"Le tensioni create in un corpo elastico dalle distorsioni di Volterra, e la conseguente doppia rifrazione accidentiale," Rend. Reale Accad. dei Lincei (5) **18** (1909), 437-444.

Physics. – *The tensions that are created in an elastic body by Volterra distortions and the consequent accidental double refraction*. Note by O. M. Corbino, presented to the society by V. Volterra.

1. The experimental verification of the theory of elasticity has led two difficulties so far: Indeed, except for a few simple cases, the external force is ordinarily exerted in a discontinuous way along the points of the surface, while the predicted distribution of tensions or deformations inside of the body is not therefore accessible to experiment, in general.

The first difficulty is eliminated by the particular deformations that were considered by prof. Volterra in his memoir "Sull'equilibrio dei corpi elastici a connessioni multiple," because it is then possible to create a system of non-simple deformations or tensions in the body without the intervention of external forces.

The predicted variations in form can be confirmed in reality by using cylindrical rings of caoutchouc such that after removing part of the substance, the faces of the cut were glued together. It is therefore natural to examine the consequences of the theory more thoroughly using a transparent, elastic body, such as gelatin, and to study the distribution of tensions that are created by the distortions by means of the accidental birefringence that is acquired.

Since the theoretical predictions of the observable effects can be complete, it will be advantageous to suppose that after a *radial or parallel cut* has been made and the faces of the cut are glued back together, a ring of small breadth is examined in parallel polarized light that *propagates in the direction of the axis of the cylinder*. The figure that one then obtains when the image of the ring is projected onto a screen and through an analyzer, and then observed through a nicol permits a very rigorous control of the theory in the most interesting part that relates to the tensions that exist in the *xy*-plane that is normal to the axis of the cylinder.

The analytic expressions for these tensions were already given in prof. Volterra's memoir. It will be the objective of this note precisely to examine the fundamental formulas and their interpretation in regard to the birefringence effects that accompany the distortions, in order to move on to the comparison with the results of the experiments that were performed on my advice by Trabacchi, who overcame noteworthy technical difficulties to succeed in also confirming some more unpredictable details of the consequences of the theory.

2. The formula of prof. Volterra that relates to the radial and parallel cuts gives the tensions t_{11}, t_{22} that are parallel to two fixed axes and the shear tension t_{12} for any point x, y of the ring. From that, one passes immediately to the dilatations γ_{11}, γ_{22} in the direction of the axis and to the shear γ_{12}, by virtue of the relations:

(1) $$t_{11} = L\,\Theta + 2K\,\gamma_{11}, \qquad t_{22} = L\,\Theta + 2K\,\gamma_{22}, \qquad t_{12} = 2K\,\gamma_{12},$$

in which Θ is the cubic dilatation, and L and K are the two elastic constants of the body.

On the other hand, if one denotes the *principal* dilatations at each point of the body (in the directions of the principal axes of deformation, which vary from point to point) by γ'_{11} and γ'_{12} then the birefringence Δ at each point will be proportional to the difference $\gamma'_{11} - \gamma'_{12}$, which is a result of all experiments on accidental double refraction. Now let α denote the angle that is formed between the principal axes above and the fixed axes, so the directions of the principal axes will be given at each point by:

(2) $$\tan 2\alpha = \frac{\gamma_{12}}{\gamma_{11} - \gamma_{12}},$$

so, from(1), one will have:

(3) $$\tan 2\alpha = \frac{t_{12}}{\frac{1}{2}(t_{11} - t_{12})}.$$

We have analogously:

$$\Delta = A(\gamma'_{11} - \gamma'_{12})$$

for the birefringence Δ, in which A denotes a quantity that is constant for a given lamina. However:

$$(\gamma'_{11} - \gamma'_{12})^2 = (\gamma_{11} - \gamma_{22})^2 + \gamma_{12}^2,$$

and one can then measure Δ (with a convenient unit) from:

(4) $$\Delta^2 = \tfrac{1}{4}(t_{11} - t_{22})^2 + t_{12}^2.$$

One obtains an obvious connection with the theory of conics from this result: One gets a right triangle (Fig. 1) that has t_{12} and $\frac{1}{2}(t_{11} - t_{22})$ for its opposite sides and an angle that is opposite to the former that equals 2α and determines the direction of the birefringence at each point, while the hypotenuse measures the intensity of that birefringence.

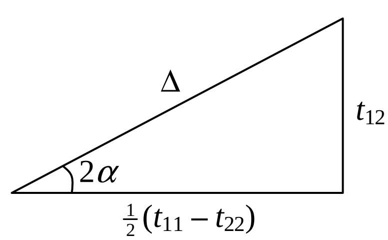

Figure 1.

Now suppose that the lamina is between two crossed polarizers, and let φ be the angle that is formed between the principal section of the polarizer and the x-direction, along which one makes the cut. The intensity I of the light that emerges from each point of the lamina will be given by:

$$I^2 = \sin^2(\alpha - \varphi)\,\sin^2(h\Delta),$$

in which h is a constant that depends upon the wave length of the incident light for a given lamina.

One will then observe a system of absolutely black lines that pass through all points for which

(5) $$\sin 2(\alpha - \varphi)\,\Delta = 0,$$

and thus the equations of the black line will be:

(6) $$\Delta = 0$$

for arbitrary values of φ, and:

(7) $$\sin 2(\alpha - \varphi) = 0$$

for arbitrary values of Δ.

By the property that is clearly exhibited by Fig. 1, the last relation will become:

(8) $$t_{12} = 0,$$

for $\varphi = 0$, and:

(9) $$t_{12} - t_{22} = 0$$

for $\varphi = 45^{\circ}$.

Therefore, the black lines that are observed by crossed polarizers, one of which is directed along the line of the cut that corresponds to equation (8), and the other of which, will be observed when the cut is 45° from polarizer that corresponds to equation (9).

3. Case of a radial cut. – From Volterra's formula, one gets:

$$t_{12} = a\frac{xy}{r^2}\left[1 - \frac{R_1^2 R_2^2(\log R_1^2 - \log R_2^2)}{R_1^2 - R_2^2}\frac{1}{r^2}\right],$$

$$t_{12} - t_{22} = a\frac{y^2 - x^2}{r^2}\left[1 - \frac{R_1^2 R_2^2(\log R_1^2 - \log R_2^2)}{R_1^2 - R_2^2}\frac{1}{r}\right]$$

in this case, in which a is a constant, r is the distance from the point x, y to the center, and R_1 and R_2 are the inner and outer radii of the cylindrical ring, respectively.

If one sets $y / x = \tan\vartheta$ then (3) will give:

(10) $$\tan 2\alpha = \tan 2\vartheta,$$

namely, the radius vector and its normal are the principal axes of the dilatation at any point, and the direction of the axis of birefringence coincides with one of the lines.

Observe that one will have a black line between crossed polarizers for arbitrary orientations of the lamina at the points where:

$$\Delta = 0,$$

i.e., where one then has collectively:

$$t_{11} - t_{22} = 0, \qquad t_{12} = 0,$$

and since the equation of that line will be:

$$r^2 = \frac{R_1^2 R_2^2 (\log R_1^2 - \log R_2^2)}{R_1^2 - R_2^2},$$

that will represent a circumference of radius:

$$r = R_1 \, R_2 \sqrt{\frac{\log R_1^2 - \log R_2^2}{R_1^2 - R_2^2}}.$$

In addition, one will get the other corresponding lines from equation (7), which, from (10), becomes:

$$\vartheta = \varphi + m \, \frac{\pi}{2},$$

in which m is an arbitrary integer.

One then has a circle and a cross whose arms are parallel to the principal sections of the polarizers; be that as it may, if one orients the lamina in its plane then the circle will persist, and the cross will keep its arms permanently oriented along the principal sections of the crossed polarizers.

One finally notes that the circumference along which the lamina remains devoid of birefringence (i.e., $\Delta = 0$) does not coincide with Volterra's *neutral axis*, which is to be expected, since the latter is defined by different conditions.

4. Case of a parallel cut. – From Volterra's formula that relates to that case, when it is transformed into polar coordinates and one lets a, b, c, d, e denote the independent coefficients of the coordinates of the points that one successively encounters in the formula that gives t_{12}, one will obtain, from a simple calculation:

$$t_{12} = \frac{a}{r} \sin \vartheta \left\{ b - 1 + 2c(4\cos^2 \vartheta - 1)\left(1 - \frac{2d}{r^2}\right) + er^2 \right\},$$

$$t_{12} - t_{22} = \frac{a}{r} \cos \vartheta \left\{ b - 1 - 2c(4\sin^2 \vartheta - 1)\left(1 - \frac{2d}{r^2}\right) + er^2 \right\},$$

in which a contains the amplitude of the cut, a, b, c are the elastic constants of the body, d is the radius R_1 or R_2 of the ring, and e is the elastic constant of the radii.

In the case of gelatin, one can set the Poisson coefficient equal to 1 / 2, and the coefficients will then become:

$$b = 0, \qquad c = \frac{1}{2}, \qquad d = \frac{R_1^2 R_2^2}{R_1^2 + R_2^2}, \qquad e = \frac{2}{R_1^2 + R_2^2}.$$

If one examines the lamina between crossed polarizers when the direction of the cut is parallel or normal to the plane of polarization then one will observe the lines that correspond to the equation:

$$t_{12} = 0,$$

which will give two lines: One corresponds to:

$$\sin \vartheta = 0,$$

which represents the direction of the cut, while the other one corresponds to the equation r and ϑ:

$$r^4 + 2\rho_1^2 \cos 2\vartheta\, r^2 - (1 + 2 \cos 2\vartheta)\, \rho^4 = 0,$$

in which, one sets:

$$\rho_1^2 = \frac{R_1^2 + R_2^2}{2}, \qquad \rho^2 = R_1 R_2 .$$

Let ε denote the ratio of the outer radius $R_1 = R$ and the inner one R_2 , and solve the preceding equation with respect to r^2 . One will obtain:

(11)
$$r^2 = \frac{1+\varepsilon^2}{2\varepsilon^2} R^2 \left[\sqrt{\cos^2 2\vartheta + (1 + 2\cos 2\vartheta) \frac{4\varepsilon^2}{(1+\varepsilon^2)^2}} - \cos 2\vartheta \right].$$

One recognizes immediately that the curve that represents r as a function of ϑ is symmetric with respect to the axes. It is constructed, as in Fig. 2, from the points for which $\varepsilon = 2, 5$, which is a condition that is close to the one that is realized in experiment. For $\vartheta = 90^{\circ}$, one has:

$$r = R$$

for any ε, and the curve will be tangent to the outer circle at that point.

A second solution, which corresponds to the negative value of the radical, gives values of r that are, in general, less than the inner radius of the envelope, and which then have no significance for us. However, $\vartheta = 90^{\circ}$ will correspond to a value $r = R$, and thus, also to points M, N that satisfy the condition $t_{12} = 0$.

If one orients the cut of the lamina at 45° with respect to the polarizers then the black lines will correspond to the equation:

$$t_{11} - t_{21} = 0,$$

which defines a line:

$$\cos \vartheta = 0$$

that is normal to the line of the cut and a curve that has the equation:

$$r^4 - 2\rho_1^2 (1 - \cos 2\vartheta)\, r^2 + (1 - 2 \cos 2\vartheta)\, \rho^4 = 0.$$

Solving this for r^2 will give:

$$r^2 = \frac{1+\varepsilon^2}{2\varepsilon^2} R^2 \left[\sqrt{(1-\cos 2\vartheta)^2 + (-1+2\cos 2\vartheta)\frac{4\varepsilon^2}{(1+\varepsilon^2)}} + 1 - \cos 2\vartheta \right]. \tag{12}$$

It is noteworthy that this relation is obtained from the preceding one (11) by replacing $\cos 2\vartheta$ with $\cos 2\vartheta - 1$. The same thing can be said for $\frac{t_{12}}{\sin\vartheta}$ and $\frac{t_{11} - t_{22}}{\cos\vartheta}$.

That will permit us to establish a simple graphical correspondence between the angles for which the two curves possess the same radius vector. The value $\vartheta = AB$ (Fig. 3) corresponds to a certain value ρ of the radius vector along the curve (11). If one sets $BC = AB$ and draws CD normal to OA then the arc OC' of the circle whose center is O' will be twice the arc ϑ' that corresponds to one-half the value of ρ of the radius vector under the curve (12). Naturally, the law of correspondence will permit one to pass to the inverse.

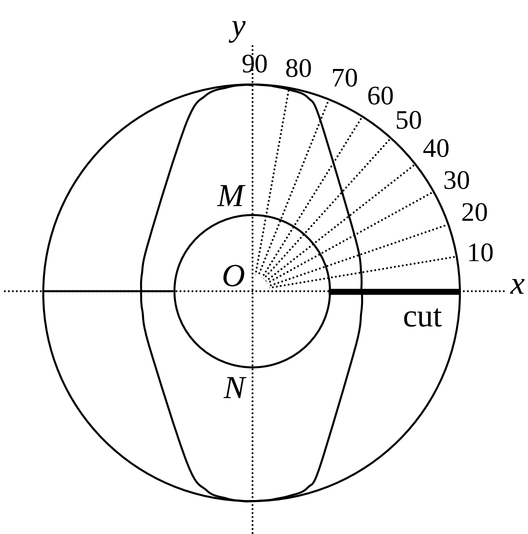

Figure 2.

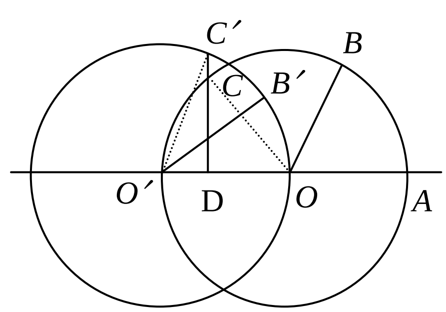

Figure 3.

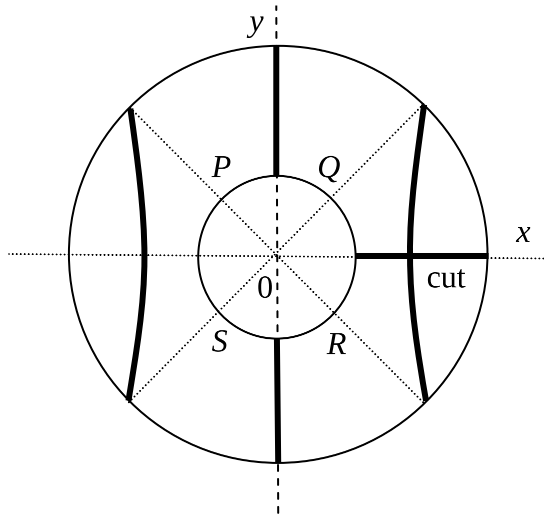

Figure 4.

One then deduces that the curve (12) begins with a value of the radius vector for $\vartheta = 0$ that is equal to that of the curve (11) for $\vartheta = 45^{\circ}$, and when one increases ϑ, the radius will increase until it becomes R (viz., the radius of the inner circle) for $\vartheta = 45^{\circ}$. Beyond 45°, the curve will no longer develop, and since it must be symmetric with respect to the two axes, it will be comprised on the right and left by two lines that form angles of $+ 45^{\circ}$ and $- 45^{\circ}$ with the x-axis (Fig. 4). For 45°, one will again have the singular solutions $r = R_1$ that correspond to the points P, Q, R, S in the figure, which are analogous to the points M, N in Fig. 2.

The two curves (10) and (11), along with the x and y axes, divide the lamina into six regions; passing from one to a contiguous one will invert the sign of t_{12} or $t_{11} - t_{22}$. The general distribution of the directions and intensities of the birefringence can be inferred from this.

The two systems of lines, when considered together, will give points in their common part where one simultaneously has:

$$t_{11} - t_{22} = 0, \qquad t_{12} = 0,$$

namely:

$$\Delta = 0.$$

Now, the two curves (11), (12) have no common point, as one easily recognizes, since one always has $\varepsilon > 1$. Therefore, the lamina does not possess any neutral line, so it has no birefringence, but only six neutral points, and they are precisely the points of intersection of the curve (11) with the x-axis of the cut.

This is a truly remarkable result.

The four regions into which the ring is divided, according to the Volterra figure, and within which the substance is alternatively compressed and dilated, behave quite differently in regard to birefringence. This apparent contradiction should not be surprising, since Volterra was concerned with the overall cubic dilatation of the four regions, while birefringence depends upon the difference between the principal dilatations in the xy-plane.

After one introduces the t_{23} in the z-direction, in addition the t that have been considered up to now, an analogous calculation must be performed in order to correctly interpret the Rolla experiment, in which the double refraction with respect to the light that propagated at 45° to the cut and normal to the z-axis was observed through tubes that were embedded into the gelatin and parallel to the xy-plane.

The direction of the birefringence at each point is given by (3) as:

$$\tan 2\alpha = \frac{1}{2}\tan\vartheta \frac{r^2 + 2\rho_1^2(\cos^2\vartheta - 1)r^2 - (4\cos^2\vartheta - 1)\rho^4}{r^4 - 4\rho_1^2(\sin^2\vartheta)r^2 + (4\sin^2\vartheta - 1)\rho^4}.$$

For each value of $\tan 2\alpha$, this equation define a curve in r, ϑ that be called isogonic, and that is the locus of all points where the direction of birefringence is equally inclined with respect to the line of the cut. Thus, (10) and (11) represent the isogonic for $\alpha = 0$ and $\alpha = 45^{\circ}$. Experimentally, if the polarizers make an angle φ with the cut then the black line that one gets will be the isogonic that relates to the angle $\alpha = \varphi$.

For $\vartheta = 90^{\circ}$, the preceding relation will give the result that was referred to before, namely, that the lamina will be birefringent along the normal line to the line of the cut and will have an axis that is directed at 45° to that line.

For Δ = *constant*, (4) will define a system of lines along which the birefringence has the same value. However, the discussion of that complicated equation is not feasible.

All of the preceding presupposed that the conditions under which the Volterra formula from which we began would be valid were realized, namely, that the cut was very small, and that the bases were taken to be planar and initially at a distance from each other. However, it is clear that if we observe the light in the direction of the z-axis, as we assumed, then the forces that must be applied to the bases in order to make them planar and at a normal distance will not have a very great influence on the observed effects, especially if the ring is reduced in the z direction to a disc of small height.

Furthermore, in truth, experiments have confirmed all of the details that were predicted above, without any need for taking the ring between two layers of glass that would render the bases absolutely planar and at the original distance.