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On the integration of the equations of hydrodynamics

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§ 1.

In a previous article (this journal, v. **54**, pp. 254), I developed a theorem that reduced the integration of the hydrodynamical equations for stationary motion to a system of two second-order partial differential equations or to the problem of finding a minimum to a certain integral, in which the function to be integrated represents the *vis viva*. That was achieved by expressing the velocities in terms of two new functions that would give integrals of the ordinary differential equations that come about and which would fulfill the equation of continuity identically. The extension of this process to the case of non-stationary motion led to very complicated equations that did not admit any reduction to a problem in the calculus of variations.

Since then, I have found that this general case can also be always reduced to such a problem, and indeed to the integral of a function that differs from the *vis viva* only by an additional term. The substitution that leads to that result is essentially different from the one that was applies in the aforementioned article. However, both of them have in common that they single out the determination of the pressure from the treatment of the rest of problem and lead to equations that represent motions of the most general nature that the fluid is capable of when it is independent of external forces. Finally, they have in common that the new dependent variables that are employed will define integrals of the resulting system of ordinary differential equations when they are set equal to constants. However, whereas that substitution leads to two second-order partial differential equations for stationary motion, *in the present problem, the problem comes down to three differential equations, two of which are first order, and one of which is second order.*

The substitution that is employed links to the usual methods of treating hydrodynamical equations. In fact, one ordinarily makes the assumption that the expression:

$$u\,dx + v\,dy + w\,dz$$

should be a complete differential. However, the u, v, w can always be arranged in such a way that this expression reduces to a two-term, differential; i.e., to the form:

$$d\varphi + m\,d\psi,$$

which yields the equations:

$$u = \frac{\partial\varphi}{\partial x} + m\frac{\partial\psi}{\partial x},$$
$$v = \frac{\partial\varphi}{\partial y} + m\frac{\partial\psi}{\partial y},$$
$$w = \frac{\partial\varphi}{\partial z} + m\frac{\partial\psi}{\partial z},$$

which are just the substitutions that are applied. I remark that this has a certain relationship to the consideration of vortex motions that *Helmholtz* (this journal, v. **55**, pp. 25) introduced into the theory. Here, the velocities split into one part that is represented by corresponding differential quotients of *one* function and a second one that does not admit such a representation in the slightest. Those vortex motions now depend upon that second part alone – i.e., upon the functions m, ψ. If one then defines the rotational velocities of a fluid particle according to the equations (formula 2) that were given there then one will have:

$$2\xi = \frac{\partial m}{\partial z}\frac{\partial\psi}{\partial y} - \frac{\partial m}{\partial y}\frac{\partial\psi}{\partial z},$$
$$2\eta = \frac{\partial m}{\partial x}\frac{\partial\psi}{\partial z} - \frac{\partial m}{\partial z}\frac{\partial\psi}{\partial x},$$
$$2\zeta = \frac{\partial m}{\partial y}\frac{\partial\psi}{\partial x} - \frac{\partial m}{\partial x}\frac{\partial\psi}{\partial y},$$

in which the function φ vanishes completely (*).

(*) In passing, that yields the problem of putting the expression $u\,dx + v\,dy + w\,dz$ into the form $d\varphi + m\,d\psi$ when u, v, w are any given functions. As is already known from the *Pfaff* problem, from the equations above, m, φ are integrals of the equations:

$$dx : dy : dz = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} : \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} : \frac{\partial v}{\partial y} - \frac{\partial w}{\partial x}.$$

Since the multiplier of the equations is 1, if one knows *one* integral ψ then one can find the second one m by the principle of the last multiplier. However, one will actually have:

$$\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = \frac{\partial m}{\partial z}\frac{\partial\psi}{\partial y} - \frac{\partial\psi}{\partial z}\frac{\partial m}{\partial y}, \text{ etc.,}$$

then, and φ will actually be a complete differential. However, φ satisfies the differential equation:

$$\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)\frac{\partial\varphi}{\partial x} + \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\frac{\partial\varphi}{\partial y} + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\frac{\partial\varphi}{\partial z} = u\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) + v\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + w\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right).$$

If one introduces m, ψ, ϑ, in place of x, y, z, as new variables then one will obtain:

I shall next turn to a general system of equations that exhibits properties that are analogous to the system of hydrodynamics.

§ 2.

Suppose that one has the system of equations:

$$(1)\qquad \left\{\begin{aligned}
\frac{\partial V}{\partial x} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + u_1\frac{\partial u}{\partial x_1} + \cdots + u_{2n}\frac{\partial u}{\partial x_{2n}}, \\
\frac{\partial V}{\partial x_1} &= \frac{\partial u_1}{\partial t} + u\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_1}{\partial x_1} + \cdots + u_{2n}\frac{\partial u_1}{\partial x_{2n}}, \\
&\cdots\cdots \\
\frac{\partial V}{\partial x_{2n}} &= \frac{\partial u_{2n}}{\partial t} + u\frac{\partial u_{2n}}{\partial x} + u_1\frac{\partial u_{2n}}{\partial x_1} + \cdots + u_{2n}\frac{\partial u_{2n}}{\partial x_{2n}},
\end{aligned}\right.$$

$$(2)\qquad \frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial x_1} + \cdots + \frac{\partial u_{2n}}{\partial x_{2n}} = 0,$$

which should be coupled with the ordinary differential equations:

$$(3)\qquad \frac{dx}{dt} = u, \qquad \frac{dx_1}{dt} = u_1, \qquad \ldots, \qquad \frac{dx_{2n}}{dt} = u_{2n}.$$

Equations (1) can be summarized in a symbolic form. Namely, if the symbol δ implies that only the x, but not t, are considered to be variable then:

$$(4)\qquad \delta V = \sum_k \delta x_k \left(\frac{\partial u_k}{\partial t} + u\frac{\partial u_k}{\partial x} + \cdots + u_{2n}\frac{\partial u_k}{\partial x_{2n}}\right),$$

or also, when one sets:

$$(5)\qquad 2T = u^2 + u_1^2 + \ldots + u_{2n}^2,$$

one will get the following:

$$(6)\qquad \delta(V - T) = \sum_k \frac{\partial u_k}{\partial t}\delta x_k + \frac{1}{2}\sum_i\sum_k \left(\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k}\right)(u_i\,\delta x_k - u_k\,\delta x_i).$$

$$\varphi = \int \frac{u\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) + v\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + w\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)}{\frac{\partial \vartheta}{\partial x}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) + \frac{\partial \vartheta}{\partial y}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + \frac{\partial \vartheta}{\partial z}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)}\, d\vartheta$$

by integration (cf., *Jacobi*, Math. W., v. I, pp. 144).

We now remark that one can always give the expression:

$$u\,\delta x + u_1\,\delta x_1 + \ldots + u_{2n}\,\delta x_{2n}$$

the following form:

$$\delta\varphi + m_1\,\delta\varphi_1 + \ldots + m_{2n}\,\delta\varphi_{2n}\,,$$

so we will be led to make the substitutions:

$$u_k = \frac{\partial\varphi}{\partial x_k} + m_1\frac{\partial\varphi_1}{\partial x_k} + m_2\frac{\partial\varphi_2}{\partial x_k} + \cdots + m_n\frac{\partial\varphi_n}{\partial x_k} \tag{7}$$

for the u. One can then represent the expressions $\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k}$ as sums of determinants with the help of these substitutions, namely:

$$\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k} = \sum_r \left(\frac{\partial m_r}{\partial x_i}\frac{\partial\varphi_r}{\partial x_k} - \frac{\partial m_r}{\partial x_k}\frac{\partial\varphi_r}{\partial x_i}\right).$$

However, if one multiplies this expression by the determinant $(u_i\,\delta x_k - u_k\,\delta x_i)$ and then sums over k, i then one will get, from known theorems:

$$\frac{1}{2}\sum_i\sum_k\left(\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k}\right)(u_i\,\delta x_k - u_k\,\delta x_i) = \sum_r \begin{vmatrix} u\frac{\partial m_r}{\partial x} + u_1\frac{\partial m_r}{\partial x_1} + \cdots & \frac{\partial m_r}{\partial x}\delta x + \frac{\partial m_r}{\partial x_1}\delta x_1 + \cdots \\ u\frac{\partial\varphi_r}{\partial x} + u_1\frac{\partial\varphi_r}{\partial x_1} + \cdots & \frac{\partial\varphi_r}{\partial x}\delta x + \frac{\partial\varphi_r}{\partial x_1}\delta x_1 + \cdots \end{vmatrix}.$$

With consideration given to equations (3), one can employ the briefer notation for this:

$$\sum_r\left\{\left(\frac{dm_r}{dt} - \frac{\partial m_r}{\partial t}\right)\delta\varphi_r - \left(\frac{d\varphi_r}{dt} - \frac{\partial\varphi_r}{\partial t}\right)\delta m_r\right\}.$$

If we now introduce this into equation (6) then the sum:

$$\sum_k \frac{\partial u_k}{\partial t}\delta u_k = \delta\frac{\partial\varphi}{\partial t} + \sum_r\left(m_r\,\delta\frac{\partial\varphi_r}{\partial t} + \frac{\partial m_r}{\partial t}\delta\varphi_r\right)$$

will combine with the part:

$$\sum_r\left(\frac{\partial\varphi_r}{\partial t}\delta m_r - \frac{\partial m_r}{\partial t}\delta\varphi_r\right)$$

in the sum above to give the complete variation of the expression:

$$\frac{\partial\varphi}{\partial t}+\sum_r m_r\frac{\partial\varphi_r}{\partial t},$$

and equation (6) will then assume the following form:

$$\delta\left\{V-T-\frac{\partial\varphi}{\partial t}-\sum_r m_r\frac{\partial\varphi_r}{\partial t}\right\} = \sum_r\left(\frac{dm_r}{dt}\delta\varphi_r-\frac{d\varphi_r}{dt}\delta m_r\right). \tag{8}$$

However, this equation contains only $2r$ variations δm, $\delta\varphi$ on the right-hand side, while $2n + 1$ variables x will be varied on the left-hand side. The expression on the left that is to be varied must then be an arbitrary function of the $2n + 1$ arguments φ, m, t, and when we once more eliminate the symbols dm / dt, $d\varphi / dt$, we can state the following theorem:

Theorem 1

Equations (1), (2) *can be replaced with the system:*

$$\left\{\begin{aligned}&\frac{\partial m_r}{\partial t}+u\frac{\partial m_r}{\partial x}+u_1\frac{\partial m_r}{\partial x_1}+\cdots=\Pi'\varphi_r,\\&\frac{\partial\varphi_r}{\partial t}+u\frac{\partial\varphi_r}{\partial x}+u_1\frac{\partial\varphi_r}{\partial x_1}+\cdots=-\Pi' m_r,\\&\frac{\partial u}{\partial x}+\frac{\partial u_1}{\partial x_1}+\cdots+\frac{\partial u_{2n}}{\partial x_{2n}}=0,\end{aligned}\right. \tag{9}$$

in which:

$$u_k = \frac{\partial\varphi}{\partial x_k}+m_1\frac{\partial\varphi_1}{\partial x_k}+m_2\frac{\partial\varphi_2}{\partial x_k}+\cdots+m_n\frac{\partial\varphi_n}{\partial x_k},$$

and in which Π *means an arbitrary function of* t, $\varphi_1, \ldots, \varphi_n, m_1, \ldots, m_n$.

This system contains $2n$ *equations of first order and one of second order. After it has been integrated, the* u *are themselves given by the equation above.* V *is determined from the equation:*

$$V = \left(\frac{\partial\varphi}{\partial t}+\sum_r m_r\frac{\partial\varphi_r}{\partial t}\right)+\tfrac{1}{2}\sum_k\left(\frac{\partial\varphi}{\partial x_k}+\sum_r m_r\frac{\partial\varphi_r}{\partial x_k}\right)^2+\Pi. \tag{10}$$

Equations (3) *finally come down to the system:*

$$\frac{d\varphi_r}{dt} = -\Pi' m_r, \qquad \frac{dm_r}{dt} = \Pi'\varphi_r. \tag{11}$$

The missing integral of the system (3), *which includes one equation more than the present one, gives the principle of the last multipler.*

One can add that theorem to the following one, which can be verified with no further discussion:

Theorem 2

When V is thought of as being expressed by equation (10), *equations* (9) *will make the integral:*

$$\int^{2n+2} V \, dx \, dx_1 \ldots dx_{2n} \, dt$$

assume a maximum or minimum.

These equations include an arbitrary function Π. Meanwhile, one can assume that the integral of a system of $2n$ equations of the first order and one equation of second order must include just as many arbitrary constants as the integral of a system of $2n + 2$ equations of first order. It then seems that equations (9) – into which Π enters, in addition – will lead to more arbitrary constants than the nature of the problem permits. This surplus of arbitrary constants can then have no effect on the dependent functions of the original problem; it must vanish from the expressions for $V, u, u_1, \ldots, u_{2n}$. I will now show that, in fact:

One can set the function Π *equal to zero without compromising the generality of the values of* $V, u, u_1, \ldots, u_{2n}$.

§ 3.

Equations (11) have the canonical form, which, as is known, allows one to give the integrals of these equations a corresponding form, and to express them by the complete solution of a partial differential equation. In fact, one can always determine a function (W) of $t, \varphi_1, \varphi_2, \ldots, \varphi_n$, and n constants $a_1, a_2, \ldots, a_n$ such that:

$$(12) \qquad \left\{ \begin{array}{l} m_1 = \left(\dfrac{\partial W}{\partial \varphi_1}\right), \quad m_2 = \left(\dfrac{\partial W}{\partial \varphi_2}\right), \ \ldots \quad m_n = \left(\dfrac{\partial W}{\partial \varphi_n}\right), \\ -\alpha_1 = \left(\dfrac{\partial W}{\partial a_1}\right), \ -\alpha_2 = \left(\dfrac{\partial W}{\partial a_2}\right), \ \ldots \ -\alpha_n = \left(\dfrac{\partial W}{\partial a_n}\right) \end{array} \right.$$

are the integrals of equations (11), while the α mean new constants, and one has:

$$(13) \qquad \left(\frac{\partial W}{\partial t}\right) = \Pi,$$

from which the partial differential equations for W will emerge when one eliminates the m from Π with the help of the first of equations (12).

One can now obviously introduce the function (W), which includes just one arbitrary constant, into the calculations in place of the function Π, in which the a, α are no longer to be regarded as constants, but as functions of $t, x, x_1, \ldots, x_{2n}$ when one goes from the ordinary differential equations to the partial ones. At the same time, one can also think of these functions a, α as dependent variables in the equations, instead of m, φ, in which the φ are also replaced with these functions in (W). We see how the functions V, u can be expressed in terms of these new dependent variables.

If one introduces equations (12) into the expressions for the u then they will go to:

$$u_k = \frac{\partial \varphi}{\partial x_k} + \left(\frac{\partial W}{\partial \varphi_1}\right)\frac{\partial \varphi_1}{\partial x_k} + \left(\frac{\partial W}{\partial \varphi_2}\right)\frac{\partial \varphi_2}{\partial x_k} + \cdots + \left(\frac{\partial W}{\partial \varphi_n}\right)\frac{\partial \varphi_n}{\partial x_k}.$$

However, if we let W denote the function (W), when we consider it to be a function of the t, x, x_1, …, x_{2n}, then we will obviously have:

$$\frac{\partial W}{\partial x_k} = \left(\frac{\partial W}{\partial \varphi_1}\right)\frac{\partial \varphi_1}{\partial x_k} + \left(\frac{\partial W}{\partial \varphi_2}\right)\frac{\partial \varphi_2}{\partial x_k} + \cdots + \left(\frac{\partial W}{\partial a_1}\right)\frac{\partial a_1}{\partial x_k} + \left(\frac{\partial W}{\partial a_2}\right)\frac{\partial a_2}{\partial x_k} + \ldots ,$$

and we can then (again with the help of equations (12)) once more replace the expression above for u with the following one:

$$(14) \qquad u_k = \frac{\partial(\varphi + W)}{\partial x_k} + \alpha_1 \frac{\partial a_1}{\partial x_k} + \alpha_2 \frac{\partial a_2}{\partial x_k} + \cdots + \alpha_n \frac{\partial a_n}{\partial x_k}.$$

If we further remark that we also have:

$$\frac{\partial W}{\partial t} = \left(\frac{\partial W}{\partial t}\right) + \left(\frac{\partial W}{\partial \varphi_1}\right)\frac{\partial \varphi_1}{\partial t} + \left(\frac{\partial W}{\partial \varphi_2}\right)\frac{\partial \varphi_2}{\partial t} + \cdots + \left(\frac{\partial W}{\partial a_1}\right)\frac{\partial a_1}{\partial t} + \left(\frac{\partial W}{\partial a_2}\right)\frac{\partial a_2}{\partial t} + \ldots$$

then the expression:

$$\Pi + \frac{\partial \varphi}{\partial t} + \sum_r m_r \frac{\partial \varphi_r}{\partial t} = \left(\frac{\partial W}{\partial t}\right) + \frac{\partial \varphi}{\partial t} + \sum_r \left(\frac{\partial W}{\partial \varphi_r}\right)\frac{\partial \varphi_r}{\partial t}$$

will go to the following one:

$$\frac{\partial(\varphi + W)}{\partial t} + \alpha_1 \frac{\partial a_1}{\partial t} + \alpha_2 \frac{\partial a_2}{\partial t} + \cdots + \alpha_n \frac{\partial a_n}{\partial t}.$$

Therefore, equation (10) will immediately assume the form:

$$(15) \qquad \begin{aligned} V = {} & \frac{\partial(\varphi + W)}{\partial t} + \alpha_1 \frac{\partial a_1}{\partial t} + \alpha_2 \frac{\partial a_2}{\partial t} + \cdots + \alpha_n \frac{\partial a_n}{\partial t} + \ldots \\ & + \tfrac{1}{2} \sum_k \left(\frac{\partial(\varphi + W)}{\partial x_k} + \alpha_1 \frac{\partial a_1}{\partial x_k} + \alpha_2 \frac{\partial a_2}{\partial x_k} + \cdots \right)^2 . \end{aligned}$$

We now compare equations (14), (15) with equations (7), (10). We see directly that the function $\varphi + W$ enters in place of φ, while the α enter in place of m, the a, in place of the φ, and finally, that the function Π vanishes. Now, equations (11) obviously correspond to the following ones, moreover:

$$\frac{da_r}{dt} = 0, \qquad \frac{d\alpha_r}{dt} = 0, \tag{16}$$

which will yield equations that are entirely similar to equations (9) when they are solved. One then recognizes that the reduced problem to which we have now arrived differs from the one that was contained in Theorem 1 only by the facts that the function Π is set to zero, and that m other symbols enter in place of the φ. However, at the same time, the ordinary differential equations will become integrable, and when we then revert to the previous notation, we can pose the following theorem:

Theorem 3

Equations (1), (2) *can be replaced with the system:*

$$\left\{\begin{aligned} \frac{\partial m_r}{\partial t} + u\frac{\partial m_r}{\partial x} + u_1\frac{\partial m_r}{\partial x_1} + \cdots + u_{2n}\frac{\partial m_r}{\partial x_{2n}} &= 0, \\ \frac{\partial \varphi_r}{\partial t} + u\frac{\partial \varphi_r}{\partial x} + u_1\frac{\partial \varphi_r}{\partial x_1} + \cdots + u_{2n}\frac{\partial \varphi_r}{\partial x_{2n}} &= 0, \\ \frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial x_1} + \cdots + \frac{\partial u_{2n}}{\partial x_{2n}} &= 0, \end{aligned}\right. \tag{17}$$

in which:

$$u_k = \frac{\partial \varphi}{\partial x_k} + m_1\frac{\partial \varphi_1}{\partial x_k} + m_2\frac{\partial \varphi_2}{\partial x_k} + \cdots + m_n\frac{\partial \varphi_n}{\partial x_k}.$$

Two of these equations are of first order, and one of them is of second order. Once they are integrated, the u will be given by the formula above, but V will be given by the formula:

$$\begin{aligned} V = {} & \frac{\partial \varphi}{\partial t} + m_1\frac{\partial \varphi_1}{\partial t} + m_2\frac{\partial \varphi_2}{\partial t} + \cdots + m_n\frac{\partial \varphi_n}{\partial t} \\ & + \tfrac{1}{2}\sum_k \left\{\frac{\partial \varphi}{\partial x_k} + m_1\frac{\partial \varphi_1}{\partial x_k} + m_2\frac{\partial \varphi_2}{\partial x_k} + \cdots + m_n\frac{\partial \varphi_n}{\partial x_k}\right\}^2, \end{aligned} \tag{18}$$

and equations (3) *will have the equations:*

$$m_r = \text{const.}, \qquad \varphi_r = \text{const.} \tag{19}$$

for their integrals, into which, a last integral enters that is obtained from equations (3) *and* (19) *with the help of the last multiplier.*

Thus, in that way, the present system is reduced to another one that includes no more arbitrary constants in its general solution than the original one did, and which has the property that the integrals yield the additional ordinary differential equations with no further assumptions. One easily infers the following theorem:

Theorem 4

Equations (17) *make the integral:*

$$\int^{(2n+2)} V\,dx\,dx_1\dots dx_{2n}dt$$

assume a maximum or a minimum, in which V is thought of as being expressed by equation (18).

§ 4.

Nothing remains for us to do but to express the results that we have arrived at in the case of hydrodynamics, for which one has $n = 1$. Let U be the force function, p, the pressure, and let q be the density, so one has the following theorem:

Theorem 5

Equations:

$$(20)\qquad \begin{cases} \dfrac{\partial}{\partial x}\left(U-\dfrac{p}{q}\right)=\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}, \\ \dfrac{\partial}{\partial y}\left(U-\dfrac{p}{q}\right)=\dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}, \\ \dfrac{\partial}{\partial z}\left(U-\dfrac{p}{q}\right)=\dfrac{\partial w}{\partial t}+u\dfrac{\partial w}{\partial x}+v\dfrac{\partial w}{\partial y}+w\dfrac{\partial w}{\partial z}, \\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0 \end{cases}$$

can be linked with the problem of finding a minimum or maximum for the integral:

$$\iiint\!\!\int\left(U-\frac{p}{q}\right)dt\,dx\,dy\,dz$$

in which one sets:

$$(21)\qquad U-\frac{p}{q}=\frac{\partial\varphi}{\partial t}+m\frac{\partial\psi}{\partial t}+\frac{u^2+v^2+w^2}{2},$$

and the following expressions are true for the u, v, w:

(22)
$$\begin{cases} u = \dfrac{\partial\varphi}{\partial x} + m\dfrac{\partial\psi}{\partial x}, \\ v = \dfrac{\partial\varphi}{\partial y} + m\dfrac{\partial\psi}{\partial y}, \\ w = \dfrac{\partial\varphi}{\partial z} + m\dfrac{\partial\psi}{\partial z}. \end{cases}$$

The integrals of the equations:

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w$$

will then be:

(23)
$$m = \text{const.}, \qquad \psi = \text{const.},$$

and a third one that the theory of the last multiplier implies.

The equations to which the problem reduces will then be:

$$0 = \frac{\partial m}{\partial t} + \left(\frac{\partial m}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial m}{\partial y}\frac{\partial\varphi}{\partial y} + \frac{\partial m}{\partial z}\frac{\partial\varphi}{\partial z}\right) + m\left(\frac{\partial m}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial m}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial m}{\partial z}\frac{\partial\psi}{\partial z}\right),$$

$$0 = \frac{\partial\psi}{\partial t} + \left(\frac{\partial\psi}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\varphi}{\partial y} + \frac{\partial\psi}{\partial z}\frac{\partial\varphi}{\partial z}\right) + m\left(\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial z}\right),$$

$$0 = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x} + m\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y} + m\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z} + m\frac{\partial\psi}{\partial z}\right).$$

It is very easy to go from these equations for the stationary state to the ones that I developed in the cited place. One only remarks that it follows from the consideration that was presented in the beginning of § 3 that one can choose *only one* of the m, ψ to be completely free of t, while the other one will be of the form $tf + F$.

I further point out that equation (21) will become that of the *vis viva* in a well-known form when one lets m vanish, which reverts to the usual assumption.

Berlin, 1 March 1858.