"Su di una reciprocità tra deformazioni e distortioni," Rend. Accad. dei Lincei **24** (1915), 404-408.

MECHANICS. – *On a reciprocity between deformation and distortions.* Note by G. COLONETTI, presented by member V. VOLTERRA.

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In a classical series of memoirs on the equilibrium of multiply-connected elastic bodies, prof. Volterra proved the possibility of creating states of equilibrium in them that differed from the natural state without the intervention of any external forces by means of certain operations that he himself gave the name of *distortion* ([1]).

We note that between two systems of forces that are generated by two given systems of distortions the exists a reciprocity that is completely analogous to the one that Betti has established between two systems of displacements that derive from two given systems of external forces ([2]).

I propose to demonstrate that a reciprocity also exists between systems of internal tensions produced by given external stresses and systems of displacements that are determined in the same elastic body from a given distortion.

Let S be the connected – indeed, multiply connected, in general – space that is occupied by an elastic body in equilibrium under the action of a given system of external forces. Let Σ indicate the closed surface that bounds it, and let s be an arbitrary diaphragm that that is drawn across that space, i.e., a surface that is completely contained in S, and does not have its contour on Σ.

If one imagines that one makes a cut in the given elastic body along that diaphragm then the original state of equilibrium in it is conserved after making the cut, whether one imagines applying two force distributions or, equivalently, internal tensions that are initially transmitted across the surface σ in the given body.

Call φ the unitary elastic potential, and let:

$$X, Y, Z$$

be the components, relative to three coordinate axes, of the external forces that are applied to the generic volume element dS, which refers to the unit of volume, and let:

$$X_n , Y_n , Z_n$$

([1]) V. Volterra, *Un teorema sulle teoria della elasticità* (Rend. R. Accademia dei Lincei, 5th ser., vol. XIV); *Sull'equilibrio dei corpi elastici più volte connessi* (ibid., 5th series, vol. XIV); *Sulla distortioni dei solidi simmetrici* (ibid., 5th series, vol. XIV); *Contributo allo studio distorsioni dei solidi elastici* (ibid., 5th series, vol. XIV); *Sulle distorsioni generate da tagli uniformi* (ibid., 5th series, vol. XIV); *Nuovi studii sulle distorsioni dei solidi elastici* (ibid., 5th series, vol. XV); *Sull'equilibrio deo corpo elastici più volte connessi* (Il Nuovo Cimento, 5th series, vol. X and XI); *Sur l'équilibre des corps élastiques multiplement connexes* (Ann. éc norm., **3**, t. XXIV).

([2]) Cf., on this, V. Volterra, *Sur l'équilibre des corps élastiques multiplement connexes* (Ann. éc norm., **3**, t. XXIV), pp. 432 et seq.

be the analogous applied pressure referred to the unit area on a generic element with normal n of the surfaces Σ and σ. The conditions of equilibrium are well-known to be summarized in the relations:

$$
(1) \qquad \begin{cases} 0 = \int_S \delta\varphi \cdot dS + \int_S (X\,\delta u + Y\,\delta v + Z\,\delta w)\,dS \\ \qquad + \int_\Sigma (X_n \delta u + Y_n \delta v + Z_n \delta z)\,d\Sigma \\ \qquad + \int_\sigma [X_n(\delta u_\alpha - \delta u_\beta) + Y_n(\delta v_\alpha - \delta v_\beta) + Z_n(\delta u_\alpha - \delta u_\beta)]\,d\sigma, \end{cases}
$$

in which:

$$\delta u,\ \delta v,\ \delta w$$

denote the components, relative to the same axes, of the displacement of a generic point of S or of Σ *under an arbitrary possible deformation of the* SLICED *elastic body*;

$$\delta u_\alpha,\ \delta v_\alpha,\ \delta w_\alpha,$$

and

$$\delta u_\beta,\ \delta v_\beta,\ \delta w_\beta,$$

are the values of:

$$\delta u,\ \delta v,\ \delta w$$

relative to a generic point of σ, when considered as belonging to one face of the cut or the other one. More precisely, one assumes, as usual, that the direction of the normal to Σ is positive and points to the interior of S, and, in contradistinction, one intends that the index α refers to the components of a displacement of a point of σ that one regards as belonging to the face of the cut such that the normal to σ points into S, while the index β refers to the points that belong to the other face of the cut, with respect to which the normal to σ points out of S.

We suppose that the relative motion of the two faces of the cut is a simple rigid motion in space: i.e., that:

$$
\begin{aligned}
\delta u_\alpha - \delta u_\beta &= l' + q'z - r'y \\
\delta v_\alpha - \delta v_\beta &= m' + r'z - p'y \\
\delta w_\alpha - \delta w_\beta &= n' + p'z - q'y
\end{aligned}
$$

with l', m', n', p', q', r' constants.

The variation of the configuration that the elastic body experiences is then, in general, a Volterra distortion, in which:

$$l',\ m',\ n',\ p',\ q',\ r'$$

are the characteristics. The only exception is the case in which the act of making the cut makes the space S cease to be continuous; in this case, the phenomenon obviously reduces to a simple rigid displacement of one of the two portions into which the body has been divided.

In any event, call:

$$u', v', w',$$

the components of the displacement thus determined at a generic point of S or Σ, and set, as usual:

$$\begin{aligned} x'_x &= \frac{\partial u'}{\partial x}, & y'_z &= \frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial z}, \\ y'_y &= \frac{\partial v'}{\partial y}, & z'_x &= \frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}, \\ z'_z &= \frac{\partial w'}{\partial z}, & x'_y &= \frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y}. \end{aligned}$$

(1) then easily transforms into the relation:

$$(2) \qquad \begin{cases} \int_S (Xu' + Yv' + Zw')\, dS + \int_\Sigma (X_n u' + Y_n v' + Z_n w')\, d\Sigma \\ + \int_\sigma [X_n (l' + q'z - r'y) + Y_n (m' + r'z - p'y) + Z_n (n' + p'z - q'y)] \\ = -\int_S \left(\frac{\partial \varphi}{\partial x_x} x'_x + \frac{\partial \varphi}{\partial y_y} y'_x + \frac{\partial \varphi}{\partial z_z} z'_z + \frac{\partial \varphi}{\partial y_z} y'_z + \frac{\partial \varphi}{\partial z_x} z'_x + \frac{\partial \varphi}{\partial x_y} x'_y \right) dS, \end{cases}$$

which we write in the form:

$$(2') \qquad \begin{cases} \int_S (Xu' + Yv' + Zw')\, dS + \int_\Sigma (X_n u' + Y_n v' + Z_n w')\, d\Sigma \\ \quad + Ll' + Mm' + Nn' + Pp' + Qq' + Rr' \\ = -\int_S \left(\frac{\partial \varphi}{\partial x_x} x'_x + \frac{\partial \varphi}{\partial y_y} y'_x + \frac{\partial \varphi}{\partial z_z} z'_z + \frac{\partial \varphi}{\partial y_z} y'_z + \frac{\partial \varphi}{\partial z_x} z'_x + \frac{\partial \varphi}{\partial x_y} x'_y \right) dS, \end{cases}$$

in which, for brevity, we let:

$$\begin{aligned} L &= \int_\sigma X_n d\sigma & P &= \int_\sigma (Z_n y - Y_n z) d\sigma \\ M &= \int_\sigma Y_n d\sigma & Q &= \int_\sigma (X_n z - Z_n x) d\sigma \\ N &= \int_\sigma X_n d\sigma & R &= \int_\sigma (Y_n x - X_n y) d\sigma \end{aligned}$$

denote the six characteristics of the system of internal tensions relative to σ.

Now, it is easy to prove that the right-hand side of equation (2′) is identically null.

The assertion is obvious in the case in which the u', v', w' describe a simple rigid motion in space, since one then has, at any point of S:

$$x'_x = y'_y = z'_z = y'_z = z'_x = x'_y = 0.$$

In the general case, in which one finds distortions, properly speaking, if one observes that, for equilibrium, one must satisfy an equation of type (1) for a force that is null on S, as well as on Σ, and under completely arbitrary variations δu, δv, δw. Now, such variations become precisely equal to the displacements u, v, w that are produced in the given elastic system by the given system of external forces, since at any point of s one arrives at:

$$\delta u_\alpha = \delta u_\beta, \qquad \delta v_\alpha = \delta v_\beta, \qquad \delta w_\alpha = \delta w_\beta,$$

which reduces the equation to:

$$0 = \int_S \left(\frac{\partial \varphi'}{\partial x'_x} x_x + \frac{\partial \varphi'}{\partial y'_y} y_y + \frac{\partial \varphi'}{\partial z'_z} z_z + \frac{\partial \varphi'}{\partial y'_z} y_z + \frac{\partial \varphi'}{\partial z'_x} z_x + \frac{\partial \varphi'}{\partial x'_y} x_y \right) dS\,,$$

and thus, from a well-known property of quadratic forms:

$$\begin{aligned}\frac{\partial \varphi'}{\partial x'_x} x_x + \frac{\partial \varphi'}{\partial y'_y} y_y + \frac{\partial \varphi'}{\partial z'_z} z_z + \frac{\partial \varphi'}{\partial y'_z} y_z + \frac{\partial \varphi'}{\partial z'_x} z_x + \frac{\partial \varphi'}{\partial x'_x} x_y &= \\ = \frac{\partial \varphi}{\partial x_x} x'_x + \frac{\partial \varphi}{\partial y_y} y'_y + \frac{\partial \varphi}{\partial z_z} z'_z + \frac{\partial \varphi}{\partial y_z} y'_z + \frac{\partial \varphi}{\partial z_x} z'_x + \frac{\partial \varphi}{\partial x_x} x'_y &\end{aligned}$$

one concludes that:

$$\int_S \left(\frac{\partial \varphi}{\partial x_x} x'_x + \frac{\partial \varphi}{\partial y_y} y'_y + \frac{\partial \varphi}{\partial z_z} z'_z + \frac{\partial \varphi}{\partial y_z} y'_z + \frac{\partial \varphi}{\partial z_x} z'_x + \frac{\partial \varphi}{\partial x_y} x'_y \right) dS = 0$$

as we wished to prove.

(2′) therefore assumes the characteristic form:

$$(3) \qquad \begin{cases} \int_S (Xu' + Yv' + Zw')\, dS + \int_\Sigma (X_n u' + Y_n v' + Z_n w')\, dS \\ \qquad + Ll' + Mm' + Nn' + Pp' + Qq' + Rr' = 0, \end{cases}$$

which expresses the theorem:

"The sum of the products of the six characteristics of the system of internal tensions that are developed in an elastic body in equilibrium, in connection with a given section, with the corresponding characteristics of a distortion is equal and opposite to the external force, when applied to that body, that brings about the change of configuration and which originates in this distortion."

If one introduces the concept of *negative unitary distortion*, which refers to any distortion whose characteristics are all null, except for one, which is given the fixed value – 1, then one may state the theorem in the form:

"Each of the six characteristics of the system of internal tensions that are developed in an elastic body in equilibrium, in connection with a given section, are measured by the work that the external force applied to the body does when one carries out the corresponding 'negative unitary' distortion."

Thus expressed, the theorem is not new: This author has already given a general proof in a brief Note that had the honor of appearing in these Rendiconti ([1]) three years ago.

It has already been many years since it was first expressed in a more general form, illustrated in some particularly intuitive cases, and usefully applied to the solution of some problems of hyperstatic equilibrium ([2]).

In a later Note, I propose to specify the import of these applications by exhibiting the way that the proof of reciprocity between deformations and distortions reconnects with the theory of Volterra, which is among the more important chapters in the science of construction.

([1]) G. Colonetti, *Sul principio di reciprocità* (Rend. R. Accademia dei Lincei, 5th series, vol. XXI); cf., also: *Introduzione teorica as un corso di statica dei corpi elastici* (lithographed lecture notes, Genova, 1912); *Sul principio di reciprocità* (Giorn. del genio civile, 1913).

([2]) Cf., on this, W. Ritter, *Anwendungen der graphischen Statik*, Part three, Zürich 1900, pp. 89 et seq.