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ON

FUNCTIONAL DETERMINANTS

AND

CURVILINEAR COORDINATES

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The principal objective of the following study is a certain extension of the oblique curvilinear coordinates of that great theory that was created by Lamé to the general case. Nevertheless, I have added some remarks that are not unimportant and touch upon orthogonal systems. Finally, I believe that all of that theory must be attached to that of functional determinants, whose properties I will recall, to the extent that they are of use to me, while deducing them from the consideration of quadratic forms.

§ I. – *Quadratic form of the product of two determinants. – Identities that it yields.*

Let two determinants whose elements are completely independent be:

$$X = \begin{vmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_n^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_n^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{(n)} & x_2^{(n)} & \cdots & x_n^{(n)} \end{vmatrix}, \quad Z = \begin{vmatrix} z_1^{(1)} & z_2^{(1)} & \cdots & z_n^{(1)} \\ z_1^{(2)} & z_2^{(2)} & \cdots & z_n^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ z_1^{(n)} & z_2^{(n)} & \cdots & z_n^{(n)} \end{vmatrix},$$

and let their product be:

$$V = \begin{vmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,n} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ v_{n,1} & v_{n,2} & \cdots & v_{n,n} \end{vmatrix},$$

in which $v_{i,j} = \sum_g x_i^{(g)} z_j^{(g)}$, where the summation extends from $g = 1$ to $g = n$, although that will always be implied for all indexed summations.

From the fact that:

$$\frac{dV}{dx_i^{(g)}} = Z\frac{dX}{dx_i^{(g)}}, \qquad \frac{dV}{dz_i^{(g)}} = X\frac{dZ}{dz_i^{(g)}},$$

one concludes immediately that:

$$\sum_g x_j^{(g)} \frac{dV}{dx_i^{(g)}} = 0 \text{ or } V, \qquad \sum_g z_j^{(g)} \frac{dV}{dz_i^{(g)}} = 0 \text{ or } V,$$

in which the right-hand sides are equal to 0 or V, according to whether the indices i, j are different or coincide, resp. Having said that, one can represent the determinant V by any of the n quadratic forms:

$$F^{(g)} = \sum_{i,j} C_{i,j}\, x_i^{(g)} x_j^{(g)}$$

that correspond to the n values of g, and in which the *coefficients* $C_{i,j}$ keep the same values. Indeed, under those hypotheses, and due to the fact that:

$$\frac{dF^{(g)}}{dx_i^{(g)}} = \sum_j C_{i,j}\, z_j^{(g)}, \quad \frac{dF^{(g)}}{dz_i^{(g)}} = \sum_i C_{i,j}\, x_i^{(g)},$$

one will have, from what precedes this immediately, that:

$$\text{(a)} \qquad \begin{cases} V \text{ or } 0 = \sum_g x_h^{(g)} \dfrac{dF^{(g)}}{dx_i^{(g)}} = \sum_j C_{i,j} \sum_g x_h^{(g)} z_j^{(g)} = \sum_j C_{i,j} v_{h,j}, \\ V \text{ or } 0 = \sum_g z_k^{(g)} \dfrac{dF^{(g)}}{dz_j^{(g)}} = \sum_i C_{i,j} \sum_g x_k^{(g)} x_i^{(g)} = \sum_i C_{i,j} v_{k,k}, \end{cases}$$

and from the n^2 equalities that are composed of the extreme expressions, one will infer the symmetric value $\dfrac{dV}{dv_{i,j}}$ for the coefficient $C_{i,j}$. One will then have:

$$F^{(g)} = \sum_{i,j} \frac{dV}{dv_{i,j}} x_i^{(g)} z_j^{(g)} = V.$$

Upon multiplying the expression for $\frac{dF^{(g)}}{dx_i^{(g)}}$ above by $\frac{dF^{(g)}}{dz_i^{(g)}}$ and then taking the $\sum_g$, one will get:

$$\sum_g \frac{dF^{(g)}}{dx_i^{(g)}} \frac{dF^{(g)}}{dz_h^{(g)}} = \sum_j C_{i,j} \sum_g z_j^{(g)} \frac{dF^{(g)}}{dz_h^{(g)}},$$

and since, by virtue of (a), all of the terms in the right-hand side will disappear, except for $j = h$, it will result that:

(b) $$\sum_g \frac{dF^{(g)}}{dx_i^{(g)}} \frac{dF^{(g)}}{dz_j^{(g)}} = V\, C_{i,j} = V\, \frac{dV}{dv_{i,j}}.$$

If one supposes, for the moment, that the two determinants X and Z coincide with each other term-by-term, and that one sets:

$$X^2 = P = \begin{vmatrix} \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,n} \\ \xi_{1,2} & \xi_{2,2} & \cdots & \xi_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{1,n} & \xi_{2,n} & \cdots & \xi_{n,n} \end{vmatrix}, \qquad Z^2 = Q = \begin{vmatrix} \zeta_{1,1} & \zeta_{1,2} & \cdots & \zeta_{1,n} \\ \zeta_{1,2} & \zeta_{2,2} & \cdots & \zeta_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ \zeta_{1,n} & \zeta_{2,n} & \cdots & \zeta_{n,n} \end{vmatrix},$$

in which:

$$\xi_{i,j} = \xi_{j,i} = \sum_g x_i^{(g)} x_j^{(g)}, \qquad \zeta_{i,j} = \zeta_{j,i} = \sum_g z_i^{(g)} z_j^{(g)}, \qquad \text{and} \qquad PQ = V^2,$$

then one will have, as a particular case of what was just said, or by some direct considerations that are entirely parallel to the preceding ones:

$$2\, f^{(g)} = \sum_{i,j} A_{i,j}\, x_i^{(g)} x_j^{(g)} = P, \quad 2\, \mathrm{f}^{(g)} = \sum_{i,j} B_{i,j}\, z_i^{(g)} z_j^{(g)} = Q,$$

in which the forms f, f verify the relations:

(a*) $$\sum_g x_j^{(g)} \frac{df^{(g)}}{dx_i^{(g)}} = 0 \text{ or } P, \qquad \sum_g z_j^{(g)} \frac{d\mathrm{f}^{(g)}}{dz_i^{(g)}} = 0 \text{ or } Q,$$

and give rise to the identities:

(b*) $$\sum_g \frac{df^{(g)}}{dx_i^{(g)}}\frac{df^{(g)}}{dx_j^{(g)}} = P\,A_{i,j}\,, \qquad \sum_g \frac{df^{(g)}}{dz_i^{(g)}}\frac{df^{(g)}}{dz_j^{(g)}} = Q\,B_{i,j}\,.$$

It is now easy to see that:

(c) $$\begin{cases} x_i^{(g)} = \dfrac{1}{Q}\sum_h v_{i,h}\dfrac{df^{(g)}}{dz_h^{(g)}} = \dfrac{1}{V}\sum_h \xi_{i,h}\dfrac{dF^{(g)}}{dx_h^{(g)}}, \\ z_i^{(g)} = \dfrac{1}{P}\sum_h v_{i,h}\dfrac{df^{(g)}}{dz_h^{(g)}} = \dfrac{1}{V}\sum_h \zeta_{i,h}\dfrac{dF^{(g)}}{dz_h^{(g)}}, \end{cases}$$

because if one successively multiplies the first row, for example, by $x_i^{(g)}$ and $z_j^{(g)}$, resp., and then sums over $\sum_g$ then one will conclude that:

$$\xi_{i,j} = \frac{1}{V}\sum_h \xi_{i,h}\sum_g x_j^{(g)}\frac{dF^{(g)}}{dx_h^{(g)}}\,, \qquad v_{i,j} = \frac{1}{Q}\sum_h v_{i,h}\sum_g z_j^{(g)}\frac{df^{(g)}}{dz_h^{(g)}}\,,$$

in which one will see all the terms on the right-hand sides disappear, by virtue of (a), (a*), except for the ones that have $h = j$.

Upon multiplying them by the first two expressions for $x_i^{(g)}, z_j^{(g)}$ and summing over $\sum_g$, one will get:

$$\xi_{i,j} = \frac{1}{Q^2}\sum_h\sum_k v_{i,h}\,v_{j,k}\sum_g \frac{df^{(g)}}{dz_h^{(g)}}\frac{df^{(g)}}{dz_k^{(g)}}\,,$$

i.e., from (b*):

(d) $$\begin{cases} \xi_{i,j} = \dfrac{1}{Q}\sum_{h,k} v_{i,h}v_{j,k}B_{h,k}, \\ \text{and similarly,} \\ \zeta_{i,j} = \dfrac{1}{P}\sum_{h,k} v_{h,i}v_{k,j}A_{h,k}. \end{cases}$$

The multiplication of the last two expressions for $x_i^{(g)}, z_j^{(g)}$ also gives:

$$v_{i,j} = \frac{1}{V}\sum_{h,k}\xi_{i,h}\,\xi_{j,k}C_{h,k}\,.$$

When the x and the z are coupled by the conditions that $v_{i,j} = 0$ or 1 (according to whether j is or is not different from i, resp.), formulas (c), (d) will reduce to:

$$(\text{c}^*)\quad \begin{cases} x_i^{(g)} = \dfrac{1}{Q}\dfrac{d\text{f}^{(g)}}{dz_i^{(g)}} = \sum\limits_j \xi_{i,j} z_j^{(g)}, \\ z_i^{(g)} = \dfrac{1}{P}\dfrac{df^{(g)}}{dx_i^{(g)}} = \sum\limits_j \zeta_{i,j} x_j^{(g)}, \end{cases} \qquad (\text{d}^*)\quad \begin{cases} \xi_{i,j} = \dfrac{1}{Q} B_{i,j}, \\ \zeta_{i,j} = \dfrac{1}{P} A_{i,j}, \end{cases}$$

in which $PQ = V^2 = 1$. The forms f, f, for their own part, amount to:

$$\frac{2f^{(g)}}{P} = \sum_{i,j} \zeta_{i,j} x_i^{(g)} x_j^{(g)} = 1, \qquad \frac{2\text{f}^{(g)}}{Q} = \sum_{i,j} \xi_{i,j} z_i^{(g)} z_j^{(g)} = 1,$$

and the expression for $v_{i,j}$ that follows from (d) will give $\sum_h \xi_{i,h}\,\zeta_{j,h} = 0$ or 1.

One will then get back to some known results from the theory of functional determinants, except for some differences in form, upon supposing that:

$$x_i^{(g)} = \frac{du_g}{d\alpha_i}, \qquad z_j^{(g)} = \frac{d\alpha_i}{du_g},$$

in which the $\alpha_1, \alpha_2, \ldots, \alpha_n$ are functions of the $u_1, u_2, \ldots, u_n$ that are taken to be independent variables, or inversely.

§ II. – *Differential relations. – Distinguishing two groups of elements.*

Upon letting t denote any independent variable and setting:

$$(\text{e})\qquad \sum_g x_i^{(g)} \frac{dx_j^{(g)}}{dt} = R_{i,j}^{(g)},$$

in general, in such a way:

$$R_{i,j}^{(g)} + R_{j,i}^{(g)} = \frac{d\xi_{i,j}}{dt},$$

it is easy to recognize that one will have:

$$(\text{f})\qquad P\frac{dx_h^{(g)}}{dt} = \sum_i R_{i,h}^{(g)} \frac{df^{(g)}}{dx_i^{(g)}}$$

identically, because upon multiplying that by $x_k^{(g)}$ and taking the sum $\sum_g$, one will conclude that:

$$P\sum_g x_k^{(g)} \frac{dx_h^{(g)}}{dt} = \sum_i R_{i,h}^{(g)} \sum_g x_k^{(g)} \frac{df^{(g)}}{dx_i^{(g)}},$$

or, by virtue of (a*), all of the terms in the right-hand side will disappear, with the exception of the ones for which $i = k$:

When one multiplies equation (f) by $\frac{df^{(g)}}{dx_j^{(g)}}$ and then sums over $\sum_g$, upon switching $\sum_g$, $\sum_i$, and taking (b*) into account, one will infer that:

(g) $$\sum_g \frac{df^{(g)}}{dx_j^{(g)}} \frac{dx_h^{(g)}}{dt} = \sum_i A_{i,j} R_{i,k}^{(g)},$$

and from this, if s is any other independent variable, upon multiplying (f) by $\frac{dx_k^{(g)}}{ds}$, one will get:

$$P\sum_g \frac{dx_h^{(g)}}{dt} \frac{dx_k^{(g)}}{ds} = \sum_i R_{i,h}^{(t)} \sum_g \frac{df^{(g)}}{dx_i^{(g)}} \frac{dx_k^{(g)}}{ds} = \sum_i R_{i,h}^{(t)} \sum_j A_{i,j} R_{j,k}^{(s)},$$

or rather:

(h) $$P\sum_g \frac{dx_h^{(g)}}{dt} \frac{dx_k^{(g)}}{ds} = \sum_{i,j} A_{i,j} R_{i,h}^{(t)} R_{j,k}^{(s)}.$$

Now, it results from the fact that:

$$R_{h,k}^{(s)} = \sum_g x_h^{(g)} \frac{dx_k^{(g)}}{ds}$$

that:

$$\frac{dR_{h,k}^{(s)}}{dt} = \sum_g \frac{dx_h^{(g)}}{dt} \frac{dx_k^{(g)}}{ds} + \sum_g x_h^{(g)} \frac{d^2 x_k^{(g)}}{dt\,ds},$$

so, upon changing t and s and taking (h) into account, one will infer that:

(k) $$\frac{dR_{h,k}^{(s)}}{dt} - \frac{dR_{h,k}^{(t)}}{ds} = \frac{1}{P} \sum_{i,j} A_{i,j} (R_{i,h}^{(t)} R_{j,k}^{(s)} - R_{i,k}^{(s)} R_{j,k}^{(t)}).$$

That multiple equation (and it seems to me that this has not been pointed out, in general) expresses, in particular, the integrability conditions for the simultaneous systems of linear equations:

(f′) $$P\frac{dx_h^{(g)}}{dt} = \sum_i R_{i,h}^{(t)} \frac{df^{(g)}}{dx_i^{(g)}}, \qquad P\frac{dx_h^{(g)}}{ds} = \sum_i R_{i,h}^{(s)} \frac{df^{(g)}}{dx_i^{(g)}},$$

from which, one proposes to deduce the x, when all of the other quantities are supposed to be known in terms of t and s. One will get some entirely analogous new conditions when one introduces an arbitrary number of independent variables t, s, …, and one can observe that the set of systems (f′) can be integrated under conditions that are analogous to (k), as a sequence of separate linear systems in ordinary differentials.

In the case of functional determinants, the R are subject to some special conditions that are provided by the relation:

$$\frac{dx_i^{(g)}}{d\alpha_j} = \frac{dx_j^{(g)}}{d\alpha_i}.$$

One then infers that:

$$\sum_g x_h^{(g)} \frac{dx_i^{(g)}}{d\alpha_j} = \sum_g x_h^{(g)} \frac{dx_j^{(g)}}{d\alpha_i};$$

i.e.:

(l) $$R_{h,i}^{(\alpha_j)} = R_{h,j}^{(\alpha_i)}.$$

When that relation is combined with $R_{i,j}^{(\alpha_h)} + R_{j,i}^{(\alpha_h)} = \frac{d\xi_{i,j}}{d\alpha_h}$, it will yield:

(m) $$R_{j,i}^{(\alpha_h)} = \frac{1}{2}\left(\frac{d\xi_{i,j}}{d\alpha_h} + \frac{d\xi_{j,h}}{d\alpha_i} - \frac{d\xi_{i,h}}{d\alpha_j}\right),$$

which permits one to introduce the ξ in place of the R everywhere, if one deems that appropriate.

With the theory of curvilinear coordinates in mind, above all, one agrees to perform a particular substitution in the preceding formulas while confining oneself to the case of functional determinants. I set:

$$\begin{cases} x_i^{(g)} = l_i\, a_i^{(g)}, \\ \xi_{i,j} = l_i\, l_j \lambda_{i,j}, \end{cases} \qquad \begin{cases} z_i^{(g)} = h_i\, \mathrm{a}_i^{(g)}, \\ \zeta_{i,j} = h_i\, h_j\, \theta_{i,j}, \end{cases}$$

under the conditions that $\lambda_{i,i} = 1$, $\theta_{i,i} = 1$, in such a way that one will have:

$$\sum_g a_i^{(g)}\, a_j^{(g)} = \lambda_{i,j}, \qquad \sum_g \mathrm{a}_i^{(g)}\, \mathrm{a}_j^{(g)} = \theta_{i,j}.$$

The two determinants:

$$\Delta = \begin{vmatrix} \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,n} \\ \lambda_{1,2} & \lambda_{2,2} & \cdots & \lambda_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{1,n} & \lambda_{2,n} & \cdots & \lambda_{n,n} \end{vmatrix}, \qquad \nabla = \begin{vmatrix} \theta_{1,1} & \theta_{1,2} & \cdots & \theta_{1,n} \\ \theta_{1,2} & \theta_{2,2} & \cdots & \theta_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ \theta_{1,n} & \theta_{2,n} & \cdots & \theta_{n,n} \end{vmatrix}$$

are coupled by the relation:

$$\mathrm{h}_1\,\mathrm{h}_2 \ldots \mathrm{h}_n \sqrt{\nabla} \cdot l_1\, l_2 \ldots l_n \sqrt{\Delta} = 1.$$

One can take *f*, f to have the following forms:

$$(\varphi)\qquad \begin{cases} 2\varphi^{(g)} = \dfrac{d\Delta}{d\lambda_{1,1}} a_1^{(g)} a_1^{(g)} + \dfrac{d\Delta}{d\lambda_{1,2}} a_1^{(g)} a_2^{(g)} + \cdots + \dfrac{d\Delta}{d\lambda_{n,n}} a_n^{(g)} a_n^{(g)} = \Delta, \\ 2\varpi^{(g)} = \dfrac{d\nabla}{d\theta_{1,1}} \mathrm{a}_1^{(g)}\, \mathrm{a}_1^{(g)} + \dfrac{d\nabla}{d\theta_{1,2}} \mathrm{a}_1^{(g)}\, \mathrm{a}_2^{(g)} + \cdots + \dfrac{d\nabla}{d\theta_{n,n}} \mathrm{a}_n^{(g)} \mathrm{a}_n^{(g)} = \nabla, \end{cases}$$

in which it is implied that the $\lambda_{i,i}$, $\theta_{i,i}$ are replaced with unity after the differentiations. They verify the relations:

$$(a^*)\qquad \sum_g a_j^{(g)} \frac{d\varphi^{(g)}}{da_i} = 0 \text{ or } \Delta, \qquad \sum_g \mathrm{a}_j^{(g)} \frac{d\varpi^{(g)}}{d\mathrm{a}_i^{(g)}} = 0 \text{ or } \nabla,$$

$$(b^*)\qquad \begin{cases} \displaystyle\sum_g \frac{d\varphi^{(g)}}{da_i^{(g)}} \frac{d\varphi^{(g)}}{da_j^{(g)}} = \frac{1}{2}\Delta \frac{d\Delta}{d\lambda_{i,j}}, \\ \displaystyle\sum_g \frac{d\varphi^{(g)}}{da_i^{(g)}} \frac{d\varphi^{(g)}}{da_i^{(g)}} = \Delta \frac{d\Delta}{d\lambda_{i,i}}, \end{cases} \qquad \begin{cases} \displaystyle\sum_g \frac{d\varpi^{(g)}}{d\mathrm{a}_i^{(g)}} \frac{d\varpi^{(g)}}{d\mathrm{a}_j^{(g)}} = \frac{1}{2}\nabla \frac{d\nabla}{d\lambda_{i,j}}, \\ \displaystyle\sum_g \frac{d\varpi^{(g)}}{d\mathrm{a}_i^{(g)}} \frac{d\varpi^{(g)}}{d\mathrm{a}_i^{(g)}} = \nabla \frac{d\nabla}{d\theta_{i,i}}. \end{cases}$$

From what the (d*) become, one will have:

$$(d^*)\qquad \begin{cases} \theta_{i,j} = \dfrac{d\Delta / d\lambda_{i,j}}{2\sqrt{\dfrac{d\Delta}{d\lambda_{i,i}} \dfrac{d\Delta}{d\lambda_{j,j}}}} \\ \mathrm{h}_i^2 = \dfrac{1}{l_i^2} \cdot \dfrac{1}{\Delta} \cdot \dfrac{d\Delta}{d\lambda_{i,i}}, \end{cases} \qquad \begin{cases} \lambda_{i,j} = \dfrac{d\nabla / d\theta_{i,j}}{2\sqrt{\dfrac{d\nabla}{d\theta_{i,i}} \dfrac{d\nabla}{d\theta_{j,j}}}} \\ l_i^2 = \dfrac{1}{\mathrm{h}_i^2} \cdot \dfrac{1}{\nabla} \cdot \dfrac{d\nabla}{d\theta_{i,i}}. \end{cases}$$

The (*c*) will be replaced by:

(c*) $$\mathrm{a}_i^{(g)} = \frac{1}{\sqrt{\Delta \dfrac{d\Delta}{d\lambda_{i,i}}}} \cdot \frac{d\varphi^{(g)}}{da_i^{(g)}}, \qquad a_i^{(g)} = \frac{1}{\sqrt{\nabla \dfrac{d\nabla}{d\theta_{i,i}}}} \cdot \frac{d\varpi^{(g)}}{\mathrm{da}_i^{(g)}}.$$

Finally, upon setting:

(*e*) $$\sum_g a_i^{(g)} \frac{da_j^{(g)}}{dt} = \mathcal{R}_{i,j}^{(t)},$$

in such a way that:

$$\mathcal{R}_{i,j}^{(t)} + \mathcal{R}_{j,i}^{(t)} = \frac{d\lambda_{i,j}}{dt},$$

one will have here that:

(*f*) $$\Delta \frac{da_i^{(g)}}{dt} = \sum_h \mathcal{R}_{h,i}^{(t)} \frac{d\varphi^{(g)}}{da_h^{(g)}},$$

(*h*) $$\Delta \sum_g \frac{da_h^{(g)}}{dt} \frac{da_k^{(g)}}{ds} = \sum_{i,j} \frac{1}{2} \frac{d\Delta}{d\lambda_{i,j}} \mathcal{R}_{j,h}^{(t)} \mathcal{R}_{i,k}^{(s)},$$

(*k*) $$\Delta \left(\frac{d\mathcal{R}_{h,k}^{(t)}}{ds} - \frac{d\mathcal{R}_{h,k}^{(s)}}{dt} \right) = \sum_{i,j} \frac{1}{2} \frac{d\Delta}{d\lambda_{i,j}} \left(\mathcal{R}_{j,k}^{(t)} \mathcal{R}_{i,h}^{(s)} - \mathcal{R}_{j,k}^{(s)} \mathcal{R}_{i,h}^{(t)} \right),$$

in which $\mathcal{R}_{i,i} = 0$. The latter condition introduces a simplification in the groups (*f*), (*h*), (*k*), in comparison to the homologous groups (f), (h), (k). By contrast, the condition (l) is replaced with this other, somewhat more complex, one:

(*l*) $$\frac{dl_i}{d\alpha_j} \lambda_{h,i} + l_i \, \mathcal{R}_{h,i}^{(\alpha_j)} = \frac{dl_j}{d\alpha_i} \lambda_{h,j} + l_j \, \mathcal{R}_{h,j}^{(\alpha_i)}.$$

If one would like to rid oneself of the $\mathcal{R}$ everywhere then it would suffice to take (m) into account, while observing that:

$$R_{i,j}^{(t)} = l_i \, l_j \, \mathcal{R}_{i,j}^{(t)} + \lambda_{i,j} \, l_i \, \frac{dl_j}{dt}.$$

Having discussed these preliminaries, I shall now begin the theory of curvilinear coordinates. However, in order to simplify the notation as much as possible, I will represent u_1, u_2, u_3 by x, y, z, $\alpha_1, \alpha_2, \alpha_3$ by α, β, γ, l_1, l_2, l_3 by l, m, n, $\lambda_{2,3}$, $\lambda_{1,3}$, $\lambda_{1,2}$ by λ, μ, ν, h_1, h_2, h_3 by $\mathfrak{l}, \mathfrak{m}, \mathfrak{n}$, $\theta_{2,3}$, $\theta_{1,3}$, $\theta_{1,2}$ by $\varepsilon, \eta, \theta$, $a_1^{(1)}$, $a_1^{(2)}$, $a_1^{(3)}$, $a_2^{(1)}$, $a_2^{(2)}$, $a_2^{(3)}$, $a_3^{(1)}$, $a_3^{(2)}$, $a_3^{(3)}$ by $a, a', a'', b, b', b'', c, c', c''$, and finally, $\mathrm{a}_1^{(1)}$, $\mathrm{a}_1^{(2)}$, $\mathrm{a}_1^{(3)}$, … by $\mathrm{a}, \mathrm{a}', \mathrm{a}''$, …

§ III. – *Curvilinear coordinates. – Development of some formulas.*

If the orthogonal, rectilinear coordinates of an arbitrary point in space are x, y, z then let:

$$F(x, y, z) = \alpha, \qquad F_1(x, y, z) = \alpha, \qquad F_2(x, y, z) = \gamma$$

be the equations of three surfaces, where α, β, γ denote three arbitrary parameters that do not enter into the left-hand sides. From Lamé's fertile vision of things, for each system of values that is attributed to those parameters, the mutual intersection of the three surfaces will determine a point M in space. The linear elements of the intersections that issue from that point are represented by $l\,d\alpha$, $m\,d\beta$, $n\,d\gamma$, and if λ, μ, ν denote the cosines of the angles between those elements then one will have this first group:

$$(\text{A})\quad\left\{\begin{aligned}
&\frac{dx^2}{d\alpha^2}+\frac{dy^2}{d\alpha^2}+\frac{dz^2}{d\alpha^2}=l^2, &&\frac{dx}{d\beta}\frac{dx}{d\gamma}+\frac{dy}{d\beta}\frac{dy}{d\gamma}+\frac{dz}{d\beta}\frac{dz}{d\gamma}=mn\lambda,\\
&\frac{dx^2}{d\beta^2}+\frac{dy^2}{d\beta^2}+\frac{dz^2}{d\chi^2}=m^2, &&\frac{dx}{d\gamma}\frac{dx}{d\alpha}+\frac{dy}{d\gamma}\frac{dy}{d\alpha}+\frac{dz}{d\gamma}\frac{dz}{d\alpha}=nl\mu,\\
&\frac{dx^2}{d\gamma^2}+\frac{dy^2}{d\gamma^2}+\frac{dz^2}{d\gamma^2}=n^2, &&\frac{dx}{d\alpha}\frac{dx}{d\beta}+\frac{dy}{d\alpha}\frac{dy}{d\beta}+\frac{dz}{d\alpha}\frac{dz}{d\beta}=lm\nu,
\end{aligned}\right.$$

to which one appends the inverse group:

$$(\text{A}')\quad\left\{\begin{aligned}
&\frac{d\alpha^2}{dx^2}+\frac{d\alpha^2}{dy^2}+\frac{d\alpha^2}{dz^2}=\mathfrak{l}^2, &&\frac{d\beta}{dx}\frac{d\gamma}{dx}+\frac{d\beta}{dy}\frac{d\gamma}{d\gamma}+\frac{d\beta}{dy}\frac{d\gamma}{d\gamma}=\mathfrak{m}\mathfrak{n}\varepsilon,\\
&\frac{d\beta^2}{dx^2}+\frac{d\beta^2}{dx^2}+\frac{d\beta^2}{dz^2}=\mathfrak{m}^2, &&\frac{d\alpha}{dx}\frac{d\gamma}{dx}+\frac{d\alpha}{dy}\frac{d\gamma}{dy}+\frac{d\alpha}{dz}\frac{d\gamma}{dz}=\mathfrak{n}\mathfrak{l}\eta,\\
&\frac{d\gamma^2}{dx^2}+\frac{d\gamma^2}{dy^2}+\frac{d\gamma^2}{dz^2}=\mathfrak{n}^2, &&\frac{d\alpha}{dx}\frac{d\beta}{dx}+\frac{d\alpha}{dy}\frac{d\beta}{dy}+\frac{d\alpha}{dz}\frac{d\beta}{dz}=\mathfrak{l}\mathfrak{m}\,\theta,
\end{aligned}\right.$$

in which $d\alpha / \mathfrak{l}$, $d\beta / \mathfrak{m}$, $d\gamma / \mathfrak{n}$ are the shortest distances from the point M to the two successive surfaces of the same families, respectively, and $\varepsilon, \eta, \theta$ are the cosines of the angles that those shortest distances subtend. Here, one takes:

$$(1)\quad\left\{\begin{aligned}
&\frac{dx}{d\alpha}=la, &&\frac{dx}{d\beta}=mb, &&\frac{dx}{d\gamma}=la,\\
&\frac{dy}{d\alpha}=la', &&\frac{dy}{d\beta}=mb', &&\frac{dy}{d\gamma}=nc',\\
&\frac{dz}{d\alpha}=la'', &&\frac{dz}{d\beta}=mb'', &&\frac{dz}{d\gamma}=nc'',
\end{aligned}\right.
\qquad
\left\{\begin{aligned}
&\frac{d\alpha}{dx}=\mathfrak{l}\mathfrak{a}, &&\frac{d\beta}{dx}=\mathfrak{m}\mathfrak{b}, &&\frac{d\gamma}{dx}=\mathfrak{n}\mathfrak{c},\\
&\frac{d\alpha}{dy}=\mathfrak{l}\mathfrak{a}', &&\frac{d\beta}{dy}=\mathfrak{m}\mathfrak{b}', &&\frac{d\gamma}{dy}=\mathfrak{n}\mathfrak{c}',\\
&\frac{d\alpha}{dz}=\mathfrak{l}\mathfrak{a}'', &&\frac{d\beta}{dz}=\mathfrak{m}\mathfrak{b}'', &&\frac{d\gamma}{dz}=\mathfrak{n}\mathfrak{c}''.
\end{aligned}\right.$$

Having said that, one will have:

$$\Delta = 1 - \lambda^2 - \mu^2 - \nu^2 + 2\lambda\mu\nu, \qquad \nabla = 1 - \varepsilon^2 - \eta^2 - \theta^2 + 2\varepsilon\eta\theta,$$

$$lmn\sqrt{\Delta}\cdot \mathfrak{lmn}\sqrt{\nabla} = 1, \qquad dV = \sqrt{\Delta}\, lmn\, d\alpha\, d\beta\, d\gamma,$$

in which dV denotes the volume element. The forms (φ) become:

$$(2)\qquad \begin{cases} \begin{cases} 2\varphi = (1-\lambda^2)\,a^2 + (1-\mu^2)\,b^2 + (1-\nu^2)\,c^2 + \Delta_\lambda bc + \Delta_\mu ac + \Delta_\nu ab = \Delta, \\ 2\varphi' = (1-\lambda^2)\,a'^2 + \cdots, \\ 2\varphi'' = (1-\lambda^2)\,a''^2 + \cdots, \end{cases} \\ \begin{cases} 2\varpi = (1-\varepsilon^2)\,\mathfrak{a}^2 + (1-\eta^2)\,\mathfrak{b}^2 + (1-\theta^2)\,\mathfrak{c}^2 + \nabla_\varepsilon \mathfrak{bc} + \nabla_\eta \mathfrak{ac} + \nabla_\theta \mathfrak{ab} = \nabla, \\ 2\varpi' = (1-\varepsilon^2)\,\mathfrak{a}'^2 + \cdots, \\ 2\varpi'' = (1-\varepsilon^2)\,\mathfrak{a}''^2 + \cdots, \end{cases} \end{cases}$$

in which the letters that are used as indices indicate partial derivatives, in such a way that $\Delta_\lambda = 2\,(\mu\nu - \lambda)$, for example. Those forms satisfy the conditions:

$$(3)\qquad \begin{cases} \sum a \dfrac{d\varphi}{da} = \Delta, \\ \sum b \dfrac{d\varphi}{da} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \end{cases} \qquad \begin{cases} \sum \mathfrak{a} \dfrac{d\varpi}{d\mathfrak{a}} = \Delta, \\ \sum \mathfrak{b} \dfrac{d\varpi}{d\mathfrak{a}} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \end{cases}$$

in which the $\sum$ always affect the primed quantities. Equations (c*) are written:

$$(4)\qquad \begin{cases} \begin{cases} \mathrm{a} = \dfrac{1}{\sqrt{\Delta(1-\lambda^2)}}\dfrac{d\varphi}{da}, \quad \mathrm{b} = \dfrac{1}{\sqrt{\Delta(1-\mu^2)}}\dfrac{d\varphi}{db}, \quad \mathrm{c} = \dfrac{1}{\sqrt{\Delta(1-\nu^2)}}\dfrac{d\varphi}{dc}, \\ \mathrm{a}' = \dfrac{1}{\sqrt{\Delta(1-\lambda^2)}}\dfrac{d\varphi'}{da'}, \quad \ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots, \\ \mathrm{a}'' = \dfrac{1}{\sqrt{\Delta(1-\lambda^2)}}\dfrac{d\varphi''}{da''}, \quad \ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots, \end{cases} \\ \begin{cases} a' = \dfrac{1}{\sqrt{\nabla(1-\varepsilon^2)}}\dfrac{d\varpi}{d\mathrm{a}}, \quad b = \dfrac{1}{\sqrt{\nabla(1-\eta^2)}}\dfrac{d\varpi}{d\mathrm{b}}, \quad c = \dfrac{1}{\sqrt{\nabla(1-\theta^2)}}\dfrac{d\varpi}{d\mathrm{c}}, \\ a' = \dfrac{1}{\sqrt{\nabla(1-\varepsilon^2)}}\dfrac{d\varpi'}{d\mathrm{a}'}, \quad \ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots, \\ a'' = \dfrac{1}{\sqrt{\nabla(1-\varepsilon^2)}}\dfrac{d\varpi''}{d\mathrm{a}''}, \quad \ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots, \end{cases} \end{cases}$$

and (d^*) are written:

$$(5)\qquad \begin{cases} l^2 = \dfrac{1}{l^2}\dfrac{1-\lambda^2}{\Delta}, \quad \varepsilon = \dfrac{\frac{1}{2}\Delta_\lambda}{\sqrt{(1-\mu^2)(1-\nu^2)}}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{cases} \qquad \begin{cases} l^2 = \dfrac{1}{l^2}\dfrac{1-\varepsilon^2}{\nabla}, \quad \lambda = \dfrac{\frac{1}{2}\nabla_\varepsilon}{\sqrt{(1-\eta^2)(1-\theta^2)}}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{cases}$$

Having set:

$$(6)\qquad \begin{cases} \sum b\dfrac{da}{dt} = \mathrm{R}^{(t)}, \quad \sum a\dfrac{dc}{dt} = \mathrm{Q}^{(t)}, \quad \sum c\dfrac{db}{dt} = \mathrm{P}^{(t)}, \\ \sum a\dfrac{db}{dt} = \mathcal{R}^{(t)}, \quad \sum c\dfrac{da}{dt} = \mathcal{Q}^{(t)}, \quad \sum b\dfrac{dc}{dt} = \mathcal{P}^{(t)}, \end{cases}$$

which implies that:

$$(6^*)\qquad \mathrm{R}^{(t)} + \mathcal{R}^{(t)} = \frac{d\nu}{dt}, \qquad \mathrm{Q}^{(t)} + \mathcal{Q}^{(t)} = \frac{d\mu}{dt}, \qquad \mathrm{P}^{(t)} + \mathcal{P}^{(t)} = \frac{d\lambda}{dt},$$

in which t is any of the variables α, β, γ, the (f) can be written:

$$(7)\qquad \begin{cases} \Delta\dfrac{da}{dt} = \mathrm{R}^{(t)}\dfrac{d\varphi}{db} + \mathcal{Q}^{(t)}\dfrac{d\varphi}{dc}, \\ \Delta\dfrac{db}{dt} = \mathrm{P}^{(t)}\dfrac{d\varphi}{dc} + \mathcal{R}^{(t)}\dfrac{d\varphi}{da}, \\ \Delta\dfrac{dc}{dt} = \mathrm{Q}^{(t)}\dfrac{d\varphi}{da} + \mathcal{P}^{(t)}\dfrac{d\varphi}{db}, \end{cases}$$

and one will get two analogous groups by first priming the letters φ, a, b, c, and then double-priming them. Here, the group (k) will become:

$$(8)\qquad \begin{cases} \Delta\left(\dfrac{d\mathrm{R}^{(t)}}{ds} - \dfrac{d\mathrm{R}^{(s)}}{dt}\right) = (1-\nu^2)(\mathrm{P}^{(s)}\mathcal{Q}^{(t)} - \mathrm{P}^{(t)}\mathcal{Q}^{(s)}) + \frac{1}{2}\Delta_\lambda(\mathrm{R}^{(s)}\mathrm{P}^{(t)} - \mathrm{P}^{(t)}\mathrm{R}^{(s)}) \\ \qquad + \frac{1}{2}\Delta_\mu(\mathcal{Q}^{(t)}\mathcal{R}^{(s)} - \mathcal{Q}^{(s)}\mathcal{R}^{(t)}) + \frac{1}{2}\Delta_\nu(\mathrm{R}^{(t)}\mathcal{R}^{(s)} - \mathrm{P}^{(s)}\mathcal{R}^{(t)}), \\ \Delta\left(\dfrac{d\mathrm{Q}^{(t)}}{ds} - \dfrac{d\mathrm{Q}^{(s)}}{dt}\right) = (1-\mu^2)(\mathrm{R}^{(s)}\mathcal{P}^{(t)} - \mathrm{R}^{(t)}\mathcal{P}^{(s)}) + \frac{1}{2}\Delta_\lambda(\mathcal{P}^{(s)}\mathcal{Q}^{(t)} - \mathcal{P}^{(t)}\mathcal{Q}^{(s)}) \\ \qquad + \frac{1}{2}\Delta_\mu(\mathrm{Q}^{(t)}\mathcal{Q}^{(s)} - \mathrm{Q}^{(s)}\mathcal{Q}^{(t)}) + \frac{1}{2}\Delta_\nu(\mathrm{Q}^{(t)}\mathrm{R}^{(s)} - \mathrm{Q}^{(s)}\mathrm{R}^{(t)}), \\ \Delta\left(\dfrac{d\mathrm{P}^{(t)}}{ds} - \dfrac{d\mathrm{P}^{(s)}}{dt}\right) = (1-\lambda^2)(\mathrm{Q}^{(s)}\mathcal{R}^{(t)} - \mathrm{Q}^{(t)}\mathcal{R}^{(s)}) + \frac{1}{2}\Delta_\lambda(\mathrm{P}^{(s)}\mathcal{P}^{(t)} - \mathrm{P}^{(t)}\mathcal{P}^{(s)}) \\ \qquad + \frac{1}{2}\Delta_\mu(\mathrm{P}^{(t)}\mathrm{Q}^{(s)} - \mathrm{P}^{(s)}\mathrm{Q}^{(t)}) + \frac{1}{2}\Delta_\nu(\mathcal{R}^{(t)}\mathcal{P}^{(s)} - \mathcal{R}^{(s)}\mathcal{P}^{(t)}). \end{cases}$$

Finally, upon isolating the partial derivatives of l, m, n, one will deduce from equations (l) that:

$$
(9)\qquad \left\{
\begin{aligned}
\frac{dl}{d\beta} &= \frac{m}{1-\nu^2}\mathcal{R}^{(\alpha)} + \frac{\nu l}{1-\nu^2}\mathrm{R}^{(\beta)},\\
\frac{dm}{d\gamma} &= \frac{n}{1-\lambda^2}\mathcal{P}^{(\beta)} + \frac{\lambda m}{1-\lambda^2}\mathrm{P}^{(\gamma)},\\
\frac{dn}{d\alpha} &= \frac{l}{1-\mu^2}\mathcal{Q}^{(\gamma)} + \frac{\mu n}{1-\mu^2}\mathrm{Q}^{(\alpha)},\\
\frac{dl}{d\gamma} &= \frac{n}{1-\mu^2}\mathrm{Q}^{(\alpha)} + \frac{\mu l}{1-\mu^2}\mathcal{Q}^{(\gamma)},\\
\frac{dm}{d\alpha} &= \frac{l}{1-\nu^2}\mathrm{R}^{(\beta)} + \frac{\nu m}{1-\nu^2}\mathcal{R}^{(\alpha)},\\
\frac{dn}{d\beta} &= \frac{m}{1-\lambda^2}\mathrm{P}^{(\gamma)} + \frac{\lambda n}{1-\lambda^2}\mathcal{P}^{(\beta)},\\
\left[(1-\nu^2)\mathcal{Q}^{(\beta)} + \tfrac{1}{2}\Delta_\lambda \mathrm{R}^{(\beta)}\right] l &= \left[(1-\nu^2)\mathrm{P}^{(\alpha)} + \tfrac{1}{2}\Delta_\mu \mathcal{R}^{(\alpha)}\right] m,\\
\left[(1-\lambda^2)\mathcal{R}^{(\gamma)} + \tfrac{1}{2}\Delta_\mu \mathrm{R}^{(\gamma)}\right] m &= \left[(1-\lambda^2)\mathrm{Q}^{(\beta)} + \tfrac{1}{2}\Delta_\nu \mathcal{P}^{(\beta)}\right] n,\\
\left[(1-\mu^2)\mathcal{P}^{(\alpha)} + \tfrac{1}{2}\Delta_\nu \mathrm{R}^{(\alpha)}\right] n &= \left[(1-\mu^2)\mathrm{R}^{(\gamma)} + \tfrac{1}{2}\Delta_\lambda \mathcal{Q}^{(\gamma)}\right] l.
\end{aligned}\right.
$$

I combine these with the equations for the lines of curvature on the coordinate surfaces. For an infinitely-small displacement that is performed on the surface α = constant, one will have:

$$\delta x = \frac{dx}{d\beta}d\beta + \frac{dx}{d\gamma}d\gamma,$$

and in turn:

$$(10)\qquad \sum b\, d\beta = m\, d\beta + n\lambda\, d\gamma, \qquad \sum c\,\delta x = m\lambda\, d\beta + n\, d\gamma.$$

If the displacement corresponds to a line of curvature then one must have:

$$\delta x = -\Theta\,\delta \mathrm{a}, \quad \delta y = -\Theta\,\delta \mathrm{a}', \quad \delta z = -\Theta\,\delta \mathrm{a}'';$$

hence, one easily concludes that:

$$
(11)\qquad \left\{
\begin{aligned}
\sum b\,\delta x &= \frac{\Theta}{\sqrt{\Delta(1-\lambda^2)}}\left[\tfrac{1}{2}\Delta_\mu \mathrm{P}^{(\beta)} + (1-\lambda^2)\mathcal{R}^{(\beta)}\right] d\beta\\
&\quad + \frac{\Theta}{\sqrt{\Delta(1-\lambda^2)}}\left[\tfrac{1}{2}\Delta_\mu \mathrm{P}^{(\gamma)} + (1-\lambda^2)\mathcal{R}^{(\gamma)}\right] d\gamma,\\
\sum c\,\delta x &= \frac{\Theta}{\sqrt{\Delta(1-\lambda^2)}}\left[\tfrac{1}{2}\Delta_\nu \mathcal{P}^{(\beta)} + (1-\lambda^2)\mathrm{Q}^{(\beta)}\right] d\beta\\
&\quad + \frac{\Theta}{\sqrt{\Delta(1-\lambda^2)}}\left[\tfrac{1}{2}\Delta_\nu \mathcal{P}^{(\gamma)} + (1-\lambda^2)\mathrm{Q}^{(\gamma)}\right] d\gamma.
\end{aligned}\right.
$$

Upon equating these expressions for $\sum b\ \delta x$, $\sum c\ \delta x$ to the preceding ones and then eliminating Θ, one will get the equations of the lines of curvature of the surface α = constant. One will also get two values for Θ that will be the expressions for the principal radii of curvature.

General remark. – The formulas of the present paragraph correspond directly to the differential question that takes the form: If the rectangular coordinates x, y, z of an arbitrary point in space are expressed arbitrarily in terms of well-defined functions of three independent variables α, β, γ then one proposes to calculate the principal geometric elements, which are either the intersection curves of the coordinate surfaces or those surfaces themselves.

However, that presents the otherwise difficult problem of finding x, y, z in terms of α, β, γ when one is given three distinct conditions between the latter variables and the l, …, ν, $\mathfrak{l}$, …, θ, or some other quantities that they can depend upon. Equations (8), concurrently with (9) and (6*), will then play the principal role. When combined with the three given conditions, they will suffice to determine completely the functions l, m, …, ν or $\mathfrak{l}$, $\mathfrak{m}$, …, θ and the auxiliary functions P, Q … Once those quantities are determined, one can present the much simpler question, which I would like to address, of deducing the cosines a, b, … from equations (7) and its analogues, such that (8) will assure their coexistence precisely. After that, x, y, z are obtained from (1) by integrating the exact differentials:

$$(c)\qquad \begin{cases} dx = la\,d\alpha + mb\,d\beta + nc\,d\gamma, \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

§ IV. – *Integration of a particular system of partial difference equations.*

If one regards the angles of the elementary trihedron $M \cdot (\alpha)(\beta)(\gamma)$ as being determined by α, β, γ then one imagines three rectangular axes $M \cdot X, Y, Z$ being drawn through its summit that make angles with $O \cdot xyz$ whose cosines are $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$, $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{B}''$, $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$. If one lets ξ, ξ', ξ'', η, η', η'', ζ, ζ', ζ'' denote the cosines of the angles that the elements of the curvilinear axes (α), (β), (γ) that issue from M make with those auxiliary axes then one will have:

$$\begin{array}{lll} a = \mathcal{A}\,\xi + \mathcal{B}\,\xi' + \mathcal{C}\,\xi'', & b = \mathcal{A}\,\eta + \mathcal{B}\,\eta' + \mathcal{C}\,\eta'', & \dots, \\ a' = \mathcal{A}'\,\xi + \mathcal{B}'\,\xi' + \mathcal{C}'\,\xi'', & \dots\dots\dots, & \dots, \\ \dots\dots\dots, & \dots\dots\dots, & \dots. \end{array}$$

with the six relations:

$$\xi^2 + \xi'^2 + \xi''^2 = 1, \quad \dots, \qquad \xi\,\eta + \xi'\,\eta' + \xi''\,\eta'' = \nu, \qquad \dots,$$

and one can arrange some of the ξ, η, … such that these relations will still be arbitrary, in such a fashion as to give the auxiliary system the position that one deems to be the most convenient relative to the elementary trihedron. Having said that, if one makes:

$$\sum \mathcal{B}\frac{d\mathcal{A}}{dt} = r^{(t)}, \qquad \sum \mathcal{A}\frac{d\mathcal{C}}{dt} = q^{(t)}, \qquad \sum \mathcal{C}\frac{d\mathcal{B}}{dt} = p^{(t)},$$
$$\sum \eta\frac{d\xi}{dt} = \upsilon^{(t)}, \qquad \sum \xi\frac{d\zeta}{dt} = \kappa^{(t)}, \qquad \sum \zeta\frac{d\eta}{dt} = \pi^{(t)}$$

then one will deduce from the preceding values of a, b, c, … that:

$$\sum b\frac{da}{dt} = (\xi\eta' - \eta\xi')\, r^{(t)} + (\eta\zeta'' - \xi\eta'')\, q^{(t)} + (\xi'\eta'' - \eta'\xi'')\, p^{(t)} + \upsilon^{(t)},$$
$$\sum a\frac{dc}{dt} = (\zeta\xi' - \xi\zeta')\, r^{(t)} + (\xi\zeta'' - \zeta\xi'')\, q^{(t)} + (\xi''\zeta' - \xi'\zeta'')\, p^{(t)} + \kappa^{(t)},$$
$$\sum c\frac{db}{dt} = (\eta\zeta' - \zeta\eta')\, r^{(t)} + (\zeta\eta'' - \eta\zeta'')\, q^{(t)} + (\eta'\zeta'' - \zeta'\eta'')\, p^{(t)} + \pi^{(t)},$$

which express the P, Q, R, and in turn, the $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, linearly in terms of the p, q, r, and some quantities that one regards as known, or *vice versa*,.

By means of that sort of coordinate transformation, one will be led to consider the canonical system:

$$(7^*)\quad \left\{\begin{aligned} \frac{d\mathcal{A}}{d\alpha} &= \mathcal{B}\, r^{(\alpha)} - \mathcal{C}\, q^{(\alpha)}, \\ \frac{d\mathcal{B}}{d\alpha} &= \mathcal{C}\, p^{(\alpha)} - \mathcal{A}\, r^{(\alpha)}, \\ \frac{d\mathcal{C}}{d\alpha} &= \mathcal{A}\, q^{(\alpha)} - \mathcal{B}\, p^{(\alpha)}, \end{aligned}\right. \qquad \left\{\begin{aligned} \frac{d\mathcal{A}}{d\beta} &= \mathcal{B}\, r^{(\beta)} - \mathcal{C}\, q^{(\beta)}, \\ \frac{d\mathcal{B}}{d\beta} &= \mathcal{C}\, p^{(\beta)} - \mathcal{A}\, r^{(\beta)}, \\ \frac{d\mathcal{C}}{d\beta} &= \mathcal{A}\, q^{(\beta)} - \mathcal{B}\, p^{(\beta)}, \end{aligned}\right. \qquad \left\{\begin{aligned} \frac{d\mathcal{A}}{d\gamma} &= \mathcal{B}\, r^{(\gamma)} - \mathcal{C}\, q^{(\gamma)}, \\ \frac{d\mathcal{B}}{d\gamma} &= \mathcal{C}\, p^{(\gamma)} - \mathcal{A}\, r^{(\gamma)}, \\ \frac{d\mathcal{C}}{d\gamma} &= \mathcal{A}\, q^{(\gamma)} - \mathcal{B}\, p^{(\gamma)}, \end{aligned}\right.$$

instead of the system (7), along with two other ones in $\mathcal{A}'$, …, $\mathcal{A}''$, … that are entirely analogous. Here, one will have:

$$\sum \mathcal{B}_t\, \mathcal{C}_s = -\, r^{(t)}\, q^{(s)}, \qquad \sum \mathcal{A}_t\, \mathcal{A}_s = r^{(t)}\, r^{(s)} + q^{(t)}\, q^{(s)},$$

and their analogues; the letters that are used as indices indicate partial derivatives, in such a way that $\mathcal{A}_t = d\mathcal{A} / dt$. The p, q, r, which are considered to be known in terms of α, β, γ, are supposed to verify the *fundamental* group:

$$(8^*)\qquad \begin{cases} r_s^{(t)} - r_t^{(s)} + q^{(t)} p^{(s)} - q^{(s)} p^{(t)} = 0, \\ q_s^{(t)} - q_t^{(s)} + p^{(t)} r^{(s)} - p^{(s)} r^{(t)} = 0, \\ p_s^{(t)} - p_t^{(s)} + r^{(t)} q^{(s)} - r^{(s)} q^{(t)} = 0, \end{cases}$$

identically. From the first group (7^*), one deduces that:

$$\mathcal{A} = \text{A}\,\sigma + \text{B}\,\tau + \text{C}\,\upsilon, \qquad \mathcal{B} = \text{A}\,\sigma' + \text{B}\,\tau' + \text{C}\,\upsilon', \qquad \mathcal{C} = \text{A}\,\sigma'' + \text{B}\,\tau'' + \text{C}\,\upsilon'',$$

in which A, B, C are arbitrary functions of β, γ, and the σ, τ, … are some well-defined functions of α, β, γ that are deduced from a complete solution to the first group (7^*) (*see* below) that is integrated under the hypothesis that only α is variable. Those functions must verify the usual relations between cosines:

$$\sigma^2 + \sigma'^2 + \sigma''^2 = 1, \quad \sigma\,\tau + \sigma'\,\tau' + \sigma''\,\tau'' = 0, \quad ...,$$

due to the fact that one must have

$$\mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2 = 1,$$

independently of A, B, C, and in turn:

$$\text{A}^2 + \text{B}^2 + \text{C}^2 = 1.$$

Upon expressing the idea that the preceding values of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ verify the second group (7^*), one will conclude that:

$$(7^{**})\qquad \text{A}_\beta = \text{B}\,r' - \text{C}\,q', \qquad \text{B}_\beta = \text{C}\,p' - \text{A}\,r', \qquad \text{C}_\beta = \text{A}\,q' - \text{B}\,p',$$

in which:

$$\begin{aligned} r' &= \sum \tau\,\sigma_\beta + \upsilon\, p^{(\beta)} + \upsilon' q^{(\beta)} + \upsilon'' r^{(\beta)}, \\ q' &= \sum \sigma\,\upsilon_\beta + \tau\, p^{(\beta)} + \tau' q^{(\beta)} + \tau'' r^{(\beta)}, \\ p' &= \sum \upsilon\,\tau_\beta + \sigma p^{(\beta)} + \sigma' q^{(\beta)} + \sigma'' r^{(\beta)}, \end{aligned}$$

in which the quantities p', q', r' are independent of the α. Equations (7^{**}), which have the same form as (7^*), but no longer refer to the trace of α, since they are integrated under the hypothesis that only β is variable, give:

$$\text{A} = \text{a}\varphi + \text{b}\psi + \text{c}\chi, \qquad \text{B} = \text{a}\varphi' + \text{b}\psi' + \text{c}\chi', \quad \text{C} = \text{a}\varphi'' + \text{b}\psi'' + \text{c}\chi'',$$

in which φ, ψ, … are well-defined functions of β, γ that verify the usual relations between cosines, and a, b, c are three arbitrary functions of γ such that $\text{a}^2 + \text{b}^2 + \text{c}^2 = 1$. Upon substituting the expressions above in the expressions for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ above, one will get results of the form:

$$\mathcal{A} = \mathrm{a}\xi + \mathrm{b}\eta + \mathrm{c}\zeta, \qquad \mathcal{B} = \mathrm{a}\xi' + \mathrm{b}\eta' + \mathrm{c}\zeta', \quad \mathcal{C} = \mathrm{a}\xi'' + \mathrm{b}\eta'' + \mathrm{c}\zeta'',$$

in only the unknowns a, b, c, and the definitive verification of the third group (7*) will give:

(7***) $\mathrm{a}_\gamma = \mathrm{b}\, r'' - \mathrm{c}\, q'', \qquad \mathrm{b}_\gamma = \mathrm{c}\, p'' - \mathrm{a}\, r'', \qquad \mathrm{c}_\gamma = \mathrm{a}\, q'' - \mathrm{b}\, p'',$

in which:

$$r'' = \sum \eta\, \xi_\beta + \zeta\, p^{(\gamma)} + \zeta' q^{(\gamma)} + \zeta'' r^{(\gamma)},$$
$$q'' = \sum \xi\, \zeta_\beta + \eta\, p^{(\gamma)} + \eta' q^{(\gamma)} + \eta'' r^{(\gamma)},$$
$$p'' = \sum \zeta\, \eta_\beta + \xi\, p^{(\gamma)} + \xi' q^{(\gamma)} + \xi'' r^{(\gamma)},$$

in which the p'', q'', r'' depend upon only γ. The integration of the group (7***) for only the independent variable γ will finally give:

$$\mathrm{a} = g\theta + h\varpi + k\omega, \qquad \mathrm{b} = g\theta' + h\varpi' + k\omega', \quad \mathrm{c} = g\theta'' + h\varpi'' + k\omega'',$$

in which g, h, k are three arbitrary constants, such that $g^2 + h^2 + k^2 = 1$, and the θ, ϖ, … verify the usual relations between cosines. One will then have, upon combining everything:

(12) $\mathcal{A} = g\,\mathfrak{A} + h\,\mathfrak{B} + k\,\mathfrak{C}, \qquad \mathcal{B} = g\,\mathfrak{A}' + h\,\mathfrak{B}' + k\,\mathfrak{C}', \qquad \mathcal{C} = g\,\mathfrak{A}'' + h\,\mathfrak{B}'' + k\,\mathfrak{C}'',$

in which $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, … are functions of α, β, γ that are currently known.

As for the expressions for $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$, $\mathcal{A}''$, $\mathcal{B}''$, $\mathcal{C}''$, one will obviously get them by replacing the constants g, h, k with some other constants g', h', k', g'', h'', k'' in the expressions that were just found for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and establishing the usual relations between the cosines:

$$g^2 + h^2 + k^2 = 1, \qquad gg' + hh' + kk' = 0, \quad \dots$$

between those constants.

It is not pointless to recall that the p', q', r' depend upon only β, γ, and the p'', q'', r'' depend upon only γ. That will result from the sequence of calculations and the integrability conditions (8*), which are supposed to be fulfilled; one can also prove that directly. I will confine myself to proving that $dr' / da = 0$. From the identities:

$$\sigma\alpha = \sigma' r^{(\alpha)} - \sigma'' q^{(\alpha)}, \qquad \sigma'_\alpha = \sigma'' p^{(\alpha)} - \sigma\, r^{(\alpha)}, \qquad \sigma''_\alpha = \sigma\, q^{(\alpha)} - \sigma' p^{(\alpha)},$$

and analogous ones in τ, …, υ, …, one will conclude that:

$$\sum \tau\, \sigma_{\alpha\beta} = -\sum \upsilon\, p_\beta^{(\alpha)} + \sum (\tau' \sigma''_\beta - \tau'' \sigma'_\beta)\, p^{(\alpha)}, \qquad \sum \tau_\alpha\, \sigma_\beta = -\sum (\tau' \sigma''_\beta - \tau'' \sigma'_\beta)\, p^{(\alpha)},$$

and in turn:

$$\left(\sum \tau\, \sigma_\beta\right)_\alpha = -\sum \upsilon\, p_\beta^{(\alpha)}, \qquad r'_\alpha = \sum \upsilon \left(p_\alpha^{(\beta)} - p_\beta^{(\alpha)}\right) + \sum \upsilon_\alpha\, p^{(\beta)},$$

in which one will see that all of the terms mutually cancel when one considers (8*) and the equations that the υ verify.

In the foregoing, one must integrate some systems that fall into the classical type three times in succession, and one encounters them in the context of the rotation of bodies:

$$\frac{da}{dt} = br - cq, \qquad \frac{db}{dt} = cp - ar, \qquad \frac{dc}{dt} = aq - bp,$$

in which p, q, r are regarded as known functions of time t. Upon making use of the imaginary substitution that was employed by Hoppe (Journal de Crelle, t. LXIII, pp. 122):

$$b \sin \theta + c \cos \theta = 1, \quad b \cos \theta - c \sin \theta = ia, \qquad \text{in which } i = \sqrt{-1},$$

which is a substitution that gives:

$$\sin \theta \frac{db}{dt} + \cos \theta \frac{dc}{dt} + ia \frac{d\theta}{dt} = 0,$$

when one considers the differential equations and sets $\tan \theta / 2 = z$, one will have:

$$2i \frac{dz}{dt} = 2rz + (q + ip)\, z - (q - ip)\,,$$

and then:

$$i \frac{da}{dt} + (r \cos q + q \sin q)\, a + \frac{d\theta}{dt} - p = 0.$$

When z is determined from the penultimate equation (which is due to Bernoulli), the last one will give a by quadratures; b and c will then follow. One immediately concludes three special systems from the knowledge of a, b, c that will be denoted by σ, σ', σ''; τ, τ', τ''; υ, υ', υ'', respectively, for the first group (7*), for example.

One can alter the integration of the system of nine equations (7*) in various ways. I will confine myself to the following one: One infers from the first horizontal row in (7*) that:

$$(13) \qquad \mathcal{B} = \frac{\mathcal{A}_\beta\, q^{(\gamma)} - \mathcal{A}_\gamma\, q^{(\beta)}}{r^{(\beta)} q^{(\gamma)} - r^{(\gamma)} q^{(\beta)}}, \qquad \mathcal{C} = \frac{\mathcal{A}_\beta\, r^{(\gamma)} - \mathcal{A}_\gamma\, r^{(\beta)}}{r^{(\beta)} q^{(\gamma)} - r^{(\gamma)} q^{(\beta)}},$$

and when one takes into account that $\mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2 = 1$, that will transform the first vertical group in (7*) into:

$$
(14)\quad \begin{cases}
[q^{(\beta)}r^{(\gamma)} - q^{(\gamma)}r^{(\beta)}]\mathcal{A}_\alpha + [r^{(\alpha)}q^{(\gamma)} - r^{(\alpha)}q^{(\beta)}]\mathcal{A}_\beta + [r^{(\beta)}q^{(\alpha)} - r^{(\alpha)}q^{(\beta)}]\mathcal{A}_\gamma = 0, \\
[q^{(\gamma)2} + r^{(\gamma)2}]\mathcal{A}_\beta^2 + [q^{(\beta)2} + r^{(\beta)2}]\mathcal{A}_\gamma^2 - 2[q^{(\beta)}q^{(\gamma)} + r^{(\beta)}r^{(\gamma)}]\mathcal{A}_\alpha\mathcal{A}_\gamma \\
\qquad\qquad - (1 - \mathcal{A}^2)[r^{(\beta)}q^{(\gamma)} - r^{(\beta)}q^{(\gamma)}]^2 = 0, \\
\mathcal{B}_\alpha = \mathcal{C}\, p^{(\alpha)} - \mathcal{A}\, r^{(\alpha)}, \quad \mathcal{C}_\alpha = \mathcal{A}\, q^{(\alpha)} - \mathcal{B}\, p^{(\alpha)},
\end{cases}
$$

in which $\mathcal{B}$ and $\mathcal{C}$ are regarded as having been replaced with their preceding values in the last two equations, one of which is superfluous, by virtue of the other three and the relations (8^*). The first of the preceding equations will make $\mathcal{A}$ depend arbitrarily upon two well-defined functions ϖ and ω. If one considers $\mathcal{A}$ to depend immediately upon ϖ and ω then the substitution in the following two equations will lead to a first-order equation and a second-order between $\mathcal{A}$, ϖ, and ω when one has eliminated β and γ, for example, by means of the forms of ϖ and ω themselves, which must make α disappear. It is easy to see that one will deduce a second-order equation from these equations that one can consider to be an ordinary differential equation in $d\omega$, for example. The constants that are introduced by its integration are considered to be arbitrary functions of ϖ, so the equations that were employed already and the integrability conditions will easily lead to two first-order equation in those constants, and ω must finally disappear.

§ V. – *Triply-orthogonal systems.*

If the cosines λ, μ, ν are equal to zero then the relations (9) will reduce to:

$$
(15)\quad \begin{cases}
l_\beta = -mR^{(\alpha)}, & m_\gamma = -nP^{(\beta)}, & n_\alpha = -l\,Q^{(\gamma)}, \\
l_\gamma = nQ^{(\alpha)}, & m_\alpha = l\,R^{(\beta)}, & n_\beta = mP^{(\alpha)},
\end{cases}
$$

$$
(15^*)\qquad -l\,Q^{(\beta)} = m\,P^{(\alpha)}, \qquad -m\,R^{(\gamma)} = n\,Q^{(\beta)}, \qquad -n\,P^{(\alpha)} = l\,R^{(\gamma)}.
$$

The multiplication of (15^*) shows that one must have:

$$
(16)\qquad P^{(\alpha)} = 0, \qquad Q^{(\beta)} = 0, \qquad R^{(\gamma)} = 0,
$$

in which l, m, n are assumed to be non-zero.

By virtue of (16), the identities (8) will reduce to:

$$
(17)\quad \begin{cases}
R_\gamma^{(\alpha)} = -Q^{(\alpha)}P^{(\gamma)}, & Q_\beta^{(\alpha)} = R^{(\alpha)}P^{(\beta)}, & P_\alpha^{(\beta)} = -R^{(\beta)}Q^{(\alpha)}, \\
R_\gamma^{(\beta)} = P^{(\beta)}Q^{(\gamma)}, & Q_\beta^{(\gamma)} = -R^{(\beta)}P^{(\gamma)}, & P_\alpha^{(\gamma)} = R^{(\alpha)}Q^{(\gamma)},
\end{cases}
$$

$$(18)\qquad \begin{cases} R_\beta^{(\alpha)} - R_\alpha^{(\beta)} + Q^{(\alpha)}P^{(\beta)} = 0, \\ Q_\alpha^{(\gamma)} - Q_\gamma^{(\alpha)} + R^{(\alpha)}P^{(\gamma)} = 0, \\ P_\gamma^{(\beta)} - P_\beta^{(\gamma)} + R^{(\beta)}Q^{(\gamma)} = 0, \end{cases}$$

and will be equivalent to only six distinct equations, as one will convince oneself by differentiating each of the last three by the variables that do not appear in them, on the surface of things.

On the other hand, upon regarding (17), (18), one will infer from (15) that:

$$l_{\beta\gamma} = -m\,R_\gamma^{(\alpha)} - m_\gamma R^{(\alpha)} = m\,Q^{(\alpha)}\,P^{(\gamma)} + n\,P^{(\beta)}\,R^{(\alpha)},$$
$$l_{\gamma\beta} = \quad n\,Q_\beta^{(\alpha)} + n_\beta\,Q^{(\alpha)} = n\,R^{(\alpha)}\,P^{(\beta)} + m\,Q^{(\alpha)}\,P^{(\gamma)},$$

in such a way that equations (15) will not lead to any new relation between the P, Q, R, as one could predict. Therefore, the only non-identical relations that the latter functions must verify will be expressed by equations (16).

Presently, the nine cosines a, b, c, … can be expressed in an infinitude of ways by means of three independent functions. For example, if one takes Euler's formulas (DUHAMEL, *Mécanique*, 2nd ed., t. I, pp. 267), which give:

$$P^{(t)} = \quad \cos\varphi\,\frac{d\theta}{dt} + \sin\varphi\sin\theta\,\frac{d\psi}{dt},$$
$$Q^{(t)} = -\sin\varphi\,\frac{d\theta}{dt} + \cos\varphi\sin\theta\,\frac{d\psi}{dt},$$
$$R^{(t)} = \quad \cos\varphi\,\frac{d\psi}{dt} + \frac{d\varphi}{dt},$$

then it will result from what was just said that the determination of the triply-orthogonal systems can be reduced to:

1. The integration of three equations:

$$(19)\qquad \frac{d\psi}{d\alpha} + \cot\varphi\,\frac{du}{d\alpha} = 0, \quad \frac{d\psi}{d\beta} - \tan\varphi\,\frac{du}{d\beta} = 0, \quad \frac{d\psi}{d\gamma} - \cot u\,\frac{d\varphi}{d\gamma} = 0,$$

in which:

$$\tan\frac{\theta}{2} = e^u, \qquad \cot u = \frac{\frac{1}{2}(e^u + e^{-u})}{\frac{1}{2}(e^u - e^{-u})}.$$

2. The determination of l, m, n from the linear equations (13).

3. The quadratures (c) (end of § III).

As for *l*, *m*, *n*, which must verify equations (15), if one focuses on *l*, for example, then the elimination of *m*, *n* will show that the function must satisfy the two simultaneous, compatible equations:

$$(20)\qquad \begin{cases} \dfrac{d^2 l}{d\alpha\, d\beta} - \dfrac{R_\alpha^{(\alpha)}}{R^{(\alpha)}} \dfrac{dl}{d\beta} + R^{(\alpha)} R^{(\beta)} l = 0, \\[2ex] \dfrac{d^2 l}{d\alpha\, d\gamma} - \dfrac{Q_\alpha^{(\alpha)}}{Q^{(\alpha)}} \dfrac{dl}{d\gamma} + Q^{(\alpha)} Q^{(\gamma)} l = 0. \end{cases}$$

so when *l* is known, equations (15) will give *m* and *n* with no other integration.

"That method, which I believed to be new when I sent the theoretical part of the present work to the Institute (June 1864), had been the object of a prior communication by Bonnet (March 1862). That geometer had made the remark that the integration of equations (19) can be reduced to that of a single third-order equation that one obtains by eliminating *u* from those three equations and then considering φ to be a function of ψ, α, β. One can obtain a single third-order equation by a somewhat different choice of variables. If one eliminates φ from the three (19) then that will give:

$$0 = \frac{d\psi}{d\alpha}\frac{d\psi}{d\beta} + \frac{du}{d\alpha}\frac{du}{d\beta}, \qquad \frac{d\psi}{d\gamma} = -\cot u\ \frac{d}{d\gamma}\left(\frac{\psi_\alpha}{u_\alpha}\right) : \left(1 + \frac{\psi_\alpha^2}{u_\alpha^2}\right),$$

and when one considers α, β to be functions of *u*, ψ, γ (which are taken to be independent variables), one will find from the formulas that relate to the change of variables that $\frac{d\alpha}{d\psi}$, $\frac{d\alpha}{du}$, $\frac{d\alpha}{d\gamma}$ are proportional to some expressions that depend upon only *u* and the derivatives of β with respect to ψ, *u*, γ up to order two, inclusively. Upon expressing the idea that those proportional quantities will satisfy the usual integrability condition, one will get the aforementioned third-order equation. That equation, which is moderately complicated, can be replaced with that of Bonnet, or *vice versa*, according to the viewpoint that one assumes (*)."

Lamé's method. – By virtue of (15) and what was said about the motion of a material point (i.e., curvilinear coordinates), the identities (17), (18) amount to those of that illustrious author in relation to the arcs of curves and their variations, and upon eliminating *P*, *Q*, *R*, one will be naturally led to the *fundamental* group [8], [9] of curvilinear coordinates (Lamé, pp. 76, 78):

(*) I have enclosed in quotes all of the parts of this article that were introduced after an editing of it that Hermite was willing to accept on March 1865, and which differs from the one that was sent to the Institute only by the preliminary considerations on determinants and the addition of some examples in §§ VI and VIII.

(17*) $$\left(\frac{l_\beta}{m}\right)_\gamma = \frac{l_\gamma}{n}\frac{n_\beta}{m}, \ldots,$$

(18*) $$\left(\frac{l_\beta}{m}\right)_\beta + \left(\frac{m_\alpha}{l}\right)_\alpha + \frac{l_\gamma}{n}\frac{m_\gamma}{n} = 0, \ldots$$

Equations (7) will become:

(21) $$\left\{\begin{array}{lll} \dfrac{da}{d\alpha} = bR^{(\alpha)} - cQ^{(\alpha)}, & \dfrac{d\alpha}{d\beta} = bR^{(\beta)}, & \dfrac{da}{d\gamma} = -cQ^{(\gamma)}, \\ \dfrac{db}{d\beta} = -aR^{(\alpha)}, & \dfrac{db}{d\beta} = cP^{(\beta)} - aR^{(\beta)}, & \dfrac{db}{d\gamma} = cP^{(\gamma)}, \\ \dfrac{dc}{d\alpha} = aQ^{(\alpha)}, & \dfrac{dc}{d\beta} = -bP^{(\beta)}, & \dfrac{dc}{d\gamma} = aQ^{(\gamma)} - bP^{(\gamma)}, \end{array}\right.$$

here, and upon introducing $\frac{1}{l}\frac{dx}{d\alpha}$, $\frac{1}{m}\frac{dx}{d\beta}$, $\frac{1}{n}\frac{dx}{d\gamma}$, in place of *a*, *b*, *c*, respectively, they will become:

(22) $$\frac{d^2x}{d\alpha\,d\beta} = \frac{l_\beta}{l}\frac{dx}{d\alpha} + \frac{m_\alpha}{m}\frac{dx}{d\beta}, \ldots,$$

(22*) $$\frac{d^2x}{d\alpha^2} = \frac{l_\beta}{l}\frac{dx}{d\alpha} - \frac{l\,l_\alpha}{m^2}\frac{dx}{d\beta} - \frac{l\,l_\gamma}{n^2}\frac{dx}{d\gamma}, \ldots,$$

i.e., the [28], [30] in *Coordinées curvilignes*. One must combine them with:

(23) $$\frac{1}{l^2}\frac{dx^2}{d\alpha^2} + \frac{1}{m^2}\frac{dx^2}{d\beta^2} + \frac{1}{n^2}\frac{dx^2}{d\gamma^2} = 1.$$

Conforming to Lamé's method, after finding *l*, *m*, *n* by means of the six equations (17*), (18*), one determines *x* (and analogously *y*, *z*) by means the three (22) and (23). Since that method is far from having to be abandoned completely, whether we take the unknowns to be the rotations or *l*, *m*, *n* [which we pass to easily by means of (15)], I would like to add the following remarks:

1. One of the three (22) is a consequence of the other two and (23), as it easy to convince oneself when one takes (17*) and (18*) into account.

2. The integration of (22), (23) comes down to that of equations (21), which one can treat in succession as ordinary differential equations, which will be developed more generally in § IV.

3. It results from the first method that was presented in this section that equations (22), (23) admit three unique, well-defined solutions (when one ignores what amounts to a rectangular coordinate transformation) that will be easier to find directly by the immediate consideration of (22), (23), which is what happens for isothermal systems, in particular.

§ VI. – *Example that relates to the preceding section.*

Euler's formulas can be replaced with those of O. de Rodrigues:

$$\Theta a = 1 + X^2 - Y^2 - Z^2, \qquad \Theta b = 2\,(XY + Z), \qquad \Theta c = 2\,(XZ - Y), \ldots,$$

$$\Theta\, P^{(t)} = 2\left(Z\frac{dY}{dt} - Y\frac{dZ}{dt} - \frac{dX}{dt}\right),$$
$$\Theta\, Q^{(t)} = 2\left(X\frac{dZ}{dt} - Z\frac{dX}{dt} - \frac{dY}{dt}\right),$$
$$\Theta\, R^{(t)} = 2\left(Y\frac{dX}{dt} - X\frac{dY}{dt} - \frac{dZ}{dt}\right),$$

in which $\Theta = 1 + X^2 + Y^2 + Z^2$. In order to re-establish homogeneity, one sets:

$$X = \frac{\xi}{H}, \qquad Y = \frac{\eta}{H}, \qquad Z = \frac{\zeta}{H}.$$

Equations (16) can be written:

$$(19^*) \qquad \begin{cases} H\dfrac{d\xi}{d\alpha} - \xi\dfrac{dH}{d\alpha} = \zeta\dfrac{d\eta}{d\alpha} - \eta\dfrac{d\zeta}{d\alpha}, \\ H\dfrac{d\eta}{d\beta} - \eta\dfrac{dH}{d\beta} = \xi\dfrac{d\zeta}{d\beta} - \zeta\dfrac{d\xi}{d\beta}, \\ H\dfrac{d\zeta}{d\gamma} - \zeta\dfrac{dH}{d\gamma} = \eta\dfrac{d\xi}{d\gamma} - \xi\dfrac{d\eta}{d\gamma}. \end{cases}$$

"One can arrange that the arbitrary denominator *H* fulfills some special condition. For example, if one makes the triple assumption that ξ does not contain α, η does not contain β, and ζ does not contain γ, which will make the terms on the extreme left vanish, then the elimination of *H* from the reduced equations will give the following three equations:

$$-\, w_{\alpha\beta} = (w_\alpha - v_\alpha)\,(w_\beta - u_\beta),$$
$$-\, v_{\alpha\beta} = (v_\alpha - w_\alpha)\,(v_\beta - u_\beta),$$
$$-\, u_{\alpha\beta} = (u_\alpha - w_\alpha)\,(u_\beta - v_\beta),$$

in which one has set u = ln ξ, v = ln η, w = ln ζ, and which introduces no new conditions, as one will see upon differentiating with respect to γ, β, α, respectively. For example, one infers from the first one that:

$$\left(\frac{w_{\alpha\beta}}{v_\alpha - w_\alpha}\right)_\alpha = w_{\alpha\beta},$$

i.e.:

$$\frac{w_{\alpha\alpha\beta}}{w_{\alpha\beta}} = v_\alpha - w_\alpha + \frac{v_{\alpha\alpha} - w_{\alpha\alpha}}{v_\alpha - w_\alpha}.$$

Upon differentiating with respect to β and considering that same equation, one will conclude that:

$$(\ln \varpi)_{\alpha\beta} = -2\varpi \qquad \text{or} \qquad \varpi = w_{\alpha\beta}.$$

From Liouville, that will give:

$$\varpi = \frac{\mathcal{A}'\mathcal{B}'}{(\mathcal{A}-\mathcal{B})^2}.$$

It will then result that:

$$\zeta = \mathcal{A}_1\,\mathcal{B}_1\,(\mathcal{A}-\mathcal{B}), \qquad \eta = \mathcal{A}_1\,\mathcal{C}_1\,(\mathcal{C}-\mathcal{A}), \qquad \xi = \mathcal{B}_1\,\mathcal{C}_1\,(\mathcal{B}-\mathcal{C}),$$

$$H = k - \int \mathcal{A}_1^2\mathcal{A}'\,d\alpha - \int \mathcal{B}_1^2\mathcal{B}'\,d\beta - \int \mathcal{C}_1^2\mathcal{C}'\,d\gamma.$$

k is an arbitrary constant, $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{B}$, $\mathcal{B}_1$, $\mathcal{C}$, $\mathcal{C}_1$ are arbitrary functions of α, β, γ, respectively, and the primes indicate derivatives. Three of these six arbitrary functions can be taken to be α, β, γ.”

Upon supposing that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are constants, one will get the very special solution:

$$X = \beta\gamma, \qquad Y = \alpha\gamma, \qquad Z = \alpha\beta,$$

which will give:

$$\Theta\,P^{(\beta)} = -2\gamma(\alpha^2 + 1), \qquad \Theta\,Q^{(\alpha)} = 2\gamma(\beta^2 - 1), \qquad \Theta\,R^{(\alpha)} = -2\beta(\gamma^2 + 1),$$
$$\Theta\,P^{(\gamma)} = 2\beta(\alpha^2 - 1), \qquad \Theta\,Q^{(\gamma)} = -2\alpha(\beta^2 + 1), \qquad \Theta\,R^{(\beta)} = 2\alpha(\gamma^2 - 1),$$

in which:

$$\Theta = 1 + \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2.$$

Equations (15) generally provide the three combinations $P^{(\gamma)}\dfrac{dl}{d\beta} + R^{(\alpha)}\dfrac{dn}{d\beta} = 0$, …, which will become immediately integrable here, when one suppresses a common factor, and one will then conclude that:

$$\begin{aligned}(\beta^2+1)\,m-(\gamma^2-1)\,n &= \varphi(\beta,\gamma),\\ (\gamma^2+1)\,n-(\alpha^2-1)\,l &= \psi(\alpha,\gamma),\\ (\alpha^2+1)\,l-(\beta^2-1)\,m &= \chi(\alpha,\beta).\end{aligned}$$

If one infers the values of l, m, n from this and substitutes them in (15) then one will get:

$$\begin{aligned}(\alpha^2-1)\frac{d\chi}{d\alpha}+(\alpha^2+1)\frac{d\psi}{d\alpha}&=0,\\ (\beta^2-1)\frac{d\varphi}{d\beta}+(\beta^2+1)\frac{d\chi}{d\beta}&=0,\\ (\gamma^2-1)\frac{d\psi}{d\gamma}+(\gamma^2+1)\frac{d\varphi}{d\gamma}&=0;\end{aligned}$$

hence, by the immediate differentiation:

$$\frac{d^2\chi}{d\alpha\,d\beta}=0,\qquad \frac{d^2\psi}{d\alpha\,d\gamma}=0,\qquad \frac{d^2\varphi}{d\beta\,d\gamma}=0.$$

If one integrates these and substitutes into the preceding equations then one will conclude the defining expressions for φ, ψ, χ, and in turn:

$$\begin{aligned}\Theta\,l &= \frac{\Theta^2}{2\alpha}\frac{d}{d\alpha}\left(\frac{\lambda(\alpha)}{\Theta^2}\right)+(\gamma^2+1)\,\mu(\beta)+(\beta^2-1)\,\nu(\gamma),\\ \Theta\,m &= \frac{\Theta^2}{2\beta}\frac{d}{d\beta}\left(\frac{\mu(\beta)}{\Theta^2}\right)+(\alpha^2+1)\,\nu(\gamma)+(\gamma^2-1)\,\lambda(\alpha),\\ \Theta\,n &= \frac{\Theta^2}{2\gamma}\frac{d}{d\gamma}\left(\frac{\nu(\gamma)}{\Theta^2}\right)+(\beta^2+1)\,\lambda(\alpha)+(\alpha^2-1)\,\mu(\beta),\end{aligned}$$

in which l, m, n are three arbitrary functions. Since the cosines a, b, … are expressed in terms of α, β, γ, moreover, one will have, upon integrating the exact differentials:

$$\begin{aligned}x &= \frac{\alpha}{\Theta}[(\gamma^2+1)\,\mu(\beta)+(\beta^2-1)\,\nu(\gamma)-(\gamma^2+\beta^2)\,\lambda(\alpha)]+\int\frac{\lambda'(\alpha)}{2\alpha}d\alpha,\\ y &= \frac{\beta}{\Theta}[(\alpha^2+1)\,\nu(\gamma)+(\gamma^2-1)\,\lambda(\alpha)-(\alpha^2+\gamma^2)\,\mu(\beta)]+\int\frac{\mu'(\beta)}{2\beta}d\beta,\\ z &= \frac{\gamma}{\Theta}[(\beta^2+1)\,\lambda(\alpha)+(\alpha^2-1)\,\mu(\beta)-(\alpha^2+\beta^2)\,\nu(\gamma)]+\int\frac{\nu'(\gamma)}{2\gamma}d\gamma.\end{aligned}$$

For example, if one takes λ, μ, ν to be linear functions of α^2, β^2, γ^2, respectively, then one will easily conclude the following combinations from the equations obtained:

$$\left(\beta+\frac{1}{\beta}\right)y-\left(\gamma-\frac{1}{\gamma}\right)z = g,$$
$$\left(\gamma+\frac{1}{\gamma}\right)z-\left(\alpha-\frac{1}{\alpha}\right)x = h,$$
$$\left(\alpha+\frac{1}{\alpha}\right)x-\left(\beta-\frac{1}{\beta}\right)y = k,$$

and in turn:

$$\pm\sqrt{\left[\left(\alpha+\frac{1}{\alpha}\right)x-k\right]^2+4y^2}\pm\sqrt{\left[\left(\alpha-\frac{1}{\alpha}\right)x+k\right]^2-4z^2} = g,$$

$$\pm\sqrt{\left[\left(\beta+\frac{1}{\beta}\right)y-g\right]^2+4z^2}\pm\sqrt{\left[\left(\beta-\frac{1}{\beta}\right)y+k\right]^2-4x^2} = h,$$

$$\pm\sqrt{\left[\left(\gamma+\frac{1}{\gamma}\right)z-h\right]^2+4x^2}\pm\sqrt{\left[\left(\gamma-\frac{1}{\gamma}\right)z+g\right]^2-4y^2} = k,$$

in which g, h, k are constants. The first of these orthogonal surfaces reduces to a family of spheres when $g = 0$. If, at the same time, $h = 0$ then the second one will also transform into a spherical family, while the third one will always remain of order four.

"It is characteristic of that example that l, m, n refer to three different arbitrary functions of one variable and their first derivatives. I am curious to know whether other orthogonal systems do not enjoy the same property. By the use of indeterminate coefficients, upon considering (15), (17), (18), one will see that the necessary and sufficient conditions for that to be true are:

$$\frac{\mathrm{R}^{(\beta)}}{\mathrm{Q}^{(\gamma)}} = a, \qquad \frac{\mathrm{P}^{(\gamma)}}{\mathrm{R}^{(\alpha)}} = b, \qquad \frac{\mathrm{Q}^{(\alpha)}}{\mathrm{P}^{(\beta)}} = c,$$

in which a, b, c are three arbitrary functions that are missing α, β, γ, respectively.

Upon taking (17) into account in the interval of transformations, those relations will give:

$$\left(\frac{\mathrm{R}_{\beta}^{(\alpha)}}{\mathrm{R}^{(\alpha)}}\right)_{\alpha} = \left(\frac{\mathrm{P}_{\alpha}^{(\gamma)}}{\mathrm{P}^{(\gamma)}}\right)_{\beta} = \left(\frac{\mathrm{R}^{(\alpha)}\,\mathrm{Q}^{(\gamma)}}{\mathrm{P}^{(\gamma)}}\right)_{\beta} = -\,\mathrm{R}^{(\alpha)}\,\mathrm{R}^{(\beta)}, \ldots$$

One then forms the triple group:

$$(\alpha)\qquad \begin{cases} -\mathrm{R}^{(\alpha)}\,\mathrm{R}^{(\beta)} = (\ln \mathrm{R}^{(\alpha)})_{\alpha\beta} = (\ln \mathrm{R}^{(\beta)})_{\alpha\beta} = (\ln \mathrm{Q}^{(\gamma)})_{\alpha\beta}, \\ -\mathrm{Q}^{(\alpha)}\mathrm{Q}^{(\gamma)} = (\ln \mathrm{Q}^{(\gamma)})_{\alpha\gamma} = \cdots, \\ -\mathrm{P}^{(\beta)}\mathrm{P}^{(\gamma)} = (\ln \mathrm{P}^{(\beta)})_{\beta\gamma} = \cdots \end{cases}$$

From a comparison of the logarithmic terms, one concludes, with a little attention that one can adopt in the most general form possible:

$$(\alpha')\qquad \mathrm{P}^{(\beta)} = \theta C, \quad \mathrm{P}^{(\gamma)} = \theta B', \quad \mathrm{Q}^{(\gamma)} = \theta A, \quad \mathrm{Q}^{(\alpha)} = \theta C', \quad \mathrm{R}^{(\alpha)} = \theta B, \quad \mathrm{R}^{(\beta)} = \theta A',$$

in which θ is an entirely indeterminate function, and A, A', … are arbitrary functions of the same type as a, b, c.

(17), (18) now generally provide the obvious combinations:

$$(\beta)\qquad \mathrm{P}^{(\gamma)}\,\mathrm{R}_\alpha^{(\beta)} - \mathrm{R}^{(\beta)}\,\mathrm{P}_\alpha^{(\gamma)} + \mathrm{R}^{(\alpha)}\,\mathrm{P}_\beta^{(\gamma)} - \mathrm{P}^{(\gamma)}\,\mathrm{R}_\beta^{(\alpha)} + \mathrm{P}^{(\beta)}\,\mathrm{R}_\gamma^{(\alpha)} - \mathrm{R}^{(\alpha)}\,\mathrm{P}_\gamma^{(\beta)} = 0,$$

and two other analogous ones:

$$(g)\qquad \begin{cases} \mathrm{P}^{(\beta)}\mathrm{P}_\alpha^{(\gamma)} - \mathrm{P}^{(\gamma)}\mathrm{P}_\alpha^{(\beta)} = \mathrm{Q}^{(\gamma)}\mathrm{Q}_\beta^{(\alpha)} - \mathrm{Q}^{(\alpha)}\mathrm{Q}_\beta^{(\gamma)} \\ \quad = \mathrm{R}^{(\alpha)}\mathrm{R}_\gamma^{(\beta)} - \mathrm{R}^{(\beta)}\mathrm{R}_\gamma^{(\alpha)} = \mathrm{P}^{(\beta)}\mathrm{Q}^{(\gamma)}\mathrm{R}^{(\alpha)} + \mathrm{P}^{(\gamma)}\mathrm{Q}^{(\alpha)}\mathrm{R}^{(\beta)}, \end{cases}$$

which, abstracting from the last expression in (γ), has the property that it remains absolutely the same when the letters that appear in it are multiplied by the same arbitrary factor.

Substituting the preceding values of $\mathrm{P}^{(\beta)}$, $\mathrm{P}^{(\gamma)}$, … in (β) will yield:

$$\frac{A'_\gamma}{A_\beta} = \frac{B}{C'}, \qquad \frac{B'_\alpha}{B_\gamma} = \frac{C}{A'}, \qquad \frac{C'_\beta}{C_\alpha} = \frac{A}{B'},$$

so, upon differentiating with respect to α, β, γ, one will conclude these more inclusive forms:

$$A = \mathcal{C}_2\,\mathcal{B}, \qquad B = \mathcal{A}_2\,\mathcal{C}, \qquad C = \mathcal{B}_2\,\mathcal{A},$$
$$B' = \mathcal{C}_2\,\mathcal{A}_1, \qquad C' = \mathcal{A}_2\,\mathcal{B}_1, \qquad A' = \mathcal{B}_2\,\mathcal{C}_1,$$

in which $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{A}_2$ are arbitrary functions of only α, etc.; however, the complete verification of the undifferentiated equations will demand that:

$$\frac{\mathcal{B}'}{\mathcal{B}_1\mathcal{B}_2} = \frac{\mathcal{C}'_1}{\mathcal{C}\,\mathcal{C}_2} = g, \qquad \frac{\mathcal{C}'}{\mathcal{C}_1\mathcal{C}_2} = \frac{\mathcal{A}'_1}{\mathcal{A}\,\mathcal{A}_2} = h, \qquad \frac{\mathcal{A}'}{\mathcal{A}_1\mathcal{A}_2} = \frac{\mathcal{B}'_1}{\mathcal{B}\,\mathcal{B}_2} = k,$$

in which g, h, k are constants that one can obviously suppose to be equal to unity. Those equations will yield $\mathcal{A}\mathcal{A}' - \mathcal{A}_1\mathcal{A}'_1 = 0$, … The three constants that the integration of the latter introduces must be equal in order for the three values of θ that are deduced from (γ) to coincide. From that, it is obvious that one can take:

$$\mathcal{A} = \alpha + \frac{1}{\alpha}, \qquad \mathcal{A}_1 = -\alpha + \frac{1}{\alpha}, \qquad \mathcal{A}_2 = -\frac{1}{\alpha}, \ldots,$$

and consequently:

$$\theta = \frac{2\alpha\beta\gamma}{1+\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2}.$$

One then comes back to the preceding example.

If one eliminates the arbitrary functions whose derivative is missing from the expression for x, y, z that relate to this example, when considered pair-wise, then one will see that all of the lines of curvature of the orthogonal surfaces will be planar. If one would like to look for all the similar systems upon starting with the equations for the rotations (17), (18) then one will first get:

$$\frac{\mathrm{R}^{(\alpha)}}{\mathrm{Q}^{(\alpha)}} = a, \qquad \frac{\mathrm{P}^{(\beta)}}{\mathrm{R}^{(\beta)}} = b, \qquad \frac{\mathrm{Q}^{(\gamma)}}{\mathrm{P}^{(\gamma)}} = c, \qquad (\textit{see } \S\ \mathrm{IX})$$

to express the idea that all of the lines of curvature are planar; one will find two groups that are analogous to (α) and (α'), except that one must switch the upper indices for $\mathrm{P}^{(\beta)}$, $\mathrm{P}^{(\gamma)}$, … One must then begin to specialize the arbitrary functions A, A', … as much as possible by means of the first two of (γ). By the consideration of some third-order linear equations and ordinary differential equations, one will get some forms for the A, A', … that are comparatively much-reduced. One will continue to circumscribe them by the group (β); however, I shall suppress that analysis, which demands attending to some details that are a little tricky."

§ VII. – *Isothermal triply-orthogonal system.*

Several eminent geometers have sought to simplify (by some considerations that were borrowed chiefly from infinitesimal geometry) the method by means of which the illustrious author of *Coordinées curvilignes* has shown that the ellipsoidal system is the only triply-orthogonal system that is isothermal. The importance of the subject has led me to indicate some modifications that seem to me to give that method all of the analytical rigor and simplicity that one might desire.

The isothermal condition (*Coordinées curvilignes*, pp. 95) gives:

$$l = BC, \qquad m = AC, \qquad n = AB,$$

in which A, B, C are arbitrary functions that are missing α, β, γ, respectively. The relations (17^*) will then give:

$$A = \frac{B}{B_\gamma} A_\gamma + \frac{C}{C_\gamma} A_\beta , \quad B = \frac{A}{A_\gamma} B_\gamma + \frac{C}{C_\alpha} A_\alpha , \quad C = \frac{A}{A_\beta} C_\beta + \frac{B}{B_\alpha} C_\alpha ,$$

in which the variables that are used as indices indicate partial derivatives with respect to those variables, as always.

Upon differentiating the first one with respect to α twice, one will deduce that:

$$0 = \left(\frac{B}{B_\gamma}\right)_\alpha A_\gamma + \left(\frac{C}{C_\gamma}\right)_\alpha A_\beta , \quad 0 = \left(\frac{B}{B_\gamma}\right)_{\alpha\alpha} A_\gamma + \left(\frac{C}{C_\gamma}\right)_{\alpha\alpha} A_\beta ,$$

so, upon completely excluding the hypothesis that either A_β or A_γ is zero:

$$\frac{\left(\frac{B}{B_\gamma}\right)_{\alpha\alpha}}{\left(\frac{B}{B_\gamma}\right)_\alpha} = \frac{\left(\frac{C}{C_\gamma}\right)_{\alpha\alpha}}{\left(\frac{C}{C_\gamma}\right)_\alpha} = \frac{d \ln \chi'(\alpha)}{d\alpha} ,$$

in which the first two ratios must be independent of β and γ, respectively. From that:

$$\frac{B}{B_\gamma} = f(\gamma)\, \chi(\alpha) + f_1(\gamma), \qquad \frac{C}{C_\gamma} = \mathrm{f}(\beta)\, \chi(\alpha) + \mathrm{f}_1(\beta),$$

in which the *f* and *c* are arbitrary functions. Upon rearranging, it will result that:

$$0 = A_\gamma f(\gamma) + A_\beta \,\mathrm{f}(\beta), \qquad A = A_\gamma f_1(\gamma) + A_\beta \,\mathrm{f}_1(\beta).$$

Upon setting:

$$\int \frac{d\gamma}{f(\gamma)} = \varpi(\gamma), \qquad \int \frac{d\beta}{\mathrm{f}(\beta)} = \omega(\beta), \qquad \varpi - \omega = v$$

and letting Φ denote an arbitrary function, the first one will give:

$$A = \Phi(v), \qquad A_\gamma = \Phi'(v)\, \varpi'(\gamma), \qquad A_\beta = -\Phi'(v)\, \omega'(\beta).$$

Hence, by virtue of the second one:

$$\Psi(v) = \frac{\Phi}{\Phi'} = \varpi'(\gamma)\, f_1(\gamma) - \varpi'(\beta)\, \mathrm{f}_1(\beta),$$

and upon differentiating this alternately by γ and β:

$$\Psi'(v)\, \varpi'(\gamma) = [\varpi'(\gamma)\, f_1(\gamma)]', \quad \Psi'(v)\, \omega'(\beta) = [\omega'(\beta)\, \mathrm{f}_1(\beta)]'.$$

As a result:

$$\Psi'(v) = \frac{[\varpi'(\gamma)\, f_1(\gamma)]'}{\varpi'(\gamma)} = \frac{[\omega'(\beta)\, \mathrm{f}_1(\beta)]'}{\omega'(\beta)} = \frac{1}{k},$$

$$\Psi = \frac{v}{k} + K, \qquad \Phi = h\,(v + H)^k,$$

in which k, H, h, H are arbitrary constants. The necessary form for A is known, and in turn, that of B, C, as well, so the simultaneous verification of (17*) will give, in the most general form possible:

$$A^2 = g\,(\mathcal{B} - \mathcal{C})^p, \qquad B^2 = g_1\,(\mathcal{C} - \mathcal{A})^p, \qquad C^2 = g_2\,(\mathcal{A} - \mathcal{B})^p,$$

in which g, g_1, g_2, p are arbitrary constants. There is a second, exponential, form that corresponds to an infinite k, but one can see with no extra effort that it cannot verify (18*).

With those values, the first of (18*) will become:

$$(\mathcal{A}) \quad \left\{ \begin{aligned} &2(\mathcal{B}-\mathcal{C})^p\, g\, \mathcal{A}'' - 2(\mathcal{C}-\mathcal{A})^p\, g_1\, \mathcal{B}'' = \left(\frac{2}{\mathcal{A}-\mathcal{B}} - \frac{p}{\mathcal{C}-\mathcal{A}}\right)(\mathcal{B}-\mathcal{C})^p\, g\, \mathcal{A}'^2 \\ &\quad + \left(\frac{2}{\mathcal{A}-\mathcal{B}} - \frac{p}{\mathcal{B}-\mathcal{C}}\right)(\mathcal{C}-\mathcal{A})^p\, g_1\, \mathcal{B}'^2 + \frac{p(\mathcal{A}-\mathcal{B})}{(\mathcal{B}-\mathcal{C})(\mathcal{C}-\mathcal{A})}(\mathcal{A}-\mathcal{B})^p\, g_2\, \mathcal{C}'^2. \end{aligned} \right.$$

If one adds this to two other equations that are obtained by a circular permutation of the letters then one will get:

$$(\mathcal{B}) \qquad (p-1)\,[(\mathcal{B}-\mathcal{C})^{p+2}\, g\, \mathcal{A}'^2 + (\mathcal{C}-\mathcal{A})^{p+2}\, g_1\, \mathcal{B}'^2 + (\mathcal{A}-\mathcal{B})^{p+2}\, g_2\, \mathcal{C}'^2] = 0.$$

The hypothesis that the second factor will be zero when one sets:

$$g\, \mathcal{A}'^2 = \mathrm{U}, \qquad g_1\, \mathcal{B}'^2 = \mathrm{V}, \qquad g_2\, \mathcal{C}'^2 = \mathrm{W},$$

and that one takes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ to be independent variables will give, upon differentiating twice with respect to $\mathcal{A}$:

$$(\mathcal{B}-\mathcal{C})^{p+2}\, \mathrm{U} + (\mathcal{C}-\mathcal{A})^{p+2}\, \mathrm{V} + (\mathcal{A}-\mathcal{B})^{p+2}\, \mathrm{W} = 0,$$

$$(\mathcal{B}-\mathcal{C})^{p+2}\, \frac{d\mathrm{U}}{d\mathcal{A}} + (p+2)(\mathcal{C}-\mathcal{A})^{p+1}\, \mathrm{V} + (p+2)(\mathcal{A}-\mathcal{B})^{p+1}\, \mathrm{W} = 0,$$

$$(\mathcal{B} - \mathcal{C})^{p+2} \frac{d^2\mathrm{U}}{d\mathcal{A}^2} + (p + 2)(p + 1)(\mathcal{C} - \mathcal{A})^p \,\mathrm{V} + (p + 2)(p + 1)(\mathcal{A} - \mathcal{B})^p \,\mathrm{W} = 0.$$

If one supposes that $p = -2$ then the first of these equations will yield constant values for U, V, W that verify equations ($\mathcal{A}$), but correspond to an imaginary isothermal system that one can consequently reject. On the other hand, one recognizes that the assumption that $p = -1$, which will give $\frac{d^2\mathrm{U}}{d\mathcal{A}^2} = 0$, cannot agree with ($\mathcal{A}$). Since $(p + 1)$ and $(p + 2)$ are non-zero, the elimination of V, W from the preceding three equations will give a second-order equation in U that will lead to the inadmissible value U = 0. Equation ($\mathcal{B}$) will then demand perforce that one must take $p = 1$. Upon recalling that:

$$g\,\mathcal{A}'^2 = \mathrm{U}, \qquad 2\,g\,\mathcal{A}'\,\mathcal{A}'' = \frac{d\mathrm{U}}{d\alpha}, \qquad \text{so} \qquad 2\,g\,\mathcal{A}'' = \frac{d\mathrm{U}}{d\mathcal{A}},$$

equation ($\mathcal{A}$) will then become:

$$(\mathcal{B} - \mathcal{C})\frac{d\mathrm{U}}{d\mathcal{A}} - (\mathcal{C} - \mathcal{A})\frac{dV}{d\mathcal{B}}$$

$$= \left(\frac{2}{\mathcal{B}-\mathcal{C}} - \frac{1}{\mathcal{C}-\mathcal{A}}\right)(\mathcal{B} - \mathcal{C})\,\mathrm{U} + \left(\frac{2}{\mathcal{A}-\mathcal{B}} - \frac{1}{\mathcal{B}-\mathcal{C}}\right)(\mathcal{C} - \mathcal{A})\,\mathrm{V} + \frac{(\mathcal{A}-\mathcal{B})^2}{(\mathcal{B}-\mathcal{C})(\mathcal{C}-\mathcal{A})}\,\mathrm{W}.$$

If one isolates W and differentiates three times in turn with respect to $\mathcal{A}$ then after suppressing the factor $(\mathcal{B} - \mathcal{C})$ that the first differentiation introduces, one will get

$$(\mathcal{A} - \mathcal{B})^2 \frac{d^2\mathrm{U}}{d\mathcal{A}^2} - 4\,(\mathcal{A} - \mathcal{B})\,\frac{d\mathrm{U}}{d\mathcal{A}} + 6\mathrm{U} = 2\,(\mathcal{A} - \mathcal{B})\,\frac{d\mathrm{V}}{d\mathcal{B}} + 6\mathrm{V},$$

$$(\mathcal{A} - \mathcal{B})^2 \frac{d^3\mathrm{U}}{d\mathcal{A}^3} - 2\,(\mathcal{A} - \mathcal{B})\frac{d^2\mathrm{U}}{d\mathcal{A}^2} + 2\frac{d\mathrm{U}}{d\mathcal{A}} = 2\,\frac{d\mathrm{V}}{d\mathcal{B}},$$

$$(\mathcal{A} - \mathcal{B})^2 \frac{d^4\mathrm{U}}{d\mathcal{A}^4} = 0.$$

Therefore, since λ, λ_1, λ_2 are arbitrary constants, one will have, upon combining these, or by symmetry:

$$\mathrm{U} = \mathcal{A}^3 + \lambda\,\mathcal{A}^2 + \lambda_1\,\mathcal{A} + \lambda_2\,,$$

$$\mathrm{V} = \mathcal{B}^3 + \lambda\,\mathcal{B}^2 + \lambda_1\,\mathcal{B} + \lambda_2\,,$$

$$\mathrm{W} = \mathcal{C}^3 + \lambda\,\mathcal{C}^2 + \lambda_1\,\mathcal{C} + \lambda_2\,,$$

and in turn, if U, V, W have those values:

$$d\alpha = \frac{d\mathcal{A}}{\sqrt{\dfrac{\mathrm{U}}{g}}}, \quad d\beta = \frac{d\mathcal{B}}{\sqrt{\dfrac{\mathrm{V}}{g_1}}}, \quad d\gamma = \frac{d\mathcal{C}}{\sqrt{\dfrac{\mathrm{W}}{g_2}}}.$$

As for the final determination of *x*, *y*, *z*, nothing can replace Lamé's calculation.

§ VIII. – *Orthogonal system that is deduced from an elliptic system.*

When one knows a particular orthogonal system, one can deduce the cosines *a*, *b*, *c*, …, and the $\mathrm{P}^{(\beta)}$, $\mathrm{P}^{(\gamma)}$, … in terms of α, β, γ. If one substitutes those particular values of P, Q, R into equations (15), and one integrates that latter equations while taking *l*, *m*, *n* to be unknowns then one will get the expressions for those unknowns with three arbitrary functions, in general. Upon preserving the particular values of *a*, *b*, *c*, …, one will then have the *x*, *y*, *z* by quadratures. Here is an example that is suitable to exhibit that remark: Upon setting:

$$G = \frac{-4}{(h-k)(k-j)(j-h)},$$

in which *h*, *k*, *j* are constants, and supposing that $\alpha > \beta > \gamma$, the formulas that relate to the elliptic system can be written:

$$\left\{\begin{aligned} x_1 &= \sqrt{G(k-j)(\alpha-h)(\beta-h)(\gamma-h)}, \\ y_1 &= \sqrt{G(j-h)(\alpha-k)(\beta-k)(\gamma-k)}, \\ z_1 &= \sqrt{G(h-k)(\alpha-j)(\beta-j)(\gamma-j)}, \end{aligned}\right. \qquad \left\{\begin{aligned} l_1 &= \sqrt{(\alpha-\beta)(\alpha-\gamma)}, \\ m_1 &= \sqrt{(\alpha-\beta)(\beta-\gamma)}, \\ n_1 &= \sqrt{(\alpha-\gamma)(\beta-\gamma)}. \end{aligned}\right.$$

From (15), the corresponding values of P, Q, R will be:

$$\mathrm{R}^{(\alpha)} = \frac{1}{2(\alpha-\beta)}\sqrt{\frac{\alpha-\gamma}{\beta-\gamma}}, \qquad \mathrm{R}^{(\beta)} = \frac{1}{2(\alpha-\beta)}\sqrt{\frac{\beta-\gamma}{\alpha-\gamma}}, \qquad \ldots$$

If one substitutes these values into the same equations (15), (20), when one currently regards *l*, *m*, *n* as three unknown functions, then (20) will give:

$$\frac{d^2 l}{d\alpha\, d\beta} + \frac{\alpha+\beta-2\gamma}{2(\alpha-\beta)(\alpha-\gamma)}\frac{dl}{d\beta} + \frac{l}{4(\alpha-\beta)^2} = 0$$

for the determination of *l*.

If one sets $l = l_1\, u$, and ultimately $m = m_1\, v$, $n = n_1\, w$, then upon observing that l_1 is one of their common solutions:

$$2\,(\alpha-\beta)\frac{d^2u}{d\alpha\,d\beta}+3\frac{du}{d\beta}-\frac{du}{d\alpha} = 0,$$

$$2\,(\alpha-\gamma)\frac{d^2u}{d\alpha\,d\gamma}+3\frac{du}{d\gamma}-\frac{du}{d\alpha} = 0.$$

These equations will admit the simple solution:

$$\frac{T}{(\alpha-t)\sqrt{(\alpha-t)(\beta-t)(\gamma-t)}},$$

in which T, t are arbitrary constants. Equations (15) yield simple, corresponding values for V, W, and one will have the very general solution:

$$u = \sum\frac{T}{(\alpha-t)\sqrt{(\alpha-t)(\beta-t)(\gamma-t)}},$$

$$v = \sum\frac{T}{(\beta-t)\sqrt{(\alpha-t)(\beta-t)(\gamma-t)}},$$

$$w = \sum\frac{T}{(\gamma-t)\sqrt{(\alpha-t)(\beta-t)(\gamma-t)}},$$

in which the $\sum$ extends over all values that one would like to give to T, t. In order to find x, y, z, one takes:

$$a = \frac{1}{l_1}\frac{dx_1}{d\alpha} = \frac{1}{2l_1}\sqrt{\frac{G(k-j)(\beta-h)(\gamma-h)}{(\alpha-h)}}, \qquad b = \ldots,$$

and upon integrating the exact differentials (c), § III, one will find that:

$$x = \sqrt{G(k-j)(\alpha-h)(\beta-h)(\gamma-h)}\sum\frac{T}{(h-t)\sqrt{(\alpha-t)(\beta-t)(\gamma-t)}},$$

$$y = \sqrt{G(k-h)(\alpha-k)(\beta-k)(\gamma-k)}\sum\frac{T}{(k-t)\sqrt{(\alpha-t)(\beta-t)(\gamma-t)}},$$

$$z = \sqrt{G(h-k)(\alpha-j)(\beta-j)(\gamma-j)}\sum\frac{T}{(j-t)\sqrt{(\alpha-t)(\beta-t)(\gamma-t)}}.$$

The systems that are included in these formulas, which are infinite in number, obviously have the same spherical image as the elliptic system.

§ IX. – *Some mappable surfaces.*

I shall introduce this last section in order to connect the general theory of curvilinear coordinates with the partial theory of the deformation of surfaces, which is impossible to deduce from Lamé's formulas. If one introduces the hypothesis that $n = 0$ into equations (15), § V then they will become $l_\gamma = 0$, $m_\gamma = 0$, $\mathrm{P}^{(\gamma)} = 0$, $\mathrm{Q}^{(\gamma)} = 0$, $\mathrm{R}^{(\gamma)} = 0$, and:

$$\mathrm{R}^{(\alpha)} = -\frac{l_\beta}{m}, \quad \mathrm{R}^{(\beta)} = \frac{m_\alpha}{l}, \quad m\,\mathrm{P}^{(\alpha)} + l\,\mathrm{Q}^{(\beta)} = 0. \tag{24}$$

One can no longer make use of relations (17), (18), or (17*), (18*), which were established under the express condition that $\mathrm{P}^{(\alpha)}$, $\mathrm{Q}^{(\alpha)}$, $\mathrm{R}^{(\alpha)}$ are equal to zero. However, upon referring to the identities (8), into which that assumption was not introduced, and in which one made $l = 0$, $m = 0$, $v = 0$, one will have, upon considering (24):

$$\left\{\begin{aligned}
&\mathrm{Q}^{(\alpha)}\,\mathrm{P}^{(\beta)} - \mathrm{P}^{(\alpha)}\,\mathrm{Q}^{(\beta)} = \left(\frac{l_\beta}{m}\right)_\beta + \left(\frac{m_\alpha}{l}\right)_\alpha, \\
&\mathrm{Q}^{(\alpha)}_\beta - \mathrm{Q}^{(\beta)}_\alpha + \frac{m_\alpha}{l}\,\mathrm{P}^{(\alpha)} + \frac{l_\beta}{m}\,\mathrm{P}^{(\beta)} = 0, \\
&\mathrm{P}^{(\alpha)}_\beta - \mathrm{P}^{(\beta)}_\alpha + \frac{l_\beta}{m}\,\mathrm{Q}^{(\beta)} - \frac{m_\beta}{l}\,\mathrm{Q}^{(\alpha)} = 0.
\end{aligned}\right. \tag{25}$$

Equations (24), (25) correspond to the problem of the *mapping* of surfaces when l and m are given in terms of α, β. In the particular case of $l = 1$, upon eliminating $\mathrm{Q}^{(\beta)}$, one will get back to the Bour's *fundamental* equations (Journal de l'École Polytechnique, Cahier XXXIX).

When one infers the values of $\mathrm{P}^{(\alpha)}$, $\mathrm{P}^{(\beta)}$, $\mathrm{Q}^{(\alpha)}$, $\mathrm{Q}^{(\beta)}$ from these equations, one will obtain the cosines by integrating equations (7), which are:

$$\left\{\begin{aligned}
&\frac{da}{d\alpha} = -b\frac{l_\beta}{m} - c\,\mathrm{Q}^{(\alpha)}, \quad &&\frac{da}{d\beta} = b\frac{m_\alpha}{l} - c\,\mathrm{Q}^{(\beta)}, \\
&\frac{db}{d\alpha} = c\,\mathrm{P}^{(\alpha)} + a\frac{l_\beta}{m}, \quad &&\frac{db}{d\beta} = c\,\mathrm{P}^{(\beta)} - a\frac{m_\alpha}{l}, \\
&\frac{dc}{d\alpha} = a\,\mathrm{Q}^{(\alpha)} - b\,\mathrm{P}^{(\alpha)}, \quad &&\frac{dc}{d\beta} = a\,\mathrm{Q}^{(\beta)} - b\,\mathrm{P}^{(\beta)},
\end{aligned}\right. \tag{26}$$

here, and which were considered more generally in § IV. One will then get x, y, z from equations (c), § III.

If one rids oneself of any sort of auxiliary variable and determines x, for example, *directly* then one will only have to eliminate $\mathrm{P}^{(\alpha)}$, $\mathrm{P}^{(\beta)}$, $\mathrm{Q}^{(\alpha)}$, $\mathrm{Q}^{(\beta)}$ from (26) and the first of (25). One will then get:

(27) $$c^2\left[\left(\frac{l_\beta}{m}\right)_\beta+\left(\frac{m_\beta}{l}\right)_\alpha\right]+\frac{da}{d\alpha}\frac{db}{d\beta}-\frac{da}{d\beta}\frac{db}{d\alpha}+\frac{m_\beta}{l}\left(a\frac{da}{d\alpha}+b\frac{db}{d\alpha}\right)+\frac{l_\beta}{m}\left(a\frac{da}{d\beta}+b\frac{db}{d\beta}\right)=0,$$

in which one replaces c^2 with $1 - a^2 - b^2$, and a and b with $\frac{1}{l}\frac{dx}{d\alpha}$ and $\frac{1}{m}\frac{dx}{d\beta}$, which will yield the second-order equation that any of the coordinates x, y, z must verify.

Another mode of solution will result from the use of Euler's formulas, which will transform (24) into:

$$\frac{d\varphi}{d\alpha}+\cos\theta\frac{d\psi}{d\alpha}=-\frac{l_\beta}{m},$$
$$\frac{d\varphi}{d\beta}+\cos\theta\frac{d\psi}{d\beta}=\frac{m_\alpha}{l},$$
$$m\left(\cos\varphi\frac{d\theta}{d\alpha}+\sin\varphi\sin\theta\frac{d\psi}{d\alpha}\right)=l\left(\sin\varphi\frac{d\theta}{d\beta}-\cos\varphi\sin\theta\frac{d\psi}{d\beta}\right).$$

Equations (25) will be simple identities then. When φ, ψ, θ have been determined in terms of α, β by those three equations, Euler's formulas will give the cosines immediately, and the quadratures (c), § III will finally yield x, y, z (*).

"From the equations for the lines of curvature of any of the surfaces considered, namely:

$$(l + \Theta\, Q^{(\alpha)})\, d\alpha + \Theta\, Q^{(\beta)}\, d\beta = 0,$$
$$(m - \Theta\, P^{(\alpha)})\, d\beta - \Theta\, P^{(\beta)}\, d\alpha = 0,$$

one will have:

$$(P^{(\alpha)}\, Q^{(\beta)} - P^{(\beta)}\, Q^{(\alpha)})\,\Theta^2 + (m\, Q^{(\alpha)} - l\, P^{(\beta)}) + ml = 0.$$

Upon comparing this with the first of (25), one will conclude that the product of the inverses of the principal radii of curvature will be:

$$\frac{1}{ml}\left[\left(\frac{l_\beta}{m}\right)_\beta+\left(\frac{m_\alpha}{l}\right)_\alpha\right].$$

Upon denoting the angles of contingency and torsion, and the inclination of the curve β = const. to the tangent plane by $\omega^{(\alpha)}\, d\alpha$, $\upsilon^{(\alpha)}\, d\alpha$, $\varepsilon^{(\alpha)}$, resp., and letting $\omega^{(\beta)}\, d\beta$, $\upsilon^{(\beta)}\, d\beta$, $\varepsilon^{(\beta)}$ denote the analogous quantities for the curve α = const. ($\varepsilon^{(\alpha)}$, $\varepsilon^{(\beta)}$ are measured by starting from the corresponding osculating plane and supposing that they turn around the tangent in the direct sense), it will be easy to see geometrically or analytically that:

$$P^{(\alpha)} = \upsilon^{(\alpha)} + \varepsilon_\alpha^{(\alpha)}, \qquad P^{(\beta)} = \omega^{(\beta)}\sin\varepsilon^{(\beta)},$$

(*) I have been informed that the results above were established by Codani, but I do not know in what precise era.

$$Q^{(\alpha)} = \omega^{(\alpha)} \sin \varepsilon^{(\alpha)}, \qquad Q^{(\beta)} = \upsilon^{(\beta)} + \varepsilon_\beta^{(\beta)},$$
$$R^{(\alpha)} = \omega^{(\alpha)} \cos \varepsilon^{(\alpha)}, \qquad R^{(\beta)} = \omega^{(\beta)} \cos \varepsilon^{(\beta)}.$$

When α = const., β = const. correspond to the lines of curvature for an individual surface, it will result from the equations above that relate to those lines that $P^{(\alpha)} = 0$, $Q^{(\beta)} = 0$. In that case, $\upsilon^{(\alpha)} + \varepsilon_\alpha^{(\alpha)} = 0$, in such a way that if the line is planar then the ratio $R^{(\alpha)} / Q^{(\alpha)}$ will not depend upon α.

Final remark. – If one must establish the simplest possible of all the various formulas that relate to either mappable surface or triply-orthogonal systems then it will obviously suffice to take the classical formulas of mechanics that give the variations of the cosines by means of the components of rotations. One will write down those equations twice in the first case and three times in the second, with the components of the two rotations (the two or three groups (7*), § IV, for example). One writes down the group (8*) (in the same section), which I presume to have been established, first, in full generality, and then upon making the moving rectangular axes coincide with the tangents to the orthogonal trajectories of the surface (the third will coincide with the normal), one will recover the formulas of the present paragraph, whereas upon making them coincide with the tangents to the curves of intersection of the three orthogonal surfaces, one will obtain the formulas that relate to that theory. The latter path is the one that was followed by Bonnet, while immediately employing Euler's formulas, in which it seems to me that he has disguised somewhat the role of partial rotations whose analytical composition (which Lamé had neglected) has, in the other hand, left the three geometrically-obvious conditions $P^{(\alpha)} = 0$, $Q^{(\beta)} = 0$, $R^{(\gamma)} = 0$ fruitless for that celebrated geometer."