"Sur les lignes de courbure of the surface des ondes," Ann. di. mat. pura ed appl. **2** (1859), 278-285; followed by "Osservazioni sulla medesima quistione," *ibid.*, 285-287.

On the lines of curvature of the wave surface

By EDOUARD COMBESCURE

Translated by D. H. Delphenich

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1. The equations of the normal to the point x, y, z of an arbitrary surface can be written:

$$x - X = R\lambda, \qquad y - Y = R\mu, \qquad z - Z = R\nu\,;$$

λ, μ, ν denote the direction cosines of that line, and R denotes the distance from the point x, y, z to another arbitrary point X, Y, Z on the interior part of that same line. If the latter point is such that when one proceeds along the surface, the infinitely-close normal meets the first one at the same point X, Y, Z then the preceding equations must persist when one varies x, y, z infinitely little, and in turn, λ, μ, ν, while the other quantities R, X, Y, Z remain constant. One will then have the following equations for the lines of curvature:

$$(1) \qquad dx = R\,d\lambda, \qquad dy = R\,d\mu, \qquad dz = R\,d\nu.$$

These equations obviously reduce to two, due to the relation:

$$\lambda\,d\lambda + \mu\,d\mu + \nu\,d\nu = 0,$$

and when one eliminates R, they will reproduce the usual equation for the lines of curvature. However, it can be advantageous to keep their present form, since it seems to relate to the particular question that I have in mind and in all questions where the equation of the surface refers to the coordinates x, y, z symmetrically.

2. Upon setting:

$$(2) \qquad \begin{gathered} x^2 + y^2 + z^2 = \alpha, \qquad ax^2 + by^2 + cz^2 = \beta, \\ a\,(b + c)\,x^2 + b\,(c + a)\,y^2 + c\,(a + b)\,z^2 = \gamma, \end{gathered}$$

the equation for the wave surface will become the following one: (see Lamé, *Th. de l'Elasticité*, page 245):

$$f = \alpha\beta - \gamma + abc = 0,$$

in which a, b, c are written here instead of a^2, b^2, c^2, for simplicity [†]. Upon considering a, b, c to be positive or negative in the relations (2), one will give rise to two new varieties of surfaces that relate to the wave surface, to some extent, in the same way that the hyperboloids with one and two sheets relate to the ellipsoid, at least, with certain affectations of form, so I shall not presently occupy myself with anything more than the fourth-order surfaces, which are such that each of the three coordinate planes cut them along two disjoint second-order curves that are situated in an arbitrary manner in those plane and whose study is not devoid of interest.

One deduces from the preceding equations that:

$$\frac{df}{dx} = \alpha\frac{d\beta}{dx} + \beta\frac{d\alpha}{dx} - \frac{dy}{dx} = 2x\,[a\alpha + \beta - a\,(b + c)],$$

and as a result:

$$(3)\qquad \begin{cases} \lambda = \dfrac{x}{D}[a\alpha + \beta - a(b+c)], \\[2mm] \mu = \dfrac{y}{D}[b\alpha + \beta - b(c+a)], \\[2mm] \nu = \dfrac{\alpha}{D}[c\alpha + \beta - c(a+b)], \end{cases}$$

in which, one necessarily has [‡]:

$$D^2 = \mathrm{S}\; x^2\,[a\alpha + \beta - a\,(b + c)]^2.$$

If one sets:

$$a + b + c = A, \qquad ab + bc + ca = B, \qquad abc = C,$$

to abbreviate, and keeps in mind the identities:

$$a^2 = aA - a\,(b + c), \qquad a^2\,(b + c) = Ba - C,$$
$$a^2\,(b + c)^2 = Ba\,(b + c) + Ca - AC,$$

which implies:

$$(\alpha)\qquad \mathrm{S}\; a^2x^2 = A\beta - \gamma, \qquad \mathrm{S}\; a^2(b + c)\,x^2 = B\beta - C\alpha, \qquad \mathrm{S}\; a^2(b + c)\,x^2 = B\gamma + C\beta - AC\alpha,$$

then one will easily find, upon taking the relation $\gamma = \alpha\beta + C$ into account, that:

$$D^2 = (\alpha\beta + C)\,(\beta - \alpha^2 + A\alpha - B).$$

It results immediately from the preceding expressions for the cosines, while keeping the identities (α) in mind, that:

[†] Translator: This practice is not generally advisable!
[‡] The "S" notation refers to a type of summation whose precise definition is not always clear.

$$\mathrm{S}\,\lambda x = \frac{\alpha\beta - C}{D}, \qquad \mathrm{S}\,a\lambda x = \frac{\beta}{D}(\beta - \alpha^2 + A\alpha - B),$$

or, upon choosing a definite sign for D:

$$\mathrm{S}\,\lambda x = \sqrt{\frac{\alpha\beta - C}{\beta - \alpha^2 + A\alpha - B}}, \qquad \mathrm{S}\,a\lambda\,x = \beta\sqrt{\frac{\beta - \alpha^2 + A\alpha - B}{\alpha\beta - C}}.$$

If one sets $p = \mathrm{S}\,\lambda x$ then p will represent the length of the perpendicular abscissa from the origin to the tangent plane, and second of the equations above will give:

$$\mathrm{S}\,a\lambda x = \frac{\beta}{p},$$

so

$$\mathrm{S}\,ax\,d\lambda = d\,\frac{\beta}{p} - \mathrm{S}\,a\lambda\,dx.$$

However, from the expressions (3), one will have:

$$\mathrm{S}\,a\lambda\,dx = \frac{1}{2D}\left(\alpha\,d\,\mathrm{S}\,a^2x^2 + \beta\,d\,\mathrm{S}\,a^2x^2 - d\,\mathrm{S}\,a^2(b+c)x^2\right),$$

and consequently, upon taking the relations (α) into account, one will obtain:

$$\mathrm{S}\,a\lambda\,dx = \frac{1}{2D}\left((\beta - \alpha^2 + A\alpha - B)\,d\beta - (\alpha\beta - C)\,d\alpha\right) = \frac{1}{2}\frac{d\beta}{p} - \frac{1}{2}\,p\,d\alpha.$$

Moreover, equations (1) immediately provide:

$$\mathrm{S}\,a\lambda\,dx = R\,\mathrm{S}\,ax\,d\lambda,$$

or:

$$\tfrac{1}{2}\,d\beta = R\,\mathrm{S}\,ax\,d\lambda,$$

so, upon substituting:

$$db = R\left(\frac{d\beta}{p} + p\,d\alpha - 2\beta\frac{dp}{p^2}\right).$$

setting $p^2 = v$, $R/\sqrt{v} = \theta$, and observing that:

$$dp = \mathrm{S}\,x\,d\lambda = \frac{1}{R}\,\mathrm{S}\,x\,dx = \frac{1}{2R}\,d\alpha,$$

one will get the following definitive group that relates to the lines of curvature of the wave surface:

$$(4) \qquad d\alpha = \theta\, dv, \qquad d\beta = \theta\left(v\, d\alpha + \frac{v\, d\beta - \beta\, dv}{v}\right), \qquad \beta = \frac{v(\alpha^2 - A\alpha + B) - C}{v - \alpha},$$

where the last equation is nothing but the one that defines p^2 or v when it is solved for β.

3. If one sets v = const. then it will result from these equations that α = const., and also that β = const. However, the simultaneous hypotheses α = const., v = const. generally determines only a limited number of points on the wave surfaces. For this double hypothesis to be true, it is necessary that it should correspond, in reality, to a line of curvature whose corresponding value of β is indeterminate. That can happen only for three simultaneous values, and in particular $v = \alpha = a, b, c$, which correspond to the three circular sections that are given by the plane coordinates. The three elliptic sections by the same planes must also satisfy equations (4), which is easy to verify. If one actually sets $\beta = bc$ (which will give $x = 0$) then equations (4) will become:

$$d\alpha = \theta\, dv, \qquad v^2\, d\alpha = bc\, dv, \qquad \text{or} \qquad d\,\frac{1}{v} = -\frac{d\alpha}{bc},$$

and the value that satisfies the third of (4) will become $v = \dfrac{bc}{b + c - \alpha}$ here. That particular solution includes the circular solution, because upon making $\alpha = c$ in it, one will get $v = c$. One then knows three particular solutions:

$$v = \frac{bc}{b + c - \alpha}, \qquad v = \frac{ca}{c + a - \alpha}, \qquad v = \frac{ab}{a + b - \alpha}.$$

From what we just said, we see that the developable surface circumscribes the wave surface and has a concentric sphere (which is a surface for which v, and therefore α, is constant) cannot touch the wave surface along a line of curvature of the latter, except under the hypotheses that were pointed out above. This confirms the inexactitude that was pointed out and established beyond any doubt by Bertrand on page 817 of Comptes Rendus (1858) and Brioschi on page 135 of Annali di Matematica (1859); Cayley, in turn, returned to that inexactitude in the May 1859 issue of the Quarterly Journal.

4. If one lets Π denote a symmetric product of three factors then one will deduce from the third of equations (4) that:

$$d\beta = \frac{-dv\,\Pi(\alpha - a) + d\alpha\{(2\alpha - A)v^2 + (B - \alpha^2) - C\}}{(v - \alpha)^2},$$

or rather:

$$d\beta = \frac{-dv\,\Pi(\alpha - a) + d\alpha\{\Pi(v-a) - v(v-\alpha)^2\}}{(v-\alpha)^2},$$

so:

$$d\beta + v\,d\alpha = \frac{d\alpha\,\Pi(v-a) - dv\,\Pi(\alpha - a)}{(v-\alpha)^2}.$$

The first and second of equations (4) give:

$$d\beta + v\,d\alpha = \theta\left\{d\beta + v\,d\alpha + \left(v - \frac{\beta}{v}\right)dv\right\},$$

and upon eliminating θ from the first one:

$$(d\beta + v\,d\alpha)\,(dv - d\alpha) = \left(v - \frac{\beta}{v}\right)\,dv\,d\alpha;$$

moreover:

$$v - \frac{\beta}{v} = \frac{(\alpha v^2 - C)(v-\alpha) - v\Pi(\alpha - a)}{\alpha v(v-\alpha)}.$$

From these values, one will have:

$$\overline{d\alpha}^2\,\Pi(v-a) + \overline{dv}^2\,\Pi(\alpha - a) + d\alpha\,dv\left\{\frac{(v-\alpha)^2(\alpha v^2 - C)}{\alpha v} - \frac{v\,\Pi(\alpha-a) + \alpha\,\Pi(v-a)}{\alpha}\right\} = 0$$

for the isolated equation of the lines of curvature, which reduces to:

$$(5)\quad \overline{d\alpha}^2\,\Pi(v-a) + \overline{dv}^2\,\Pi(\alpha - a) + \frac{d\alpha\,dv}{v}\{\alpha(-2v^3 + Av^2 - C) + Av^3 - 2Bv^2 + 3Cv\} = 0,$$

or, if one prefers, upon setting $v = 1 / u$, to:

$$\overline{d\alpha}^2\,u\,\Pi(1 - au) + \overline{du}^2\,\Pi(\alpha - a) + du\,d\alpha\,\{\alpha(Cu^3 - Au + 2) - 2Cu^2 + 2Bu - A\} = 0.$$

One can consider the Euler equation that relates to the addition of elliptic functions to be a particular case of this or the other more general equation:

$$F(u)\,\overline{d\alpha}^2 + F_1(\alpha)\,\overline{du}^2 + f(u, \alpha)\,du\,d\alpha = 0,$$

in which F, F_1, f denote fourth-degree functions. That agreement and the existence of the particular solutions that I pointed out, as well as another solution that I will soon say a few words about, can lead one to think that the general integral is an algebraic function;

however, until we have more ample information, that must be considered to be a simplistic argument.

Upon dividing the differential equation in α and v above by $\Pi(v - a)$ and then decomposing the coefficient of $d\alpha\, dv$ into simple fractions, one will easily give that equation the following form, whose *superior* and *inferior* symmetry (if I may be permitted to call it that) makes it simple to remember:

$$\left(\frac{d\alpha}{dv}\right)^2 - \left(\mathrm{S}\frac{\alpha - a}{v - a} - \frac{\alpha}{v}\right)\frac{d\alpha}{dv} + \Pi\frac{\alpha - a}{v - a} = 0. \tag{6}$$

If one considers v and α to be the abscissa and ordinate of a plane curve, resp., and one takes three points A, B, C on the bisector of the angle between the positive coordinate axes Ov and $O\alpha$, such that $OA = a\sqrt{2}$, $OB = b\sqrt{2}$, $OC = c\sqrt{2}$, if one lets M denote an arbitrary point of the locus that corresponds to the preceding differential equation, and joins M to the points O, A, B, C then that equation can be written:

$$\left(\frac{d\alpha}{dv}\right)^2 (\tan\varphi_1 + \tan\varphi_2 + \tan\varphi_3 - \tan\varphi) + \frac{d\alpha}{dv} + \tan\varphi_1 \tan\varphi_2 \tan\varphi_3 = 0,$$

in which φ, φ_1, φ_2, φ_3 denote the angles that the four lines subtend with the v-axis. If, in addition, one lets $\tan\omega_1$, $\tan\omega_2$ denote the two values of $d\alpha / dv$ that relate to the point M then one will get the following geometric relations:

$$\tan\omega_1 + \tan\omega_2 + \tan\varphi = \tan\varphi_1 + \tan\varphi_2 + \tan\varphi_3,$$
$$\tan\omega_1 \tan\omega_2 = \tan\varphi_1 \tan\varphi_2 \tan\varphi_3 .$$

That will result in a certain geometric means of constructing the points of the locus of points M.

In the case where $b = a$, equation (6) will become illusory, as far as the lines of curvature of the wave surface are concerned. However, upon imagining that it refers to a plane curve and supposing, in addition, that $c = 0$, it will give:

$$\frac{d\alpha}{dv} = \frac{\alpha - a}{v - a}\left(1 \pm \sqrt{1 - \frac{\alpha}{v}}\right),$$

from which, one will easily deduce that:

$$\frac{\alpha}{v} = \frac{1}{\eta}\left(2 - \frac{1}{\eta}\right) \qquad \text{upon taking} \qquad \eta = H\sqrt{v(a - v)} + \frac{v}{a},$$

in which H is an arbitrary constant. Finally, one can point out the particular solution $\alpha = v$, which verifies equation (6) and has no relationship to the lines of curvature of the wave surface.

5. The equations of the normal will give:

$$X^2 = x^2 - 2R\lambda x + R^2\lambda^2, \qquad \text{etc.},$$

if one then sets:

$$\text{S}\, X^2 = A, \qquad \text{S}\, X^2 = B, \qquad \text{S}\, a\,(b + c)\, X^2 = C,$$

so one will have:

$$A = \alpha - 2p\,R + 3R^2,$$
$$B = \beta - 2R\,\text{S}\, ax\lambda + AR^2,$$
$$C = \gamma - 2R\,\text{S}\, a(b + c)\, x\lambda + 2BR^2.$$

From the identities (α) and the expressions λ, μ, ν, one finds at once that:

$$\text{S}\, a\,(b + c)\,\lambda\, x = \frac{\alpha(\alpha\beta - C) - C(\alpha^2 - A\alpha + B - \beta)}{D} = \beta p + \frac{C}{p};$$

upon recalling the value of S $a\alpha\lambda$ in no. **2**, one will get:

$$A = \alpha - 2v\theta + 3v\theta^2,$$
$$B = \beta - 2\beta\theta + Av\theta^2,$$
$$C = \gamma - 2(\beta v + C)\,\theta + 2Bv\theta^2.$$

Upon combining these with equation (6), or:

$$\theta^2 - \left\{\text{S}\frac{\alpha - a}{v - a} - \frac{\alpha}{v}\right\}\theta + \Pi\frac{\alpha - a}{v - a} = 0$$

and

$$\beta = \frac{v(\alpha^2 - A\alpha + B) - C}{v - \alpha},$$

the elimination of β, α, v, θ from these five equations will give an equation in A, B, C for the locus of centers of principal curvature.

6. Here, I will add some differential relations that can be useful. If one sets:

$$\Delta = (a - b)(b - c)(c - a)$$

then one will know, and one can easily deduce from equations (1), that:

$$x^2 = \frac{b - c}{\Delta}(\alpha - a)(\beta - bc), \qquad y^2 = \frac{c - a}{\Delta}(\alpha - b)(\beta - ca), \qquad z^2 = \frac{a - b}{\Delta}(\alpha - c)(\beta - ab).$$

Upon differentiating these expressions and considering the identities:

$$\text{S}\, a^2\,(b-c) = -\,\Delta, \qquad \text{S}\, bc\,(b-c) = -\,\Delta, \qquad \text{S}\, a\,(b^2-c^2) = \Delta,$$
$$\text{S}\, bc\,(b^2-c^2) = -\,A\,\Delta, \qquad \text{S}\, a^3\,(b-c) = -\,A\,\Delta,$$
$$\text{S}\, b^2c^2\,(b-c) = -\,B\,\Delta, \qquad \text{S}\, a^3\,(b^2-c^2) = -\,B\,\Delta,$$

one will easily find that the expression for the square $\overline{ds}^2$ of an arbitrary curve element that is traced on the wave surface:

$$4\,\overline{ds}^2 = \frac{\alpha^2 - A\alpha + B - \beta}{\Pi(\alpha - a)}\,\overline{d\alpha}^2 + \frac{C - \alpha\beta}{\Pi(\beta - bc)}\,\overline{d\beta}^2 .$$

If one now observes that the equations of the geodesic lines of an arbitrary surface can be written:

$$d^2x = N\lambda\, ds^2, \qquad d^2y = N\mu\, ds^2, \qquad d^2z = N\nu\, ds^2,$$

in which N is an indeterminate, and that here the relations $\text{S}\, x^2 = \alpha$, $\text{S}\, ax^2 = \beta$ will give successively:

$$\text{S}\, x\, dx = \tfrac{1}{2}\, d\alpha, \qquad \text{S}\, x\, d^2x = \tfrac{1}{2}\, d^2\alpha - dz^2,$$
$$\text{S}\, ax\, dx = \tfrac{1}{2}\, d\beta, \qquad \text{S}\, ax\, d^2x = \tfrac{1}{2}\, d^2\beta - \text{S}\, a\, dz^2,$$

then one will first have:

$$\text{S}\, x\, d^2x = N\, ds^2\, \text{S}\, \lambda x = Np\, ds^2,$$
$$\text{S}\, ax\, d^2x = N\, ds^2\, \text{S}\, a\lambda x = N\, \frac{\beta}{p}\, ds^2,$$

and then:

$$\tfrac{1}{2}\, d^2\alpha - ds^2 = N\, p\, ds^2,$$
$$\tfrac{1}{2}\, d^2\beta - \text{S}\, a\, dx^2 = N\, \frac{\beta}{p}\, ds^2,$$

and thus:

$$d^2\alpha - 2\, ds^2 = (d^2\beta - 2\, \text{S}\, a\, dx^2),$$

for the equation of the geodesic lines, in which one substitutes the preceding expressions for ds^2 and $\text{S}\, a\, dx^2$, and in which one then takes whatever one likes to be the independent variable.

Paris, 25 August 1859.

Some observations on the latter question

By Prof. FRANCESCO BRIOSCHI

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1. In an interesting paper by Combescure, it results that if one considers the point of the wave surface whose coordinates are *x*, *y*, *z*, and lets *r* denote the square of the radius vector for that point, and lets *p* denote the square of the length of the perpendicular to the tangent plane then the equation between the variables of the lines of curvature of that surface will be the following one [see equation (5)]:

$$\varphi(p)\left(\frac{dr}{dp}\right)^2 + \varphi(r) + \frac{1}{p}\{\varphi'(p)\ p\,(p-r) - \varphi(p)(3p-r\}\frac{dr}{dp} = 0, \tag{1}$$

in which:

$$\varphi(p) = (t-a)(t-b)(t-c).$$

One easily passes from the equation above to:

$$\varphi(p)\left(\frac{dr}{dp}\right)^2 - \varphi(r) + 2\varphi(p)\,(p-r)\,\frac{d}{dp}\left(\log\frac{(p-r)\sqrt{p}}{\sqrt{\varphi(p)}}\right)\cdot\frac{dr}{dp} = 0,$$

and from this, if one replaces *r* with the variable ω, which is coupled to the latter by the equation:

$$\log\frac{(p-r)\sqrt{p}}{\sqrt{\varphi(p)}} = \omega,$$

to:

$$\psi(p)\left(\frac{d\omega}{dp}\right)^2 - (e^{\omega} + e^{-\omega}) + \frac{d^2\psi(p)}{dp^2} = 0, \tag{2}$$

in which, one has set $\psi(p) = \sqrt{p\,\varphi(p)}$, for brevity.

2. Let V_1, V_2 denote the two propagation velocities of a plane wave, and let λ, μ, ν denote the cosines of the angles that the normal to that wave makes with the axes, so one can consider the wave surface to be the surface that is enveloped by the planes (Lamé, *Théorie de l'Elasticité*, page 242):

$$\lambda x + \mu y + \nu z = V_2\,. \tag{3}$$

Set $u = V_1^2$, $v = V_2^2$; equation (3) will obviously give $p = u$, and one will have, as is known (Lamé, page 243):

$$x = \lambda\left(\sqrt{u} + \frac{\rho}{u-a}\right), \quad y = \mu\left(\sqrt{u} + \frac{\rho}{u-b}\right), \quad z = \nu\left(\sqrt{u} + \frac{\rho}{u-c}\right),$$

in which:

$$p = \frac{\varphi(u)}{(v-u)\sqrt{u}},$$

and if one observes that [Lamé, page 238, eqs. (37), (39)]:

$$\mathrm{S}\,\frac{\lambda^2}{u-a} = 0, \qquad \mathrm{S}\,\frac{\lambda^2}{(u-a)^2} = \frac{v-u}{\varphi(u)}$$

then one will have:

$$r = u + \frac{\varphi(u)}{u\,(v-u)}.$$

The variables r, p are then coupled to u, v by:

$$r = u + \frac{\varphi(u)}{u\,(v-u)}, \qquad p = u.$$

From this, one deduces the relations:

$$\frac{(p-r)\sqrt{p}}{\sqrt{\varphi(p)}} = \frac{\sqrt{\varphi(u)}}{(u-v)\sqrt{u}}, \qquad \sqrt{p\,\varphi(p)} = \sqrt{u\,\varphi(u)}\,,$$

or if one sets:

$$\log\frac{(u-v)\sqrt{u}}{\sqrt{\varphi(u)}} = \theta$$

then the following ones:

$$\omega = -\,\theta, \qquad \psi(p) = \psi(u).$$

If one now substitutes these values in equation (3) then one will obtain the equation for the lines of curvature of the wave surface in the variables u, v. However, that equation obviously has the same form as (2). Therefore, one has the singular property that if one replaces the variables r, p in Combescure's equation (1) with the variables u, v then the resulting equation will be that of the lines of curvature of the wave surface in those latter variables (*).

(*) Rouché presented a note to the session of the Academia della Scienze (Institut. no. 1228) on 13 June 1859 in which he announced that he had found the equation for the lines of curvature of the wave surface in a finite, algebraic form when one assumes that the velocities of propagation are the variables. A retraction then followed, due to the fact that the paper included an error in calculation. Observe that the lines on the wave surface for which u = const., v = const. are not orthogonal, as was asserted in the note on page 135 of year 2 in these Annali. Equation (3) in that note is the condition for the orthogonality only in the case for

If one sets $\frac{\mu(p)}{p(r-p)} = k$, for brevity, then one will get the following formulas for the values of the coordinates x, y, z of an arbitrary point on the wave surface as functions of the quantities r, p:

$$x = \lambda\sqrt{p}\cdot\frac{r-a}{p-a}, \qquad y = \mu\sqrt{p}\cdot\frac{r-b}{p-b}, \qquad z = \nu\sqrt{p}\cdot\frac{r-c}{p-c},$$

in which:

$$\lambda^2 = \frac{p-a}{\varphi'(a)}(p-a+k), \qquad \mu^2 = \frac{p-b}{\varphi'(b)}(p-b+k), \qquad \nu^2 = \frac{p-c}{\varphi'(c)}(p-c+k).$$

Pavia, November 1859.

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which equation (2) is verified. Therefore, (3) cannot be valid for lines u = const., v = const. that do not verify the condition (2).