"Sur les équations générales de l'élasticité," Bull. Soc. math. Fr. **30** (1902), 108-110.

On the general equations of elasticity

By M. COMBEBIAC

Translated by D. H. Delphenich

The components of the effort that are exerted on the surface element at a point that belongs to an elastic medium are expressed by a homogeneous, linear substitution as functions of the direction cosines of the normal to the element.

The nine coefficients that must therefore intervene have their number reduced to six by virtue of the condition that the pressures that are exerted on the surface of an infinitesimal portion of matter must give rise to a unique resultant in order to produce equilibrium between the inertial forces and external forces that, when they are assumed to be applied to each elementary mass, indeed result in a unique resultant, up to infinitesimals of higher order than that of the volume of the portion of matter considered.

Now, this latter hypothesis fails for a magnetic substance that is placed in a magnetic field. In that case, a particle is always subjected to a couple if it is small.

By denoting the intensity of magnetization at the point considered by I, the magnetic force at that point by F, and the angle that is defined by the two directions of the force and magnetization by θ, a particle in the volume $d\varpi$ is subjected to a couple of the same order as $d\varpi$ whose moment has:

$$FI \sin \theta \, d\varpi$$

for its expression, and whose representative axis has a direction that is rectangular to that of the force and magnetization.

One can see no reason to exclude this case from the theory of elasticity *a priori*.

In order to establish the equations of equilibrium or motion for the medium, apply the Helmholtz decomposition to the homogeneous, linear substitution that gives the expressions for the components T_x, T_y, T_z of the tension as functions of the direction cosines α, β, γ of the normal to the surface element, in such a manner that this substitution takes the following form:

$$(1) \qquad \begin{cases} T_x = N_1\alpha + T_3\beta + T_2\gamma + M\gamma - N\beta, \\ T_y = T_3\alpha + N_2\beta + T_1\gamma + N\alpha - L\gamma, \\ T_z = T_2\alpha + T_1\beta + N_3\gamma + L\beta - M\alpha. \end{cases}$$

The equations that relate to a virtual displacement are:

$$
(2) \qquad \begin{cases}
\rho(X - J_x) = \dfrac{\partial N_1}{\partial x} + \dfrac{\partial T_3}{\partial y} + \dfrac{\partial T_2}{\partial z} + \dfrac{\partial M}{\partial z} - \dfrac{\partial N}{\partial y}, \\
\rho(Y - J_y) = \dfrac{\partial T_3}{\partial x} + \dfrac{\partial N_2}{\partial y} + \dfrac{\partial T_1}{\partial z} + \dfrac{\partial N}{\partial x} - \dfrac{\partial L}{\partial z}, \\
\rho(Z - J_z) = \dfrac{\partial T_2}{\partial x} + \dfrac{\partial T_1}{\partial y} + \dfrac{\partial N_3}{\partial z} + \dfrac{\partial L}{\partial y} - \dfrac{\partial M}{\partial x},
\end{cases}
$$

where ρ represents the density at the point considered, J_x, J_y, J_z, the projections of the acceleration onto the coordinate axes, and X, Y, Z are those of the external force, referred to the unit of mass.

As for the equations that relate to a virtual displacement around the origin, after all reductions have been made, they are expressed by following the classical calculations step-by-step, but while taking into account the existence of a couple, such that the components of the representative axis of that couple are equal to:

$$2L, 2M, 2N,$$

respectively.

It then results that the quantities L, M, N in equations (2) represent one-half the external couple, and must be considered to be given functions of the coordinates x, y, z.

One sees that equations (2) differ from the general equations of elasticity, when they are written in the usual form, only by the presence of terms that depend upon L, M, N.

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On the general equations of elasticity

By M. COMBEBIAC

Translated by D. H. Delphenich

In a note that was recently published in the *Bulletin*, I have indicated some terms that it is necessary to introduce into the general equations of elasticity in order to take into account external couples that can act on the particles of a body (e.g., magnetic couples).

This case differs from the one where couples do not exist in that the determinant of the coefficients that intervene in the determination of the efforts at a point is no longer symmetric, in such a way that the coefficients are nine in number, instead of six.

In the usual case, these six coefficients are expressed by some known relations as functions of the six parameters that characterize the deformation of the point considered, at least when that deformation is sufficiently small, and there is a one-to-one correspondence between these two systems of six quantities, which we call, in order to simply the language, the *deformation* and *state of tension.*

There is good reason to demand to know what this relation must be in the case that we are occupied with.

In the usual case, that relation can be obtained by starting with the expression for the elastic energy, or more simply, that of the elastic work done by a virtual displacement.

We investigate what that expression would be in the general case (viz., the case where there exist elementary couples).

By reason of the state of tension, each element of the body is subjected a force and a couple whose magnitude is of order that of the volume of the element, and one must observe that this accounting consists of the efforts that are exerted on the external surface of the body.

In order to have an expression for the elastic work done, it will suffice to subtract the work that is done by the surface forces whose origin is obviously external to the solid body from the work that is done by the forces and couples that relate to the volume elements.

Let ξ, η, ζ be the components of the virtual displacement at an arbitrary point of the body along the coordinate axes.

Let N_1, N_2, N_3, T_1, T_2, T_3, L, M, N be the nine quantities that determine the state of tension at the same point in question, according to the notation that was employed in the note that was mentioned above.

Let X_e, Y_e, Z_e be the components of the effort per unit area that is exerted at a point of the external surface along the coordinate axes.

The elastic work that is done by the virtual displacement ξ, η, ζ has the expression:

$$\delta T = \iiint (A\xi + B\eta + C\zeta)\, d\tau + \iiint (2Lp + 2Mq + 2Nr)\, d\tau - \iint (X_e\xi + Y_e\eta + Z_e\zeta)\, d\sigma\,,$$

where $d\tau$ denotes the volume element and $d\sigma$, a surface element of the body, and in which one has set:

$$A = -\left(\frac{\partial N_1}{\partial x} + \frac{\partial T_3}{\partial y} + \frac{\partial T_2}{\partial z} + \frac{\partial M}{\partial z} - \frac{\partial N}{\partial y}\right),$$

$$B = -\left(\frac{\partial T_3}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial T_1}{\partial z} + \frac{\partial N}{\partial x} - \frac{\partial L}{\partial z}\right),$$

$$C = -\left(\frac{\partial T_2}{\partial x} + \frac{\partial T_1}{\partial y} + \frac{\partial N_3}{\partial z} + \frac{\partial L}{\partial y} - \frac{\partial M}{\partial x}\right),$$

$$X_e = -(N_1\alpha + T_3\beta + T_2\gamma + M\gamma - N\beta),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$p = \frac{1}{2}\left(\frac{\partial \eta}{\partial z} - \frac{\partial \zeta}{\partial y}\right),$$

$$q = \frac{1}{2}\left(\frac{\partial \zeta}{\partial x} - \frac{\partial \xi}{\partial z}\right),$$

$$r = \frac{1}{2}\left(\frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial x}\right).$$

As one knows, the last term in the expression for δT, which consists of a surface integral preceded by a minus sign, can be transformed into a volume integral, namely:

$$\begin{aligned}\iiint\Big[& N_1\frac{\partial \xi}{\partial x} + (T_3 - N)\frac{\partial \xi}{\partial y} + (T_2 + M)\frac{\partial \xi}{\partial z} \\ & + (T_3 + N)\frac{\partial \eta}{\partial x} + N_2\frac{\partial \eta}{\partial y} + (T_1 - L)\frac{\partial \eta}{\partial z} \\ & + (T_2 - M)\frac{\partial \zeta}{\partial x} + (T_1 + L)\frac{\partial \zeta}{\partial y} + N_3\frac{\partial \zeta}{\partial z}\Big]d\tau \\ - \iiint & (A\xi + B\eta + C\zeta)\,d\tau .\end{aligned}$$

The expression for δT becomes:

$$\delta T = \iiint\left[N_1\frac{\partial \xi}{\partial x} + N_2\frac{\partial \eta}{\partial y} + N_3\frac{\partial \zeta}{\partial z}\right.$$

$$+\ T_1\left(\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}\right)+T_2\left(\frac{\partial\xi}{\partial z}+\frac{\partial\zeta}{\partial x}\right)+T_3\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right)\Bigg]d\tau\,.$$

This expression is independent of L, M, N, and is the same as in the case where L, M, N are zero.

The quantities N_1, N_2, N_3, T_1, T_2, T_3 must therefore be expressed in the same manner as in the latter case as functions of the six parameters that characterize the deformation:

$$\frac{\partial\xi}{\partial x},\ \frac{\partial\eta}{\partial y},\ \frac{\partial\zeta}{\partial z},\ \frac{1}{2}\left(\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}\right),\ \frac{1}{2}\left(\frac{\partial\xi}{\partial z}+\frac{\partial\zeta}{\partial x}\right),\ \frac{1}{2}\left(\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\right).$$

The analytical determination of the problem is the same as in the usual case.

The equations of the problem are:

$$\begin{aligned}
\frac{\partial N_1}{\partial x}+\frac{\partial T_3}{\partial y}+\frac{\partial T_2}{\partial z} &= \rho(X-J_x)+\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z},\\
\frac{\partial T_3}{\partial x}+\frac{\partial N_2}{\partial y}+\frac{\partial T_1}{\partial z} &= \rho(Y-J_y)+\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x},\\
\frac{\partial T_2}{\partial x}+\frac{\partial T_1}{\partial y}+\frac{\partial N_3}{\partial z} &= \rho(Z-J_z)+\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y},
\end{aligned}$$

$$\begin{aligned}
N_1\alpha+T_3\beta+T_2\gamma &= -X_e+N\beta-M\gamma,\\
T_3\alpha+N_2\beta+T_1\gamma &= -Y_e+L\gamma-N\alpha,\\
T_2\alpha+T_1\beta+N_3\gamma &= -Z_e+M\alpha-L\beta,
\end{aligned}$$

where the last three relate to the points of the external surface.

N_1, N_2, N_3, T_1, T_2, T_3 must be expressed as functions of the six parameters that represent the deformation, while L, M, N must be regarded as given functions of x, y, z on the surface of the body, as well as in the interior.

In the case of an isotropic body, one can account for the fact that the quantities L, M, N must be independent of the deformation. This amounts to the fact that there cannot exist a relation between an ellipsoid and *one* vector that is independent of the displacements without deformation; i.e., here, that would be rotations around the center of the ellipsoid.

One sees this by considerations that are analogous to the ones that permit one to prove that two concentric quadrics (for example, the ellipsoid of dilatations and the indicatrix quadric of the distribution of the efforts at a point) that have a *geometric* relationship between them that determines the one in terms of the other must have the same principal planes and the same planes of circular sections ([1]).

In the case that we are concerned with of a vector that is geometrically determined with respect to a quadric, one easily proves that its direction must be given by one of the points at infinity that are determined as points of intersection of the common chords to

([1]) Cf., APPELL, *Traité de Mécanique*, t. III, pp. 508; Paris, Gauthier-Villars, 1901.

the section of the quadric by the plane at infinity and the imaginary circle at infinity; i.e., this direction must be that of one of the axes. However, since this relation will not be invariant under a permutation of the axes of the quadric under these conditions, one sees that it cannot exist.

I am compelled to point out some things that were obligingly brought to my attention by APPELL since the publication of my first note, in order to make known two papers that treated the same subject.

Love ([1]), in his treatise, briefly pointed out that in the case where the medium is subject to elementary external couples there is good reason to make nine coefficients intervene, instead of six, in order to determine the efforts at a point. Moreover, that author referred to a paper by Larmor.

The latter author ([2]) studied the propagation of waves in a medium, each point of which was endowed with a *directed inertia*, in addition to the usual inertia, as one will have, for example, in a body that is contained in a cavity of a gyroscope. Under these conditions, an element in motion presents a couple of inertia that intervenes in the equations of motion. This couple is expressed as a function of the velocities and accelerations.

Such a medium is nothing but the hypothetical Maxwell medium, and Larmor applied its properties to the theory of the propagation of polarized light.

One sees that the question that was treated by Larmor differs appreciably from the one that we have proposed in the present note in order to attract the attention of mathematicians. Light is far from resolving the difficulties (the reader will be, without a doubt, aware of them) of a paradoxical nature that are presented by the determination of the deformation at each point in a body that is subject to external elementary couples.

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([1]) *Treatise on the mathematical theory of elasticity*, Cambridge University Press, 1892.

([2]) "On the propagation of a disturbance in a gyrostatically loaded medium," Proc. Lond. Math. Soc., Nov., 1891.