these parameters (whose ratios are all that is important) can be written:

$$\lambda = \frac{\partial \varphi_3}{\partial p} - p'\frac{\partial \varphi_1}{\partial p} - q'\frac{\partial \varphi_2}{\partial p},$$

$$\mu = \frac{\partial \varphi_3}{\partial q} - p'\frac{\partial \varphi_1}{\partial q} - q'\frac{\partial \varphi_2}{\partial q},$$

$$\nu = \left(\frac{d\varphi_3}{dx}\right) - p'\left(\frac{d\varphi_1}{dx}\right) - q'\left(\frac{d\varphi_2}{dx}\right),$$

$$\rho = \left(\frac{d\varphi_3}{dy}\right) - p'\left(\frac{d\varphi_1}{dy}\right) - q'\left(\frac{d\varphi_2}{dy}\right),$$

where p', q' denote arbitrary variables. The lines that generate the surface $F = 0$ correspond to a relation between the parameters p', q':

$$\Phi(x, y, z, p, q; p', q') = 0.$$

It seems immediate that the proposed equation is derived from the transformation that is defined by the system (9).

The transformations (9) do not differ essentially from the ones that we considered above (no. 4), since all of the properties that we proved by supposing that the transformations are defined by a system such as (2), for which the condition (7) is satisfied, also apply to the transformations that just studied.

6. From the preceding developments, it results that under certain conditions equation (2) leads to second-order, partial differential equations for z; one can naturally repeat what was said for z for z'. If z and z' simultaneously satisfy two second-order, partial differential equations then we say that equations (2) define a *Bäcklund transformation.*

Several cases are possible. If each element of one of the two systems corresponds to the union of ∞^1 elements of the other one then the integrals of the two equations correspond in a one-to-one manner. The transformation will then be called *a Bäcklund transformation of the first kind*, or, more briefly, *a* (B_1) *transformation*. When a transformation is defined by four equations that are not solved for x', y', p', q':

$$V_i(x, y, z, p, q; x', y', z', p', q') = 0 \qquad (i = 1, 2, 3,4),$$

one sees with no difficulty that an element (x, y, z, p, q) corresponds to the union of ∞^1 elements of (E') if the determinant:

$$\left| \left(\frac{dV_i}{dx^i}\right) \left(\frac{dV_i}{dy^i}\right) \frac{\partial V_i}{\partial p^i} \; \frac{\partial V_i}{\partial q^i} \right|$$

is zero. Moreover, it is not necessary for this determinant to be zero identically; it suffices that this betrue when one takes the equations of the transformations into account.

Suppose that an element (x, y, z, p, q) corresponds to the union of ∞^1 elements (x', y', z', p', q'), without the elements of (E) that correspond to an element of (E') enjoying the same property. If z' satisfies a Monge-Ampère equation then the transformation will be a Bäcklund transformation. An integral $z(x, y)$ will correspond to only one integral $z'(x', y')$. On the contrary, if one is given an integral $z'(x', y')$ then it will correspond to an infinitude of integrals $z(x, y)$ that depend upon an arbitrary constant. The transformation will be called *Bäcklund transformations of the second kind* or (B_2) *transformations*.

Finally, $z(x, y)$ and $z'(x', y')$ can simultaneously satisfy two Monge-Ampère equations in such a way that each integral of either of these two equations will corresponds to ∞^1 integrals of the other one. We refer to these transformations as *Bäcklund transformations of the third kind* or (B_3) *transformations*.

It is easy to give examples of these three kinds of transformations.

For the (B_3) transformations, it suffices to consider a system of four equations:

$$F_i(x, y, z, p, q; x', y', z', p', q') = 0 \qquad (i = 1, 2, 3,4).$$

Two contact transformations will always permit one to replace two equations of this system with:

$$x' = x, \qquad y' = y.$$

Now consider three arbitrary functions f_1, f_2, f_3 of x, y, z, p, q, z', f_2 that depend essentially on z', and a function f_4 that satisfies the equation:

$$f_3 \frac{\partial f_1}{\partial z'} + f_4 \frac{\partial f_2}{\partial z'} = 1.$$

The equations:

$$x' = f_1(x, y, z, p, q; z'), \qquad y' = f_2(x, y, z, p, q; z'),$$
$$p' = f_3(x, y, z, p, q; z'), \qquad q' = f_4(x, y, z, p, q; z')$$

define a (B_2) transformation. Let a second-order equation admit a system of first-order characteristics. Suppose that this equation does not contain the variable z, so the operations that were pointed out above (no. 4) permit us to determine four independent functions of z that define a (B_2) transformation that is analogous to the ones that we just considered. In other words, if the proposed equation remains invariant under an infinitesimal contact transformation then it will always be possible to make it correspond to a Monge-Ampère equation by a (B_2) transformation, since the equations of the (B_2) transformation remain likewise invariant under the infinitesimal contact transformation.

We propose to show how one can write the equations of certain (B_1) transformations with no integration. In the space (E), consider a family of ∞^1 multiplicities that are each composed of a union of ∞^1 elements. It suffices to consider a family of ∞^1 curves, and to append to that curve a developable surface that passes through that curve, so the equations, when solved for the constants, are written:

$$\varphi_i(x, y, z, p, q) = \text{const.} \qquad (i = 1, 2, 3, 4);$$

likewise, if one lets:

$$\varphi_i'(x', y', z', p', q') = \text{const.} \qquad (i = 1, 2, 3, 4)$$

be the equations of a family that is analogous to (E') then the equations:

$$\varphi_i(x, y, z, p, q) = \varphi_i'(x', y', z', p', q') \qquad (i = 1, 2, 3, 4)$$

define a (B_1) transformation.

7. We have shown above (no. 4) how, when one is given a transformation:

$$(13) \qquad \begin{cases} x' = f_1(x, y, z, p, q; z'), & y' = f_2(x, y, z, p, q; z'), \\ p' = f_3(x, y, z, p, q; z'), & q' = f_4(x, y, z, p, q; z'), \end{cases}$$

$$\left(f_3 \frac{\partial f_1}{\partial z'} + f_4 \frac{\partial f_2}{\partial z'} = 1 \right),$$

it corresponds to a second-order equation:

$$(14) \qquad F(x, y, z, p, q, r, s, t) = 0$$

that admits a system of first-order characteristics. z' is determined when one knows an integral of that equation:

$$z' = f(x, y, z, p, q, r, s, t),$$

and equations (13) define x', y', p', q'. One obtains the value of z' by solving the second of equations (5) and it is simple to deduce the value of the ratio $\frac{\partial f}{\partial s} \Big/ \frac{\partial f}{\partial t}$ from this. That value is α / β, on the condition that one replaces z' with f in that expression. From the convention that we made on the sense that one attributes to the variables r, s, t, that ratio is, up to sign, equal to the angular coefficient of the projection of the generator of the surface (14) that passes through the point r, s, t onto the plane $r = 0$.

Now consider equation (14). It corresponds to an infinitude of transformations such as (11), namely:

$$(15) \qquad \begin{cases} x'' = f_1'(x, y, z, p, q; z''), & y'' = f_2'(x, y, z, p, q; z''), \\ p'' = f_3'(x, y, z, p, q; z''), & q'' = f_4'(x, y, z, p, q; z''), \end{cases}$$

and one of these transformations z'' will be given by:

$$z'' = f'(x, y, z, p, q, s, t),$$

and, from what we just said, we will have:

$$\frac{\partial f}{\partial s}\frac{\partial f'}{\partial t} - \frac{\partial f'}{\partial s}\frac{\partial f}{\partial t} = 0.$$

The preceding equality shows that if one eliminates the quantities x, y, z, p, q, s, t from (13), (15), and the equations $z' = f$, $z'' = f'$ then there will always be at least four relations between the coordinates x', y', z', p', q'; x'', y'', z'', p'', q'' of two corresponding elements in the systems (E'), (E''):

(16) $$\psi_i (x', y', z', p', q'; x'', y'', z'', p'', q'') = 0 \qquad (i = 1, 2, 3, 4).$$

In particular, if the transformations (13) and (15) are Bäcklund transformations then the transformation (16) will likewise be one. One easily verifies what sort of transformation (16) will be from the nature of the transformations (13) and (15).

When one makes the preceding elimination, it might happen that one obtains five relations between the quantities x', y', z', p', q'; x'', y'', z'', p'', q'', and not just four. In this case, the transformations (13) and (15) are not distinct, but are deduced from each other by a transformation (T'). This latter result can be attached to some general theorems that were proved by Bäcklund. Upon confining ourselves to the particular case that is of concern to us, it is easy to give a very simple proof. Since we do not consider the case where one can deduce an equation between just the variables x', y', z', p', q' in the system (13), for example, we can suppose that the five relations are written in the form:

(17) $$\begin{cases} x'' = g_1(x', y', z', p', q'), \\ y'' = g_2(x', y', z', p', q'), \\ z'' = g_3(x', y', z', p', q'), \\ p'' = h_1(x', y', z', p', q'), \\ q'' = h_2(x', y', z', p', q'). \end{cases}$$

The surfaces of (E') that correspond to the surfaces that are generated by the elements (x'', y'', z'', p'', q'') are defined by the system:

$$\frac{dg_3}{dx'} - h_1\frac{dg_1}{dx'} - h_2\frac{dg_2}{dx'} = 0,$$

$$\frac{dg_3}{dy'} - h_1\frac{dg_1}{dy'} - h_2\frac{dg_2}{dy'} = 0.$$

This system is identically zero if equations (17) define a contact transformation. In all other cases, it cannot admit an integral that passes through an arbitrary curve and tangent along that curve with an arbitrarily given developable; rather, that is what happens for the equation or system of equations that defines $z'(x', y')$.

Recalling the case in which equation (14) does not contain z, it will be possible to make it correspond to a transformation (13) such that the equations do not depend upon z and f no longer depends upon z. That transformation will be unique, because of there exists a second one such as (15) then upon eliminating x, y, p, q, s, t there will be five relations between x', y', z', p', q'; x'', y'', z'', p'', q''. In other words, if equation (14) admits an infinitesimal contact transformation (θ) that generates a group (Θ) then it corresponds to one and only Monge-Ampère equation (ε) under a (B_2) transformation, such that an integral of (ε) corresponds to ∞^1 integrals of the proposed one, which are deduced from each other by transformations of the group (Θ).

Now suppose that the given equation admits not only the transformation (θ), but also other infinitesimal transformations. If (θ_1) is one of the transformations that leave (θ) invariant then I say that (θ_1) corresponds to an infinitesimal transformation (θ') that does not change (ε). Indeed, if one performs a transformation of the group (Θ_1) that is generated by (θ_1) on the equations of the Bäcklund transformation then the equations of the new transformation remain invariant under the transformed group of (Θ) under the transformation considered (Θ_1) – i.e., under the group (Θ) itself. We will thus have a (B_2) transformation that will determine, on the one hand, the proposed equation, and on the other hand, the Monge-Ampère equation that we already obtained. Consequently, it will be deduced from the original transformation by a transformation (T') that leaves (ε) invariant. All of the transformations (T') that correspond to the transformations of (Θ_1) form a one-parameter group.

The transformations of the group of the proposed equations that leave (θ) invariant themselves form a group (γ), and the Monge-Ampère equation admits a holomorphic group that is isomorphic to (γ). Moreover, it might happen that equation (ε) admits other contact transformations that do not correspond to transformations that are analogous to the proposed ones, but to a subgroup of the group of the original equation that does not leave (θ) invariant, so it never corresponds to a group of transformations that do not change (ε).

One can further point out an interesting application of the preceding results. Let an equation (14) that admits two infinitesimal transformations (θ) and (θ_1) be given. It is possible to make two Monge-Ampère equations (ε) and (ε_1) correspond to that equation by employing the particular (B_2) transformations that we spoke of in no. 6, and successively considering the transformations (θ) and (θ_1). The two Monge-Ampère equations that one thus obtains naturally correspond under a (B_2) transformation, but, in general, that (B_2) transformation does not belong to the particular class that was in question above; this will the case only if (θ) and (θ_1) commute.

If (θ) and (θ_1) are homologous to the interior of the group of the proposed equation then the two equations (ε) and (ε_1) are identical. There then exists a Bäcklund transformation that makes any integral of equation (ε) correspond to an infinitude of integrals of the same equation.

It is easy to give an example that shows very neatly how things happen. Consider the equation that studied by Goursat ([1]):

([1]) Bull. Soc. math. t. XXV, pp. 36. – *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, t. II, pp. 252.

$$s = 2\lambda(x, y)\sqrt{pq}\,. \tag{18}$$

If one takes the unknown to be $\sqrt{p} = z'$ then one finds that z' is defined by the linear equation:

$$s' - \frac{\partial \log \lambda}{\partial x} q' - \lambda^2 z = 0. \tag{19}$$

The transformation that permits one to pass from (18) to (19) is defined by the system:

$$z' = \sqrt{p}\,, \qquad q' = \lambda^2 q. \tag{20}$$

Equation (18) always admits two infinitesimal contact transformations that we denote by their characteristic functions, which are 1 and z. One can say that the linear equation that we just obtained for z' corresponds to the transformation 1. That transformation remains invariant under the transformation z. That latter transformation must therefore correspond to an infinitesimal transformation of (19), which is the transformation that has z' for its characteristic function. Upon taking $p / z = z''$ to be the new unknown, one finds a new second-order equation for z'' that corresponds to the transformation z. One must eliminate z from:

$$z'' = \frac{p}{z}, \qquad q'' = \frac{2\lambda\sqrt{pq}}{z} - \frac{pq}{z^2}. \tag{21}$$

The product of two transformations (20) and (21) is a (B_3) transformation whose equations one obtains by writing down the values of z', p', q', z'', p'', q'' as functions of x, y, z, p, q, r and eliminating z, p, q, r. One thus finds:

$$p'' = 2z''\frac{p'}{z'} - z''^2, \qquad q'' = 2z''\frac{q'}{z'} - z''^2\frac{q'^2}{\lambda^2 z'^2}\,. \tag{22}$$

A value of z'' then corresponds to an infinitude of values of z that one obtains by multiplying any one of them by an arbitrary constant, so the values of z' that correspond to these values of z, and which, consequently, correspond to a value of z'', are likewise obtained by multiplying one of them by an arbitrary constant, which likewise appears in formulas (22). On the contrary, the transformation 1 does not leave the transformation z invariant, so the values of z that correspond to a value of z' and differ only by an additive constant correspond to values of z'' such that there exists no one-parameter group of contact transformations that will give all of the integrals z'' that correspond to the same solution of (19) when they are applied to an arbitrary integral z''.

8. The Monge-Ampére equations admit two systems of first-order characteristics. They will correspond to two families of transformation that are analogous to the ones that we studied above, but in order to apply the result that was established at the beginning of the preceding paragraph, one must simultaneously consider only two transformations that are deduced from this same system of characteristics. It is indeed clear that our arguments apply to only this case.

Meanwhile, it might happen that the product of two Bäcklund transformations that are each deduced from one of the systems of characteristics is likewise a Bäcklund transformation. For example, suppose that we have a Monge-Ampère equation that does not contain z. Upon successively considering the two systems of characteristics, one can make it correspond to two other Monge-Ampère equations by some (B_2) transformation whose equations do not depend upon z.

One will thus obtain eight equations between x, y, z, p, q, and the coordinates of two other systems of elements x', y', z', p', q'; x'', y'', z'', p'', q''. Upon eliminating x, y, z, p, q, one gets four equations that obviously define a (B_1) transformation.

The Goursat equation will again provide us with an example. If one is given the equation:

$$s = 2\lambda(x, y)\sqrt{pq}\,,$$

while successively taking the unknowns to be $\sqrt{p}$ and $\sqrt{q}$, then one obtains two linear equations that will correspond under a Laplace transformation.

We propose to apply the results of the study that we made (nos. 4 and 5) to the Monge-Ampère equation. With no loss of generality, we consider only equations that are linear in r, s, t. Let:

$$r + (m + \mu)\, s + m\mu\, t + M = 0$$

be such an equation, where m, μ, M denote functions of x, y, z, p, q. This equation possesses two systems of first-order characteristics: The system (C) corresponds to the differential equation $dy = m\, dx$ and the system (Γ) corresponds to $dy = \mu\, dx$. From the system (C), one will deduce a transformation such as (13) by taking f_1 and f_2 to be two integrals of the linear equation:

$$X(f) = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + \mu\left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right) - (M + \lambda m)\frac{\partial f}{\partial p} + \lambda\frac{\partial f}{\partial q} = 0,$$

where λ is an arbitrary function of x, y, z, p, q, z'.

For f_3 and f_4, one will have:

$$f_3 = \frac{\dfrac{\partial f_2}{\partial q} - m\dfrac{\partial f_2}{\partial p}}{\left(\dfrac{\partial f_2}{\partial q} - m\dfrac{\partial f_2}{\partial p}\right)\dfrac{\partial f_1}{\partial z'} - \left(\dfrac{\partial f_1}{\partial q} - m\dfrac{\partial f_1}{\partial p}\right)\dfrac{\partial f_2}{\partial z'}},$$

$$f_4 = -\frac{\dfrac{\partial f_1}{\partial q} - m\dfrac{\partial f_1}{\partial p}}{\left(\dfrac{\partial f_2}{\partial q} - m\dfrac{\partial f_2}{\partial p}\right)\dfrac{\partial f_1}{\partial z'} - \left(\dfrac{\partial f_1}{\partial q} - m\dfrac{\partial f_1}{\partial p}\right)\dfrac{\partial f_2}{\partial z'}}.$$

We remark that if f is an integral of $X(f) = 0$ then one has:

$$X\left(\frac{\partial f}{\partial z'}\right) = \frac{\partial \lambda}{\partial z'}\left(m\frac{\partial f}{\partial p} - \frac{\partial f}{\partial q}\right);$$

so, upon appealing to the identity $f_3\frac{\partial f_1}{\partial z'} + f_4\frac{\partial f_2}{\partial z'} = 1$, one will deduce that:

$$\frac{\partial f_1}{\partial z'}X(f_3) + \frac{\partial f_2}{\partial z'}X(f_4) = 0. \tag{23}$$

The transformation considered will be a (B_1) transformation if the determinant:

$$\left|\left(\frac{df_i}{dx}\right)\left(\frac{df_i}{dy}\right)\frac{\partial f_i}{\partial p}\frac{\partial f_i}{\partial q}\right| \qquad (i = 1, 2, 3, 4)$$

is zero (no. 6). By virtue of (23) and the equations $X(f_1) = X(f_2) = 0$, one must therefore have:

$$X(f_3) = X(f_4) = 0.$$

In the second part of this paper, we shall determine the (B_1) transformations that correspond to a very extended class of Monge-Ampère equations. Here, I will confine myself to pointing out a result whose proof is quite simple, from what we just said. If two equations (ε), (ε_1) correspond to one Monge-Ampère equation under two (B_1) transformations that are deduced from the same system of characteristics then the product of these two transformations is never a Bäcklund transformation.

Indeed, let:

$$(\alpha) \qquad \begin{cases} x' = f_1(x, y, z, p, q; z'), & y' = f_2(x, y, z, p, q; z'), \\ p' = f_3(x, y, z, p, q; z'), & q' = f_4(x, y, z, p, q; z'), \end{cases}$$

be the equations of a (B_1) transformation that is deduced from the system (C) of characteristics of the given Monge-Ampère equation. z' is given by:

$$z' = \psi(x, y, z, p, q, r, s, t),$$

and one has:

$$m\frac{\partial \psi}{\partial s} - \frac{\partial \psi}{\partial t} = 0,$$

in which f_1, f_2, f_3, f_4 satisfy the equation $X(f) = 0$.

A (B_1) that is deduced from the system (Γ) will be defined by:

$$(\beta) \qquad \begin{cases} x'' = f_1'(x, y, z, p, q; z''), & y'' = f_2'(x, y, z, p, q; z''), \\ p'' = f_3'(x, y, z, p, q; z''), & q'' = f_4'(x, y, z, p, q; z''); \end{cases}$$

in order to determine z'', one has:

$$z'' = \psi'(x, y, z, p, q, s, t),$$

with

$$\mu \frac{\partial \psi'}{\partial s} - \frac{\partial \psi'}{\partial t} = 0.$$

On the one hand, by an argument that was explained for the first transformation, one verifies that f_1', f_2', f_3', f_4' must satisfy an equation such as:

$$X_1(f) = \left(\frac{df'}{dx}\right) + m\left(\frac{df'}{dy}\right) - (M + \rho\mu)\frac{\partial f'}{\partial p} + \rho\frac{\partial f'}{\partial q} = 0,$$

ρ being a function of x, y, z, p, q, z''.

In order for the product of the transformations (α) and (β) to be a Bäcklund transformation, it is necessary and sufficient that upon eliminating x, y, z, p, q, s, t from (α), (β) and the equations $z' = \psi$, $z'' = \psi'$, one finds four relations between $x', y', z', p', q', x'', y'', z'', p'', q''$. Since the determinant:

$$\frac{\partial \psi}{\partial s}\frac{\partial \psi'}{\partial t} - \frac{\partial \psi}{\partial t}\frac{\partial \psi'}{\partial s}$$

is non-zero, it must therefore be the case that one obtains four relations by eliminating x, y, z, p, q from (α) and (β). In other words, one must have that all of the fifth-order functional determinants of the functions f_1, f_2, f_3, f_4, f_1', f_2', f_3', f_4' with respect to x, y, z, p, q are zero; i.e., these functions must satisfy the same equation:

$$A\frac{\partial f}{\partial x} + B\frac{\partial f}{\partial y} + C\frac{\partial f}{\partial z} + D\frac{\partial f}{\partial p} + E\frac{\partial f}{\partial q} = 0.$$

However, f_1, f_2, f_3, f_4 satisfy $X(f) = 0$, and no other equation, since they are four independent functions. Likewise, f_1', f_2', f_3', f_4' satisfy just the equation $X_1(f') = 0$, and it is clear that these two equations cannot be identical.

In order to find the transformations of the form (9) that are derived from the proposed Monge-Ampère equation, one must take $\varphi_1, \varphi_2, \varphi_3$ to be three distinct integrals of the equation $X(f) = 0$, where λ denotes an arbitrary function of just x, y, z, p, q. The fourth equation that defines the transformation is then:

$$m\left(\frac{\partial \varphi_3}{\partial p} - p'\frac{\partial \varphi_1}{\partial p} - q'\frac{\partial \varphi_2}{\partial p}\right) - \frac{\partial \varphi_3}{\partial q} + p'\frac{\partial \varphi_1}{\partial q} + q'\frac{\partial \varphi_2}{\partial q} = 0.$$

9. As we have already remarked, the transformations that we just studied permit us to calculate x', y', z', p', q' as functions of x, y, z, p, q, s, t. One has:

$$x' = \psi_1(x, y, z, p, q, s, t),$$
$$y' = \psi_2(x, y, z, p, q, s, t),$$
$$z' = \psi_3(x, y, z, p, q, s, t),$$
$$p' = \psi_5(x, y, z, p, q, s, t),$$
$$q' = \psi_6(x, y, z, p, q, s, t).$$

It is obvious that one will have to prolong the preceding transformation and calculate the derivatives of order n arbitrary as functions of x, y, z, and the derivatives of z up to order $n + 1$, but it is necessary to specify them even more, and the calculation offers no difficulty, moreover. At the same time, we establish certain formulas that will very useful to us later on.

We write the second-order equation in the form:

$$r + F(x, y, z, p, q, s, t) = 0, \tag{24}$$

and if m and μ are the two roots of the second-degree equation in λ:

$$\lambda^2 - \lambda \frac{\partial F}{\partial s} + \frac{\partial F}{\partial t} = 0$$

then we suppose that the system (C) of characteristics that satisfy the differential equation:

$$dy = m\,dx$$

is of first order and that the transformation considered is deduced from the system (C).

All of the functions ψ, ψ_1, ψ_2, ψ_3, ψ_4 satisfy the equation:

$$m\frac{\partial f}{\partial s} - \frac{\partial f}{\partial t} = 0.$$

Having said this, we further remark that it suffices to take the derivative of (24) with respect to y in order to obtain the following equality, which we shall make use of:

$$p_{2,1} + m\,p_{1,2} + \mu\,(p_{1,2} + mp_{0,3}) + \left(\frac{dF}{dy}\right) = 0. \tag{25}$$

When one replaces x, y, z, p, q, s, t with the expressions that correspond to a characteristic of (24) in ψ, ψ_1, ψ_2, ψ_3, ψ_4, one obtains a system of ∞^1 elements of (E'). Let m' denote the value of the ratio dy' / dx' for the systems that correspond to a characteristic of (C). In order to find the expression for m', it will suffice to replace dy with $m\,dx$ in the equation:

$$d\psi_2 - m'\,d\psi_1 = 0,$$

and one thus arrives at the formula:

$$m' = \frac{\left(\frac{d\psi_2}{dx}\right) + m\left(\frac{d\psi_2}{dy}\right) + (m-\mu)\frac{\partial\psi_2}{\partial s}(p_{1,2}+, p_{0,3}) - \frac{\partial\psi_2}{\partial s}\left(\frac{dF}{dy}\right)}{\left(\frac{d\psi_1}{dx}\right) + m\left(\frac{d\psi_1}{dy}\right) + (m-\mu)\frac{\partial\psi_1}{\partial s}(p_{1,2}+, p_{0,3}) - \frac{\partial\psi_1}{\partial s}\left(\frac{dF}{dy}\right)}.$$

Now consider the system (Γ) that is defined by the equation:

$$dy = \mu\, dx,$$

where the value of μ' – i.e., the value of the ratio dy' / dx' for the system of elements of (E') that correspond to the characteristics of (Γ) – is obtained by operating as one did for m'. One finds that:

$$\mu' = \frac{\left(\frac{d\psi_2}{dx}\right) + \mu\left(\frac{d\psi_2}{dy}\right) - \frac{\partial\psi_2}{\partial s}\left(\frac{dF}{dy}\right)}{\left(\frac{d\psi_1}{dx}\right) + \mu\left(\frac{d\psi_1}{dy}\right) - \frac{\partial\psi_1}{\partial s}\left(\frac{dF}{dy}\right)}.$$

We now concern ourselves with the calculation of the successive derivatives of z'. We consider only the derivatives $p'_{0,n}$, $p'_{1,n-1}$, since the knowledge of the expressions for these derivatives will suffice. If z' satisfies a second-order equation then one can always, as we did for equation (24), suppose that a change of variables has been performed such that all of the derivatives of z', starting with the second-order ones, are expressed as functions of x', y', z', and the derivatives of order less than or equal to two, where the first index is equal to at most one.

In order to get s' and t', one must annul the coefficient of dx and that of dy in the equation:

$$d\psi_4 - s'\, dy_1 - t'\, d\psi_2 = 0,$$

where we replace $p_{1,2} + m\, p_{4,2}$ with its value that is deduced from (25). The solution to the equations thus obtained is simple: If we set:

$$\Delta = \left(\frac{d\psi_2}{dy}\right)\left[\left(\frac{d\psi_1}{dx}\right) - \frac{\partial\psi_1}{\partial s}\left(\frac{dF}{dy}\right)\right] - \left(\frac{d\psi_1}{dy}\right)\left[\left(\frac{d\psi_2}{dx}\right) - \frac{\partial\psi_2}{\partial s}\left(\frac{dF}{dy}\right)\right]$$

$$+ (p_{1,2} + m\, p_{0,3}) \left\{\frac{\partial\psi_2}{\partial s}\left[\left(\frac{d\psi_1}{dx}\right) + \mu\left(\frac{\partial\psi_1}{\partial y}\right)\right] - \frac{\partial\psi_1}{\partial s}\left[\left(\frac{d\psi_2}{dx}\right) + \mu\left(\frac{\partial\psi_2}{\partial y}\right)\right]\right\},$$

then one gets:

$$\Delta s' = \left(\frac{d\psi_2}{dy}\right)\left[\left(\frac{d\psi_4}{dx}\right) - \frac{\partial\psi_4}{\partial s}\left(\frac{dF}{dy}\right)\right] - \left(\frac{d\psi_4}{dy}\right)\left[\left(\frac{d\psi_2}{dx}\right) - \frac{\partial\psi_2}{\partial s}\left(\frac{dF}{dy}\right)\right]$$

$$+ (p_{1,2} + m\,p_{0,3}) \left\{\frac{\partial \psi_2}{\partial s}\left[\left(\frac{d\psi_4}{dx}\right)+\mu\left(\frac{\partial \psi_4}{\partial y}\right)\right]-\frac{\partial \psi_4}{\partial s}\left[\left(\frac{d\psi_2}{dx}\right)+\mu\left(\frac{\partial \psi_2}{\partial y}\right)\right]\right\},$$

$$\Delta t' = -\left(\frac{d\psi_1}{dy}\right)\left[\left(\frac{d\psi_4}{dx}\right)-\frac{\partial \psi_4}{\partial s}\left(\frac{dF}{dy}\right)\right]-\left(\frac{d\psi_4}{dy}\right)\left[\left(\frac{d\psi_1}{dx}\right)-\frac{\partial \psi_1}{\partial s}\left(\frac{dF}{dy}\right)\right]$$

$$+ (p_{1,2} + m\,p_{0,3}) \left\{\frac{\partial \psi_4}{\partial s}\left[\left(\frac{d\psi_1}{dx}\right)+\mu\left(\frac{\partial \psi_1}{\partial y}\right)\right]-\frac{\partial \psi_1}{\partial s}\left[\left(\frac{d\psi_4}{dx}\right)+\mu\left(\frac{\partial \psi_4}{\partial y}\right)\right]\right\}.$$

As before (no. 3), one is assured that there is no reason to examine the case where Δ is zero. We see that s' and t' do not depend upon the derivatives of z' of order higher than three, but we see, in addition, that the third derivatives figure only in the combination $p_{1,2} + m\,p_{0,3}$. Assume that the same thing is true up to order $n - 1$ – i.e., that $p'_{1,n-2}$ and $p'_{0,n-1}$ are functions of $x, y, z, p, q, \ldots, p_{1,n-2}, p_{0,n-1}, p_{1,n-1} + m\,p_{0,n}$ – and show that the property persists for the derivatives of order n – i.e., that it is general. Upon taking the derivative of order $n - 1$ of the given equation with respect to y, one finds:

$$(25') \qquad p_{2,n-1} + m\,p_{1,n} + m(p_{1,n} + m\,p_{0,n+1}) + \left(\frac{d^{n-1}F}{dy^{n-1}}\right) = 0.$$

By hypothesis, one has:

$$p'_{0,n-1} = K(x, y, z, \ldots, u),$$

upon setting $p_{1,n-1} + m\,p_{0,n} = u$, and in order to calculate $p'_{0,n}$, $p'_{1,n-1}$ one must equate the coefficients of dx and dy in:

$$p'_{1,n-1}\,d\psi_1 + p'_{0,n}\,d\,\psi_2 = dK$$

to zero. One finds two equations:

$$
(26)\quad \begin{cases}
p'_{1,n-1}\left[\left(\dfrac{d\psi_1}{dy}\right)+\dfrac{\partial\psi_1}{\partial s}(p_{1,2}+m\,p_{0,3})\right]+p'_{0,n}\left[\left(\dfrac{d\psi_2}{dy}\right)+\dfrac{\partial\psi_2}{\partial s}(p_{1,2}+m\,p_{0,3})\right] \\
\qquad =\left(\dfrac{dK}{dy}\right)+\dfrac{\partial K}{\partial y}(p_{1,n}+m\,p_{0,n+1}), \\
\\
p'_{1,n-1}\left[\left(\dfrac{d\psi_1}{dx}\right)-\dfrac{\partial\psi_1}{\partial s}\left(\dfrac{dF}{dy}\right)-\mu\dfrac{\partial\psi_1}{\partial s}(p_{1,2}+m\,p_{0,3})\right] \\
+p'_{0,n}\left[\left(\dfrac{d\psi_2}{dy}\right)+\dfrac{\partial\psi_2}{\partial s}\left(\dfrac{dF}{dy}\right)-\mu\dfrac{\partial\psi_2}{\partial s}(p_{1,2}+m\,p_{0,3})\right] \\
\qquad =\left(\dfrac{dK}{dy}\right)-\left(\dfrac{\partial K}{\partial y}\right)\left(\dfrac{d^{n-1}F}{dy^{n-1}}\right)-\mu\dfrac{\partial K}{\partial u}(p_{1,n}+m\,p_{0,n+1}).
\end{cases}
$$

One sees immediately that $p'_{1,n-1}$ and $p'_{0,n}$ are expressed as functions of $x, y, z, \ldots, p_{1,n-1}, p_{0,n}, p_{1,n} + m\,p_{0,n+1}$, which we would like to prove. One sees further that if n is greater than two then $p'_{1,n-1} + \mu'\,p'_{0,n}$ depends upon only x, y, z, and the derivatives of z up to order n. One proves this by multiplying the first of equations (26) by m and adding both sides of the two equations. On the contrary, $p'_{1,n-1} + m'\,p'_{0,n}$ depends upon derivatives of order $n + 1$. When the equation considered is a linear equation in r, s, t:

$$r + (m + \mu)\,s + m\,\mu\,t + M = 0,$$

x', y', z', p', q' are functions of $x, y, z, p, q, s + mt$, so the last-stated property will be true if n is equal to two: $s' + m'\,t'$ will depend upon only the third derivatives of z.

10. If one is given a Bäcklund transformation of arbitrary type and two integral surfaces that correspond under that transformation then the characteristics correspond on these two surfaces ([1]). We consider two cases in turn: We first assume that an integral of the second-order equation (24) corresponds to just one integral of the transformed one.

We first remark that if one replaces x, y, z, p, q, s, t in the functions $\psi, \psi_1, \psi_2, \psi_3, \psi_4$ with the expressions that correspond to a simply infinite multiplicity that is a union of second-order elements then the elements (x', y', z', p', q') are not united, in general, but they will be if the elements (x, y, z, p, q, s, t) belong to an integral of (24), since (x', y', z', p', q') then belongs an integral of the transformed equation. In particular, this will be true if the second-order elements generate a characteristic of (24).

If a characteristic of order n of the proposed equation is known then this amounts to saying that one knows ∞^1 systems of values for $x, y, z, p, q, \ldots, p_{1,n-1}, p_{0,n}$, and that there exists an infinitude of integral surfaces that depend upon an infinitude of arbitrary constants that pass through this multiplicity of n^{th}-order elements. In the first place,

([1]) *See* GOURSAT, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre,* t. II, pp. 290.

examine the case where the characteristic considered belongs to the system (C) – i.e., it satisfies the equation:

$$dy = m\, dx.$$

The equality:

$$dp_{0,n} = (p_{1,n} + m\, p_{0,n+1})\, dx$$

is verified all along this characteristic, and $p_{1,n} + m\, p_{0,n+1}$ has a well-defined value at each point. From what we said in the preceding paragraph, the given characteristic of order n corresponds to a multiplicity of ∞^1 n^{th}-order elements, through which pass all of the transformed integral surfaces of the integrals of (24) that contain the given characteristic. These surfaces thus have a contact of order at least n along that multiplicity, but it can happen that this contact is of higher order. We confirm that if the transformation is a transformation of the first kind then the transformed surfaces have a contact of order n + 1 at all of the points of this multiplicity. A characteristic of order n thus corresponds to a characteristic of order n or n + 1. The preceding argument only applies, in general, if n is greater than one. It applies to the case where n is equal to one if (24) is a Monge-Ampère equation – i.e., as one can always assume, an equation that is linear in r, s, t.

One easily sees, by always reasoning in the same manner, that a characteristic of order n of the system (Γ) corresponds to a characteristic of order n – 1 of the transformed one.

It remains for us to study the case where, equation (24) being a Monge-Ampère equation, each of its integrals correspond to an infinitude of integrals of the transformed one. As we have done before several times, we assume that the given equation is linear in r, s, t.

If one is given a first-order characteristic then upon replacing x, y, z, p, q with their values in equation (3)″ (no. 2), one finds ζ by an integration. If we determine ζ by giving its value ζ_0 for an element (x_0, y_0, z_0, p_0, q_0) for the characteristic then we define a multiplicity of (E') that corresponds to the given characteristic, since x', y', z', p', q' depends only upon x, y, z, p, q, ζ. There exist an infinitude of integrals of (24) that depend upon an infinitude of arbitrary constants that contain the first-order characteristic considered. There is likewise an infinitude of integrals of the transformed equation that pass through the transformed multiplicity, which is, consequently, a characteristic multiplicity.

If one is given a characteristic of order n then it contains a first-order characteristic that one can make it correspond to, since, as we just said, it is a characteristic of the transformed equation. From the remark that was made in no. 3, one will know the values of the derivatives of z' up to order n along this characteristic. Therefore, the given characteristic of order n will correspond to a characteristic whose order is also n.

One proves in the same fashion that if one knows how to solve the Cauchy problem for a second-order, partial differential equation (24) then one knows how to solve this problem for all equations that correspond to (24) under a Bäcklund transformation.

If two second-order equations are such that one can make each integral of one of them correspond to the integrals of the other one by algebraic operation, and if one of them is integrable by the Darboux method then the other is, as well; this theorem has been proved by Goursat. In particular, it results that if an equation is integrable by Darboux's method then all of the equations that correspond to it under a Bäcklund transformation enjoy the same property. Furthermore, we shall return to this equation in what follows in order to point out some more precise results.

11. Up to now, we have neglected the case in which one can deduce an equation that contains only the coordinates of one of the two systems of first-order elements from the four given equations in $x, y, z, p, q, x', y', z', p', q'$. We shall rapidly examine that special case.

Suppose that one of the systems of equations contains only x', y', z', p', q'. It is always possible, by a contact transformation, to make that equation reduce to $y' = 0$ in some way. The equations of the transformation that we just studied will then be:

$$x' = f(x, y, z, p, q; z'), \qquad y' = 0,$$
$$p' = \varphi(x, y, z, p, q; z'), \qquad q' = \psi(x, y, z, p, q; z').$$

The condition:

$$dz' - p'\,dx' - q'\,dy' = 0,$$

which expresses the idea that (x', y', z', p', q') generates a multiplicity of ∞^2 united elements, becomes:

$$\left(\varphi\frac{\partial f}{\partial z'} - 1\right)dz' + \varphi\left[\left(\frac{df}{dx}\right) + r\frac{\partial f}{\partial p} + s\frac{\partial f}{\partial q}\right]dx + \varphi\left[\left(\frac{df}{dy}\right) + s\frac{\partial f}{\partial p} + t\frac{\partial f}{\partial q}\right]dy = 0$$

here.

The preceding integrability condition is written:

$$\frac{df}{dy}\frac{d\varphi}{dx} - \frac{d\varphi}{dy}\frac{df}{dx} = 0.$$

If this condition does not depend upon z' then it reduces to a second-order equation that is integrable by the Monge method and which admits the intermediate integral:

$$\varphi = \text{arbitrary function of } f,$$

whose existence is almost obvious, *a priori*.

If one has:

$$\varphi\frac{\partial f}{\partial z'} = 1$$

then one cannot repeat this argument. The equation:

$$dz' - p'\,dx' - q'\,dy' = 0$$

will only be verified if one has:

$$\left(\frac{df}{dx}\right) + r\frac{\partial f}{\partial p} + s\frac{\partial f}{\partial q} = 0,$$

$$\left(\frac{df}{dy}\right) + s\frac{\partial f}{\partial p} + t\frac{\partial f}{\partial q} = 0.$$

This system is equivalent to a second-order equation that admits a first-order intermediate integral:

$$f(x, y, z, p, q, \lambda) = \mu,$$

where λ and μ denote two arbitrary constants. As before, this result can be anticipated with no difficulty.

It can, moreover, happen that two of the equations of the transformation contain only the coordinates x', y', z', p', q', or that one has one equation that depends upon only x', y', z', p', q' and another one that depends upon only x, y, z, p, q, but these special cases hold no interest for the theory of second-order equations.

PART TWO

12. In this second part, we shall study the Bäcklund transformations of the first kind, or (B_1) transformations, according to the terminology that we have adopted; i.e., the Bäcklund transformations that make the integrals of the transformed equations correspond in a one-to-one way.

The fundamental results are provided by the use of the formulas that were proved above (no. 9), which I rapidly recall. All of the symbols employed have the same significance as before.

Let there be given a second-order equation:

$$r + F(x, y, z, p, q, s, t) = 0, \tag{27}$$

which admits two systems of characteristics (C) and (Γ) that are defined by the equations:

$$(C) \qquad dy = m\,dx,$$
$$(\Gamma) \qquad dy = \mu\,dx.$$

We assume that m and μ have different values, and we do not concern ourselves with the special case in which the two systems of characteristics coincide.

As in the paragraph that was already cited, we assume that the characteristics of the system (C) are of first order and that the (B_1) transformation is deduced from that system.

The transformed equation:

$$r' + F'(x', y', z', p', q', s', t') = 0 \tag{28}$$

likewise admits two systems of characteristics: The one – viz., the system (C'), which corresponds to (C) – satisfies the equations:

$$dy' = m'\,dx',$$

while the other one – viz., the system (Γ'), which corresponds to (Γ) – satisfies:

$$dy' = \mu'\,dx'.$$

We have seen that $p'_{1,n-1} + \mu' p'_{0,n}$ depends upon only $x, y, z, p, q, \ldots, p_{1,n-1}, p_{0,n}$, while the derivatives of z of order $n + 1$ appear in the expression for $p'_{1,n-1} + m' p'_{0,n}$, n being an arbitrary whole number that is greater than two. We thus have the means to distinguish the two systems of characteristics, which allows us to make a very important remark.

It is quite obvious that, conversely, we can express $x, y, z, p, q, \ldots, p_{1,n-1}, p_{0,n}$ as functions of $x', y', z', p', q', \ldots, p'_{1,n}, p'_{0,n+1}$, and by replacing $x, y, z,$ and the derivatives of z up to order n in the first of equations (26) with their expressions as functions of x', y', z', and the derivatives of z', one sees that $p_{1,n} + m\, p_{0,n+1}$ depends upon only the derivatives of z' of order at most $n + 1$. Consequently, relative to equation (28), the (B_1) transformation considered is deduced from the system of characteristics (Γ'); i.e., from the system that does not correspond to the system (C), which is deduced from the transformation that relates to equation (27).

We have seen (no. 10) that a family (α) of integrals of (27) that has a contact of order n along a characteristic of the system (C) will correspond to a family of integrals of (28) that have a contact of order at least n at all points of a characteristic of (C'). I say that along that characteristic the integrals of (28) considered have a contact of $(n + 1)^{\text{th}}$ order. Indeed, if the contact is of lower order then since the transformation is, relative to equation (28), deduced not from the system (C'), but from system (Γ'), one confirms that by applying a result of the paragraph cited above the corresponding integrals of (27) – i.e., the integrals (α) – will admit a contact of order less than n along the characteristics considered, which is contrary to the hypothesis.

13. If we appeal to the result that we just obtained then we shall show that if one is given a second-order, partial differential equation (27) then there exist only one equation that corresponds to it under a (B_1) transformation that is deduced from the system (C), or, more precisely, that all of the equations that correspond to it in this manner are deduced from one of them by a contact transformation.

Indeed, let equation (28), which corresponds to (27) by a (B_1) transformation, be deduced from (C). Suppose that there exists another equation:

$$\Psi(x'', y'', z'', p'', q'', r'', s'', t'') = 0 \tag{29}$$

that enjoys the same property. Let (C'') and (Γ'') be two systems of characteristics of (29), where (C'') corresponds to (C) and (Γ'') corresponds to (Γ). From what was proved in Part One (no. 7), equations (28) and (29) can be converted into each other by either a Bäcklund transformation or a contact transformation. We shall see that the second hypothesis is the only admissible one. The transformation that establishes a uniform correspondence between the integrals of (27) and those of (29) is deduced, on the one hand, from the system (C), and on the other hand, from the system (Γ''), which would result from the proposition that was made the object of the preceding paragraph, while the transformation that permits one to pass from (27) to (28) is deduced from the system (C) and the system (Γ').

If there exists a Bäcklund transformation that changes equation (28) into equation (29) then it will be a (B_1) transformation that is deduced from the system (Γ') and the system (Γ''). This cannot happen, since these two systems of characteristics correspond to each other, so equations (28) and (29) are identical, up to a contact transformation.

The preceding proof can give rise to an objection: Suppose that (28) and (29) correspond under a Bäcklund transformation. It is not absolutely obvious that this transformation, which can only be a (B_1) transformation, is deduced from the systems (Γ') and (Γ''). It is very easy to eliminate this difficulty by remarking that if that transformation is, for example, deduced from the system (C') then since equation (28) has two systems of first-order characteristics it will be a Monge-Ampère equation, and from the remark that was made at the end of no. 8, equations (27) and (28) cannot correspond by a Bäcklund transformation, which is contrary to hypothesis.

We shall point out an application of the preceding theorem that is particularly interesting. Imagine that equation (27) remains invariant under a certain continuous group of contact transformations (g). We shall show that equation (28) admits a group of holomorphic contact transformation that is isomorphic to (g).

Let (Θ) be a transformation of the group (g). The equations that define (Θ) permit us to express x, y, z, p, q as functions of the new variables X, Y, Z, P, Q. Upon replacing x, y, z, p, q with their values in the equations of the Bäcklund transformation, one obtains a new (B_1) transformation that leads to equation (28) for $z'(x', y')$ and to the transform of (27) under (Θ) for $Z(X, Y)$; i.e., to equation (27) itself. From the theorem that was proved at the beginning of this paragraph, one can pass from the original (B_1) transformation to the new transformation by a transformation (T') that naturally does not change equation (28). When the parameters that the equations that define the group (g) depend upon vary, one obtains a continuous family of transformations (T') that corresponds to the transformations of (g) in the manner that was just explained and generates a holomorphic group that is isomorphic to (g). The stated proposition is thus proved.

It is important to remark that the argument does not apply to an isolated transformation or to a discontinuous group of contact transformations that leave the proposed equation invariant ([1]). To fix ideas, consider the equation that one encounters in the theory of surfaces of constant total curvature:

$$s = \sin z. \tag{30}$$

If one takes ρ to be a new unknown, with the independent variables being preserved, then one obtains the equation:

$$s' = z' \sqrt{1 - q'^2} \tag{31}$$

that corresponds to the preceding one by a (B_1) transformation whose equations are:

$$z' = p, \qquad q' = \sin z. \tag{32}$$

Equation (30) does not change if one adds an arbitrary multiple of 2π to z, but no transformation of (31) corresponds to these transformations. No matter what the determination of z, one finds the same value for z'. The transformation (32) thus remains invariant under a discontinuous group of transformations (T), while it is impossible that this is true for a continuous group. We have seen (no. 1) that the transformation will not be of the first kind. Moreover, one knows that from the standpoint of the theory that concerns us only the continuous groups of contact transformations are important.

([1]) In a Note that was inserted into the *Comptes rendus* (t. CXXXII, pp. 305), I stated the proposition that was just proved without making these restrictions, whose necessity I recognized only later on.

14. Suppose that one of the two transformed equations – for example, equation (28) – is such that one of its systems of characteristics admits an invariant of order n – i.e., that there exists a function ϖ' of x', y', z', …, $p'_{1,n-1}$, $p'_{0,n}$ such that the equation:

$$\varpi' = k, \tag{33}$$

where k denotes an arbitrary constant, possesses an infinitude of integrals that depend upon an infinitude of arbitrary constants that simultaneously satisfy equation (28). By replacing all of the quantities that appear in ϖ' with their expressions as functions of x, y, z, and the derivatives of z, one finds a function ϖ that is an invariant for one of the systems of characteristics of (27), because all of the transforms of the integrals common to (27) and (33) satisfy both (27) and:

$$\varpi = k.$$

Conversely, if one considers the invariant ϖ, which belongs to one of the systems of characteristics of (27) then one sees that there exists an invariant for the corresponding system of characteristics of (28), and that invariant can only be ϖ', since the integrals correspond uniformly. In the last Part, we shall confirm that if the integrals correspond according to a more complicated law then things happen in a completely different manner.

If we are given an invariant then we propose to find what the order is for the invariant that it corresponds to under a (B_1) transformation, as was just explained. We first consider the invariants of order higher than two.

Suppose that there exists such an invariant ϖ' that belongs to the system (C') of characteristics of (28). ϖ' is a function of x', y', z', …, $p'_{1,n-1} + \mu' p'_{0,n}$, and upon replacing these quantities with their expressions, one finds, from the results of no. 9, that ϖ does not depend upon derivatives of z of order higher than n. On the contrary, if the invariant considered belongs to the system (Γ') then it corresponds to an invariant of order n + 1, since $p'_{1,n-1} + m' p'_{0,n}$ depends upon derivatives of z of order n + 1.

One finds some entirely similar results when n is equal to two. A second-order invariant of the system (C') satisfies the equation:

$$\mu' \frac{\partial \varpi'}{\partial s'} - \frac{\partial \varpi'}{\partial t'} = 0.$$

A direct calculation permits one to verify that if one replaces x', y', z', p', q', s', t' in the preceding equation with their values that were found in the paragraph cited above then one obtains a function of only x, y, z, p, q, t, since the derivative of ϖ' with respect to $p_{1,2} + m\, p_{0,3}$ is zero. The order of the invariant that corresponds to ϖ' does not exceed two. If the second-order invariant that we consider belongs to the system (Γ') then it corresponds to a third-order invariant. One verifies quite easily that the third-order derivatives cannot be omitted.

It is quite obvious that one will arrive at identical results by first considering the invariants of the system of characteristics (27) and looking for the order of the invariants of the corresponding systems of characteristics of (28). Finally, one sees that if n denotes an arbitrary number that is greater than or equal to two then the (B_1) transformation is deduced from an invariant of order n of the system of characteristics (C) of the original equation, so that transformation will make an invariant of order $n + 1$ correspond to the system of characteristics (C') of the transformed equation, while an invariant of order n of the system (Γ) will correspond to an invariant of order $n - 1$ of the system (Γ').

One proves in the same fashion that an equation of order n that forms a system in involution with (27) will correspond to an equation of order $n - 1$ or $n + 1$ will form a system in involution with (28).

It remains to examine the case where there exists a first-order invariant for one of the systems of characteristics of (28). Suppose, to fix ideas, that one has performed a contact transformation such that this invariant is y'. It suffices to refer to the expression for y in order to see that, in general, the corresponding invariant is of second order, but we shall show that in certain cases that invariant is of only first order.

If that is true then one of the equations of the (B_1) transformation reduces to:

$$y' = f_2(x, y, z, p, q),$$

and consequently, after a suitable transformation (T), it becomes:

$$y' = y.$$

Having said this, we write down the equations of the Bäcklund transformation:

$$\begin{aligned} y' &= f(x, y, z, p, q; z'), &\qquad y' &= y, \\ p' &= \varphi(x, y, z, p, q; z'), &\qquad q' &= \psi(x, y, z, p, q; z'), \end{aligned}$$

so since that transformation is a (B_1) transformation, one has:

$$(34) \qquad \begin{cases} \varphi = \dfrac{1}{\dfrac{\partial f}{\partial z'}}, \\[2ex] \begin{vmatrix} \left(\dfrac{df}{dx}\right) & \dfrac{\partial f}{\partial p} & \dfrac{\partial f}{\partial q} \\[2ex] \left(\dfrac{d\varphi}{dx}\right) & \dfrac{\partial \varphi}{\partial p} & \dfrac{\partial \varphi}{\partial q} \\[2ex] \left(\dfrac{d\psi}{dx}\right) & \dfrac{\partial \psi}{\partial p} & \dfrac{\partial \psi}{\partial q} \end{vmatrix} = 0. \end{cases}$$

In addition, equation (27) does not contain t, since y is an invariant, so one sees immediately that f cannot depend upon q, and equations (34) show that the same is true for φ and ψ [we exclude the case where equation (27) reduces to $df / dx = 0$]. The equations that define the transformation can thus be written:

$$(35)\qquad \begin{cases} x' = f(x, y, z, p; z'), & y' = y, \\ p' = \dfrac{1}{\dfrac{\partial f}{\partial z'}}, & q' = \psi(x, y, z, p; z'). \end{cases}$$

If f does not depend upon z' then one can no longer argue as we just did: The transformation must then belong to the special class of transformations that was pointed out by Goursat, and the equations that satisfy the defining ones will be:

$$(35)'\qquad \begin{cases} x' = f(x, y, z, p, q), & y' = y, \\ z' = f(x, y, z, p, q), & p' = g(x, y, z, p, q; q'). \end{cases}$$

Upon proceeding as above, one finds that g does not contain q' and that the determinant:

$$\begin{vmatrix} \left(\dfrac{d\varphi}{dx}\right) & \dfrac{\partial\varphi}{\partial p} & \dfrac{\partial\varphi}{\partial q} \\ \left(\dfrac{d\varpi}{dx}\right) & \dfrac{\partial\varpi}{\partial p} & \dfrac{\partial\varpi}{\partial q} \\ \left(\dfrac{dg}{dx}\right) & \dfrac{\partial g}{\partial p} & \dfrac{\partial g}{\partial q} \end{vmatrix}$$

must be zero. Transformations such as (35)′ do not differ from transformations as in (35), so one easily perceives that one passes from one to the other by replacing the primed variables with the ones that are not, and conversely.

Upon operating on equations (35)′ in the way that we explained (no. 4 and 5), one finds that z satisfies a Monge-Ampère equation in which t does not appear, while z' satisfies an equation of the following form:

$$s' + q'\, G(x', y', z', p', q') + K(x', y', z', p', r') = 0.$$

15. We just confirmed that in certain cases a (B_1) transformation can make a first-order invariant correspond to an invariant of the same order. We shall see that the same thing is always true for the equations of the same form as the ones that we obtained. In order to this, we shall show that one can always find (B_1) transformations that correspond to these equations by algebraic calculations and by the integration of first-order, partial differential equations in just one unknown function. Moreover, it will suffice to find one of these transformations, since all of the other ones are deduced from it by contact transformations.

Consider a Monge-Ampère equation such that one of its systems of characteristics admits the invariant y:

$$r + m\, s + M = 0;$$

this equation is derived from the transformation that is defined by the equations:

$$x' = \varphi(x, y, z, p, q), \qquad y' = y,$$

$$z' = \varpi(x, y, z, p, q), \qquad p' = \frac{m\dfrac{\partial\varpi}{\partial p} - \dfrac{\partial\varpi}{\partial q}}{m\dfrac{\partial\varphi}{\partial p} - \dfrac{\partial\varphi}{\partial q}} = g(x, y, z, p, q),$$

where φ and ϖ are two integrals of:

(36)
$$X(f) = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} - (M + \lambda m)\frac{\partial f}{\partial p} + \lambda\frac{\partial f}{\partial q} = 0,$$

in which λ denotes an arbitrary function of x, y, z, p, q (no. 8).

In order for the transformation considered to be a (B_1) transformation, it is necessary that g likewise satisfy (36). This condition is easily expressed by remarking that if f is a solution of (36) then the derivative of $X(f)$ with respect to an arbitrary variable is zero. It results from this remark that one has:

$$X\left(\frac{\partial f}{\partial p}\right) = -\frac{\partial f}{\partial z} + \frac{\partial(M + \lambda m)}{\partial p}\frac{\partial f}{\partial p} - \frac{\partial\lambda}{\partial p}\frac{\partial f}{\partial q},$$

$$X\left(\frac{\partial f}{\partial q}\right) = \frac{\partial(M + \lambda m)}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial\lambda}{\partial q}\frac{\partial f}{\partial q}.$$

Upon developing the calculations, one finds that g satisfies (36) if φ and ϖ are two distinct integrals of the system:

(37)
$$\begin{cases} \dfrac{\partial f}{\partial x} + [p(\xi + mH) - M - \lambda m]\dfrac{\partial f}{\partial p} + (\lambda - pH)\dfrac{\partial f}{\partial q} = 0, \\ \dfrac{\partial f}{\partial z} - (\xi + mH)\dfrac{\partial f}{\partial p} + H\dfrac{\partial f}{\partial q} = 0, \end{cases}$$

where H denotes a function of x, y, z, p, q, and where one has set, to abbreviate:

$$m\,\xi = \frac{\partial m}{\partial x} + p\frac{\partial m}{\partial z} - M\frac{\partial m}{\partial p} + m\frac{\partial M}{\partial p} - \frac{\partial M}{\partial q}.$$

The system (37) must be complete. This condition determines the two functions H and λ. Upon combining the two equations that express the idea that the system enjoys this property, one sees that λ is defined by the equality:

$$\lambda\left(\frac{\partial m}{\partial z} - \xi\frac{\partial m}{\partial p} + m\frac{\partial\xi}{\partial p} - \frac{\partial\xi}{\partial q}\right) = \frac{\partial\xi}{\partial x} + p\frac{\partial\xi}{\partial z} - M\frac{\partial\xi}{\partial p} - \xi^2 + \xi\frac{\partial M}{\partial p} - \frac{\partial M}{\partial z},$$

and the fact that H must verify the equation:

$$\frac{\partial H}{\partial x}+p\frac{\partial H}{\partial z}+(M+\lambda m)\frac{\partial H}{\partial p}-\lambda\frac{\partial H}{\partial q}+(\xi+mH)\left(H-\frac{\partial \lambda}{\partial p}\right)+H\frac{\partial \lambda}{\partial q}+\frac{\partial \lambda}{\partial z}=0.$$

If φ and ϖ are calculated in this way then it results from the developments of the preceding paragraph that the equation that defines z' is of the form:

$$s' + q'\,G(x', y', z', p', r') + K(x', y', z', p', r') = 0.$$

One immediately sees that, conversely, when one is given such an equation, the transformation:

$$x = h(x', y', z', p'; z), \qquad y = y',$$

$$p = \frac{1}{\dfrac{\partial h}{\partial z}}, \qquad q = k(x', y', z', p'; z)$$

makes it correspond to a Monge-Ampère equation of the type indicated if h sand k are determined by the two equations:

$$\frac{\partial h}{\partial z'}=\frac{\partial h}{\partial p'}G\left(x',y',z',p',-\frac{\dfrac{\partial h}{\partial x'}+p'\dfrac{\partial h}{\partial z'}}{\dfrac{\partial h}{\partial p'}}\right),$$

$$\frac{\partial h}{\partial y'}+k\frac{\partial h}{\partial z}=\frac{\partial h}{\partial p'}K\left(x',y',z',p',-\frac{\dfrac{\partial h}{\partial x'}+p'\dfrac{\partial h}{\partial z'}}{\dfrac{\partial h}{\partial p'}}\right).$$

Gomes Teixeira has studied some transformations that are analogous to the ones that we just considered, but limited to the special case where the first two equations that define the transformation reduce to:

$$x' = x, \qquad y' = y.$$

In order for this to be possible, it is necessary that the coefficient of λ in the equation that determines this quantity be zero, so one finds a relation between m and M. As for the equation that determines z', G is then independent of r'. Conversely, if G does not contain r' then that equation is reducible to a Monge-Ampère equation by a transformation of Gomes Teixiera.

In order to solve the problem that we studied in this paragraph, it is not necessary to completely integrate the equation that gives H, but it suffices to determine a particular integral.

In the proposed Monge-Ampère equation, we have supposed that the coefficient of r was unity in order to simplify the calculations a little, but that hypothesis is not essential. We shall not develop the calculations in the case of an equation of the form:

$$s + g(x, y, z, p, q) = 0,$$

since one only has to operate as one does in the general case. Here, one can deduce a (B_1) transformation from the two systems of characteristics. In particular, one confirms that in order for a transformation of Gomes Teixeira to be applicable to the preceding equation, it is necessary and sufficient that g be linear with respect to p or q.

16. One of the systems of characteristics of the Monge-Ampère equation:

$$r + ms + M = 0$$

admits the first-order invariant y. We shall now consider the case where the same systems of characteristics likewise admits a second-order invariant. From what we proved above (no. 14), one of the systems of characteristics of the transformed equation will possess two first-order invariants, one of which will be y'. This transformed equation will be integrable by the Monge method, and along with being linear with respect to s' and q', it will necessarily be of the following form:

$$\frac{d}{dx'}[q'\,\rho(x', y', z', p') + \sigma(x', y', z', p')] = 0. \tag{38}$$

We shall first show that the transformation that permits one to pass from the original Monge-Ampère equation to equation (38) differs from the Teixera transformation only by transformations (T) and (T'). Define a transformation (T) by the equations:

$$X' = \xi(x', y', z', p'),$$

$$Y' = y',$$

$$Z' = \zeta(x', y', z', p'),$$

$$P' = \frac{\dfrac{\partial \zeta}{\partial p'}}{\dfrac{\partial \xi}{\partial p'}},$$

$$Q' = q'\rho(x', y', z', p') + \frac{\partial \zeta}{\partial y'} - \frac{\dfrac{\partial \zeta}{\partial p'}}{\dfrac{\partial \xi}{\partial p'}} \frac{\partial \xi}{\partial y'},$$

in which ζ satisfies the equation:

$$\rho \frac{\partial \zeta}{\partial z'} - \rho^2 + [\rho, \zeta] = 0,$$

and ξ is an integral of the complete system:

$$\frac{\partial \xi}{\partial x'} - \frac{\dfrac{\partial \zeta}{\partial x'} + p'\rho}{\dfrac{\partial \zeta}{\partial p'}} \frac{\partial \xi}{\partial p'} = 0,$$

$$\frac{\partial \xi}{\partial z'} - \frac{\dfrac{\partial \zeta}{\partial z'} - \rho}{\dfrac{\partial \zeta}{\partial p'}} \frac{\partial \xi}{\partial p'} = 0.$$

Equation (38) then becomes:

(38)′ $$\frac{d}{dX'}[Q' + \Theta(X', Y', Z', P')] = 0$$

and no longer contains Q'. Under these conditions, the (B_1) transformation that will change equation (38)′ into the proposed equation, or at least into a equation that is identical to the proposed one up to a contact transformation, will be a Teixeira transformation that one can write down by calling the independent variables that do not change x, y, instead of X'¸Y', namely:

$$z = P', \qquad q + p\frac{\partial \Theta}{\partial P'} + z\frac{\partial \Theta}{\partial L'} = 0.$$

z is defined by a Monge-Ampère equation, one of whose systems of characteristics possesses a first-order invariant and a second-order invariant, and when the function Θ takes on all of the possible forms, one obtains all of the Monge-Ampère equations that enjoy that property.

PART THREE

17. When two second-order, partial differential equations can be converted into each other by a Bäcklund transformation of the first kind, and when one can, consequently, being given the general integral of one of the equations, calculate the general integral of the other one with no integration, it is quite obvious that if an equation has an explicit general integral then the same is true for the other one. More generally, if one of the two equations admits a general integral of the first kind then the other one enjoys the same property. When one considers the (B_2) and (B_3) transformations, one cannot see how things behave as easily. Here, we consider only the general case. We only show that if a Bäcklund transformation makes each integral of one equation (ε) correspond to an infinitude of integrals of an equation (ε'), and if (ε) admits an explicit general integral then the equation (ε') is of the first kind.

In order to do this, we prove the following proposition, from which the stated property will ensue in an obvious way.

If the total differential equation:

$$du = \varphi(\alpha, \beta, A, A', \ldots, A^{(m)}, B, B', \ldots, B^{(n)}, u)\, d\alpha + \psi(\alpha, \beta, A, A', \ldots, A^{(m)}, B, B', \ldots, B^{(n)}, u)\, d\beta, \tag{39}$$

where A denotes a function of α and B denotes a function of β, is completely integrable then for any functions A and B one can give this equation the following form:

$$R(\alpha, \beta, A, A', \ldots, A^{(m)}, B, B', \ldots, B^{(n)}, u) = \int \Phi(\alpha, A, A', \ldots, A^{(m)})\, d\alpha + \int \Psi(\beta, B, B', \ldots, B^{(n)})\, d\beta. \tag{40}$$

Goursat has already proved this theorem in the case where φ and ψ do not contain u ([1]). We shall study the general case.

By hypothesis, the equality:

$$\frac{\partial \varphi}{\partial \beta} + \frac{\partial \varphi}{\partial B} B' + \cdots + \frac{\partial \varphi}{\partial B^{(n)}} B^{(n+1)} + \frac{\partial \varphi}{\partial u}\psi = \frac{\partial \psi}{\partial \alpha} + \frac{\partial \psi}{\partial A} A' + \cdots + \frac{\partial \psi}{\partial A^{(m)}} A^{(m+1)} + \frac{\partial \psi}{\partial u}\varphi, \tag{41}$$

which expresses the idea that equation (39) is completely integrable, is verified identically if one regards $\alpha, A, A', \ldots, A^{(m+1)}, \beta, B, B', \ldots, B^{(n+1)}$ as independent variables.

([1]) Bulletin de la Société mathématique, t. XXV, 1897. *Leçons sur l'intégration des équations aux dérivées partielles du second ordre,* t. II, pp. 233.

It then results immediately that φ does not depend upon $B^{(n)}$ and ψ does not depend upon $A^{(m)}$.

Having said this, upon differentiating the two sides of the equality (41) twice with respect to $A^{(n)}$ and twice with respect to $B^{(n)}$, one gets:

$$\psi''\frac{\partial\varphi}{\partial u} = \varphi''\frac{\partial\psi}{\partial u}.$$

If neither of the two quantities φ'', ψ'' is zero then, if one refers to a conveniently-chosen function of $\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, u$ as N, one can write that:

$$\frac{\dfrac{\partial\varphi''}{\partial u}}{\varphi''} = \frac{\dfrac{\partial\psi''}{\partial u}}{\psi''} = \frac{\dfrac{\partial N}{\partial u}}{N},$$

or, upon integrating:

$$\begin{aligned}\varphi = {} & A^{(m)}\lambda(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, u) \\ & + \rho(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, u) \\ & + f(\alpha, \beta, A, A', \ldots, A^{(m)}, B, B', \ldots, B^{(n-1)}, u)\, N,\end{aligned}$$

$$\begin{aligned}\psi = {} & B^{(n)}\mu(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, u) \\ & + \sigma(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, u) \\ & + g(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n)}, u)\, N,\end{aligned}$$

Define a new variable v by the equation:

$$\frac{du}{N} = dv.$$

Equation (39) then becomes:

$$\text{(39)}' \qquad \begin{aligned}dv = {} & [A^{(m)}\lambda_0(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, v) \\ & + \rho_0(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, v) \\ & + f(\alpha, \beta, A, A', \ldots, A^{(m)}, B, B', \ldots, B^{(n-1)})]\, d\alpha \\ & + [B^{(n)}\mu_0(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, v) \\ & + \sigma_0(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, v) \\ & + g(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n)})]\, d\beta,\end{aligned}$$

in which λ_0, ρ_0, μ_0, σ_0 are functions whose explicit calculation presents no difficulty. I will not indicate their expressions, any more than those of the new functions that we will consider in the course of that proof. If will suffice for us to know what the quantities are that appear in each of these functions, which are indicated between parentheses.

Upon differentiating the integrability condition for the preceding equation twice with respect to $A^{(m)}$ and once with respect to $B^{(n)}$, one finds:

$$\frac{\partial f''}{\partial B^{(n-1)}} - \frac{\partial \mu_0}{\partial v} f'' = 0.$$

Since f'' is non-zero, one deduces from this that μ_0 is linear with respect to v, so the same must naturally be true for λ_0, and one can write:

$$\begin{aligned} dv \;=\; & [v\, A^{(m)} \omega(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}) \\ & \quad + \rho_0(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, v) \\ & \quad + f_0(\alpha, \beta, A, A', \ldots, A^{(m)}, B, B', \ldots, B^{(n-1)})]\, d\alpha \\ & + [v\, B^{(n)} \varpi(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}) \\ & \quad + \sigma_0(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, v) \\ & \quad + g_0(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n)})]\, d\beta . \end{aligned}$$

This equation is completely integrable. Upon differentiating both sides of the equality that expresses this property with respect to $A^{(m)}$ and $B^{(n)}$, it becomes:

$$v \frac{\partial \omega}{\partial B^{(n-1)}} + \frac{\partial f_0'}{\partial B^{(n-1)}} + \omega g_0' = v \frac{\partial \varpi}{\partial A^{(m-1)}} + \frac{\partial g_0'}{\partial A^{(m-1)}} + \varpi f_0' . \tag{42}$$

In order for this condition to be verified, it is necessary that one first have:

$$\frac{\partial \omega}{\partial B^{(n-1)}} = \frac{\partial \varpi}{\partial A^{(m-1)}},$$

and if M denotes a suitably-chosen function of $\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}$ then one will thus have:

$$\omega = \frac{1}{M} \frac{\partial M}{\partial A^{(m-1)}}, \qquad \varpi = \frac{1}{M} \frac{\partial M}{\partial B^{(n-1)}};$$

i.e., equation (39)′ is written:

$$dv - \frac{v}{M} \frac{\partial M}{\partial A^{(m-1)}}\, dA^{(m-1)} - \frac{v}{M} \frac{\partial M}{\partial B^{(n-1)}}\, dB^{(n-1)} = (\rho_0 + f_0)\, d\alpha + (\sigma_0 + g_0)\, d\beta.$$

Upon setting:

$$\frac{v}{M} = w_1,$$

one finds, after a simple calculation, that:

$$\begin{aligned} dw_1 = \; & [\rho_1(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, w_1) \\ & + f_1(\alpha, \beta, A, A', \ldots, A^{(m)}, B, B', \ldots, B^{(n-1)})]\, d\alpha \\ & + [\sigma_1(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, w_1) \\ & + g_1(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, v)]\, d\beta. \end{aligned}$$

The integrability condition for this new equation, when differentiated twice with respect to $A^{(m)}$, gives:

$$\frac{df_1''}{d\beta} = \frac{\partial \sigma_1}{\partial w_1} f_1''.$$

σ_1 and ρ_1 are thus linear with respect to w_1, and we can replace the equation that we just wrote with the following one:

$$\begin{aligned} dw_1 = \; & [w_1\theta_1(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}) \\ & + h_1(\alpha, \beta, A, A', \ldots, A^{(m)}, B, B', \ldots, B^{(n-1)})]\, d\alpha \\ & + [w_1\chi_1(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}) \\ & + k_1(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, v)]\, d\beta, \end{aligned}$$

which is also completely integrable. Only A and its derivatives up to order $m - 1$ appear in θ_1 and χ_1, while only B and its derivatives up to order $n - 1$ appear, and w_1 is a function of u, α, β, A, A', …, $A^{(m-1)}$, B, B', …, $B^{(n-1)}$. In a more general manner, suppose that one has converted the integration of the proposed equation into the integration of an equation such as:

$$\begin{aligned} dw_i = \; & [w_i\,\theta_i(\alpha, \beta, A, A', \ldots, A^{(m-i)}, B, B', \ldots, B^{(n-i)}) \\ & + \quad h_i(\alpha, \beta, A, A', \ldots, A^{(m)}, B, B', \ldots, B^{(n-1)})]\, d\alpha \\ & + [w_i\chi_i(\alpha, \beta, A, A', \ldots, A^{(m-i)}, B, B', \ldots, B^{(n-i)}) \\ & + \quad k_i(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n)})]\, d\beta, \end{aligned} \tag{43}$$

in which i denotes an integer. θ_i and χ_i do not depend upon derivatives of A of order higher than $m - i$ or derivatives of B of order higher than $n - i$. w_i is expressed with the aid of α, β, A, A', …, $A^{(m-1)}$, B, B', …, $B^{(n-1)}$. I say that one can replace equation (43) with an equation of a similar form, but in which i is replaced by $i + 1$.

By hypothesis, (43) is completely integrable. Upon differentiating the integrability condition twice with respect to $A^{(m)}$, it becomes:

$$\frac{dh^i}{d\beta} = \chi_i h_i''.$$

One concludes from this that h_i'' does not contain derivatives of B of order greater than or equal to $n - i$, because at least one of the derivatives of order higher than $n - i$ will figure in χ_i, which we did not assume. Moreover, χ_i is linear with respect to $B^{(n-1)}$. Set:

$$\theta_i = A^{(m-i)}\,\xi + \zeta,$$
$$\chi_i = B^{(n-i)}\,\eta + \varepsilon.$$

Equation (43) becomes:

$$dw_i = w_i\,\xi\, dA^{(m-i-1)} + w_i\,\eta\, dB^{(n-i-1)} + (\zeta\, w_i + \eta_i)\, d\alpha + (\varepsilon\, w_i + k_i)\, d\beta. \tag{44}$$

In addition, upon writing down the integrability condition, one perceives immediately that one can set:

$$\xi = \frac{\partial K}{\partial A^{(m-i-1)}}, \qquad \eta = \frac{\partial K}{\partial B^{(n-i-1)}},$$

in which K denotes a function of $\alpha, \beta, A, A', \ldots, A^{(m-i-1)}, B, B', \ldots, B^{(n-i-1)}$. ζ and ε no longer contain the derivatives of $A^{(m-i)}$ and $B^{(n-i)}$.

Upon operating as before – i.e., upon taking a new variable to be:

$$w_{i+1} = \frac{w_i}{K},$$

one finds an equation of the stated form:

$$dw_{i+1} = (w_{i+1}\theta_{i+1} + h_{i+1})\, d\alpha + (w_{i+1}\chi_{i+1} + k_{i+1})\, d\beta.$$

Upon calculating the functions $\theta_{i+1}, \chi_{i+1}, h_{i+1}, k_{i+1}$, one verifies that θ_{i+1} and χ_{i+1} do not depend upon either the derivatives of A of order higher than $m - i - 1$ or on the derivatives of B of order higher than $n - i - 1$. Upon operating in this way step-by-step ([1]), one arrives at an equation:

$$\begin{aligned} dw = {} & H(\alpha, \beta, A, A', \ldots, A^{(m)}, B, B', \ldots, B^{(n-1)})\, d\alpha \\ & + L(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n)})\, d\beta, \end{aligned}$$

in which w denotes a function of $\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}, u$.

From the result of Goursat, this latter equation can be put into the form:

$$\begin{aligned} w = R_1(\alpha, \beta, A, A', \ldots, A^{(m-1)}, B, B', \ldots, B^{(n-1)}) + \int \Phi(\alpha, A, A', \ldots, A^{(m)})\, d\alpha \\ + \int \Psi(\beta, B, B', \ldots, B^{(n)})\, d\beta. \end{aligned}$$

If one replaces w with its expression then one finds that equation (39) can indeed be put into the form (40); this is the result that we shall prove.

We have neglected the case where one of the quantities φ'', ψ'' is zero. It is very easy to see that if ψ'', for example, is zero then equation (39) can be converted into an analogous equation in which only n is replaced with $n - 1$. Let:

$$(45) \qquad du = \varphi\, d\alpha + (B^{(n)}\, \psi_0 + \psi_1)\, d\beta$$

be the equation that would like to study. We write it in the following manner:

([1]) The consideration of the integrability condition indicates the modifications – which are important, moreover – that one must make to the preceding arguments when either one of the numbers $m - i$, $n - i$ is zero or both of them are.

(45)′ $$du = \varphi\, d\alpha + \psi_0\, dB^{(n-1)} + \psi_1\, d\beta.$$

One can find two functions χ and π of $\alpha, \beta, A, \ldots, A^{(m-1)}, B, \ldots, B^{(n-1)}$ such that one has:

$$du - \psi_0\, dB^{(n-1)} = \pi\left(\frac{\partial\chi}{\partial u}du + \frac{\partial\chi}{\partial B^{(n-1)}}dB^{(n-1)}\right);$$

i.e., such that one has:

$$\begin{aligned} du - \psi_0\, dB^{(n-1)} = \pi\Bigg[d\chi &- \left(\frac{\partial\chi}{\partial\alpha} + \frac{\partial\chi}{\partial A}A' + \cdots + \frac{\partial\chi}{\partial A^{(m-1)}}A^{(m)}\right)d\alpha \\ &- \left(\frac{\partial\chi}{\partial\beta} + \frac{\partial\chi}{\partial B}B' + \cdots + \frac{\partial\chi}{\partial B^{(n-2)}}B^{(n-1)}\right)d\beta\Bigg], \end{aligned}$$

and if we take χ to be the new variable then we can replace equation (45)′ with:

$$d\chi = \lambda\, d\alpha + \mu\, d\beta,$$

in which λ, μ are functions of $\alpha, \beta, A, A', \ldots, A^{(m)}, B, B', \ldots, B^{(n-1)}), \chi$. If the product $\frac{\partial^2\lambda}{\partial(A^{(m)})^2}\frac{\partial^2\mu}{\partial(B^{(n-1)})^2}$ is zero then one continues in the same manner, except that one will apply the reduction process that we first presented.

It is not without interest to remark that the preceding proposition is not only useful for the theory of Bäcklund transformations, but also in other question in the theory of partial differential equations. For example, it can indeed happen that the application of the Darboux method leads to the integration of an equation such as (39) that one can simplify in the manner that we just explained.

We make another remark: Imagine that an equation that is linear in r, s, t admits an explicit general integral that is expressed with the aid of two auxiliary variables α, β and a function A of α and its derivatives up to order n, or, in a more precise manner, that x, y, z, p, q are expressed as functions of these quantities ([1]). If a Bäcklund transformation makes each integral of (ε) correspond to an infinitude of integrals of another equation (ε') then one must calculate the quantity that we have called ζ (no. 2), and one sees that $A^{(m+1)}$ and $B^{(n+1)}$ enter into equation (3)″ linearly. When one has simplified the expression for ζ, $A^{(m+1)}$ and $B^{(n+1)}$ will therefore no longer enter in. It has been convenient for us to assume that (ε) is an equation that is linear in r, s, t, but it immediately emerges that if (ε) contains a term in $rt - s^2$ then this proposition is still true.

18. We have shown (no. 14) that if a Bäcklund transformation makes an integral of an equation (27) correspond to just one integral of an equation (28) then each invariant of one of the characteristic systems of (28) will correspond to an invariant of one of the characteristic systems of (27). An invariant of (C') of order n will correspond to an

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([1]) There is no reason to consider the expressions for x, y, z separately, since for us a partial differential equation is defined only up to a contact transformation.

invariant of (C) order at most n, and an invariant of (Γ') of order n will correspond to an invariant of (Γ) of order $n + 1$. The arguments that led to these results did not suppose that the transformation considered is a (B_1) transformation. They still persist when an integral of (28) corresponds to ∞^1 integrals of (27).

We shall now assume that a Bäcklund transformation makes each integral of the Monge-Ampère equation:

$$r + (m + \mu)\, s + m\,\mu\, t + M = 0 \tag{46}$$

correspond to an infinitude of integrals of the equation:

$$r' + G(x', y', z', p', q', s', t') = 0, \tag{47}$$

and investigate what the system of characteristics of (47) becomes under this transformation. The results that we shall prove are much less precise than the ones that were obtained in the study of (B_1) transformations. I shall at least attempt to point out what sort of cases might present themselves.

If one is given an arbitrary function F' of $x', y', z', \ldots, p'_{1,n-1}, p'_{0,n}$ then upon replacing these quantities by their values (no. 3), one finds a certain function F of $x, y, z, \ldots, p_{1,n-1}, p_{0,n}, z'$. It is very easy to see that if the transformation considered is of the third kind and if n is equal to at most 2 then this latter function depends essentially on derivatives of order n. Indeed, if things were otherwise then one would have a relation of the form:

$$F'(x', y', z', \ldots, p'_{1,n-1}, p'_{0,n}) = F(x, y, z, \ldots, p_{1,n-1}, p_{0,n}, z'),$$

and upon replacing $x, y, z, \ldots, p_{1,n-1}, p_{0,n}, z'$ in F with their values as functions of z, x', y', z', and the derivatives of z', one would find a relation such as:

$$F'(x', y', z', \ldots, p'_{1,n-1}, p'_{0,n}) = F_1(x', y', z', \ldots, p'_{1,n-2}, p'_{0,n-1}, z).$$

It is impossible that F_1 might contain z, since an integral of (47) will correspond to only one integral of (46) that is determined by the preceding equation and the equations of the Bäcklund transformation, which will therefore not be of the third kind. On the other hand, if F_1 does not depend upon z then one will have a relation between x', y', z', and the derivatives of z' whose first index is 0 or 1, which is likewise impossible, since equation (47) has been solved for r'. We have seen (no. 9) that if we have a Bäcklund transformation of the second kind then it is possible that a function such as F' is expressed as a function of only x, y, z, and the derivatives of z up to order $n - 1$.

Having made that remark, first suppose that there exists an equation:

$$F_1(x', y', z', \ldots, p'_{1,n-2}, p'_{0,n-1}) = 0, \tag{48}$$

that forms a system in involution with (47). By replacing x', y', and the derivatives of z' in the left-hand side of that equation with their values as functions of z', x, y, z, and the derivatives of z, one finds an equation:

(49) $$F_0(x, y, z, \ldots, p_{1,n-1}, p_{0,n}, z') = 0$$

that is verified if one replaces x, y, z, …, $p_{1,n-1}$, $p_{0,n}$, z' with the expressions that correspond to two integrals that are transforms of each other, while z' satisfies (48).

If F_0 does not depend upon z' then equation (49) obviously corresponds to equation (48), and forms a system in involution with (46). On the contrary, if the variable z' does enter into F_0 then one appends to equation (49) its derivative with respect to one of the two independent variables – y, for example – which is written:

$$\frac{dF_0}{dy} = \frac{\partial F_0}{\partial y} + \frac{\partial F_0}{\partial z} + \cdots + p_{1,n}\frac{\partial F_0}{\partial p_{1,n-1}} + p_{0,n+1}\frac{\partial F_0}{\partial p_{0,n}} + (C + \alpha s + \beta t)\frac{\partial F_0}{\partial z'} = 0,$$

in which we preserve the notations that were employed before (no. 1).

We may eliminate z' from equation (49) and the derived equation, and what then remains is an equation of $(n + 1)^{\text{th}}$ order that forms a system in involution with (46). We can summarize by saying that an equation of order n that forms a system in involution with (47) will correspond to an equation of order at most equal to $n + 1$ that forms a system in involution with (46).

Now, suppose that one of the systems of characteristics of (47) admits an invariant Φ_1 of n^{th} order, i.e., that no matter what the constant K is, the equation:

(50) $$\Phi_1(x', y', z', \ldots, p'_{1,n-1}, p'_{0,n}) = K$$

will admit an infinitude of integrals that depend upon an infinitude of arbitrary constants that simultaneously satisfy (47). We operate as we just did in the study of the preceding question. Replace x', y', z', …, $p'_{1,n-1}$, $p'_{0,n}$ with their values. We find a function Φ_0 of z', x, y, z, …, $p_{1,n-1}$, $p_{0,n}$ such that the equation:

(51) $$\Phi_0(x, y, z, \ldots, p_{1,n-1}, p_{0,n}, z') = K$$

is verified by corresponding integrals of equation (46) and (47), on the condition that z' must likewise satisfy equation (50).

As we did just now, one must distinguish several cases. If the left-hand side of equation (51) does not depend upon z' then one has an invariant of n^{th} order or less that corresponds to the invariant Φ_1 . On the contrary, if z' enters into Φ_0 then one differentiates equation (51) with respect to y, and it becomes:

(52) $$\frac{d\Phi_0}{dy} = 0.$$

It can happen that the function $d\Phi_0 / dy$ does not depend upon z'. Equation (52) then forms a system in involution with (46). No matter what the value of K, all of the integrals that are common to equations (47) and (50) correspond to integrals that simultaneously satisfy equations (46) and (52). Finally, if $d\Phi_0 / dy$ contains z' then one can deduce that quantity from equation (52) by substituting in Φ_0, and one will find a function Ψ_0 of x, y, z, …, $p_{1,n}$, $p_{0,n+1}$ such that the equation:

$$\Psi_0 = K$$

forms a system in involution with (46) for all values of K.

By definition, if there exists an invariant of order n for one of the characteristic systems of (47) then the corresponding system of characteristics of (46) admits an invariant of order less than or equal to $n + 1$, or there exists a unique equation of order at most $n + 1$ that forms a system in involution with (46).

We just found an upper bound on the order of the invariant or the equation that corresponds to the known invariant Φ_1. The remark that we made at the beginning of this paragraph permits us to give some indications on the lower bound. Suppose that the transformation considered is a (B_3) transformation: Φ_0 depends essentially on the derivatives of z of order n. Φ_1 corresponds to either an invariant of order n or $n + 1$ or to a unique equation of order $n + 1$. If the transformation that we are considering is a (B_2) transformation, and if Φ_0 contains derivatives of order n, then we only need to repeat what we just said, but it can happen that these derivatives do not appear in Φ_0. When this is true, the invariant Φ_1 corresponds to either an invariant of order $n – 1$ or n or to an equation of order n that forms a system in involution with (46).

One easily sees that there can be an exception if n is equal to one or two, and only in this case. Indeed, if Φ_0 depends upon the variables x, y, z, p, q, z', and not on the derivatives of z of higher order then one has:

$$\frac{d\Phi_0}{dy} = \frac{\partial\Phi_0}{\partial y} + q\frac{\partial\Phi_0}{\partial z} + s\frac{\partial\Phi_0}{\partial p} + t\frac{\partial\Phi_0}{\partial q} + (C + \alpha s + \beta t)\frac{\partial\Phi_0}{\partial z'} = 0.$$

It is possible that upon reducing this one finds an equation that is independent of s and t.

We shall now examine the case in which one of the systems of characteristics of equations (47) admits two distinct invariants of order m and n ($m \leq n$). Let u_1 be the invariant of order m and let v_1 be the invariant of order n. If one is given an arbitrary integral of equation (47) then there will exist a function f such that this integral simultaneously satisfies the equation:

$$v_1 = f(u_1). \tag{53}$$

In addition, if one is given an equation of the preceding form and any sort of function f then this equation admits an infinitude of integrals that depend upon an infinitude of arbitrary constants that likewise satisfy equation (47). By proceeding as we did before, we make each equation such as (53) correspond to an equation of order $n + 1$ or less that forms a system in involution with (46). Moreover, if one is given an arbitrary integral of (46) then there exists an equation of order at most $n + 1$ that admits that integral and which forms a system in involution with (46). Indeed, consider an integral of (47) that corresponds to the given integral of (46). We can choose the function f in such a way that equation (53) possesses that integral, and upon transforming that equation one finds an equation of order less than or equal to $n + 1$ that admits, at the same time as the given integral, an infinitude of integrals that depend upon an infinitude of arbitrary constants and satisfy (46). It results from this that one of the systems of characteristics of (46) possesses two invariant of order at most equal to $n + 1$.

At the basis of the preceding, we have simply proved that the invariants of the systems of characteristics of equation (47) can be transformed in several different manners. It is not pointless to indicate some examples that show that these different kinds of transformations can actually present themselves.

Consider the very simple equation:

$$q' r' - p' s' + f(x, y, z, p', q') = 0. \tag{54}$$

One of the systems of characteristics (Γ') admits the first-order invariant y and the other one (C') admits the invariant z'. If one preserves the independent variables then the transformation:

$$z = p', \qquad q' p - z q + f(x, y, z, q') = 0 \tag{55}$$

replaces equation (54) with another Monge-Ampère equation. If one imagines that equations (55) are solved for p' and q':

$$p' = z, \qquad q' = \psi(x, y, z, p, q) \tag{55$'$}$$

then equation (ε), which defines z, is identical to the integrability condition of:

$$dz' - z\,dx - \psi(x, y, z, p, q)\,dy = 0. \tag{56}$$

The invariant y, which belongs to the system (Γ'), likewise belongs to the corresponding system of characteristics (Γ) of (ε), but the invariant z' of the system (C') corresponds to the equation:

$$\psi(x, y, z, p, q) = 0,$$

or, if one prefers, the equation:

$$z = 0,$$

which forms a system in involution with (ε).

Now, consider the equation:

$$r' q' - s' p' = p'^3 e^x. \tag{57}$$

The system of characteristics (C) admits the invariant z' and the second-order invariant:

$$1 = \frac{2r' + p'}{p'^3}.$$

Consequently, it admits a sequence of invariants dI / dz', $d\left(\frac{dI}{dz'}\right) / dz'$, … When one applies the transformation (55), the invariant z' will correspond to a unique equation, but the invariant I will correspond to an invariant that is of first order. Indeed, equation (56) shows that p' and r' are equal to z and p, respectively. Moreover, dI / dz' and all of the invariants that follow correspond to invariants. The second system of characteristics admits a first-order invariant, a second-order one, and so on. The transformation (55) makes them correspond to invariants of the same order.

We have considered only (B_2) transformations, but it is very easy to find (B_3) transformations that transform the invariants as we described in the general theory. Imagine that the function f in equation (54) does not depend upon y; i.e., that equation (54) remains invariant under a translation that is parallel to Oy, or furthermore that the equation admits an infinitesimal contact transformation whose characteristic function is q'. We have already explained that there exists a Monge-Ampère equation (ε_1) that is the transform of (54), such that the equations of the transformation do not contain y, which corresponds to (ε) by a (B_3) transformation. In particular, one finds that one of the systems of characteristics of (ε_1) admits a first-order invariant that is the transform of z' and corresponds to the unique equation $z = 0$.

It is, moreover, important to make the following remark: In general, the system (Γ) admits no invariant. Meanwhile, the system (Γ') admits one that is z'. Consequently, when one studies a Bäcklund transformation that is not a (B_1) transformation, one is not assured of finding all of the invariants of a system of characteristics of one of the equations when one has only looked for the invariants that are the transforms of the invariants of the corresponding systems of characteristics of the second equation.

19. I conclude this paper with the study of an interesting Bäcklund transformation ([1]). I have already recalled the results that were obtained by Lie, Bäcklund, and Darboux in the study of surfaces of constant curvature. A system of two first-order elements (x, y, z, p, q), (x', y', z', p', q') admits four invariants relative to the group of motions in space. Upon equating these invariants to constants, one obtains the equations of the transformation that was studied by Darboux that is applied to the surfaces that are parallel to the surfaces of constant total curvature.

We shall show that there exists an analogous transformation in non-Euclidian geometry. For this, we employ a method that is entirely similar to that of Darboux, which will provide us with an occasion to recall the elegant manner by which that geometer had proved the theorems that were found by Bäcklund (no. 1).

If we are given a quadric (S), which we call the *fundamental quadric*, and which we suppose has a reduced equation of the simple form:

$$x^2 + y^2 + z^2 + 1 = 0,$$

then by equating the constants of the four invariants of a system of two elements (x, y, z, p, q), (x', y', z', p', q') relative to the group of projective transformations that leave the quadric (S) invariant, we find a system of four equations:

([1]) Bulletin des Sciences mathématiques, t. XXIV, pp. 284; 1900.

$$(58)\quad \begin{cases} F_1 = \dfrac{[xx' + yy' + zz' + 1]^2}{[x^2 + y^2 + z^2 + 1][x'^2 + y'^2 + z'^2 + 1]} = k^2, \\ F_2 = \dfrac{[p(x - x') + q(y - y') - (z - z')]^2}{[p^2 + q^2 + (px + qy - z)^2 + 1][x'^2 + y'^2 + z'^2 + 1]} = m^2, \\ F_3 = \dfrac{[p'(x' - x) + q'(y' - y) - (z' - z)]^2}{[x^2 + y^2 + z^2 + 1][p'^2 + q'^2 + (p'x' + q'y' - z')^2 + 1]} = n^2, \\ F_4 = \dfrac{[pp' + qq' + (px + qy - z)(p'x' + q'y' - z') + 1]^2}{[p^2 + q^2 + (px + qy - z)^2 + 1][p'^2 + q'^2 + (p'x' + q'y' - z')^2 + 1]} = l^2, \end{cases}$$

which defines the transformation that shall study, and in which k, m, n, l denote constants.

Upon differentiating the preceding system, one finds:

$$\frac{dF_i}{dx}dx + \frac{dF_i}{dy}dy + \left(\frac{dF_i}{dx'}\right)dx' + \left(\frac{dF_i}{dy'}\right)dy' + \frac{\partial F_i}{\partial p'}dp' + \frac{\partial F_i}{\partial q'}dq' = 0 \qquad (i = 1, 2, 3, 4).$$

If one solves the preceding equations with respect to dp', dq', dx, dy then if one lets H, K, L, M, N denote certain functions of x, y, z, p, q one gets:

$$H\,dp' = K\,dx' + L\,dy',$$
$$H\,dq' = M\,dx' + N\,dy',$$

and the elements (x', y', z', p', q') generate a surface if one has:

$$L = M.$$

One can see that in the particular case that we are occupied with the preceding condition reduces to a Monge-Ampère equation that determines z. However, the calculation will be somewhat lengthy, and one would prefer to proceed otherwise. Consider the element $x = y = z = p = q = 0$. Upon performing the calculations that we just indicated, one finds that the values of r, s, t that correspond to that element must satisfy the equation:

$$(1 - n^2 - k^2)(rt - s^2) - (nl + km)(r + t) + 1 - m^2 - l^2 = 0.$$

The group considered admits two second-order, differential invariants:

$$I = \frac{[x^2 + y^2 + z^2 + 1]^2}{[p^2 + q^2 + (px + qy - z)^2 + 1]}(rt - s^2),$$

$$J = \frac{[x^2 + y^2 + z^2 + 1]^{1/2}}{[p^2 + q^2 + (px + qy - z)^2 + 1]^{3/2}} \times \{[(1 + q^2)\,r - 2pqs + (1 + p^2)\,t][x^2 + y^2 + z^2 + 1] - (y + qz)^2\,r + 2(y + qz)(x + pz)\,s - (x + px)^2\,t\},$$

and if one annuls x, y, z, p, q then the values of these two invariants are $(rt - s^2)$ and $r + t$.

On the other hand, there always exists an infinitude of transformations of the group considered that permit one to convert an arbitrary element (x, y, z, p, q) to the origin element, on the condition that the point (x, y, z) does not belong to the fundamental quadric. The transformation that is defined by the system (58) thus applied to the integral surfaces of the equation:

$$(1 - n^2 - k^2)\, I - (nl + km)\, J + (1 - m^2 - l^2) = 0. \tag{59}$$

As for the transformed surfaces, they satisfy the equation:

$$(1 - m^2 - k^2)\, I' - (ml + nk)\, J' + (1 - n^2 - l^2) = 0,$$

in which I' and J' are deduced from I and J, respectively, by accenting all of the symbols that appear in them.

Let D denote the line perpendicular to the point $M(x,y, z)$ on the plane (p, q); i.e., the line that passes through M and the pole of the plane with respect to the quadric (S). Likewise, let D' denote perpendicular to the point $M'(x', y', z')$ on the plane (p', q'). Furthermore, let Δ denote the line conjugate to D and Δ', the line conjugate to D', and suppose that one of the lines that meet D, D', Δ, Δ' is drawn, where N and N' are the points of intersection of that line with D and D'. The points N, N' thus determined belong to an invariable system that is defined by the two points M, M', and the two planes that pass through these points, respectively. If the point M describes a surface that is tangent to the plane (p, q) then the point N will describe a surface whose tangent plane will contain the line Δ. Indeed, let A, B be the intersection points of D with (S). Suppose that the line D is displaced infinitely little and goes to D_1, so the points A, B, M, N go to A_1, B_1, M_1, N_1, and the points A_1, B_1 will be in the planes defined by Δ and the points A, B, M, respectively. Since the anharmonic ratio (A, B, M, N) is equal to the ratio (A_1, B_1, M_1, N_1), the point N will be in the plane that passes through Δ and N.

It is useless to write down the equations of the contact transformation that permits one to replace each first-order element by another element, as we just explained. We only remark that this transformation is completely analogous to the dilatation. When the points M, M' are then replaced with the points N and N', resp., in equations (58) ([1]), the constants m and n reduce to zero, since the line N, N' is the line of intersection of the planes tangent to the surfaces that are described by these points. The surfaces that are described by the point N, for example, satisfy the equation:

$$rt - s^2 + c\,\frac{[p^2 + q^2 + (px + qy - z)^2 + 1]^2}{[x^2 + y^2 + z^2 + 1]^2} = 0, \tag{60}$$

in which c denotes a constant. The integrals of that equation are the surfaces of constant total curvature of non-Euclidian geometry.

([1]) We now denote the coordinates of the transformed elements by $x, y, z, p, q, x', y', z', p', q'$.

20. It might be interesting to study the transformations of these surfaces in a detailed manner, as one does in non-Euclidian geometry. We will not enter into the examination of the questions that are posed on that subject here, and we confine ourselves to showing that the preceding theory is susceptible to applications in ordinary geometry.

Let ρ', ρ'' denote the principal radii of curvature of a surface, r, the radius vector, and d, the distance from the origin to the tangent plane. The integrals of (60) satisfy the relation:

$$\rho'\rho''(1 + d^2)^2 + \frac{1}{C}(1 + r^2)^2 = 0.$$

This equation belongs to a type of equation that was considered by Weingarten. It results from the work of that geometer that the integration of the preceding equation is equivalent to the search for the surfaces that admit the linear element:

$$\frac{\lambda^2}{c^2}\frac{du^2}{(1+u^2)^4}+\frac{2\lambda^2}{c}\frac{u\,du}{(1+u^2)^2}\frac{dv}{(1+2v)^2}+\frac{2\lambda^2 v\,dv^2}{(1+2v)^4},$$

in which λ denotes an arbitrary constant ([1]).

That linear element is convenient to the paraboloid:

$$2i\lambda z = cx^2 + y^2.$$

One can always choose λ and c in such a way that the preceding paraboloid is equal to an arbitrary paraboloid. In particular, if one gives c the value one then one has a paraboloid of revolution, so equation (60) is integrable by the Monge method. That equation was integrated by Lie ([2]), who studied it in a slightly different form. One easily puts equation (60) into this form by supposing that a homographic transformation has been performed such that the fundamental quadric has the equation:

$$z = xy.$$

([1]) WEINGARTEN, "Sur la déformation des surfaces," Acta Math., **XX** (1895), 159. - Likewise, see DARBOUX, *Leçons sur la théorie général des surfaces*, t. IV, pp. 317.

([2]) "Beiträge zur allgemeinen Transformationstheorie," Leipziger Berichte **XLVII** (1895), 494.