"Su certi stati di coazione elastica che non dipendono da azioni esterne," Rend. Accad. dei Lincei (5) **26** (1917), 43-47.

On certain states of elastic co-action that do not depend upon external actions.

Note by GUSTAVO COLONETTI
Presented to the Society by GIAN ANTONIO MAGGI ([1])

Translated by D. H. Delphenich

In his recent note ([2]), prof. G. A. MAGGI most aptly noted the form that the theory of elastic distortions has reached: It was even more apt in that he definitively specified that putting limits on that theory is the essential condition by which one can finally hope to give a certain preference towards our idea, as opposed to the other states of elastic coaction that are not susceptible to being classified, either by the ordinary deformations of the classical theory or by the modern distortions.

In reality, the term "distortions" includes all of the states of elastic equilibrium that are established in response to any arbitrary external action, and that in practice one intends to generate a cut in the solid whose form and position is arbitrary, and to impose relatively small displacements on the two faces of the cut and to re-attach those two faces, naturally, after having filled in the space that the displacement produced with new material and removing a convenient portion of matter where the displacement itself tends to go, instead of determining the superposition of parts.

I do not know if Weingarten, who was the progenitor of the concept, had glimpsed something more general that it; this is, however, at least possible. Indeed, at the time, he wrote this in those Rendiconti ([3]):

> "In the theory of equilibrium of elastic solids, it seems to me that up to now the only case to be considered is that of a body in which particles are subjected to displacements of the their natural position that vary from point to point continuously in all of the space that is occupied by that body. Under those hypotheses, when no external force acts, either on the contour or the interior of the space, the body will not be subject to internal tensions."
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> "Therefore, there should certainly not exist bodies that are subject to internal tensions that are not subjected to external forces, either on their contour or their interiors."

([1]) Received by the Academy on 19 June 1917.

([2]) G. A. MAGGI, "Posizione e soluzione di alcune questioni attinenti alla teoria delle distorsioni elastiche," Rend. della E. Accad. dei Lincei (5) **26** (1st sem. 1917).

([3]) J. Weingarten, "Sulle superficie di discontinuità nella teori della elasticità dei corpi solidi," Rend. della R. Accad. dei Lincei (5) **10** (1st sem. 1901).

> "In order to have an example of one that does exist internal tensions, it is enough to imagine a ring that is not completely closed, and in which one brings the two free, planar sections together and attaches them with an infinitely-thin layer that restores equilibrium."

However, this example of Weingarten, not the more general concept that was expressed in the preceding and was intended to illustrate it, incontestably dominated the rest of the brief note that was cited. That same example continued to dominate all of the studies that the note gave rise to.

It is quite true that the example has survived the test of time beyond any legitimate expectation, and through the magisterial work of Volterra ([1]), it has in a short span of years found its place as one of the more elegant theories of modern mathematical physics.

The more recent research of Somigliana ([2]) has, however, shown the opportunity of conferring full generality to the treatment by abandoning all of the restrictions that are not rigorously necessary; namely, one does not impose any consideration of a mechanical nature absolutely.

Now, it seems to me that once one has embarked upon this path, one must go all the way and even wonder if the artifice of the cut and the successive re-attaching of its shifted faces to which one appeals in order to generate the distortions is susceptible to all of the desirable generality: That is, if one can use it to justify effectively *all* of the systems of internal tensions that daily experience presents in elastic solids in equilibrium in the absence of external forces.

Some degree of doubt is suggested by the examination of those states of elastic co-action to which I have alluded, in principle, whose existence has been observed by engineers, but which have not found their place in the mathematical theory of elasticity, despite some isolated attempts to treat them ([3]).

Here is a more characteristic example: A piece of steel is heated to a sufficiently elevated temperature and suddenly made to contact cold bodies. The sudden and unequal cooling of its various parts will then produce a contraction in volume that varies from point to point. The parts of the piece that cool more rapidly will acquire a greater consistency in a brief period of time, which does not then permit them to follow the relaxation phenomena that are determined in the other parts of the piece by their cooling without some reaction.

All of that has been proved by a body of evidence that is regularly verified in the tempering operations of steel, in which the diminishing density in the tempered piece (which is greater when the rate of cooling is larger) and the basic changes in form that are observed in that piece depend upon some defect in the uniformity of the cooling process ([4]).

The internal tensions that one tends to create in the tempered piece are very difficult to exhibit directly: However, one can consider a proof of their existence to be the ease

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([1]) V. Volterra, "Sur l'équilibre des corps élastiques multiplement connexes," Annales de l'École Normale (3) **24** (September 1907).

([2]) C. Somigliana, "Sulle teoria delle distorsioni elastiche," Rend. della R. Accad. dei Lincei (5) **23** (1st se. 1914). "Sulle discontinuita dei potenziali elastici," Atti della R. Accad. delle Scienze Torino **51** (June 1916).

([3]) Cf., e.g.: A. Föppl, *Vorlesungen über Technische Mechanik,* volume five, Leipzig, 1907.

([4]) Cf., e.g.: F. Reiser, *Théorie et pratique de la trempe de l'acier*, French trans., Paris, 1905.

with which the sudden, unforeseen breaks in the piece of tempered steel are verified, which sometimes happen under the action of stresses that are, in themselves, absolutely insufficient to overcome the proper resistance of the material, and sometimes also happen without any appreciable cause ([1]). It is then noteworthy that this type of characteristic fragility in tempered pieces disappears (with the diminishing of density that was mentioned before, along with the other things) if one makes a suitable period of annealing follow the tempering and then a period of cooling that is sufficiently slow that one can practically maintain uniformity.

Entirely analogous facts are verified in the fusion of cast iron pieces, the molding of concrete castings ([2]), the fabrication of glass, etc.

The case of glass is particularly interesting – at least, from the experimental viewpoint – since the internal tension in it can be exhibited (and also measured, in some cases) by means of the accidental double refraction that accompanies it. Now, it is a fact that when an arbitrary piece of glass is examined with polarized light, it will, in general, reveal the presence of internal tensions that are distributed in a rather complicated way. Those tensions are not eliminated completely, even after a very accurate process of annealing and slow cooling in an environment that maintains a temperature that is as uniform as possible.

This position is easy to see, compared to the impossibility of justifying that all of these states of elastic co-action are distortions: Indeed, it becomes obvious that, in the first place, their presence is not in any way related to the degree of connectivity of the space that is occupied by the solid, and on the other hand, one does not see, when one examines the question from the physical viewpoint, the necessity of a discontinuity surface of the type that Somigliana encountered in his study of the distortions of simply-connected solids.

The reason for this apparent contradiction between the results of daily experience and those of the theoretical research that was cited must, as I have said, be sought in the constant preoccupation that dominates all of that research with specifying in every case – i.e., for any state of tension under examination – the way in which that state can be reproduced *mechanically*, and more precisely, by means of an ordinary deformation of the given solid when one brings about a cut along a certain surface that is suitably chosen, case by case. This preoccupation, which is, on the one hand, the justification for the advantage of putting the new problems into terminology that is less diverse than the ones that are capable of characterizing the classical theory of elasticity, can, on the other hand, be the cause of a limitation in the scope of our research.

Since the mechanism that was described for generating the state of internal tension is, in fact, practicable, it is obviously necessary that the deformations of the individual elements that comprise the solid must give a unique system of *generally* continuous displacements; i.e., ones that do not imply continuous solutions of overlapping matter that do not correspond at that particular surface along which intends to make the cut. Moreover, one should note that this concept translates analytically into the condition that

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([1]) Cf., my note, "Sopra un caso di frattura spontanea di un acciaio temprato," Atti della R. Accad. delle Scienze do Torino **52** (January 1917).

([2]) Interesting discussions of this argument have been published in the recent Congresso di Milano della Società Ital. per il progresso delle scienze dall'Ing. L. Luigi, in its conference on "L'evoluzione delle dighe per laghi artificiali in alto montagna," Atti, ninth meeting, Rome 1917.

the six components of the deformation satisfy a certain sextuple of second-order differential equations – namely, the equations of congruence, or Saint-Venant – at all of the points of the solid, except for the ones that belong to the particular surface that was mentioned above.

Is this condition physically necessary? Obviously, it is not.

Indeed, if we limit ourselves to considering the neighborhood of a generic point of the solid to be isolated and independent of the rest of the solid then that will not prevent us from assuming a system – however arbitrary – of values for the components of the deformation in order to represent an effectively realizable state of the particle considered.

Moreover, if we disregard the manner of generating the deformation then that will not prevent us from thinking that the solid is composed of many elements that are deformed in the way we are presently doing, in the most general fashion, so that one maintains it in a state of elastic co-action by virtue of the mutual connection of the elements, in such a way that it is not possible to return everything to its respective natural, undeformed state without first destroying the constraints that are imposed upon each element due to the presence of all the other ones.

In a later note, I will go on to a summary examination of the foundations upon which a general treatment of this argument can be based.

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