"Über eine allgemeine Transformation der hydrodynamischen Gleichungen," J. f. d. reine u. angew. Math. **54** (1857), 293-312.

On a general transformation of the hydrodynamical equations

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§ 1.

The equations upon which theory of the motion of a fluid depends are represented in two ways, in general: Either one considers the unknown quantities of the problem to be the velocities that exist at a certain time at a place in the fluid, so one treats them as functions of position and time, and arrives in that way at a system of four first-order partial differential equations that go by the name of *Euler's equations*, and in order to solve that system, one must solve a system of three ordinary differential equations that determine the motion of the individual fluid particles, or one introduces the coordinates of a fluid particle as dependent variables and arrives once more at a system of four partial differential equations by means of which one can determine each coordinate as functions of time and the initial state. That is the path that *Lagrange* pursued in his analytical mechanics. One will then bypass the solution of an additional system of ordinary differential equations, but the equations of the partial system would be of second order, except for one of them; it also seems regrettable that the characteristic properties of the all-important *stationary* motions do not emerge very clearly in that form.

Meanwhile, one can treat the problem in yet a third way that offers some special advantages in precisely the aforementioned case. In fact, for stationary motion, one can replace the differential equations with the equations of the following problem:

Find a minimum for a triple integral that is extended over space, for which the function to be integrated is the *vis viva* of a particle, increased by an arbitrary quantity that remains the same only for all of the particles that traverse the path in question. The aforementioned function will thus be expressed by means of the functions that will give the curves of motion for the particles when they are set equal to constant, and the first partial derivatives of those functions.

By that theorem, which will be derived in what follows, along with some other theorems, and which exhibits a remarkable analogy with the principle of least action, one then obtains a system of second-order partial differential equations for the stationary motion, and one will find the curves of motion immediately by integration. The corresponding result will generally be much more complicated for the non-stationary state; still, I would not like to fail to present the general development that also subsumes that case. I will even temporarily consider a general system of partial differential equations, and present the result of a transformation that corresponds to the one that was suggested.

In a treatise that appeared in *Poggendorf's* Annalen, Bd. 95, *Meissel* has already made an attempt at giving a transformation of the greatest possible generality for stationary motion, in which he excluded the usual assumption by which the velocity of a molecule is set equal to the partial derivatives of *one* function. One even easily remarks that *Meissel* (*loc. cit.*) was led to a false conclusion that, in turn, almost completely eliminated the generality of the result. In fact, the equation that *Meissel* (pp. 278, 5) gave must be replaced with another one that I will present in what follows (§ 7, 59).

The source of the present investigations was defined by the theory of functional determinants, which allow one to immediately recognize the true form under which the velocities can be represented in general.

I next turn to the following general system of equations.

§ 2.

Let $u_1, u_2, \ldots, u_n$, and V be functions of the variables $x_1, x_2, \ldots, x_n$, and t that are determined by the equations:

$$(1)\qquad \begin{cases} \dfrac{\partial V}{\partial x_1} = \dfrac{\partial u_1}{\partial t} + u_1 \dfrac{\partial u_1}{\partial x_1} + u_2 \dfrac{\partial u_1}{\partial x_2} + \cdots + u_n \dfrac{\partial u_1}{\partial x_n}, \\ \dfrac{\partial V}{\partial x_2} = \dfrac{\partial u_2}{\partial t} + u_1 \dfrac{\partial u_2}{\partial x_1} + u_2 \dfrac{\partial u_2}{\partial x_2} + \cdots + u_n \dfrac{\partial u_2}{\partial x_n}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \dfrac{\partial V}{\partial x_n} = \dfrac{\partial u_n}{\partial t} + u_1 \dfrac{\partial u_n}{\partial x_1} + u_2 \dfrac{\partial u_n}{\partial x_2} + \cdots + u_n \dfrac{\partial u_n}{\partial x_n}, \end{cases}$$

$$(2)\qquad 0 = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \cdots + \frac{\partial u_n}{\partial x_n}.$$

The meaning of the last equation can be immediately interpreted analytically. If one forms the functional determinant of the n functions:

$$a, \quad a', \; a^{(2)}, \ldots, a^{(n-1)}$$

with respect to the variables $x_1, x_2, \ldots, x_n$, and orders it in terms of the derivatives of a, such that it assumes the form:

$$(3)\qquad R = \Delta_1 \frac{\partial a}{\partial x_1} + \Delta_2 \frac{\partial a}{\partial x_2} + \cdots + \Delta_n \frac{\partial a}{\partial x_n},$$

then the Δ will be, in turn, functional determinants, from which a and always *one* of the x variables are excluded. As is known, one will then have the identity equation (cf., *Jacobi*, "Theoria novi multiplicatoris," this journal, Bd. 27, pp. 203, or *Mathem. Werke* I, pp. 51):

$$\frac{\partial \Delta_1}{\partial x_1} + \frac{\partial \Delta_2}{\partial x_2} + \cdots + \frac{\partial \Delta_n}{\partial x_n} = 0, \tag{4}$$

and the expressions $\Delta_1, \Delta_2, \ldots, \Delta_n$, which include $n-1$ arbitrary functions ($a, a', a^{(2)}, \ldots, a^{(n-1)}$), are likewise the most general ones that will satisfy equation (4). One can then examine what will follow from equations (1) when one introduces the functions $a, a', a^{(2)}, \ldots, a^{(n-1)}$ into them as dependent variables, and one sets:

$$u_1 = \Delta_1, \qquad u_2 = \Delta_2, \qquad \ldots, \qquad u_n = \Delta_n\,. \tag{5}$$

Since we have fulfilled equation (2) identically, only equations (1) remain, which can be summarized in the following symbolic equation:

$$\delta V = \sum_{k=1}^{n} \frac{\partial \Delta_k}{\partial t}\,\delta x_k + \sum_{i=1}^{n}\sum_{k=1}^{n} \Delta_i \frac{\partial \Delta_k}{\partial x_i}\,\delta x_k\,, \tag{6}$$

in which only the $x_1, x_2, \ldots, x_n$, but not t, are considered to be variable in the variation.

If one lets $2T$ denote the expression:

$$2T = \Delta_1^2 + \Delta_2^2 + \cdots + \Delta_n^2 \tag{7}$$

then one will get this expression

$$\delta T = \sum_{i=1}^{n}\sum_{k=1}^{n} \Delta_i \frac{\partial \Delta_i}{\partial x_k}\,\delta x_k \tag{8}$$

by variation, and instead of equation (6), one can consider the following one:

$$\delta(V - T) = \sum_{k=1}^{n} \frac{\partial \Delta_i}{\partial t}\,\delta x_k + \sum_{i=1}^{n}\sum_{k=1}^{n} \Delta_i \left(\frac{\partial \Delta_k}{\partial x_i} - \frac{\partial \Delta_i}{\partial x_k}\right)\delta x_k\,. \tag{9}$$

If one sets, for brevity:

$$M_k = \sum_{k=1}^{n} \Delta_i \left(\frac{\partial \Delta_k}{\partial x_i} - \frac{\partial \Delta_i}{\partial x_k}\right) \tag{10}$$

then equation (9) will go to:

$$\delta(V - T) = \sum_{k=1}^{n} \left(\frac{\partial \Delta_i}{\partial t} + M_k\right)\delta x_k\,. \tag{10a}$$

That form will already allow us to recognize some properties of the transformation. Namely, if one defines the sum:

$$M_1\,\Delta_1 + M_2\,\Delta_2 + \ldots + M_n\,\Delta_n = \sum_{i=1}^{n}\sum_{k=1}^{n} \Delta_i\Delta_k\left(\frac{\partial\Delta_k}{\partial x_i} - \frac{\partial\Delta_i}{\partial x_k}\right)$$

then one will see that it changes its sign under the exchange of i and k and must then vanish identically. It emerges from that, and the well-known properties of determinants, that the expression M must assume the form:

$$(11)\qquad \left\{\begin{array}{l} M_1 = A^{(1)}\dfrac{\partial a'}{\partial x_1} + A^{(2)}\dfrac{\partial a^{(2)}}{\partial x_1} + \cdots + A^{(n-1)}\dfrac{\partial a^{(n-1)}}{\partial x_1}, \\ M_2 = A^{(1)}\dfrac{\partial a'}{\partial x_2} + A^{(2)}\dfrac{\partial a^{(2)}}{\partial x_2} + \cdots + A^{(n-1)}\dfrac{\partial a^{(n-1)}}{\partial x_2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ M_n = A^{(1)}\dfrac{\partial a'}{\partial x_n} + A^{(2)}\dfrac{\partial a^{(2)}}{\partial x_n} + \cdots + A^{(n-1)}\dfrac{\partial a^{(n-1)}}{\partial x_n}, \end{array}\right.$$

in which the A include the first and second derivatives of the a in a still-unknown way. The form of the expression A will be given below.

Equation (10a), however, goes to the following one immediately:

$$(12)\qquad \delta(V-T) = \sum_{k=1}^{n}\frac{\partial\Delta_k}{\partial t}\,\delta x_k + A^{(1)}\,\delta a' + A^{(2)}\,\delta a^{(2)} + \ldots + A^{(n-1)}\,\delta a^{(n-1)}.$$

If we next consider the case in which the quantities Δ are thought of as independent of t (which corresponds to the case of stationary motion) then we can integrate equation (12) immediately; it will then follow, with no further discussion, from the equation:

$$(13)\qquad \delta(V-T) = A^{(1)}\,\delta a' + A^{(2)}\,\delta a^{(2)} + \ldots + A^{(n-1)}\,\delta a^{(n-1)},$$

in which only $n - 1$ variations appear on the right and n appear on the left, that:

$$(14)\qquad \left\{\begin{array}{r} A^{(1)} = \Pi'(a'), \\ A^{(2)} = \Pi'(a^{(2)}), \\ \ldots\ldots\ldots\ldots \\ A^{(n-1)} = \Pi'(a^{(n-1)}), \end{array}\right.$$

$$(14a)\qquad V - T = \Pi(a', a^{(2)}, \ldots, a^{(n-1)}).$$

In these equations, Π is an arbitrary function of a, and $\Pi'(a^{(i)})$ denotes the partial derivative of that function with respect to $a^{(i)}$. In this case, one can then *replace equation* (1), (2) *with the* $n - 1$ *equations* (14), *which are of order two.*

If the Δ are independent of t then one can even exhibit a system of $n - 1$ differential equations, which are, however, of degree three and very complicated. Namely, equation (12) represents the system:

$$(15)\qquad \begin{cases} \dfrac{\partial(V-T)}{\partial x_1} = A^{(1)}\dfrac{\partial a'}{\partial x_1} + A^{(2)}\dfrac{\partial a^{(2)}}{\partial x_1} + \cdots + A^{(n-1)}\dfrac{\partial a^{(n-1)}}{\partial x_1} + \dfrac{\partial \Delta_1}{\partial t}, \\ \dfrac{\partial(V-T)}{\partial x_2} = A^{(1)}\dfrac{\partial a'}{\partial x_2} + A^{(2)}\dfrac{\partial a^{(2)}}{\partial x_2} + \cdots + A^{(n-1)}\dfrac{\partial a^{(n-1)}}{\partial x_2} + \dfrac{\partial \Delta_2}{\partial t}, \\ \dotfill, \\ \dfrac{\partial(V-T)}{\partial x_n} = A^{(1)}\dfrac{\partial a'}{\partial x_n} + A^{(2)}\dfrac{\partial a^{(2)}}{\partial x_n} + \cdots + A^{(n-1)}\dfrac{\partial a^{(n-1)}}{\partial x_n} + \dfrac{\partial \Delta_n}{\partial t}. \end{cases}$$

If one now forms the $(n - 2)$-order functional determinant:

$$\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_k}}$$

and the sum:

$$S_h^{(m)} = \frac{\partial(V-T)}{\partial x_1}\cdot\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_1}} + \frac{\partial(V-T)}{\partial x_2}\cdot\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_2}} + \cdots + \frac{\partial(V-T)}{\partial x_n}\cdot\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_n}}$$

then $S_h^{(m)}$ is also a functional determinant, and indeed, I will obtain it when I replace the $a^{(m)}$ in Δ_h with the function of $V - T$; I can also consider $S_h^{(m)}$ to be coefficients of $\partial a^{(m)} / \partial x_h$ in the functional determinant of order n of the functions a', $a^{(2)}$, …, $a^{(n-1)}$, $(V - T)$. I will thus have, from the aforementioned theorem:

$$\frac{\partial S_1^{(m)}}{\partial x_1} + \frac{\partial S_2^{(m)}}{\partial x_2} + \cdots + \frac{\partial S_n^{(m)}}{\partial x_n} = 0$$

for every value of m from 1 to $n - 1$, which is an equation that goes to:

$$0 = \sum_{h=1}^{n}\sum_{k=1}^{n}\frac{\partial}{\partial x_h}\left\{\frac{\partial(V-T)}{\partial x_k}\cdot\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_k}}\right\}$$

with the help of the expression for $S_h^{(m)}$.

If one also replaces $\partial\,(V - T) / \partial x_k$ with its value then equation will read like an identity, and one will obtain equations for the determination of the a. In fact, one has:

$$(16) \qquad 0 = \sum_{h=1}^{n}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left\{\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_k}}\left(A^{(1)}\frac{\partial a'}{\partial x_k} + A^{(2)}\frac{\partial a^{(2)}}{\partial x_k} + \cdots + A^{(n-1)}\frac{\partial a^{(n-1)}}{\partial x_k} + \frac{\partial \Delta_k}{\partial t}\right)\right\}.$$

That equation then simplifies even more appreciably. $A^{(1)}$ is multiplied by:

$$\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_1}}\cdot\frac{\partial a'}{\partial x_1} + \frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_2}}\cdot\frac{\partial a'}{\partial x_2} + \cdots + \frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_n}}\cdot\frac{\partial a'}{\partial x_n},$$

under the differential sign $\partial / \partial x_k$, and that is nothing but the functional determinant that arises from Δ_h when one replaces $a^{(m)}$ with a' in it. It will then contain two equal functions, and must vanish when m is not equal to 1, when it will coincide with Δ_h. The coefficient of $A^{(1)}$ is then zero, and similarly, that of $A^{(2)}$, etc., up to $A^{(m)}$, which will be Δ_h, and equation (16) will be converted into:

$$(17) \qquad 0 = \sum_{h=1}^{n}\frac{\partial}{\partial x_h}(\Delta_h \cdot A^{(m)}) + \sum_{h=1}^{n}\sum_{k=1}^{n}\frac{\partial}{\partial x_h}\left(\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_k}}\frac{\partial \Delta_k}{\partial t}\right).$$

If one then adds that, from the repeatedly-applied theorem:

$$\sum_{h=1}^{n}\frac{\partial \Delta_h}{\partial x_h} = 0, \qquad \sum_{h=1}^{n}\frac{\partial}{\partial x_h}\left(\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_h}}\right) = 0$$

then equation (17) will finally assume the following form:

$$(18) \qquad 0 = \Delta_1\frac{\partial A^{(m)}}{\partial x_1} + \Delta_2\frac{\partial A^{(m)}}{\partial x_2} + \cdots + \Delta_n\frac{\partial A^{(m)}}{\partial x_n} + Q^{(m)},$$

where we have set, for brevity:

$$(18a) \qquad Q^{(m)} = \sum_{h=1}^{n}\sum_{k=1}^{n}\frac{\partial^2 \Delta_k}{\partial t\,\partial x_h}\cdot\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_k}}.$$

Equations (18) serve to determine the a. If one thinks of these equations as having been solved, and then a', $a^{(2)}$, …, $a^{(n-1)}$, and another arbitrary function a being introduced in place of x_1, x_2, …, x_n in $V - T$, then, from equations (15), one will have:

$$\Delta_1 \frac{\partial(V-T)}{\partial x_1} + \Delta_2 \frac{\partial(V-T)}{\partial x_2} + \cdots + \Delta_n \frac{\partial(V-T)}{\partial x_n} = \frac{\partial T}{\partial t},$$

and, at the same time, the left-hand side is nothing but $R\left[\frac{\partial(V-T)}{\partial a}\right]$. Namely, we let R denote the determinant of all a with respect to the x, and let square brackets suggest that the x in it are thought of as being expressed in terms of the a. One will then have:

$$R\left[\frac{\partial(V-T)}{\partial a}\right] = \frac{\partial T}{\partial t},$$

so, by integration:

(19) $$V - T = \int \left[\frac{1}{R}\cdot\frac{\partial T}{\partial t}\right] da + \Pi\,(a', a'', \ldots, a^{(n-1)}, t),$$

where Π is an arbitrary function. Equations (18), (19) will then give the functions a, V. Differentiating equation (19) will then lead to condition equations for the arbitrary functions, which would yield the complete integration of equations (18).

I only remark that as long as *any* of the expressions $Q^{(m)}$ vanishes, the corresponding equation (18) will give the integral:

(20) $$A^{(m)} = \Omega\,(a', a'', \ldots, a^{(n-1)}, t),$$

in which Ω is an arbitrary function. *All of these* expressions will vanish in the case that was consider above.

§ 3.

Before I proceed, it will be necessary to actually develop the expressions for the A. They will be given by equations (9), (10), (11); namely, one must have:

(21) $$A^{(1)} \frac{\partial a'}{\partial x_k} + A^{(2)} \frac{\partial a^{(2)}}{\partial x_k} + \cdots + A^{(n-1)} \frac{\partial a^{(n-1)}}{\partial x_k} = \sum_{i=k}^{n} \Delta_i \left(\frac{\partial \Delta_k}{\partial x_i} - \frac{\partial \Delta_i}{\partial x_k}\right).$$

The right-hand side must be converted into the left-hand side. To that end, I consider the triple sum:

(22) $$S_k = \sum_{i=1}^{n}\sum_{h=1}^{n}\sum_{m=1}^{n-1} \frac{\partial \Delta_h}{\partial x_i}\cdot\frac{\partial a^{(m)}}{\partial x_k}\cdot\frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}}.$$

The sum:

$$\sum_{m=1}^{n-1} \frac{\partial a^{(m)}}{\partial x_k} \cdot \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}},$$

which is multiplied by $\partial\Delta_h / \partial x_i$ in S_k , obviously represents a functional determinant, and indeed the one that arises from Δ_h when one replaces the derivatives with respect to x_i in it with ones with respect to x_k . However, that determinant will generally include two rows in which one differentiates with respect to x_k , so it will vanish. Only when the indices k and i are equal will it remain unchanged by that exchange, namely, Δ_h, and when $k = h$, Δ_h will itself already no longer contain derivatives with respect to x_k , and the value of the resulting determinant will be $-\Delta_i$, since, from a well-known theorem:

$$\frac{\partial \Delta_k}{\partial \frac{\partial a^{(m)}}{\partial x_i}} = - \frac{\partial \Delta_i}{\partial \frac{\partial a^{(m)}}{\partial x_k}}.$$

Therefore, the sum (22) will reduce to:

$$S_k = \sum_{h=1}^{n} \Delta_h \frac{\partial \Delta_h}{\partial x_k} - \sum_{i=1}^{n} \frac{\partial \Delta_k}{\partial x_i} \Delta_i ,$$

and one will then see that S_k differs from the right-hand side of equation (21) only in sign.

However, equation also immediately assumes the form:

$$S_k = \sum_{m=1}^{n-1} \frac{\partial a^{(m)}}{\partial x_k} \left\{ \sum_{i=1}^{n} \sum_{h=1}^{n} \frac{\partial \Delta_h}{\partial x_i} \cdot \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}} \right\}.$$

That is the form that was already achieved in (21), since the index k no longer enters into the bracket; one can then set:

$$A^{(m)} = - \sum_{i=1}^{n} \sum_{h=1}^{n} \frac{\partial \Delta_h}{\partial x_i} \cdot \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}}. \tag{23}$$

One can give this expression a more suitable form when one remarks that:

$$0 = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}} \right);$$

one can then write:

$$A^{(m)} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left\{\sum_{h=1}^{n} \Delta_h \frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}}\right\}.$$

However, the relation:

$$2T = \Delta_1^2 + \Delta_2^2 + \cdots + \Delta_n^2$$

was introduced above. If we apply this then we will finally obtain $A^{(m)}$ in its simplest form:

$$-A^{(m)} = \frac{\partial}{\partial x_1}\frac{\partial T}{\partial \frac{\partial a^{(m)}}{\partial x_1}} + \frac{\partial}{\partial x_2}\frac{\partial T}{\partial \frac{\partial a^{(m)}}{\partial x_2}} + \cdots + \frac{\partial}{\partial x_n}\frac{\partial T}{\partial \frac{\partial a^{(m)}}{\partial x_n}}. \tag{24}$$

That is then the expression for $A^{(m)}$ into which second derivatives enter, as one sees; equation (18) will then lead to the third derivatives. We remark that equation (23) allows us to put equations (18) into the form:

$$\Delta_1 \frac{\partial A^{(m)}}{\partial x_1} + \Delta_2 \frac{\partial A^{(m)}}{\partial x_2} + \cdots + \Delta_n \frac{\partial A^{(m)}}{\partial x_n} + \frac{\partial A^{(m)}}{\partial t} = R^{(m)}, \tag{25}$$

in which we have set, for brevity:

$$R^{(m)} = \sum_{i=1}^{n}\sum_{h=1}^{n} \frac{\partial \Delta_h}{\partial x_i} \cdot \frac{\partial}{\partial t}\frac{\partial \Delta_h}{\partial \frac{\partial a^{(m)}}{\partial x_i}}, \tag{25a}$$

which is a form that will be used in what follows.

However, for the simpler case in which the Δ (or the *u*) are independent of *t*, equation (24) will give, by way of (14):

$$\frac{\partial}{\partial x_1}\frac{\partial T}{\partial \frac{\partial a^{(m)}}{\partial x_1}} + \frac{\partial}{\partial x_2}\frac{\partial T}{\partial \frac{\partial a^{(m)}}{\partial x_2}} + \cdots + \frac{\partial}{\partial x_n}\frac{\partial T}{\partial \frac{\partial a^{(m)}}{\partial x_n}} + \Pi'(a^m) = 0. \tag{26}$$

The system of equations that is thus represented is nothing but the one that one obtains from the problem of *finding a minimum for the integral:*

$$\int^{(n)} (T - \Pi)\, dx_1\, dx_2 \ldots dx_n, \tag{27}$$

where:

$$2T = \Delta_1^2 + \Delta_2^2 + \cdots + \Delta_n^2. \tag{28}$$

That theorem is important for the transformation of the differential equations in question, and it is interesting for the fact that it allows one to recognize the identity of the problem that is given by equations (1), (2), and (27).

§ 4.

If we now couple the system of differential equations that was presented with the system of complete differential equations:

$$\frac{dx_1}{dt} = u_1, \qquad \frac{dx_2}{dt} = u_2, \qquad \ldots, \qquad \frac{dx_n}{dt} = u_n, \tag{29}$$

for which we can now also set:

$$\frac{dx_1}{dt} = \Delta_1, \qquad \frac{dx_2}{dt} = \Delta_2, \qquad \ldots, \qquad \frac{dx_n}{dt} = \Delta_n. \tag{30}$$

The identity equation:

$$\frac{\partial \Delta_1}{\partial x_1} + \frac{\partial \Delta_2}{\partial x_2} + \cdots + \frac{\partial \Delta_n}{\partial x_n} = 0$$

then shows that *a multiplier of the equations is equal to one*, and one can find the last integral when one knows the first $n - 1$.

One further sees that as long as the Δ are free of t, *the integrals of the equations will immediately be the following ones:*

$$a' = \text{const.}, \qquad a^{(2)} = \text{const.}, \quad \ldots, \qquad a^{(n-1)} = \text{const.} \tag{31}$$

It will then follow from equations (30) that:

$$\frac{da^{(m)}}{dt} = \frac{\partial a^{(m)}}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial a^{(m)}}{\partial x_2} \cdot \frac{dx_2}{dt} + \ldots = \Delta_1 \frac{\partial a^{(m)}}{\partial x_1} + \Delta_2 \frac{\partial a^{(m)}}{\partial x_2} + \ldots,$$

which is identically zero. *When the A are set equal to constants, they will also be integrals of the present equations*, since, from (14), they are functions of the *a*.

One can add the following general theorem for the general case:

1. *As long as one of the a is independent of t, a* = const. *will be an integral of equations* (30).

2. *As long as the expression* (25a):

$$R^{(m)} = \sum_{i=1}^{n}\sum_{h=1}^{n} \frac{\partial \Delta_h}{\partial x_i}\cdot\frac{\partial}{\partial t}\frac{\partial \Delta_h}{\partial \dfrac{\partial a^{(m)}}{\partial x_i}}$$

vanishes:

$$A^{(m)} = \text{const.}$$

will be an integral of the present equations.

From (30), (25), one will then have:

$$\frac{dA^{(m)}}{dt} = \Delta_1\frac{\partial A^{(m)}}{\partial x_1}+\Delta_2\frac{\partial A^{(m)}}{\partial x_2}+\ldots+\Delta_n\frac{\partial A^{(m)}}{\partial x_n}+\frac{\partial A^{(m)}}{\partial t} = R^{(m)} = 0.$$

§ 5.

If we set $n = 3$, moreover, then equations (1), (2) will go to *Euler's* equations, namely:

$$(32)\qquad \left\{\begin{aligned}
\frac{\partial V}{\partial x_1} &= \frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3},\\
\frac{\partial V}{\partial x_2} &= \frac{\partial u_2}{\partial t}+u_1\frac{\partial u_2}{\partial x_1}+u_2\frac{\partial u_2}{\partial x_2}+u_3\frac{\partial u_2}{\partial x_3},\\
\frac{\partial V}{\partial x_3} &= \frac{\partial u_3}{\partial t}+u_1\frac{\partial u_3}{\partial x_1}+u_2\frac{\partial u_3}{\partial x_2}+u_3\frac{\partial u_3}{\partial x_3},
\end{aligned}\right.$$

$$0 = \frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3},$$

where u_1, u_2, u_3 are the velocities that one finds at the location (x_1, x_2, x_3) at time t, and $V = U - p / q$, when U denotes the force function, p denotes the pressure, and q denotes the constant density. Using equations, I now pose:

$$(33)\qquad \left\{\begin{aligned}
u_1 = \Delta_1 &= \frac{\partial a'}{\partial x_2}\cdot\frac{\partial a^{(2)}}{\partial x_3}-\frac{\partial a^{(2)}}{\partial x_2}\cdot\frac{\partial a'}{\partial x_3},\\
u_2 = \Delta_2 &= \frac{\partial a'}{\partial x_3}\cdot\frac{\partial a^{(2)}}{\partial x_1}-\frac{\partial a^{(2)}}{\partial x_3}\cdot\frac{\partial a'}{\partial x_1},\\
u_3 = \Delta_3 &= \frac{\partial a'}{\partial x_1}\cdot\frac{\partial a^{(2)}}{\partial x_2}-\frac{\partial a^{(2)}}{\partial x_1}\cdot\frac{\partial a'}{\partial x_2}.
\end{aligned}\right.$$

From a known transformation, one will then have:

(34) $$2T = \Delta_1^2 + \Delta_2^2 + \Delta_3^2 = P' P^{(2)} - PP,$$

when one sets, for brevity:

(35) $$\begin{cases} P' = \left(\dfrac{\partial a'}{\partial x_1}\right)^2 + \left(\dfrac{\partial a'}{\partial x_2}\right)^2 + \left(\dfrac{\partial a'}{\partial x_3}\right)^2, \\ P^{(2)} = \left(\dfrac{\partial a^{(2)}}{\partial x_1}\right)^2 + \left(\dfrac{\partial a^{(2)}}{\partial x_2}\right)^2 + \left(\dfrac{\partial a^{(2)}}{\partial x_3}\right)^2, \\ P = \dfrac{\partial a'}{\partial x_1} \cdot \dfrac{\partial a^{(2)}}{\partial x_1} + \dfrac{\partial a'}{\partial x_2} \cdot \dfrac{\partial a^{(2)}}{\partial x_2} + \dfrac{\partial a'}{\partial x_3} \cdot \dfrac{\partial a^{(2)}}{\partial x_3}, \end{cases}$$

and the expressions for A will be:

(36) $$\begin{cases} -A^{(1)} = \displaystyle\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left\{ P^{(2)} \frac{\partial a'}{\partial x_i} - P \frac{\partial a^{(2)}}{\partial x_i} \right\}, \\ -A^{(2)} = \displaystyle\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left\{ P' \frac{\partial a^{(2)}}{\partial x_i} - P \frac{\partial a'}{\partial x_i} \right\}. \end{cases}$$

The differential equations upon which the problem depends in general will then be:

(37) $$\begin{cases} 0 = \Delta_1 \dfrac{\partial A^{(1)}}{\partial x_1} + \Delta_2 \dfrac{\partial A^{(1)}}{\partial x_2} + \Delta_3 \dfrac{\partial A^{(1)}}{\partial x_3} + \dfrac{\partial A^{(1)}}{\partial t} - R^{(1)}, \\ 0 = \Delta_1 \dfrac{\partial A^{(2)}}{\partial x_1} + \Delta_2 \dfrac{\partial A^{(2)}}{\partial x_2} + \Delta_3 \dfrac{\partial A^{(2)}}{\partial x_3} + \dfrac{\partial A^{(2)}}{\partial t} - R^{(2)}, \end{cases}$$

where:

(37a) $$\begin{cases} R^{(1)} = \dfrac{\partial^2 a^{(2)}}{\partial x_1\, \partial t}\left(\dfrac{\partial \Delta_3}{\partial x_2} - \dfrac{\partial \Delta_2}{\partial x_3}\right) + \dfrac{\partial^2 a^{(2)}}{\partial x_2\, \partial t}\left(\dfrac{\partial \Delta_1}{\partial x_3} - \dfrac{\partial \Delta_3}{\partial x_1}\right) + \dfrac{\partial^2 a^{(2)}}{\partial x_3\, \partial t}\left(\dfrac{\partial \Delta_2}{\partial x_1} - \dfrac{\partial \Delta_1}{\partial x_2}\right), \\ -R^{(2)} = \dfrac{\partial^2 a'}{\partial x_1\, \partial t}\left(\dfrac{\partial \Delta_3}{\partial x_2} - \dfrac{\partial \Delta_2}{\partial x_3}\right) + \dfrac{\partial^2 a'}{\partial x_2\, \partial t}\left(\dfrac{\partial \Delta_1}{\partial x_3} - \dfrac{\partial \Delta_3}{\partial x_1}\right) + \dfrac{\partial^2 a'}{\partial x_3\, \partial t}\left(\dfrac{\partial \Delta_2}{\partial x_1} - \dfrac{\partial \Delta_1}{\partial x_2}\right). \end{cases}$$

The integrals of the equations:

(38) $$\frac{dx_1}{dt} = \Delta_1, \qquad \frac{dx_2}{dt} = \Delta_2, \qquad \frac{dx_3}{dt} = \Delta_3$$

represent a varying system of curves on which the particles move. From the theorems that were discussed in § 4, *one must now have the integral:*

$$A^{(m)} = \text{const.}$$

for the integrals when the term $R^{(m)}$ *vanishes in one of the equations* (37). That is the case, e.g., when $a^{(2)}$ is independent of time; one will then have that:

$$A^{(1)} = \text{const.}$$

is an integral. However:

$$a^{(2)} = \text{const.}$$

will also be one. If we connect that with the principle of the last multiplier *then we can integrate equations* (38), *as long as one of surfaces in the system of surfaces upon which the motion takes place is independent of time.* Let the last integral be $\varphi = \text{const.}$ φ must then satisfy the equation:

$$W = \frac{\partial\psi}{\partial t} + \Delta_1\frac{\partial\psi}{\partial x_1} + \Delta_2\frac{\partial\psi}{\partial x_2} + \Delta_3\frac{\partial\psi}{\partial x_3} = 0,$$

and since the multiplier is 1, this expression will also be equal to the determinant:

$$W \equiv \begin{vmatrix} \dfrac{\partial\psi}{\partial x_1} & \dfrac{\partial\varphi}{\partial x_1} & \dfrac{\partial a^{(2)}}{\partial x_1} & \dfrac{\partial A^{(1)}}{\partial x_1} \\ \dfrac{\partial\psi}{\partial x_2} & \dfrac{\partial\varphi}{\partial x_2} & \dfrac{\partial a^{(2)}}{\partial x_2} & \dfrac{\partial A^{(1)}}{\partial x_2} \\ \dfrac{\partial\psi}{\partial x_3} & \dfrac{\partial\varphi}{\partial x_3} & \dfrac{\partial a^{(2)}}{\partial x_3} & \dfrac{\partial A^{(1)}}{\partial x_3} \\ \dfrac{\partial\psi}{\partial t} & \dfrac{\partial\varphi}{\partial t} & \dfrac{\partial a^{(2)}}{\partial t} & \dfrac{\partial A^{(1)}}{\partial t} \end{vmatrix}.$$

If one now introduces $a^{(2)}$, $A^{(1)}$, and any new variable v in place of x_1, x_2, x_3, and denotes the new derivatives by a bracket then that identity equation will go to:

$$\left(\frac{\partial\psi}{\partial t}\right) + \left(\frac{\partial\psi}{\partial v}\right)\left\{\frac{\partial v}{\partial t} + \Delta_1\frac{\partial v}{\partial x_1} + \Delta_2\frac{\partial v}{\partial x_2} + \Delta_3\frac{\partial v}{\partial x_3}\right\} \equiv D\cdot\begin{vmatrix} \left(\dfrac{\partial\psi}{\partial v}\right) & \left(\dfrac{\partial\varphi}{\partial v}\right) \\ \left(\dfrac{\partial\psi}{\partial t}\right) & \left(\dfrac{\partial\varphi}{\partial t}\right) \end{vmatrix},$$

where D is the determinant of v, $a^{(2)}$, $A^{(1)}$ with respect to x_1, x_2, x_3 . If one then sets, for brevity:

(39) $$w = \frac{\partial v}{\partial t} + \Delta_1\frac{\partial v}{\partial x_1} + \Delta_2\frac{\partial v}{\partial x_2} + \Delta_3\frac{\partial v}{\partial x_3},$$

then it will follow from the identity equation that:

$$-1 = D\left(\frac{\partial\varphi}{\partial v}\right), \qquad w = D\left(\frac{\partial\varphi}{\partial t}\right),$$

and the desired last integral will then be:

(40) $$\varphi = \int \frac{w\,dt - dv}{D}.$$

This case will occur, e.g., *when the motion is the same on all sides of a vertical*, so in that case, one can give all of the integrals, since *one* integral (viz., the plane of motion that goes through that vertical) is independent of t.

If the motion is stationary then, from (26), etc., *one will have to solve the equations that define the minimum of the integral:*

(41) $$\iiint \left(\frac{P'P^{(2)} - PP}{2} - \Pi(a', a^{(2)})\right) dx_1 dx_2 dx_3\,,$$

namely, the equations:

(42) $$A^{(1)} = \Pi'(a'), \qquad A^{(2)} = \Pi'(a^{(2)}).$$

The functions a', $a^{(2)}$ will then give the system of surfaces in whose intersection the motion takes place:

$$a' = \text{const.}, \qquad a^{(2)} = \text{const.}$$

That is the theorem that was mentioned in the introduction. The pressure will be finite, and as a result of equations (14a) and (19), it will be given, in general, by the formula:

$$U - \frac{p}{q} - T = \int \left[\frac{1}{R}\frac{\partial T}{\partial t}\right] da + \Pi\,(a', a^{(2)}, t),$$

and for the stationary motion, in particular, it will be given by:

$$U - \frac{p}{q} - T = \Pi\,(a', a^{(2)})\,.$$

This is likewise the true form that the equation of the vis viva assumes.

§ 6.

The introduction of new variables into the present equations will raise no new difficulties. For the stationary state, one will have nothing to do but to transform the expression T, which obviously assumes only a knowledge of the form that the square of the line element assumes. If that is:

(43) $$ds^2 = u_{11}dy_1^2 + u_{22}dy_2^2 + 2u_{12}dy_1dy_2 \;\ldots,$$

in which y_1, y_2, y_3 are the new variables (which are thought of as independent of t), then the transformation determinant will be:

(44) $$D = \sqrt{\begin{vmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{vmatrix}},$$

and furthermore:

$$2T = P' P^{(2)} - P\, P,$$

where:

(45) $$P = -\begin{vmatrix} u_{11} & u_{12} & u_{13} & \dfrac{\partial a^{(2)}}{\partial y_1} \\ u_{21} & u_{22} & u_{23} & \dfrac{\partial a^{(2)}}{\partial y_2} \\ u_{31} & u_{32} & u_{33} & \dfrac{\partial a^{(2)}}{\partial y_3} \\ \dfrac{\partial a'}{\partial y_1} & \dfrac{\partial a'}{\partial y_2} & \dfrac{\partial a'}{\partial y_3} & 0 \end{vmatrix} \cdot \frac{1}{D^2},$$

and in which P' will be obtained from P when one replaces $a^{(2)}$ with a', and $P^{(2)}$ will be obtained P when one replaces a' with $a^{(2)}$. From the theory of determinants, one easily infers that $2T$ also assumes the form:

(45a) $$2T = \frac{1}{D^2} \cdot \begin{vmatrix} u_{11} & u_{12} & u_{13} & \dfrac{\partial a'}{\partial y_1} & \dfrac{\partial a^{(2)}}{\partial y_1} \\ u_{21} & u_{22} & u_{23} & \dfrac{\partial a'}{\partial y_2} & \dfrac{\partial a^{(2)}}{\partial y_2} \\ u_{31} & u_{32} & u_{33} & \dfrac{\partial a'}{\partial y_3} & \dfrac{\partial a^{(2)}}{\partial y_3} \\ \dfrac{\partial a'}{\partial y_1} & \dfrac{\partial a'}{\partial y_2} & \dfrac{\partial a'}{\partial y_3} & 0 & 0 \\ \dfrac{\partial a^{(2)}}{\partial y_1} & \dfrac{\partial a^{(2)}}{\partial y_2} & \dfrac{\partial a^{(2)}}{\partial y_3} & 0 & 0 \end{vmatrix}.$$

[cf., *Hesse*, "Über Determinanten in der Geometrie," this journal, Bd. 49, pp. 248, formulas (6), (7).]

The integral that is to have a minimum will then become:

$$\iiint (T - \Pi) \cdot D\, dy_1\, dy_2\, dy_3 ,$$

and the equations:

(46)
$$\begin{cases} 0 = D \cdot \Pi'(a') + \sum_{i=1}^{3} \dfrac{\partial}{\partial y_i}\left(D \cdot \dfrac{\partial T}{\partial \dfrac{\partial a'}{\partial y_i}} \right), \\[2ex] 0 = D \cdot \Pi'(a^{(2)}) + \sum_{i=1}^{3} \dfrac{\partial}{\partial y_i}\left(D \cdot \dfrac{\partial T}{\partial \dfrac{\partial a^{(2)}}{\partial y_i}} \right) \end{cases}$$

will flow from that by a known method. That equation also gives the transformation formula for the A in the general case:

(47)
$$-A^{(m)} = \frac{1}{D} \sum_{i=1}^{3} \frac{\partial}{\partial y_i}\left(D \cdot \frac{\partial T}{\partial \dfrac{\partial a^{(m)}}{\partial y_i}} \right).$$

We require this formula for the transformation of equation (37). If we further remark that the first part of that equation, namely:

$$U = \Delta_1 \frac{\partial A^{(m)}}{\partial x_1} + \Delta_2 \frac{\partial A^{(m)}}{\partial x_2} + \Delta_3 \frac{\partial A^{(m)}}{\partial x_3},$$

is nothing but the functional determinant of a', $a^{(2)}$, $A^{(m)}$ with respect to the x then it will follow immediately that $D \cdot U$ is the functional determinant of a', $a^{(2)}$, $A^{(m)}$ with respect to the y, so when one sets:

(48)
$$\begin{cases} \nabla_1 = \dfrac{\partial a'}{\partial y_2}\dfrac{\partial a^{(2)}}{\partial y_3} - \dfrac{\partial a'}{\partial y_3}\dfrac{\partial a^{(2)}}{\partial y_2}, \\[2ex] \nabla_2 = \dfrac{\partial a'}{\partial y_3}\dfrac{\partial a^{(2)}}{\partial y_1} - \dfrac{\partial a'}{\partial y_1}\dfrac{\partial a^{(2)}}{\partial y_3}, \\[2ex] \nabla_3 = \dfrac{\partial a'}{\partial y_1}\dfrac{\partial a^{(2)}}{\partial y_2} - \dfrac{\partial a'}{\partial y_2}\dfrac{\partial a^{(2)}}{\partial y_1}, \end{cases}$$

the identity equation will come about:

(49)
$$\sum_{i=1}^{3} \Delta_i \frac{\partial A^{(m)}}{\partial x_i} = \frac{1}{D} \sum_{k=1}^{3} \nabla_k \frac{\partial A^{(m)}}{\partial y_k}.$$

If one takes the coefficients of $\frac{\partial A^{(m)}}{\partial x_i}$ on both sides of this then one will obtain the transformation of Δ_i, namely:

(50) $$\Delta_i = \sum_{k=1}^{3} \frac{\nabla_k}{D} \cdot \frac{\partial x_i}{\partial y_k}.$$

This equation, in whose derivation the nature of the a', $a^{(2)}$, $A^{(m)}$ is completely irrelevant, involves the general equation:

$$\frac{\partial \varphi}{\partial x_2}\frac{\partial \psi}{\partial x_3} - \frac{\partial \varphi}{\partial x_3}\frac{\partial \psi}{\partial x_2} = \frac{1}{D} \cdot \begin{vmatrix} \frac{\partial \varphi}{\partial y_1} & \frac{\partial \psi}{\partial y_1} & \frac{\partial x_1}{\partial y_1} \\ \frac{\partial \varphi}{\partial y_2} & \frac{\partial \psi}{\partial y_2} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial \varphi}{\partial y_3} & \frac{\partial \psi}{\partial y_3} & \frac{\partial x_1}{\partial y_3} \end{vmatrix},$$

and when one applies this equation to (37a), one will get:

(51) $$D\,R^{(1)} = \sum_{i=1}^{3} \begin{vmatrix} \frac{\partial^2 a^{(2)}}{\partial t\,\partial y_1} & \frac{\partial}{\partial y_1}\left(\frac{\nabla_1}{D}\frac{\partial x_i}{\partial y_1} + \frac{\nabla_2}{D}\frac{\partial x_i}{\partial y_2} + \frac{\nabla_3}{D}\frac{\partial x_i}{\partial y_3}\right) & \frac{\partial x_i}{\partial y_1} \\ \frac{\partial^2 a^{(2)}}{\partial t\,\partial y_2} & \frac{\partial}{\partial y_2}\left(\frac{\nabla_1}{D}\frac{\partial x_i}{\partial y_1} + \frac{\nabla_2}{D}\frac{\partial x_i}{\partial y_2} + \frac{\nabla_3}{D}\frac{\partial x_i}{\partial y_3}\right) & \frac{\partial x_i}{\partial y_2} \\ \frac{\partial^2 a^{(2)}}{\partial t\,\partial y_3} & \frac{\partial}{\partial y_3}\left(\frac{\nabla_1}{D}\frac{\partial x_i}{\partial y_1} + \frac{\nabla_2}{D}\frac{\partial x_i}{\partial y_2} + \frac{\nabla_3}{D}\frac{\partial x_i}{\partial y_3}\right) & \frac{\partial x_i}{\partial y_3} \end{vmatrix}.$$

In this, $\frac{\partial^2 a^{(2)}}{\partial t\,\partial y_1}$ is multiplied by:

$$\sum_{i=1}^{3}\left[\frac{\partial x_i}{\partial y_3}\frac{\partial}{\partial y_2}\left(\frac{\nabla_1}{D}\frac{\partial x_i}{\partial y_1} + \frac{\nabla_2}{D}\frac{\partial x_i}{\partial y_2} + \frac{\nabla_3}{D}\frac{\partial x_i}{\partial y_3}\right) - \frac{\partial x_i}{\partial y_2}\frac{\partial}{\partial y_3}\left(\frac{\nabla_1}{D}\frac{\partial x_i}{\partial y_1} + \frac{\nabla_2}{D}\frac{\partial x_i}{\partial y_2} + \frac{\nabla_3}{D}\frac{\partial x_i}{\partial y_3}\right)\right],$$

or

$$\sum_{i=1}^{3}\sum_{k=1}^{3}\left\{\frac{\partial x_i}{\partial y_3}\frac{\partial}{\partial y_2}\left(\frac{\nabla_k}{D}\frac{\partial x_i}{\partial y_k}\right) - \frac{\partial x_i}{\partial y_2}\frac{\partial}{\partial y_3}\left(\frac{\nabla_k}{D}\frac{\partial x_i}{\partial y_k}\right)\right\},$$

or, when one performs the differentiation:

$$\sum_{k=1}^{3}\left\{\frac{\partial}{\partial y_2}\left(\frac{\nabla_k}{D}\right) \cdot \sum_{k=1}^{3}\frac{\partial x_i}{\partial y_3}\frac{\partial x_i}{\partial y_k} - \frac{\partial}{\partial y_3}\left(\frac{\nabla_k}{D}\right)\sum_{i=1}^{3}\frac{\partial x_i}{\partial y_2}\frac{\partial x_i}{\partial y_k}\right\} +$$

$$\sum_{k=1}^{3}\frac{\nabla_3}{D}\left\{\sum_{i=1}^{3}\frac{\partial x_i}{\partial y_3}\frac{\partial^2 x_i}{\partial y_2\,\partial y_k}-\sum_{i=1}^{3}\frac{\partial x_i}{\partial y_2}\frac{\partial^2 x_i}{\partial y_3\,\partial y_k}\right\},$$

and when one remarks that, from (43), one has:

$$\sum_{i=1}^{3}\frac{\partial x_i}{\partial y_k}\frac{\partial x_i}{\partial y_h} = u_{kh}\,, \tag{52}$$

the coefficient in question will then go to:

$$\sum_{k=1}^{3}\left\{\frac{\partial}{\partial y_2}\left(\frac{\nabla_k}{D}\right)\cdot u_{3k}-\frac{\partial}{\partial y_3}\left(\frac{\nabla_k}{D}\right)\cdot u_{2k}+\frac{\nabla_k}{D}\left(\frac{\partial u_{3k}}{\partial y_2}-\frac{\partial u_{2k}}{\partial y_3}\right)\right\},$$

and the desired form of R(1) will finally become:

$$(53)\quad DR^{(1)} = \sum_{k=1}^{3}\frac{\nabla_k}{D}\left[\frac{\partial^2 a^{(2)}}{\partial t\,\partial y_1}\left(\frac{\partial u_{3k}}{\partial y_2}-\frac{\partial u_{2k}}{\partial y_3}\right)+\frac{\partial^2 a^{(2)}}{\partial t\,\partial y_2}\left(\frac{\partial u_{1k}}{\partial y_3}-\frac{\partial u_{3k}}{\partial y_1}\right)+\frac{\partial^2 a^{(2)}}{\partial t\,\partial y_3}\left(\frac{\partial u_{2k}}{\partial y_1}-\frac{\partial u_{1k}}{\partial y_2}\right)\right]$$

$$+\sum_{k=1}^{3}\begin{vmatrix}\dfrac{\partial^2 a^{(2)}}{\partial t\,\partial y_1} & \dfrac{\partial}{\partial y_1}\left(\dfrac{\nabla_k}{D}\right) & u_{1k}\\ \dfrac{\partial^2 a^{(2)}}{\partial t\,\partial y_2} & \dfrac{\partial}{\partial y_2}\left(\dfrac{\nabla_k}{D}\right) & u_{2k}\\ \dfrac{\partial^2 a^{(2)}}{\partial t\,\partial y_3} & \dfrac{\partial}{\partial y_3}\left(\dfrac{\nabla_k}{D}\right) & u_{3k}\end{vmatrix}.$$

This equation and equation (49) together complete the transformation of equations (37), which then likewise goes back to the transformation of only the line element.

In particular, if the y are three systems of surfaces that intersect at right angles then u_{12}, u_{23}, u_{31} will vanish, and one will get the new expression for $R^{(1)}$:

$$(54)\quad R^{(1)} = \left\{\frac{\partial}{\partial y_2}\left(u_{33}\frac{\nabla_3}{D}\right)-\frac{\partial}{\partial y_3}\left(u_{22}\frac{\nabla_2}{D}\right)\right\}\frac{\partial^2 a^{(2)}}{\partial t\,\partial y_1}+\left\{\frac{\partial}{\partial y_3}\left(u_{11}\frac{\nabla_1}{D}\right)-\frac{\partial}{\partial y_1}\left(u_{33}\frac{\nabla_3}{D}\right)\right\}\frac{\partial^2 a^{(2)}}{\partial t\,\partial y_3}$$
$$+\left\{\frac{\partial}{\partial y_1}\left(u_{22}\frac{\nabla_2}{D}\right)-\frac{\partial}{\partial y_2}\left(u_{11}\frac{\nabla_1}{D}\right)\right\}\frac{\partial^2 a^{(2)}}{\partial t\,\partial y_2},$$

and the corresponding expressions for $R^{(2)}$ is obtained by switching a' and $a^{(2)}$. Finally, the transformation of equations (38) is contained in equation (50). If one multiplies it by $\partial y_h / \partial x_i$ and sums over i then one will get:

(55) $$\frac{dy_h}{dt} = \frac{\nabla_h}{D}.$$

§ 7.

In the individual cases, one is in a position to determine *one* of the functions a', $a^{(2)}$ from the outset just from the mechanical nature of the problem. One will then obtain a differential equation for the remaining function a that will generally be of order three, but of order two for stationary motion.

Let the motion be such that all particles are required to move in parallel planes whose equations might be represented by x_3 = const. One can then set:

(56) $$a^{(2)} = x_3\,, \qquad a' = f(x_1, x_2, t).$$

Moreover, equations (35), (36) go to:

(57) $$\begin{cases} P' = \left(\dfrac{\partial a'}{\partial x_1}\right)^2 + \left(\dfrac{\partial a'}{\partial x_2}\right)^2, \quad P^{(2)} = 1, \quad P = 0, \\ -A^{(1)} = \dfrac{\partial^2 a'}{\partial x_1^2} + \dfrac{\partial^2 a'}{\partial x_2^2}, \qquad A^{(2)} = 0, \end{cases}$$

and, from (33), the expressions for the velocities will be:

(58) $$u_1 = \Delta_1 = \frac{\partial a'}{\partial x_2}, \qquad u_2 = \Delta_2 = -\frac{\partial a'}{\partial x_1}, \qquad u_3 = 0.$$

Therefore, of equations (37), the second one will vanish identically, and one will have the single equation:

(59) $$\frac{\partial a'}{\partial x_2}\cdot\frac{\partial A^{(1)}}{\partial x_1} - \frac{\partial a'}{\partial x_1}\cdot\frac{\partial A^{(1)}}{\partial x_2} + \frac{\partial A^{(1)}}{\partial t} = 0,$$

in which A is defined by (57). In the case of stationary motion, equations (42), (57) will give:

(60) $$\frac{\partial^2 a'}{\partial x_1^2} + \frac{\partial^2 a'}{\partial x_2^2} + \Pi'(a') = 0.$$

The integrals of the differential equations that belong to (59) are:

(61) $$A^{(1)} = \text{const.}$$

and a second one that one obtains from the principle of the last multiplier, namely (40):

$$\text{const.} = \int \frac{\left(\frac{\partial v}{\partial t} + \frac{\partial a'}{\partial x_2}\frac{\partial v}{\partial x_1} - \frac{\partial a'}{\partial x_1}\frac{\partial v}{\partial x_2}\right)dt - dv}{\frac{\partial v}{\partial x_2}\frac{\partial A^{(1)}}{\partial x_1} - \frac{\partial v}{\partial x_1}\frac{\partial A^{(1)}}{\partial x_2}}, \tag{62}$$

in which $A^{(1)}$ and the arbitrary function v were introduced under the integral sign in place of x_1, x_2 as variables

A second case in which the differential equations likewise reduce to *one* is given by the motion that is same in all directions around an axis. Let that axis be x_3 . One can then introduce the coordinates:

$$x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, \quad x_3 = z. \tag{63}$$

The square of the line element is then known to be:

$$ds^2 = dr^2 + r^2\,d\varphi^2 + dz^2. \tag{64}$$

One likewise obtains:

$$\left\{\begin{aligned} P' &= \left(\frac{\partial a'}{\partial r}\right)^2 + \left(\frac{\partial a'}{\partial z}\right)^2 + \frac{1}{r^2}\left(\frac{\partial a'}{\partial \varphi}\right)^2, \\ P^{(2)} &= \left(\frac{\partial a^{(2)}}{\partial r}\right)^2 + \left(\frac{\partial a^{(2)}}{\partial z}\right)^2 + \frac{1}{r^2}\left(\frac{\partial a^{(2)}}{\partial \varphi}\right)^2, \\ P &= \frac{\partial a'}{\partial r}\frac{\partial a^{(2)}}{\partial r} + \frac{\partial a'}{\partial z}\frac{\partial a^{(2)}}{\partial z} + \frac{1}{r^2}\frac{\partial a'}{\partial \varphi}\frac{\partial a^{(2)}}{\partial \varphi}. \end{aligned}\right. \tag{65}$$

The transformation is r. Now, since $2T = P'\,P^{(2)} - PP$, equations (47) will give:

$$\left\{\begin{aligned} -A^{(1)} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\left| P^{(2)}\frac{\partial a'}{\partial r} - P\frac{\partial a^{(2)}}{\partial r}\right|\right) + \frac{\partial}{\partial z}\left(P^{(2)}\frac{\partial a'}{\partial z} - P\frac{\partial a^{(2)}}{\partial z}\right) \\ &\quad + \frac{1}{r^2}\frac{\partial}{\partial \varphi}\left(P^{(2)}\frac{\partial a'}{\partial \varphi} - P\frac{\partial a^{(2)}}{\partial \varphi}\right), \\ -A^{(2)} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\left| P'\frac{\partial a^{(2)}}{\partial r} - P\frac{\partial a'}{\partial r}\right|\right) + \frac{\partial}{\partial z}\left(P'\frac{\partial a^{(2)}}{\partial z} - P\frac{\partial a'}{\partial z}\right) \\ &\quad + \frac{1}{r^2}\frac{\partial}{\partial \varphi}\left(P'\frac{\partial a^{(2)}}{\partial \varphi} - P\frac{\partial a'}{\partial \varphi}\right). \end{aligned}\right. \tag{66}$$

When considering motion that is symmetric around x_3, one can now attempt to set:

$$a^{(2)} = \varphi, \qquad a' = f(r, z, t).$$

Equations (65) will then go to:

$$P' = \left(\frac{\partial a'}{\partial r}\right)^2 + \left(\frac{\partial a'}{\partial z}\right)^2, \qquad P^{(2)} = \frac{1}{r^2}, \qquad P = 0,$$

and equations (66) will give:

(67) $$-A^{(1)} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a'}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial a'}{\partial z}\right), \qquad A^{(2)} = 0.$$

Moreover, equations (48) go to:

(68) $$\nabla_1 = -\frac{\partial a'}{\partial z}, \qquad \nabla_2 = 0, \qquad \nabla_3 = \frac{\partial a'}{\partial r}.$$

If one adds that $D = r$ then one will see from (54) that, in the first place, both expressions for R will vanish, and the equation (49) will give:

(69) $$\frac{\partial a'}{\partial r}\cdot\frac{\partial A^{(1)}}{\partial z} - \frac{\partial a'}{\partial z}\cdot\frac{\partial A^{(1)}}{\partial r} + r\frac{\partial A^{(1)}}{\partial t} = 0,$$

in which $A^{(1)}$ is defined by (67), and for stationary motion:

(70) $$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial a'}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial a'}{\partial z}\right) + \Pi'(a') = 0.$$

The differential equations that ultimately have to be integrated will be:

(71) $$\frac{dr}{dt} = -\frac{1}{r}\frac{\partial a'}{\partial z}, \qquad \frac{dz}{dt} = \frac{1}{r}\frac{\partial a'}{\partial r}.$$

One integral is, in turn, $A^{(1)}$ = const.; the other one will be obtained from the theory of multipliers, namely:

(72) $$\text{const.} = \int \frac{\left(r\frac{\partial v}{\partial t} - \frac{\partial a'}{\partial z}\frac{\partial v}{\partial r} + \frac{\partial a'}{\partial r}\frac{\partial v}{\partial z}\right)dt - dv}{\frac{\partial v}{\partial r}\frac{\partial A^{(1)}}{\partial z} - \frac{\partial v}{\partial z}\frac{\partial A^{(1)}}{\partial r}},$$

in which $A^{(1)}$, which remains constant during the integration and v, which is an arbitrary function of r and z are introduced under the integral sign in place of r, z.

Berlin, 26 May 1857.