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On Plücker complexes

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If we denote the coordinates of two planes by $u_1, u_2, \ldots, v_1, v_2, \ldots$ and the coordinates of two points that lie upon the line of intersection of the planes by $x_1, x_2, \ldots, y_1, y_2, \ldots$ then it is known that one can represent the coordinates of the line of intersection of the two planes, or – what amounts to the same thing – the connecting line of the two points, by the quantities:

(1) $$p_{ik} = u_i\, v_k - v_i\, u_k\,,$$

or the quantities:

(2) $$q_{ik} = x_i\, y_k - y_i\, x_k\,.$$

One the following equations between the p or the q:

(3) $$\begin{cases} P = p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0, \\ Q = q_{12}q_{34} + q_{13}q_{42} + q_{14}q_{23} = 0, \end{cases}$$

respectively, and the p, q depend upon each other by means of the equations:

(4) $$\rho\, p_{ik} = \frac{\partial Q}{\partial q_{ik}}, \qquad \sigma\, q_{ik} = \frac{\partial P}{\partial p_{ik}},$$

in which ρ, σ mean arbitrary quantities that can be set to 1, as will be done in what follows.

The equation of a complex of order n can then be written in two ways when one denotes the variables by p, in one case, and q, in the other, and one then obtains the two forms:

(5) $$\begin{cases} 0 = F(p) = \sum a_{ih,kl,mn,\cdots}\, p_{ih}\, p_{kl}\, p_{mn} \cdots \\ 0 = \Phi(q) = \sum \alpha_{ih,kl,mn,\cdots}\, q_{ih} q_{kl} q_{mn} \cdots \end{cases}$$

for the equation $F = 0$ of such a complex.

We make the following remarks on these expressions:

1) The value of a coefficient remains unchanged when one permutes any two of the index pairs *ih*, *kl*, etc. with each other.

2) A coefficient (like the corresponding *p*, *q*) changes its sign when two indices of one pair are permuted.

3) When one carries out the sum on the right, any coefficient will take on a polynomial factor that gives the number of *distinct* forms that it will assume by permuting the index pairs.

If one symbolically sets:

$$\text{(6)} \qquad \begin{cases} a_{ih,kl,mn,\cdots} = a_{ik}\,a_{kl}\,a_{mn}\cdots, \\ \alpha_{ih,kl,mn,\cdots} = \alpha_{ik}\,\alpha_{kl}\,\alpha_{mn}\cdots \end{cases}$$

then $F(p)$ or $\Phi(q)$ will appear as the n^{th} power of a linear complex:

$$\text{(7)} \qquad \begin{aligned} F(p) &= \left\{\sum a_{ih}\,p_{ih}\right\}^n, \\ \Phi(q) &= \left\{\sum \alpha_{ih}\,q_{ih}\right\}^n. \end{aligned}$$

The symbols a_{ih} , α_{ih} then have only the property that their signs will change when one permutes the two indices. They likewise depend upon each other very simply; if one then sets:

$$\text{(8)} \qquad \begin{aligned} A &= a_{12}a_{31} + a_{13}a_{42} + a_{14}a_{23}, \\ \mathrm{A} &= \alpha_{12}\alpha_{31} + \alpha_{13}\alpha_{42} + \alpha_{14}\alpha_{23} \end{aligned}$$

then one will have:

$$\text{(9)} \qquad \alpha_{ih} = \frac{\partial A}{\partial a_{ih}}, \qquad a_{ih} = \frac{\partial \mathrm{A}}{\partial \alpha_{ih}},$$

and the equations (6) can then also be written:

$$\text{(10)} \qquad \begin{cases} a_{ih,kl,mn,\cdots} = \dfrac{\partial \mathrm{A}}{\partial \alpha_{ih}}\dfrac{\partial \mathrm{A}}{\partial \alpha_{kl}}\dfrac{\partial \mathrm{A}}{\partial \alpha_{mn}}\cdots, \\[2ex] \alpha_{ih,kl,mn,\cdots} = \dfrac{\partial A}{\partial a_{ih}}\dfrac{\partial A}{\partial a_{kl}}\dfrac{\partial A}{\partial a_{mn}}\cdots, \end{cases}$$

with which, the coupling of the two real notations with the two symbolic ones is exhibited completely.

However, the following remark leads to a simplification of the symbolic notation, and at the same time has an essential influence on its use:

When $n > 1$, the coefficients of a complex are in no way determined completely, although they can be modified when one calls upon $P = 0$ or $Q = 0$, respectively. In fact, one can always set:

$$\text{(11)} \qquad F + MP = 0$$

in place of $F(p) = 0$, where M is a function of order $n - 2$ of the p, and thus a function that carries with it:

$$\frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$

arbitrary coefficients. I will now prove the following theorem:

The equation $F(p) = 0$ can always be modified, and in only one way, by applying $P = 0$, in such a way that one can consider it to be a symbolic power of a special linear complex – i.e., one whose lines all intersect a fixed line. The form that $F(p)$ then assumes shall be called the ***normal form*** *of the complex equation.*

We examine the conditions that the demands of the theorem will imply. If the symbolic coefficients a_{ih} (and analogously for the α) are to be the coefficients of a special complex then one can introduce quantities:

$$a_1, a_2, a_3, a_4, \qquad b_1, b_2, b_3, b_4$$

such that:

(12) $$a_{ih} = a_i\, b_h - b_i\, a_h\,.$$

In this case, the first equation (6) then goes to:

(13) $$a_{ih,kl,mn,\dots} = (a_i\, b_h - b_i\, a_h)\,(a_k\, b_l - b_k\, a_l)\,(a_m\, b_n - b_m\, a_n)\dots$$

The question is then that of whether, and under what circumstances, it is permissible for the coefficients of the complex to be set equal to the symbolic expressions on the right-hand side of this equation. Both sides of equation (13) have the aforementioned general properties of the invariability or change of sign, respectively, in common with each other. The only question, upon whose answer the possibility of any symbolism will depend, is then: *Do linear relationships exist between the right-hand parts of equations* (13) *that are not generally fulfilled by the right-hand sides?* In fact, this is the case. The symbols $a_i\, b_h - b_i\, a_h$ are then the coordinates of a line. However, the identity:

(14) $$0 = (a_1\, b_2 - b_2\, a_1)\,(a_3\, b_4 - b_3\, a_4) + (a_1\, b_3 - b_1\, a_3)\,(a_4\, b_2 - b_4\, a_2) + (a_1\, b_4 - b_1\, a_4)\,(a_2\, b_3 - b_2\, a_3)$$

exists between them, and no other one. All conceivable linear relations that exist between the right-hand parts of equations (13) must then arise from this under multiplication by $n - 2$ expressions of the form:

$$a_i\, b_h - b_i\, a_h\,.$$

One then sees immediately that the symbolic equations (13) between the coefficients of the complex assume the following equations, which are not fulfilled, in general:

(15) $$a_{12,\,34,\,ih,\dots} + a_{13,\,42,\,ih,\dots} + a_{14,\,23,\,ih,\dots} = 0.$$

Since all of the index pairs that follow the first two are completely arbitrary, the number of these relations is precisely as large as the number of combinations of $n - 2$ index pairs that are possible, or the number of coefficients that a complex M of order $n - 2$ contains.

If one now replaces the coefficients of F in (15), which these equations generally do not satisfy, by any means, with the coefficients of the complex $F + MP = 0$ (11), which is identical with $F = 0$, then one will obtain a system of first-degree equations in which the coefficients of M are the unknowns, and in which precisely as many unknowns as equations then appear. It then follows from this that either these unknowns can be determined completely from these equations or that (when the determinant of the system vanishes) the unknowns must remain undetermined. The determinant of the system then does not need to be examined when one can show that the unknowns are completely determinate, in some way. However, this happens in the following way:

$$\Delta F = \frac{\partial^2 F}{\partial p_{12}\,\partial p_{34}} + \frac{\partial^2 F}{\partial p_{13}\,\partial p_{43}} + \frac{\partial^2 F}{\partial p_{14}\,\partial p_{23}}. \tag{16}$$

One will obtain a complex $\Delta F = 0$ of order $n - 2$ from a complex $F = 0$ of order n by this process, and when one again applies the same operation to them, etc., one will obtain a sequence of complexes of order $n - 4$, $n - 6$, etc., which shall be denoted by:

$$\Delta^2 F = 0, \qquad \Delta^3 F = 0, \qquad \ldots,$$

respectively. However, should the modified complex equation $F + MP = 0$ now be representable in the symbolic form:

$$F + MP = \{\sum (a_i\, b_k - b_i\, a_k)\, p_{ik}\}^n$$

then one would obtain zero identically by applying the process Δ to the right-hand side of equation (14), and the function M must then possess the property that the equations:

$$\begin{aligned} \Delta\ (F + MP) &= 0, \\ \Delta^2\ (F + MP) &= 0, \\ \Delta^3\ (F + MP) &= 0 \end{aligned} \tag{17}$$

must exist identically for $F + MP$. In order to develop these equations, I remark that:

$$\begin{aligned} \Delta(MP) &= P\,\Delta M + M\,\Delta P + \left\{\frac{\partial M}{\partial p_{12}}\frac{\partial P}{\partial p_{34}} + \frac{\partial M}{\partial p_{13}}\frac{\partial P}{\partial p_{42}} + \cdots + \frac{\partial M}{\partial p_{34}}\frac{\partial P}{\partial p_{12}}\right\} \\ &= P\,\Delta M + 3\,M \quad + \left\{p_{12}\frac{\partial M}{\partial p_{12}} + p_{13}\frac{\partial M}{\partial p_{13}} + \cdots\right\} \\ &= P\,\Delta M + (n + 1)\,M. \end{aligned} \tag{18}$$

This will immediately yield the result of repeated application of this operation, when one first defines $\Delta(P\,\Delta M)$, $\Delta(P\,\Delta^2 M)$, etc. One likewise obtains these expressions when one replaces M with ΔM, $\Delta^2 M$, etc., in (18), and replaces n with $n - 2$, $n - 4$, … One will then have:

$$\begin{aligned}
\Delta\,(PM) &= P\,\Delta\,M + (n-1)\,M, \\
\Delta\,(P\,\Delta M) &= P\,\Delta^2 M + (n-1)\,M, \\
\Delta\,(P\Delta^2 M) &= P\,\Delta^3 M + (n-3)\,M, \\
&\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

and in order to determine $P\,\Delta^k M$, one will need only to subject the first $k - 1$ of these equations to the operations Δ^{k-1}, Δ^{k-2}, …, Δ and take their sum, along with the k^{th} one. In this way, one will obtain the following system:

$$\begin{aligned}
\Delta\;\;(PM) &= P\,\Delta\,M + (n+1)\,M, \\
\Delta^2\,(PM) &= P\,\Delta^2 M + 2n\cdot\Delta M, \\
\Delta^3\,(PM) &= P\,\Delta^3 M + 3\,(n-1)\,\Delta^2 M, \\
\Delta^4\,(PM) &= P\,\Delta^4 M + 3\,(n-2)\,\Delta^3 M, \\
&\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

whose defining rule is clear. However, equations (17) are converted into the following ones under the introduction of these values:

(19)
$$\begin{aligned}
&\Delta\;\;F + P\,\Delta\;\;M + (n+1)\,M = 0, \\
&\Delta^2\,F + P\,\Delta^2 M + 2\cdot n\cdot\Delta M = 0, \\
&\Delta^3\,F + P\,\Delta^3 M + 3(n-1)\,\Delta^2 M = 0, \\
&\Delta^4\,F + P\,\Delta^4 M + 4(n-2)\,\Delta^3 M = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

One can successively calculate M, ΔM, etc., from these equations when one starts with the last one, which contains just one $\Delta^k M$, while $\Delta^{k+1} M$ vanishes identically. However, since it comes down to not so much the determination of these functions as much as to the expression for the modified complex:

(20)
$$F + PM = F_1,$$

one can add this equation to equations (19), and then sum over all of them, after one has multiplied equations (19) by:

$$-\frac{P}{1\cdot n+1}, \qquad -\frac{P^2}{1\cdot 2\cdot n+1\cdot n}, \qquad -\frac{P^3}{1\cdot 2\cdot 3\cdot n+1\cdot n\cdot n-1}, \qquad \text{etc.,}$$

in sequence. All of the terms that are affected with M, ΔM, etc., will then drop out upon addition, and what remains will be the *normal form of the complex:*

(21) $$F_1 = F - \frac{P}{1\cdot n+1}\Delta F + \frac{P^2}{1\cdot 2\cdot n+1\cdot n}\Delta^2 F - \frac{P^3}{1\cdot 2\cdot 3\cdot n+1\cdot n\cdot n-1}\Delta^3 F - \dots,$$

which is determined completely and uniquely in this way.

The normal form of the complex allows one to give its equation the symbolic form:

(22) $$F_1 = \left\{\sum (a_i\, b_k - b_i\, a_k)\, p_{ik}\right\}^n = 0.$$

The symbol $a_i\, b_k - b_i\, a_k$ then seem to be the coordinates of a line that is given by two points a, b. If one thinks of it as being expressed by two planes α, β instead, and likewise introduces the q, in place of p, then one will obtain the second symbolic form that is equivalent to the one above:

(23) $$F_1 = \left\{\sum (\alpha_i\, \beta_k - \beta_i\, \alpha_k)\, q_{ik}\right\}^n = 0.$$

As an application of this symbolism, I will present the equation for the surface of order four and class four, which is described by the vertices of decomposing complex cones and the planes of decomposing complex curves for a second-order complex. (cf., Plücker's *Neue Geometrie des Raumes*, pp. 307, *et seq.*) For the second-degree complex, the normal form brings with it a determination of the coefficients. If $F = 0$ is the equation of the complex in arbitrary form then one will have:

$$F_1 = F - \frac{Q}{3}(a_{12,\,34} + a_{13,\,42} + a_{14,\,23}).$$

If one now sets, symbolically:

$$F_1 = \left\{\sum (a_i\, b_k - b_i\, a_k)\, p_{ik}\right\}^2,$$
$$= \left\{\sum (\alpha_i\, \beta_k - \beta_i\, \alpha_k)\, q_{ik}\right\}^2$$

then one will obtain the equation of a *complex cone* that emanates from the point y when one expresses the p or the q in terms of the quantities $x_i\, y_k - y_i\, x_k$, and thus considers the y to be given. The equation of this complex cone will then become:

(24) $$(a\ b\ x\ y)^2 = 0, \qquad \text{or} \qquad (\alpha_i\, \beta_k - \beta_i\, \alpha_k)^2 = 0.$$

One likewise gets the equation for the complex curve that is contained in the plane v when one expresses the p or the q in terms of the quantities $u_i\, v_k - v_i\, u_k$, and thus considers the v to be constant. The equation of the complex curve is then:

(25) $$(u_a\, v_b - v_a\, u_b)^2 = 0, \qquad \text{or} \qquad (\alpha\ \beta\ u\ v)^2 = 0.$$

One obtains the complex surfaces that correspond to the connecting lines between two points y, z when one replaces the y in (24) with the coordinates of a variable point $y + \lambda z$, and thus, the family of complex cones whose vertices lie on these lines, and then defines the locus of the intersections of successive complex cones – i.e., lets the discriminant of the quadratic equation in λ that thus arises vanish. One now gets the following expression for this quadratic equation with the use of the second formula in (24):

$$(\alpha_x \beta_y - \beta_x \alpha_y)^2 + 2\lambda (\alpha_x \beta_y - \beta_x \alpha_y)(\alpha_x \beta_z - \beta_x \alpha_z) + \lambda^2 (\alpha_x \beta_z - \beta_x \alpha_z)^2 = 0;$$

if one then forms the discriminant and distinguishes the different symbolic series by a prime then one will have:

$$(26) \qquad 0 = \left\{(\alpha_x\beta_y - \beta_x\alpha_y)(\alpha'_x\beta'_z - \beta'_x\alpha'_z) - (\alpha_x\beta_z - \beta_x\alpha_z)(\alpha'_x\beta'_y - \beta'_x\alpha'_y)\right\}^2$$

for the equation of the complex surface.

One will likewise obtain the equation for the complex surface in plane coordinates when one thinks of the line y, z as the intersection of two planes v, w, then replaces v with $v + \lambda w$ in (25), and again forms the discriminant in λ. The first form (25) then gives the form:

$$(27) \qquad 0 = \{(u_a v_b - v_a u_b)(u_{a'} w_{b'} - w_{a'} u_{b'}) - (u_a w_b - w_a u_b)(u_{a'} v_{b'} - v_{a'} u_{b'})\}^2$$

to the equation for the complex surface in plane coordinates.

The equations of these complex surfaces can be written more simply when one now replaces the quantities $y_i z_k - z_i y_k$ with the ones that correspond to the $v_i w_k - w_i v_k$, and conversely in (27). These two equations then assume the forms:

$$(28) \qquad \left\{ \begin{array}{l} (\alpha_x\beta - \beta_x\alpha, \alpha'_x\beta' - \beta'_x\alpha', v, w)^2 = 0, \\ (u_a\, b - u_b\, a, u_{a'}\, b' - u_{b'}\, a', y, z)^2 = 0. \end{array} \right.$$

One can also obtain these equation forms immediately. The first one says that the complex surface of a line v, w is the locus of the points whose complex cones contact the line; the second one says that the tangent planes of the complex surface contain complex curves that have the lines for the common secant.

If one consider the second equation (24):

$$(\alpha_x \beta_y - \beta_x \alpha_y)^2 = 0$$

to be the equation of the complex cone for the point x and then denotes it by:

$$(29) \qquad \gamma_y^2 = 0 \qquad (\gamma_i = \alpha_x \beta_i - \beta_x \alpha_i, \text{ then } \gamma_x = 0)$$

then from the theory of second-order surfaces:

$$(\gamma \gamma' \gamma'' u)^2 = 0$$

is the equation of the surface (29) in plane coordinates; i.e., since (29) is a cone, the square is the equation of its vertex. One must then have:

$$(\gamma\gamma'\gamma''u)^2 = M \cdot u_x^2, \tag{30}$$

where M now depends upon only the x, but no longer on the u. Now, the condition for the cone to decompose, and thus for its vertex to be indeterminate, is $M = 0$. Therefore, $M = 0$ is the equation of a surface that is the geometric locus of all points whose complex cone resolves into a pair of planes.

Now, in order to define M, one needs only to replace a series of γ with their values in the left-hand side of (30). One then has:

$$(\gamma\gamma'\gamma''u)^2 = (\alpha_x\,\beta - \beta_x\,\alpha,\ \gamma',\ \gamma'',\ u)^2 = \{\alpha_x\,(\beta\,\gamma'\,\gamma''u) - \beta_x\,(\alpha\,\gamma'\,\gamma''u)\}^2,$$

or, when one applies the known identity, and considers that $\gamma'_x = \gamma''_x = 0$:

$$= (\alpha\,\beta\,\gamma'\,\gamma'')^2\ u_x^2.$$

The factor u_x^2 has been separated in this, and one then has:

$$\begin{aligned} M &= (\alpha\,\beta\,\gamma'\,\gamma'')^2, \\ &= (\alpha,\ \beta,\ \alpha'_x\beta' - \beta'_x\gamma',\ \alpha''_x\beta'' - \beta''_x\gamma'')^2. \end{aligned}$$

The surface $M = 0$ *then has order four, and its equation consists of the equation of the complex surface* (28), *when one lets the coordinates of the guiding line symbolically mean the coefficients of the given complex.*

One likewise contains the (known from the previous identity) surface of class four whose tangent planes contain decomposing complex curves, when one defines the equation:

$$N = 0 = (a,\ b,\ u_{a'}\,b' - u_{b'}\,a',\ u_{a''}\,b'' - u_{b''}\,a'')^2,$$

or when one lets the coordinates of the guiding line in the equation for the complex surface in plane coordinates mean the symbolic coefficients of the given complex.

I will give other applications of the method of notation that was set down above on some other occasions.

Göttingen, 5 April 1869.