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NOTE

ON THE

THEORY OF EUCLIDIAN ACTION

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INTRODUCTION

Mechanics, like any science that makes perceptible facts its object of study, is, above all, experimental and inductive, and that is the character that it possesses in a classical treatise. However, one can also try to attach to it some unique general concept and give it a deductive form. In that manner, one confers upon it a new power to discover, and one finds the explanation for some notions that were already acquired inductively. Such was the work of Lagrange a century ago in his *Mécanique analytique*.

In our epoch, an attempt of this type deserves to be renewed, because the domain of phenomena that are found to depend more or less completely upon mechanics has been enlarged considerably. One of the paths that one can follow was pointed out by Helmholtz: He took his point of departure to be the method of Hamilton's variable action, in such a way that the notion from which one must deduce all of the inductive principles of mechanics is that of *action*, conveniently chosen. However, Helmholtz did not clarify precisely just what was fundamental in that original concept that would permit generalization. In order to arrive at a completely constructive definition, one can observe that the action that Maupertuis introduced into mechanics is *invariant under the group of Euclidian displacements.* This same character is also found in the statics of deformable bodies, which rests upon the consideration of the ds^2 of space. In physics, the theory of phenomena that are due to gravitation, heat, and electricity depends upon the study of differential parameters that are likewise invariant under the group of displacements, as was first shown by Laplace, Fourier, and Maxwell.

H. Poincaré once wrote that the notion of group already exists in our spirit, at least, potentially, and is imposed upon us, not as a form that we perceive, but as a form that we understand. Following that philosophical idea, all of classical mechanics and all of theoretical physics seems to be deducible from the single notion of a *Euclidian action*. That is what we propose to establish in our present note, at least, insofar as it concerns the questions that belong to the usual scope of mechanics. We therefore present the *theory of Euclidian action for extension and motion.* Since we shall have no need to employ the word "matter," our considerations likewise will apply to the *ether*. In order to have a more complete idea of the notion of matter, one must deal with the concept of *entropy*, taking the profound viewpoint that Lippmann introduced into electric theory; that aspect of the question cannot be contained with the limits of our discussion here.

I. – STATICS OF THE DEFORMABLE LINE AND THE DYNAMICS OF TRIHEDRA.

1. Deformable line. Natural state and deformed state. – Consider a curve (M_0) that is described by a point M_0 whose coordinates x_0, y_0, z_0 with respect to the fixed rectangular axes *Ox*, *Oy*, *Oz* are functions of the same parameter – for example, the arc length s_0 of the curve, when measured by starting from a well-defined origin and proceeding in a well-defined sense. Attach a tri-rectangular trihedron to each point M_0 of the curve (M_0) whose axes $M_0x'_0$, $M_0y'_0$, $M_0z'_0$ have direction cosines with respect to the axes *Ox*, *Oy*, *Oz* that are α_0, α'_0, α''_0, β_0, β'_0, β''_0, γ_0, γ'_0, γ''_0, and which are functions of the same parameter s_0. The continuous one-dimensional set of all such trihedra $M_0x'_0y'_0z'_0$ will be what we call a *deformable line*.

Give a displacement M_0M to the point M_0. Let x, y, z be the coordinates of the point M with respect to the fixed axes *Ox*, *Oy*, *Oz*. In addition, give a rotation to the trihedron $M_0x'_0y'_0z'_0$ that ultimately takes its axes to those of a trihedron $Mx'y'z'$ that we attach to the point M. We define that rotation by giving the direction cosines α, α', α'', β, β', β'', γ, γ', γ'' of the axes Mx', My', Mz' with respect to the fixed axes *Ox*, *Oy*, *Oz*. The continuous one-dimensional set of trihedra $Mx'y'z'$ will be what we call the *deformed state* of the deformable line considered, which will be called the *natural state* in its original state.

Suppose that s_0 varies, and that we shall let it play the role of time for the moment. We then let ξ_0, η_0, ζ_0 denote the projections of the velocity of the origin M_0 of the axes $M_0x'_0$, $M_0y'_0$, $M_0z'_0$ onto those axes, and let p_0, q_0, r_0 be the projections of the instantaneous rotational velocity of the trihedron $M_0x'_0y'_0z'_0$ onto the same axes. We let ξ, η, ζ and p, q, r be the analogous quantities for the trihedra $Mx'y'z'$ when one refers them, like the trihedron $M_0x'_0y'_0z'_0$, to the fixed trihedron *Oxyz*. The elements that we just introduced have values that are related as follows:

$$
(1)\qquad \begin{cases}
\xi = \alpha \dfrac{dx}{ds_0} + \alpha' \dfrac{dy}{ds_0} + \alpha'' \dfrac{dz}{ds_0}, \\[2ex]
\eta = \beta \dfrac{dx}{ds_0} + \beta' \dfrac{dy}{ds_0} + \beta'' \dfrac{dz}{ds_0}, \\[2ex]
\zeta = \gamma \dfrac{dx}{ds_0} + \gamma' \dfrac{dy}{ds_0} + \gamma'' \dfrac{dz}{ds_0},
\end{cases}
$$

(2)
$$\left\{\begin{aligned} p &= \sum \gamma \frac{d\beta}{ds_0} = -\sum \beta \frac{d\gamma}{ds_0}, \\ q &= \sum \alpha \frac{d\gamma}{ds_0} = -\sum \gamma \frac{d\alpha}{ds_0}, \\ r &= \sum \beta \frac{d\alpha}{ds_0} = -\sum \alpha \frac{d\beta}{ds_0}. \end{aligned}\right.$$

With these quantities, the linear element ds of the curve that is described by the point M is defined by the formula:

(3)
$$ds^2 = (\xi^2 + \eta^2 + \zeta^2)\ ds_0^2 .$$

Let x', y', z' be the projections of the line segment OM on the axes Mx', My', Mz', in such a way that the coordinates with respect to these axes of the *fixed point O* will be $-x'$, $-y'$, $-z'$. We will have the following known formulas:

(4)
$$\left\{\begin{aligned} &\xi - \frac{dx'}{ds_0} - qz' + ry' = 0, \\ &\eta - \frac{dy'}{ds_0} - rx' + pz' = 0, \\ &\zeta - \frac{dz'}{ds_0} - py' + qx' = 0, \end{aligned}\right.$$

which gives new expressions for ξ, η, ζ.

Suppose that we give each of the trihedra of the deformed state an infinitely small displacement that can vary with these trihedra in a continuous fashion. Let δx, δy, δz, $\delta x'$, $\delta y'$, $\delta z'$, …, $\delta\alpha$, $\delta\alpha'$, …, $\delta\gamma''$ be the variations of x, y, z, x', y', z', …, α, α', …, γ'', respectively. The variations $\delta\alpha$, $\delta\alpha'$, …, $\delta\gamma''$ are expressed by formulas such as the following ones:

(4)
$$\delta\alpha = \beta\,\delta k' - \gamma\,\delta j',$$

by means of three auxiliary functions $\delta i'$, $\delta j'$, $\delta k'$, which are the components along Mx', My', Mz' of the well-known instantaneous rotation that is attached to the infinitely small displacement in question. To abbreviate the notation, introduce the projections $\delta' x$, $\delta' y$, $\delta' z$ of the displacement δx, δy, δz onto Mx', My', Mz'. We will have:

(6)
$$\left\{\begin{aligned} \delta' x &= \delta x' + z'\delta j' - y'\delta k', \\ \delta' y &= \delta y' + x'\delta k' - x'\delta i', \\ \delta' z &= \delta z' + y'\delta i' - x'\delta j'. \end{aligned}\right.$$

Having said that, we have, from (2) and (5):

(7)
$$\left\{\begin{aligned} \delta p &= \frac{d\delta i'}{ds_0} + q\,\delta k' - r\,\delta j', \\ \delta q &= \frac{d\delta j'}{ds_0} + r\,\delta i' - p\,\delta k', \\ \delta r &= \frac{d\delta k'}{ds_0} + q\,\delta j' - r\,\delta i'. \end{aligned}\right.$$

Similarly, from (4), (6), and (7), we have:

(8)
$$\left\{\begin{aligned} \delta\xi &= \eta\,\delta k' - \zeta\,\delta j' + \frac{d\delta' x}{ds_0} + q\,\delta' z - r\,\delta' y, \\ \delta\eta &= \zeta\,\delta i' - \xi\,\delta k' + \frac{d\delta' y}{ds_0} + r\,\delta' x - p\,\delta' z, \\ \delta\zeta &= \xi\,\delta j' - \eta\,\delta i' + \frac{d\delta' z}{ds_0} + p\,\delta' y - q\,\delta' x. \end{aligned}\right.$$

2. Euclidian action of deformation on a deformable line. External force and moment. Effort and moment of deformation at a point of a deformed line. – Consider a function W of *two infinitely close positions* of the trihedron $Mx'y'z'$ – i.e., a function s_0 of $x, y, z, \alpha, \alpha', \ldots, \gamma''$ and their first derivatives with respect to s_0. We seek the form that W must take in order for the integral $\int W\,ds_0$, when taken over an arbitrary portion of the line (M_0), to have a zero variation when one subjects the set of all trihedra of the deformable line, when taken in its deformed state, *to the same arbitrary infinitesimal displacement of the Euclidian displacement group.* By definition, it amounts to determining W in such a fashion that one has $\delta W = 0$, when, on the one hand, the origin M of the trihedron $Mx'y'z'$ is subjected to the following infinitely small displacement:

(9)
$$\left\{\begin{aligned} \delta x &= (a_1 + \omega_2 z - \omega_3 y)\,\delta t, \\ \delta y &= (a_2 + \omega_3 x - \omega_1 z)\,\delta t, \\ \delta z &= (a_3 + \omega_1 y - \omega_2 x)\,\delta t, \end{aligned}\right.$$

where $a_1, a_2, a_3, \omega_1, \omega_2, \omega_3$ are arbitrary constants, and δt is an infinitely small quantity that is independent of s_0, and when, on the other hand, the trihedron $Mx'y'z'$ is subjected to an infinitely small rotation whose components along the axes Ox, Oy, Oz are $\omega_1\,\delta t$, $\omega_2\,\delta t$, $\omega_3\,\delta t$. In the present case, the variations $\delta\xi$, $\delta\eta$, $\delta\zeta$, δp, δq, δr of the six expressions $\xi, \eta, \zeta, p, q, r$ are zero, since that would result from the well-known theory of the moving trihedron. We thus obtain a solution to the question by taking W to be an arbitrary function of s_0 and the six expressions $\xi, \eta, \zeta, p, q, r$. We thus have the general solution

([1]). Indeed, relations (2) permit one to express the first derivatives of the nine cosines α, α', …, γ'' with respect to s_0 in terms of these cosines and p, q, r. On the other hand, the formulas (1) give the nine cosines α, α', …, γ'' in terms of ξ, η, ζ, and the first derivatives of x, y, z with respect to s_0. We can therefore finally write:

$$W = W\left(s_0, x, y, z, \frac{dx}{ds_0}, \frac{dy}{ds_0}, \frac{dz}{ds_0}, \xi, \eta, \zeta, p, q, r\right).$$

Since the variations $\delta\xi$, $\delta\eta$, $\delta\zeta$, δp, δq, δr are zero, by virtue of formulas (9), and for all a_1, a_2, a_3, ω_1, ω_2, ω_3, we must have:

$$\frac{\partial W}{\partial x}\delta x + \frac{\partial W}{\partial y}\delta y + \frac{\partial W}{\partial z}\delta z + \frac{\partial W}{\partial \frac{dx}{ds_0}}\delta\frac{dx}{ds_0} + \frac{\partial W}{\partial \frac{dy}{ds_0}}\delta\frac{dy}{ds_0} + \frac{\partial W}{\partial \frac{dz}{ds_0}}\delta\frac{dz}{ds_0} = 0.$$

Replace δx, δy, δz with their values in (9) and $\delta\frac{dx}{ds_0}$, $\delta\frac{dy}{ds_0}$, $\delta\frac{dz}{ds_0}$ with the values that one deduces by differentiation. One then gets the following conditions:

$$\frac{\partial W}{\partial x} = 0, \qquad \frac{\partial W}{\partial y} = 0, \qquad \frac{\partial W}{\partial z} = 0,$$

$$\frac{\partial W}{\partial \frac{dy}{ds_0}}\frac{dz}{ds_0} - \frac{\partial W}{\partial \frac{dz}{ds_0}}\frac{dy}{ds_0} = 0,$$

$$\frac{\partial W}{\partial \frac{dz}{ds_0}}\frac{dx}{ds_0} - \frac{\partial W}{\partial \frac{dx}{ds_0}}\frac{dz}{ds_0} = 0,$$

$$\frac{\partial W}{\partial \frac{dx}{ds_0}}\frac{dy}{ds_0} - \frac{\partial W}{\partial \frac{dy}{ds_0}}\frac{dx}{ds_0} = 0.$$

The first three show that W is independent of x, y, z. The last three express the idea that W depends upon dx / ds_0, dy / ds_0, dz / ds_0 only by the intermediary of the quantity

([1]) In the sequel, we suppose that the deformable line is *susceptible to all possible deformations*, so *the deformed state can be taken to be absolutely arbitrary*. One can express this by saying that the deformable is *free*.

$\left(\frac{dx}{ds_0}\right)^2 + \left(\frac{dy}{ds_0}\right)^2 + \left(\frac{dz}{ds_0}\right)^2$, which, from (3), is equal to $\xi^2 + \eta^2 + \zeta^2$. We finally see that *the desired function W has the remarkable form* $W(s_0, \xi, \eta, \zeta, p, q, r)$.

Just as the value of the integral $\int \frac{ds}{ds_0} ds_0$, when taken between two points A_0 and B_0 of the curve (M_0), will determine the *length* of the arc between the corresponding points A and B of the curve (M), similarly, upon associating, in the same spirit, the notion of *action* for the passage from the natural state (M_0) to the deformed state (M), we attach the function W to the defining elements of the deformable line, and we say that the integral $\int W\, ds_0$, when taken between the same points A_0 and B_0 of (M_0), is the *action of deformation* on the deformed line between the points A and B. We also say that W is the *density* of the action of deformation *at a point* of the deformed line, when referred to the unit of length of the undeformed line. $W\, ds_0 / ds$ will be that density of action at a point when referred to the unit of length of the deformed line.

Consider an *arbitrary* variation of the action of deformation between two points A and B of the line (M), namely:

$$\delta\int_{A_0}^{B_0} W\, ds_0 = \int_{A_0}^{B_0}\left(\frac{\partial W}{\partial \xi}\delta\xi + \frac{\partial W}{\partial \eta}\delta\eta + \frac{\partial W}{\partial \zeta}\delta\zeta + \frac{\partial W}{\partial p}\delta p + \frac{\partial W}{\partial q}\delta q + \frac{\partial W}{\partial r}\delta r\right) ds_0 .$$

By virtue of formulas (7) and (8) of no. 1, and upon integrating by parts the terms that refer to a derivative with respect to s_0 explicitly, we can write:

$$\delta\int_{A_0}^{B_0} W\, ds_0 = \left[F'\delta' x + G'\delta' y + H'\delta' z + I'\delta i' + J'\delta j' + K'\delta k'\right]_{A_0}^{B_0}$$
$$- \int_{A_0}^{B_0} (X_0'\,\delta' x + Y_0'\,\delta' y + Z_0'\,\delta' z + L_0'\,\delta i' + M_0'\delta j' + N_0'\,\delta k')\, ds_0 ,$$

where we have set:

$$(16)\quad \begin{cases} F' = \dfrac{\partial W}{\partial \xi},\ G' = \dfrac{\partial W}{\partial \eta},\ H' = \dfrac{\partial W}{\partial \zeta},\ I' = \dfrac{\partial W}{\partial p},\ J' = \dfrac{\partial W}{\partial q},\ K' = \dfrac{\partial W}{\partial r}, \\ X_0' = \dfrac{d}{ds_0}\dfrac{\partial W}{\partial \xi} + q\dfrac{\partial W}{\partial \zeta} - r\dfrac{\partial W}{\partial \eta}, \quad L_0' = \dfrac{d}{ds_0}\dfrac{\partial W}{\partial p} + q\dfrac{\partial W}{\partial r} - r\dfrac{\partial W}{\partial q} + \eta\dfrac{\partial W}{\partial \zeta} - \zeta\dfrac{\partial W}{\partial \eta}, \\ Y_0' = \dfrac{d}{ds_0}\dfrac{\partial W}{\partial \eta} + r\dfrac{\partial W}{\partial \xi} - p\dfrac{\partial W}{\partial \zeta}, \quad M_0' = \dfrac{d}{ds_0}\dfrac{\partial W}{\partial q} + r\dfrac{\partial W}{\partial p} - p\dfrac{\partial W}{\partial r} + \zeta\dfrac{\partial W}{\partial \xi} - \xi\dfrac{\partial W}{\partial \zeta}, \\ Z_0' = \dfrac{d}{ds_0}\dfrac{\partial W}{\partial \zeta} + p\dfrac{\partial W}{\partial \eta} - q\dfrac{\partial W}{\partial \xi}, \quad N_0' = \dfrac{d}{ds_0}\dfrac{\partial W}{\partial r} + p\dfrac{\partial W}{\partial q} - q\dfrac{\partial W}{\partial p} + \xi\dfrac{\partial W}{\partial \eta} - \eta\dfrac{\partial W}{\partial \xi}. \end{cases}$$

If we first consider the integral that appears in the expression for $\delta\int_{A_0}^{B_0} W\, ds_0$ then we can call the line segments that issue from M whose projections onto the axes Mx', My',

Mz' are X_0', Y_0', Z_0', and L_0', M_0', N_0', respectively, the *external force and external moment at the point M, when referred to the undeformed unit of length*, resp. If we then consider the partially-integrated part of $\delta\int_{A_0}^{B_0} W\,ds_0$ then we can call the line segments that issue from B whose projections onto the axes Mx', My', Mz' are the values that the expressions $-F'$, $-G'$, $-H'$, and $-I'$, $-J'$, $-K'$, respectively, take at the point B_0 the *external effort and external moment of deformation at the point B*, resp. We call the analogous line segments that are composed of the values that the expressions $+F'$, $+G'$, $+H'$, and $+I'$, $+J'$, $+K'$ take at the point A_0 the *external effort and external moment of deformation at the point A*, resp. The points A and B do not present themselves in the same fashion here, because we agreed to define the arc s_0 in the sense of A_0B_0.

Suppose that one cuts the deformed line AB at the point M and then one mentally separates the two parts AM and MB. One can regard the two line segments ($-F'$, $-G'$, $-H'$) and ($-I'$, $-J'$, $-K'$) that are determined for the point M as the external effort and moment of deformation of the part AM at the point M and regard the two line segments ($+F'$, $+G'$, $+H'$) and ($+I'$, $+J'$, $+K'$) that are determined for the same point M as the external effort and moment of the part MB at the point M. It will be the same thing if, instead of considering AM and MB, one imagines two portions of the deformable line that belong to AM and MB, respectively, and have one extremity at M. By reason of these remarks, we say that $-F'$, $-G'$, $-H'$, and $-I'$, $-J'$, $-K'$ are the components along the axes Mx', My', Mz' of the *effort and moment of deformation at the point M that are exerted upon AM and any portion of AM that is bounded by M*, and that $+F'$, $+G'$, $+H'$, and $+I'$, $+J'$, $+K'$ are the components along the axes Mx', My', Mz' of the *effort and moment of deformation at the point M that are exerted upon MB and any portion of MB that is bounded by M.*

2. Equations of Lord Kelvin and Tait. Varignon's theorem. Notions of energy of deformation and the natural state of the deformable line. – The various elements that were introduced in the preceding section are coupled by the following relations, which result immediately from comparing the formulas that serve as their definition:

$$(11)\quad \left\{\begin{aligned} &\frac{dF'}{ds_0}+qH'-rG'-X_0'=0, && \frac{dI'}{ds_0}+qK'-rJ'+\eta H'-\zeta G'-L_0'=0,\\ &\frac{dG'}{ds_0}+rF'-pH'-Y_0'=0, && \frac{dJ'}{ds_0}+rI'-pK'+\zeta F'-\xi H'-M_0'=0,\\ &\frac{dH'}{ds_0}+pG'-qF'-Z_0'=0, && \frac{dK'}{ds_0}+pJ'-qI'+\xi G'-\eta F'-N_0'=0.\end{aligned}\right.$$

One can propose to transform the relations that we just wrote *independently of the values of the quantities that appear in them that are calculated by using W.* Indeed, instead of defining the line segments that we have attached to the point M by their projections onto Mx', My', Mz', we can just as well define them by their projections onto other axes.

First, consider the fixed axes *Ox*, *Oy*, *Oz*. Let X_0, Y_0, Z_0 and L_0, M_0, N_0 denote the projections onto these axes of the external force and moment at an arbitrary point *M* of the deformed line, and let *F*, *G*, *H* and *I*, *J*, *K* denote the projections of the effort and moment of deformation, whose projections onto the axes Mx', My', Mz' are F', G', H' and I', J', K', resp. We can regard the external force and moment as distributed in a continuous manner along the line and referred to the unit of length of the undeformed line. In order to have the external force and moment referred to the unit of length of the deformed line, it suffices to multiply X_0', Y_0', Z_0', L_0', M_0', N_0' or X_0, Y_0, Z_0, L_0, M_0, N_0 by ds_0 / ds, where *ds* is the linear element of the deformed line that corresponds to the linear element of the undeformed line. Introduce the projections *X*, *Y*, *Z*, *L*, *M*, *N* onto the fixed axes *Ox*, *Oy*, *Oz* of the external force and moment, thus referred to the unit of length of the deformed line. We will then have the relations:

$$\text{(12)} \qquad \left\{ \begin{aligned} &\frac{dF}{ds} - X = 0, \quad && \frac{dI}{ds} + H\frac{dy}{ds} - G\frac{dz}{ds} - L = 0, \\ &\frac{dG}{ds} - Y = 0, && \frac{dJ}{ds} + F\frac{dz}{ds} - H\frac{dx}{ds} - M = 0, \\ &\frac{dH}{ds} - Z = 0, && \frac{dK}{ds} + G\frac{dx}{ds} - F\frac{dy}{ds} - N = 0, \end{aligned} \right.$$

which are identical to the ones that were considered by Lord Kelvin and Tait. However, the latter are obtained by applying what one calls in classical mechanics *the principle of solidification*, and by starting with the *a priori* notion of forces and couples, which are then expressed as functions of the deformation *a posteriori* and by virtue of the hypotheses. Moreover, Lord Kelvin and Tait have imagined only an infinitely small deformation, whereas we have presently placed ourselves in the general case.

Now, let there be a tri-rectangular trihedron $M\,x_1'\,y_1'\,z_1'$ that moves with *M*, and whose axis Mx_1' is subjected to being directed along the tangent to the curve (*M*) and pointing in the sense of increasing arc length. Let l, l', l'' denote the direction cosines of Mx_1' relative to the trihedron $Mx'y'z'$, let m, m', m'' denote those of My_1', and let n, n', n'' denote those of Mz_1'. Upon setting $\varepsilon = \sqrt{\xi^2 + \eta^2 + \zeta^2}$, we will have $l = \xi / \varepsilon$, $l' = \eta / \varepsilon$, $l'' = \zeta / \varepsilon$. Moreover, $m\xi + m'\eta + m''\zeta = 0$, $n\xi + n'\eta + n''\zeta = 0$. If the trihedron $M\,x_1'\,y_1'\,z_1'$ is referred to the fixed trihedron *Oxyz*, and s_0 plays the role of time then the projections p_1, q_1, r_1 of the instantaneous rotation of the trihedron $M\,x_1'\,y_1'\,z_1'$ into the axes Mx_1', My_1', Mz_1' will be given by formulas such as the following ones:

$$p_1 = lp + l'q + l''r + \sum n\frac{dm}{ds_0}.$$

On the other hand, let X_1', Y_1', Z_1', L_1', M_1', N_1' denote the projections onto Mx_1', My_1', Mz_1' of the external force and moment at an arbitrary point *M* of the deformed line, when referred to the unit of length of the undeformed line, and let F_1', G_1', H_1', I_1', J_1', K_1' denote the projections of the effort and moment of deformation. The transforms of equations (11) are obviously:

$$(13)\quad \begin{cases} \dfrac{dF_1'}{ds_0}+q_1H_1'-r_1G_1'-X_1'=0, & \dfrac{dI_1'}{ds_0}+q_1K_1'-r_1J_1' \qquad\qquad -L_1'=0, \\ \dfrac{dG_1'}{ds_0}+r_1F_1'-p_1H_1'-Y_1'=0, & \dfrac{dJ_1'}{ds_0}+r_1I_1'-p_1K_1'-\varepsilon H_1'-M_1'=0, \\ \dfrac{dH_1'}{ds_0}+p_1G_1'-q_1F_1'-Z_1'=0, & \dfrac{dK_1'}{ds_0}+p_1J_1'-q_1I_1'+\varepsilon G_1'-N_1'=0. \end{cases}$$

If one has $L_1' = 0$ and $q_1 = 0$ in the fourth equation of (13) then one gets:

$$\frac{dI_1'}{ds_0}-r_1J_1' = 0,$$

which implies the proposition that was established by Poisson that for $L_1' = 0$, $M_1' = 0$, $N_1' = 0$, $q_1 = 0$ *if* $J_1' = 0$ *then one will have that* $I_1' =$ const.

Among the theorems that one can deduce from the systems (11) and (12), we insist upon the following fundamental proposition of statics, whose main idea, but not its present form, is due to Varignon, and which one encounters again on the interpretation that Saint-Guilhem gave for the relations that couple the external forces and quantities of motion in dynamics. Assign the effort and moment of deformation at a point *M* of the line (*M*) to the resultant and resultant moment of a system of vectors that relate to the point *M*. Let $P\nu$, $P\sigma$ be the general resultant and resultant moment that relate to a point *P* in space. Likewise, assign the external force and moment at a point M_1, when referred to the unit of length of (*M*), to the resultant and resultant moment of a system of vectors that relate to the point *M*. Let *PN* and *PS* be the resultant and resultant moment that relate to a point *P* of space. One has the proposition: *If the arc length s is regarded as time then the velocities of the geometric points* ν *and* σ *are equal and parallel to the segments PN and PS*. This proposition is obviously a translation of equations (12). We can further arrive at it in the following manner: We give the name of *exterior work* done on the deformed line *AB* under an arbitrary virtual deformation to the equivalent expressions:

$$\begin{aligned} \delta T_e = &- \left[F'\delta'x+G'\delta'y+H'\delta'z+I'\delta i'+J'\delta j'+K'\delta'k'\right]_A^B \\ &+ \int (X'\delta'x+Y'\delta'y+Z'\delta'z+L'\delta i'+M'\delta j'+N'\delta k')\,ds \\ = &- \left[F\,\delta x+G\,\delta y+H\,\delta z+I\,\delta i+J\,\delta j+K\,\delta k\right]_A^B \\ &+ \int (X\,\delta x+Y\,\delta y+Z\,\delta z+L\,\delta i+M\,\delta j+N\,\delta k)\,ds\,, \end{aligned}$$

where δi, δj, δk are the projections onto the fixed axes of the line segment whose projections onto Mx', My', Mz' are $\delta i'$, $\delta j'$, $\delta k'$, resp. We thus have:

$$\int_A^M \delta W \, ds_0 = - \delta \mathcal{T}_e ,$$

where $\delta \mathcal{T}_e$ is taken between A and M. Since δW must be identically zero, by virtue of the invariance of W under the group of Euclidian displacements, when the variations δx, δy, δz are given by formulas (9) and when $\delta i = \omega_1 \, \delta t$, $\delta j = \omega_2 \, \delta t$, $\delta k = \omega_3 \, \delta t$, and this must be true for any values of the constants a_1, a_2, a_3, ω_1, ω_2, ω_3, we conclude that we must have:

$$[F]_A^M - \int_A^M X \, ds = 0, \quad [G]_A^M - \int_A^M Y \, ds = 0, \quad [H]_A^M - \int_A^M Z \, ds = 0,$$

$$[I + yH - zG]_A^M - \int_A^M (L + yZ - zY) \, ds = 0,$$

and two analogous formulas. *In these relations, one can regard M as variable*, and they are thus equivalent to equations (12). One remarks that these formulas are deduced easily from the ones that one ordinarily writes by means of the principle of solidification.

Imagine two states (M_0) and (M) of a deformable line, and consider an arbitrary sequence of states that begin with (M_0) and arrive at (M). To that effect, it suffices to consider functions x, y, z, α, α', …, γ'' of s_0 and a variable h that reduce to x_0, y_0, z_0, α_0, α'_0, …, γ''_0, respectively, when h has the value zero, and reduce to values x, y, z, α, α', …, γ'' that relate to (M) when h equals h. Upon making the parameter h vary from 0 to h in a continuous fashion, we obtain a continuous deformation that permits us to pass from the state (M_0) to the state (M). During this continuous deformation, the *total work* that is performed by the external forces and moments that are applied to the various elements of the line and by the efforts and moments of deformation that are applied to its extremities is obtained by integrating from 0 to h the differential that is obtained by replacing the variations of x, y, z, α, α', …, γ'' in $\delta \mathcal{T}_e$ with the partial differentials that correspond to the increase dh in h:

$$- \int_0^h \left(\int_{A_0}^{B_0} \frac{\partial W}{\partial h} ds_0 \right) dh$$

$$= - \int_{A_0}^{B_0} [W(s_0, \xi, \eta, \zeta, p, q, r) - W(s_0, \xi_0, \eta_0, \zeta_0, p_0, q_0, r_0)] \, ds_0 .$$

The work considered is independent of the intermediate states and depends upon only the extreme states (M_0) and (M). This leads us to introduce the notion of the *energy of deformation*, which must be distinguished from that of the action that was previously envisioned. We say that *– W is the density of the energy of deformation*.

The natural state that we just considered was the initial state of a sequence of deformed states. The external force and the analogous elements that relate to them are not necessarily zero. It is important to remark that, in addition, they are not essentially distinguished from the other states and that one make an arbitrary deformed state play the role of natural state. Let $(M_{(0)})$ be that state, where we denote the arc length by $s_{(0)}$. Moreover, upon letting $\xi^{(0)}$, $\eta^{(0)}$, $\zeta^{(0)}$, $p^{(0)}$, $q^{(0)}$, $r^{(0)}$ represent what ξ, η, ζ, p, q, r, resp., become when one lets $s_{(0)}$ play the role that is played by s_0 – in such a way that one has, for example, $\xi = \xi^{(0)}\, ds_{(0)} / ds_0$ – it will suffice for us to consider the function:

$$W^{(0)}(s_0, \xi^{(0)}, \eta^{(0)}, \zeta^{(0)}, p^{(0)}, q^{(0)}, r^{(0)}),$$

which is the expression for:

$$W\left(s_0, \xi^{(0)}\frac{ds_{(0)}}{ds_0}, \eta^{(0)}\frac{ds_{(0)}}{ds_0}, \ldots, r^{(0)}\frac{ds_{(0)}}{ds_0}\right)\frac{ds_0}{ds_{(0)}},$$

in which s_0, $\dfrac{ds_0}{ds_{(0)}}, \dfrac{ds_{(0)}}{ds_0}$ are replaced as functions of $s_{(0)}$. Instead of making the notion of the natural state correspond simply to the idea of a particular state, we can therefore, and in a more general fashion, make it correspond to the idea of an arbitrary state that we start with in order to study the deformation.

4. Normal form for the equations of the deformable line. Castigliano's principle of minimum work. – We can consider equations (10) of no. 2 to be differential equations that relate to the unknowns x, y, z and three parameters λ_1, λ_2, λ_3, by means of which, one expresses α, α', …, γ''. Assume that X_0, Y_0, Z_0, L_0, M_0, N_0 are given functions of s_0, x, y, z, λ_1, λ_2, λ_3. The expression W will be, as far as it is concerned, a well-defined function of s_0, $\dfrac{dx}{ds_0}, \dfrac{dy}{ds_0}, \dfrac{dz}{ds_0}$, λ_1, λ_2, λ_3, $\dfrac{d\lambda_1}{ds_0}, \dfrac{d\lambda_2}{ds_0}, \dfrac{d\lambda_3}{ds_0}$, and from the relation:

$$\delta\int_{A_0}^{B_0} W\, ds_0 + \delta T_e = 0\,, \tag{14}$$

we can replace the system (10) with the equivalent equations:

$$\left\{\begin{array}{lll}
\dfrac{d}{ds_0}\dfrac{\partial W}{\partial \dfrac{dx}{ds_0}} - X_0 = 0, & \dfrac{d}{ds_0}\dfrac{\partial W}{\partial \dfrac{dy}{ds_0}} - Y_0 = 0, & \dfrac{d}{ds_0}\dfrac{\partial W}{\partial \dfrac{dz}{ds_0}} - Z_0 = 0, \\
\dfrac{d}{ds_0}\dfrac{\partial W}{\partial \dfrac{d\lambda_1}{ds_0}} - \dfrac{\partial W}{\partial \lambda_1} - \mathcal{L}_0 = 0, & \dfrac{d}{ds_0}\dfrac{\partial W}{\partial \dfrac{d\lambda_2}{ds_0}} - \dfrac{\partial W}{\partial \lambda_2} - \mathcal{M}_0 = 0, & \dfrac{d}{ds_0}\dfrac{\partial W}{\partial \dfrac{d\lambda_3}{ds_0}} - \dfrac{\partial W}{\partial \lambda_3} - \mathcal{N}_0 = 0,
\end{array}\right. \tag{15}$$

in which $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ denote expressions that have the same nature as L_0, M_0, N_0, and which are known at the same time as them.

Introduce the auxiliary variables:

$$
(15')\qquad
\begin{cases}
F = \dfrac{\partial W}{\partial \dfrac{dx}{ds_0}}, \quad G = \dfrac{\partial W}{\partial \dfrac{dy}{ds_0}}, \quad H = \dfrac{\partial W}{\partial \dfrac{dz}{ds_0}}, \\[2ex]
\mathcal{I} = \dfrac{\partial W}{\partial \dfrac{d\lambda_1}{ds_0}}, \quad \mathcal{J} = \dfrac{\partial W}{\partial \dfrac{d\lambda_2}{ds_0}}, \quad \mathcal{K} = \dfrac{\partial W}{\partial \dfrac{d\lambda_3}{ds_0}}.
\end{cases}
$$

From these six relations, if we suppose that the Hessian of W with respect to $\dfrac{dx}{ds_0}, \dfrac{dy}{ds_0}, \dfrac{dz}{ds_0}, \dfrac{d\lambda_1}{ds_0}, \dfrac{d\lambda_2}{ds_0}, \dfrac{d\lambda_3}{ds_0}$ is non-zero then we can infer some values for these latter six derivatives as functions of F, G, H, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$. Transport these values into the expression:

$$
\mathcal{E} = \frac{dx}{ds_0}\frac{\partial W}{\partial \dfrac{dx}{ds_0}} + \frac{dy}{ds_0}\frac{\partial W}{\partial \dfrac{dy}{ds_0}} + \frac{dz}{ds_0}\frac{\partial W}{\partial \dfrac{dz}{ds_0}} + \sum \frac{d\lambda_i}{ds_0}\frac{\partial W}{\partial \dfrac{d\lambda_i}{ds_0}} - W.
$$

After substitution, we obtain a function of s_0, λ_1, λ_2, λ_3, F, G, H, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ that we continue to denote by the letter $\mathcal{E}$. Now, the total differential of the latter function is obviously:

$$
\frac{dx}{ds_0}dF + \frac{dy}{ds_0}dG + \frac{dz}{ds_0}dH + \frac{d\lambda_1}{ds_0}d\mathcal{I} + \frac{d\lambda_2}{ds_0}d\mathcal{J} + \frac{d\lambda_3}{ds_0}d\mathcal{K} - \frac{\partial W}{\partial s_0}ds_0 - \sum \frac{dW}{d\lambda_i}d\lambda_i ,
$$

and one has, in turn, the following form for the system that defines x, y, z, λ_1, λ_2, λ_3, F, G, H, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$:

$$
\frac{dx}{ds_0} = \frac{\partial \mathcal{E}}{\partial F}, \quad \frac{dy}{ds_0} = \frac{\partial \mathcal{E}}{\partial G}, \quad \frac{dz}{ds_0} = \frac{\partial \mathcal{E}}{\partial H}, \quad \frac{d\lambda_1}{ds_0} = \frac{\partial \mathcal{E}}{\partial \mathcal{I}}, \quad \frac{d\lambda_2}{ds_0} = \frac{\partial \mathcal{E}}{\partial \mathcal{J}}, \quad \frac{d\lambda_3}{ds_0} = \frac{\partial \mathcal{E}}{\partial \mathcal{K}},
$$

$$
\frac{dF}{ds_0} - X_0 = 0, \qquad \frac{dG}{ds_0} - Y_0 = 0, \qquad \frac{dH}{ds_0} - Z_0 = 0,
$$

$$
\frac{d\mathcal{I}}{ds_0} + \frac{\partial \mathcal{E}}{\partial \lambda_1} - \mathcal{L}_0 = 0, \qquad \frac{d\mathcal{J}}{ds_0} + \frac{\partial \mathcal{E}}{\partial \lambda_2} - \mathcal{M}_0 = 0, \qquad \frac{d\mathcal{K}}{ds_0} + \frac{\partial \mathcal{E}}{\partial \lambda_3} - \mathcal{N}_0 = 0.
$$

We have supposed that we can express X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ as functions of s_0, x, y, z, λ_1, λ_2, λ_3 by virtue of the formulas that define x, y, z, λ_1, λ_2, λ_3 as functions of s_0. This is possible in an infinitude of manners, and one can always choose the new forms for X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ in such a fashion that they are the partial derivatives (with the sign changed) $\frac{\partial\mathcal{O}}{\partial x}, \frac{\partial\mathcal{O}}{\partial y}, \frac{\partial\mathcal{O}}{\partial z}, \frac{\partial\mathcal{O}}{\partial\lambda_1}, \frac{\partial\mathcal{O}}{\partial\lambda_2}, \frac{\partial\mathcal{O}}{\partial\lambda_3}$, respectively, of the same function $\mathcal{O}$, and are either independent of s_0 or not. Suppose that this is true, and let $\mathcal{Q}$ denote the function of x, y, z, λ_1, λ_2, λ_3 (and possibly s_0) that is defined by the formula $\mathcal{Q} = \mathcal{E} + \mathcal{O}$. The preceding system then takes the form:

$$\frac{dx}{ds_0} = \frac{\partial\mathcal{Q}}{\partial F}, \quad \frac{dy}{ds_0} = \frac{\partial\mathcal{Q}}{\partial G}, \quad \frac{dz}{ds_0} = \frac{\partial\mathcal{Q}}{\partial H},$$
$$\frac{d\lambda_1}{ds_0} = \frac{\partial\mathcal{Q}}{\partial\mathcal{I}}, \quad \frac{d\lambda_2}{ds_0} = \frac{\partial\mathcal{Q}}{\partial\mathcal{J}}, \quad \frac{d\lambda_3}{ds_0} = \frac{\partial\mathcal{Q}}{\partial\mathcal{K}},$$

$$\frac{dF}{ds_0} = -\frac{\partial\mathcal{Q}}{\partial x}, \quad \frac{dG}{ds_0} = -\frac{\partial\mathcal{Q}}{\partial y}, \quad \frac{dH}{ds_0} = -\frac{\partial\mathcal{Q}}{\partial z},$$
$$\frac{d\mathcal{I}}{ds_0} = -\frac{\partial\mathcal{Q}}{\partial\lambda_1}, \quad \frac{d\mathcal{J}}{ds_0} = -\frac{\partial\mathcal{Q}}{\partial\lambda_2}, \quad \frac{d\mathcal{K}}{ds_0} = -\frac{\partial\mathcal{Q}}{\partial\lambda_3}.$$

Here, we have equations that are presented in the form of Hamilton's equations of dynamics. If we suppose, in particular, that the new forms for X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are chosen in such a fashion (and this is always possible) that s_0 does not appear and that they are the partial derivatives of a function $\mathcal{O}$ of x, y, z, λ_1, λ_2, λ_3, and if, in addition, we suppose that $W(s_0, \xi, \eta, \zeta, p, q, r)$ does not depend upon s_0 then we will have, more particularly, a system of canonical equations.

Equations (14), in the case where the external forces and moments are zero, corresponds to Castigliano's principle of minimum work, which was already considered by Vène, Courant, Menabres, and others. The normal form for the equations of the deformable line likewise leads to what one calls the *Castigliano theorems*. Indeed, one has, for example:

$$x_B - x_A = \int_{A_0}^{B_0} \frac{\partial\mathcal{E}}{\partial F} ds_0, \qquad F_B - F_A = \int_{A_0}^{B_0} X_0\, ds_0.$$

If one supposes that $X_0 = Y_0 = Z_0 = 0$ then the effort F, G, H is independent of s_0, and one can write:

$$x_B - x_A = \frac{\partial}{\partial F}\int_{A_0}^{B_0} \mathcal{E}\, ds_0,$$

along with analogous formulas.

5. Notions of hidden trihedron and hidden action. The flexible and inextensible line of Lagrange. The flexible and inextensible line of classical mechanics. – In the study of the deformable line, it is natural to devote special attention to the curve that is traced out by the summit of the trihedron and to consider $\alpha, \alpha', \ldots, \gamma''$ to be auxiliary functions. One is then led to introduce the notion of *hidden trihedron*, and to make a classification of the various circumstances that can present themselves in the elimination of $\alpha, \alpha', \ldots, \gamma''$. One can also abstract from the deformation that permits one to pass from the state (M_0) to the state (M); one often adopts the latter viewpoint in classical mechanics. Finally, one can make some particular hypotheses on the trihedron that is attached to the point M, and similarly, on the curve (M). This amounts to imagining some particular deformations of a deformable line that is entirely free. If the relations that one imposes are true between simply $\xi, \eta, \zeta, p, q, r$ then one can account for them in the calculation of W and deduce some more particular functions from W. The question that is then posed will be that of the direct introduction of the particular forms and the consideration of the general action that serves as the point of departure as being, in some way, *hidden*. We shall show that one can therefore summarize the equations that have been studied up to now by way of some particular cases, since they arise from the same origin.

First suppose that W depends upon only s_0, ξ, η, ζ. Equations (15) then reduce to the following ones:

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dx}{ds_0}} - X_0 = 0, \qquad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{dy}{ds_0}} - Y_0 = 0, \qquad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{dz}{ds_0}} - Z_0 = 0,$$

$$\frac{\partial W}{\partial \lambda_1} + \mathcal{L}_0 = 0, \qquad \frac{\partial W}{\partial \lambda_2} + \mathcal{M}_0 = 0, \qquad \frac{\partial W}{\partial \lambda_3} + \mathcal{N}_0 = 0.$$

Imagine the case where the functions $\mathcal{L}_0, \mathcal{M}_0, \mathcal{N}_0$ are zero. The equations $\frac{\partial W}{\partial \lambda_1} = 0$, $\frac{\partial W}{\partial \lambda_2} = 0$, $\frac{\partial W}{\partial \lambda_3} = 0$ then amount to:

$$\frac{F}{\frac{dx}{ds}} = \frac{G}{\frac{dy}{ds}} = \frac{H}{\frac{dz}{ds}},$$

and upon denoting the common value of these ratios by $-T$, the result of eliminating $\lambda_1, \lambda_2, \lambda_3$ can be written:

$$(16) \qquad \frac{d}{ds}\left(T\frac{dx}{ds}\right) + X = 0, \qquad \frac{d}{ds}\left(T\frac{dy}{ds}\right) + Y = 0, \qquad \frac{d}{ds}\left(T\frac{dz}{ds}\right) + Z = 0$$

with respect to the deformed line. The effort now reduces to an *effort of tension* T. Let the two states of the line, (M_0) and (M), be given. When the functions $\mathcal{L}_0, \mathcal{M}_0, \mathcal{N}_0$ are zero, this result can present itself accidentally; however, it can also happen that it presents itself for any deformed line (M) and as a consequence of the form of W. The function W then depends simply upon s_0 and $\xi^2 + \eta^2 + \zeta^2$, or – what amounts to the same thing – upon s_0 and $\mu = ds / ds_0 - 1$ (where m represents the linear dilatation at the point M), and one has $T = - \partial W / \partial \mu$. If we consider the particular case in which s_0 does not appear explicitly then we arrive in this manner at Lagrange's theory of the *flexible filament that is, at the same time, extensible and contractible,* which was then taken up again by Lamé and Duhem. At no point do we have to appeal – even indirectly – to the notion of dynamical force, which was introduced into mechanics by Lagrange only because of a remark of J. Bertrand. We have argued on the basis of the static force that is *measured by means of the deformation.*

How can we conceive of the *flexible and inextensible filament* while maintaining the same viewpoint? It will suffice for us to follow the path that is usually adopted, *but in the opposite sense.* We impose the condition upon the general deformable line of no. 2 that an arbitrary portion of (M) must have the same length as the *corresponding* portion of (M_0), which amounts to saying that one subjects x, y, z to the condition that $ds = ds_0$. Instead of considering an arbitrary deformation (M) of the natural state (M_0), we direct our attention to the deformed (M) for which one has $ds = ds_0$. We use the formulas of no. 2 as the definition of force, and apply them to the positions of the deformable line that coincide with those of the given inextensible line. In particular, if we imagine the flexible and inextensible line then we can define the force X, Y, Z by the system (16), where T represents the function of s that is defined by the formula $T = -\left(\dfrac{\partial W}{\partial \mu}\right)_{\mu=0}$. In order to obtain a *determinate* problem, it will not be necessary, moreover, to suppose that the function T is known; it will suffice to adjoin some convenient conditions at the extremities to the system (16).

We shall not insist upon the case in which $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are non-zero. It corresponds to the case that was imagined by Darboux in which the line is subject to an external moment that is analogous to a magnetic moment.

6. Deformable line where the axis Mx' is tangent to (M) at M. Deformable line of Lord Kelvin and Tait. Equations of Binet and Wantzel. – Consider just the deformations (M) of the general deformable line for which the axis Mx' is tangent to the curve (M) at each point, and also suppose that $M_0x'_0$ is tangent to (M_0) at M_0, so that these deformations will define a continuous sequence that starts at (M_0). This amounts to the conditions $\alpha_0 = \dfrac{dx_0}{ds}$, $\alpha'_0 = \dfrac{dy_0}{ds}$, $\alpha''_0 = \dfrac{dz_0}{ds}$; $\alpha = \dfrac{dx}{ds}$, $\alpha' = \dfrac{dy}{ds}$, $\alpha'' = \dfrac{dz}{ds}$, where or $\eta_0 = \eta = 0$, $\zeta_0 = \zeta = 0$. The fact that we have limited the study of deformations of (M0) to those deformed (M) that verify the latter conditions and the fact that we have admitted the new concept of a line that is susceptible to only the deformations in question can be regarded as identical here. This conforms absolutely to the principle of solidification that is introduced by classical authors *in the opposite order* to the one that we followed.

Let β_1, β_1', β_1'' be the direction cosines of the principal normal to the curve (M) at M with respect to the fixed axes Ox, Oy, Oz, resp., let γ_1, γ_1', γ_1'' those of the binormal, and let ω be the angle that the axis My' makes with the principal normal. One gets the formulas:

$$p\frac{ds_0}{ds} = \frac{1}{\tau} - \frac{d\omega}{ds}, \qquad q\frac{ds_0}{ds} = -\frac{\sin\omega}{\rho}, \qquad r\frac{ds_0}{ds} = \frac{\cos\omega}{\rho},$$

upon setting $\frac{1}{\rho} = \sum \beta_1 \frac{d\alpha}{ds}$, $\frac{1}{\tau} = \sum \gamma_1 \frac{d\beta_1}{ds}$, and recalling that $\sum \alpha \frac{d\gamma_1}{ds} = 0$. The expressions $1 / \rho$ and $1 / \tau$ are equal in absolute value to the curvature and torsion (*cambrure* to Barré de Saint-Venant, *tortuosity* to Lord Kelvin and Tait) at M.

When one takes into account the conditions that are imposed upon the deformed (M), one can imagine that the action W is partially hidden and defined simply by the knowledge of $W(s_0, \xi, \eta, \zeta, p, q, r)$. Upon setting $\xi = ds / ds_0 = 1 + \mu$, F, G, H become three auxiliary variables, in regard to which one knows only that one has:

$$F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds} = \frac{\partial W}{\partial \mu} = -T.$$

One can propose to eliminate them from the system (12), and one then has the four equations:

$$(17)\qquad \begin{cases} \dfrac{d}{ds}\left[-T\dfrac{dx}{ds} + \left(\dfrac{dK}{ds} - N\right)\dfrac{dy}{ds} - \left(\dfrac{dJ}{ds} - M\right)\dfrac{dz}{ds}\right] - X = 0, \\ \dfrac{d}{ds}\left[-T\dfrac{dy}{ds} + \left(\dfrac{dI}{ds} - L\right)\dfrac{dz}{ds} - \left(\dfrac{dK}{ds} - N\right)\dfrac{dz}{ds}\right] - Y = 0, \\ \dfrac{d}{ds}\left[-T\dfrac{dz}{ds} + \left(\dfrac{dJ}{ds} - M\right)\dfrac{dx}{ds} - \left(\dfrac{dI}{ds} - L\right)\dfrac{dy}{ds}\right] - Z = 0, \\ \left(\dfrac{dI}{ds} - L\right)\dfrac{dx}{ds} + \left(\dfrac{dJ}{ds} - M\right)\dfrac{dy}{ds} + \left(\dfrac{dK}{ds} - N\right)\dfrac{dz}{ds} = 0, \end{cases}$$

in which one has replaced I, J, K, T with their values as functions of the direction cosines of the axes of the trihedron $Mx'y'z'$ and the derivatives of the partially-hidden action W with respect to p, q, r, μ. If s does not figure explicitly in the givens then one can appeal to the relation $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1$ in order to eliminate ds, and the relations (17) will provide four differential equations that define x, y, z, ω as functions of s_0.

It is remarkable that the system (17) can be converted into a form that one can deduce from the calculus of variations. We shall not develop the calculations that lead to that reduction here, but we shall give only the result. Here, the expression $W(s_0, 1 + \mu, 0, 0, p,$

q, r) depends upon s_0, in addition, by the intermediary of μ, p, q, r, the arguments ω, $\frac{d\omega}{ds_0}$, and the derivatives of the first three orders of x, y, z with respect to s_0 . One arrives at a system that can be summarized in the equation:

$$\int_{s_0}^{s_0'} (\delta W + \mathcal{X}_0 \delta x + \mathcal{Y}_0 \delta y + \mathcal{Z}_0 \delta z - L_0' \delta\omega)\, ds_0 = 0 \tag{18}$$

upon setting:

$$\mathcal{X}_0 = X_0 + \frac{d^2}{ds_0^2}\left[\frac{\gamma_1 \rho}{\left(\frac{ds}{ds_0}\right)^2} L'\right] + \frac{d}{ds_0}\left[\frac{\gamma_1 \rho \frac{d^2 s}{ds_0^2}}{\left(\frac{ds}{ds_0}\right)^2} L'\right] + \frac{d}{ds_0}\left[\frac{ds_0}{ds}(\alpha' N_0 - \alpha'' M_0)\right],$$

with two analogous formulas, and upon writing $L_0' = L_0 \frac{dx}{ds} + M_0 \frac{dy}{ds} + N_0 \frac{dz}{ds}$, conforming to our earlier notations.

The preceding considerations are attached to the deformable line that was studied by Lord Kelvin and Tait, which one can deduce from the preceding or imagine directly. It will suffice to add the following condition $\mu = 0$ – i.e., $\xi = 1$ – to the ones that were considered. Equation (18) is presently true, by virtue of the fact that $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1$, and one has, more simply:

$$\mathcal{X}_0 = X_0 + \frac{d^2}{ds_0^2}(\gamma_1 \rho L_0') + \frac{d}{ds_0}(\alpha' N_0 - \alpha'' M_0),$$

with two analogous formulas.

Instead of employing equations (17), it can be more convenient to return to the starting equations. For example, suppose that X_0, Y_0, Z_0 are zero. One concludes from this that F, G, H are constants that are equal to the values F_{A_0}, G_{A_0}, H_{A_0} at one of the extremities A_0 , and one then has the three equations:

$$\begin{aligned}
\frac{dI}{ds_0} + H_{A_0} \frac{dy}{ds_0} - G_{A_0} \frac{dz}{ds_0} - L_0 &= 0, \\
\frac{dJ}{ds_0} + F_{A_0} \frac{dz}{ds_0} - H_{A_0} \frac{dx}{ds_0} - N_0 &= 0, \\
\frac{dK}{ds_0} + G_{A_0} \frac{dx}{ds_0} - F_{A_0} \frac{dy}{ds_0} - N_0 &= 0,
\end{aligned}$$

which are the original equations, and which *presently* result from the elimination of T from the system (17). If one has L_0, M_0, N_0 equal to zero, in addition – i.e., if the deformed (M) is subject to only forces that are applied to the extremities – then one will have:

$$
(19) \qquad \left\{ \begin{aligned} & I + H_{A_0} y - G_{A_0} z = \text{const.}, \\ & J + F_{A_0} y - H_{A_0} z = \text{const.}, \\ & K + G_{A_0} y - F_{A_0} z = \text{const.} \end{aligned} \right.
$$

Having made these remarks, consider the case in which the function W of s_0, p, q, r is of the form $\frac{1}{2}\,A(q^2 + r^2) + Bp + C$, where A, B, C are constants. One will have $I' = B$, $J' = Aq$, $K' = Ar$. The vector (I', J', K') or (I, J, K) is the resultant of a constant vector that is equal to B and directed along the tangent Mx' and a vector that is directed along the binormal that has the same absolute value as A / ρ. The three equations (19) are, up to notations, identical to the equations:

$$
\begin{aligned}
\varpi \frac{dy\,d^2z - dz\,d^2y}{ds^2} &= \theta \frac{dx}{ds} + cy - bz + a_1, \\
\varpi \frac{dz\,d^2x - dx\,d^2z}{ds^2} &= \theta \frac{dy}{ds} + az - cx + a_2, \\
\varpi \frac{dx\,d^2y - dy\,d^2x}{ds^2} &= \theta \frac{dz}{ds} + bx - ay + a_3,
\end{aligned}
$$

that were considered by Binet, Wantzel, and Hermite, and in which ϖ, θ, a, b, c, a_1, a_2, a_3 are constants. Lagrange has considered the case in which $\theta = 0$, and J. Bertrand has treated the one in which the three equations:

$$
cy - bz + a_1 = 0, \qquad az - cx + a_2 = 0, \qquad bx - ay + a_3 = 0
$$

represent a line – i.e., the case in which the couple $(I_{A_0}, J_{A_0}, K_{A_0})$ and the effort $(F_{A_0}, G_{A_0}, H_{A_0})$ have a unique resultant.

One can present the foregoing as follows: If the effort of deformation of the line considered at the beginning of this section is perpendicular to the principal normal then one will have $r\dfrac{\partial W}{\partial q} - q\dfrac{\partial W}{\partial r} = 0$. If one supposes that this condition results from the nature of the line then W must depend upon q and r only by the intermediary of $q^2 + r^2$. If we suppose that this condition is verified then, from the remark of Poisson that was mentioned in no. 3, them the equations of the problem will imply that $I' = \text{const}$. If we suppose that this condition results from the nature of the line then this will amount to the condition that:

$$
\frac{\partial W}{\partial p} = B,
$$

B being a constant, and we will find that:

$$W = Bp + \varphi,$$

in which φ is a function of $q^2 + r^2 = 1 / \rho^2$. Upon supposing that φ is of first degree with respect to $q^2 + r^2$, we recover the action that was envisioned above.

7. Deformable line for which the plane $Mx'y'$ osculates (M) at M. Lagrange's line, as generalized by Binet and studied by Poisson. – Instead of simply supposing that Mx' is the tangent to the curve (M), we can imagine the case in which the plane $Mx'y'$ is the osculating plane to that curve. We then have the relations $\eta_0 = \eta = 0$, $\zeta_0 = \zeta = 0$, $q_0 = q = 0$, to which, in order to simplify the question, we append the inextensibility condition $\mu = 0$, or $\xi = 1$. We continue to let W denote the partially-hidden action:

$$W(s_0, 1, 0, 0, p, 0, r)$$

and suppose that this single function is known. The quantities F', G', H' become four *auxiliary functions*, and we have simply the relations $I' = \partial W / \partial p$, $J' = \partial W / \partial r$. The three equations (11) of the line give, in particular, $\dfrac{dI'}{ds_0} - rJ' = 0$ when the given functions L'_0, M'_0, N'_0 are zero. If the function W does not depend upon p then we will have $I' = 0$, and in turn, $J' = 0$, if we suppose that $r \neq 0$. Therefore, in the present case the moment of deformation is directed along the binormal of the curve (M). In this manner, we have a line such as the one that Lagrange considered. The result that was obtained on the moment of deformation and equations (12) for the line permit us to set:

$$F = \lambda\frac{dx}{ds} - d(\mathcal{I}\, d^2x), \quad G = \lambda\frac{dy}{ds} - d(\mathcal{I}\, d^2y), \quad H = \lambda\frac{dz}{ds} - d(\mathcal{I}\, d^2z).$$

If we substitute these in the left-hand side of equations (12) then we will have the same equations as Lagrange:

$$X\, ds - d\frac{\lambda\, dx}{ds} + d^2(\mathcal{I}\, d^2x) = 0,$$
$$Y\, ds - d\frac{\lambda\, dy}{ds} + d^2(\mathcal{I}\, d^2y) = 0,$$
$$Z\, ds - d\frac{\lambda\, dz}{ds} + d^2(\mathcal{I}\, d^2z) = 0.$$

For this type of deformable line, the moment of deformation is normal to the osculating plane. Binet proposed to consider the case in which the moment of deformation is simply perpendicular to the principal normal. The hypothesis $J' = 0$ implies that $\dfrac{dI'}{ds_0} = 0$, and if we assume that this result depends upon the specialization of W then we will have, as a consequence, that $W = \varphi(s_0, r) + mp$, in which m is a constant.

Upon supposing, in particular, that $\partial\varphi / \partial r$ reduces to an expression of the form $n (r - r_0)$, where n is a constant, one has, by replacing r_0 as a function of s_0, the hypothesis that was made explicitly by Binet and then developed by Poisson. If, in addition, the curve (M_0) is a straight line and if X, Y, Z are zero, in such a way that changing (M_0) into (M) comes about solely from the efforts and moments of deformation that are applied to the extremities, then one will recover the problem of Binet and Wantzel that we were occupied with in the preceding section. Upon supposing that $m = 0$, one falls back upon the Lagrange case above.

8. Deformable line that is subject to constraints. Canonical equations. – In all of the foregoing, we have considered a deformable line that we qualified with the word *free*; i.e., the theory was developed without making external elements intervene and by means of a function W that was composed of the proper elements of the line in its natural state and deformed state.

By directing our attention to certain deformations, using the notion of a *hidden W*, we have recovered the equations that were proposed for various lines by the authors.

One can replace that exposition with another one in which one envisions a line that is deformable *sui generis*, *for which the definition already takes into account* some well-defined conditions that are verified by the particular deformations that we studied previously.

First, observe that the conditions that are imposed upon the functions $x, y, z, \alpha, \alpha', \ldots, \gamma''$ can be of two kinds:

1. Conditions between these functions and their derivatives, where s_0 is arbitrary.
2. Conditions that are verified for certain values of s_0 .

If we limit ourselves to conditions of the first kind and if, to fix ideas, $f_1 = 0$, $f_2 = 0$ are two conditions or constraint equations then we will agree that the identity of no. 2 that introduces the definitions of the forces and efforts must presently be true by virtue of the two constraint equations, or we further envision a deformable line whose theory results, by definition, from a function $W(s_0, \xi, \eta, \zeta, p, q, r)$ and two auxiliary functions λ_1 , λ_2 of s_0 , by means of the identity:

$$\int_{A_0}^{B_0} (\delta W + \lambda_1\, \delta f_1 + \lambda_2\, \delta f_2)\, ds_0 = \left[F'\delta' x + G'\delta' y + \ldots\right]_{A_0}^{B_0} - \int_{A_0}^{B_0} (X_0'\,\delta' x + Y_0'\,\delta' y + \ldots)\, ds_0 ,$$

in which all of the variations are arbitrary, this time.

We further remark that in the case where some of the left-hand sides $f_1, f_2, \ldots$ of the constraint equations refer to only the arguments that appear in W, one can conceive of either a process such as we just spoke of or, by a change of auxiliary variables, one introduces the given of these particular constraint equations into *W a priori*; this once more brings us to the notion of a *hidden W*. This new way of looking at things is especially interesting in the study of the particular lines that we studied previously, and it leads notably to an extension of the results of Clebsch to all cases, as well as the reduction to the canonical form that was obtained for the flexible and inextensible line by Appell, Legoux, and Marcolongo.

9. States that are infinitely close to the natural state. Hooke's moduli of deformation and general moduli. Critical values of the general moduli. – Let us return to the general deformable line. Suppose that the action is zero in the natural state, as well as the effort and moment of deformations, and similarly, the external force and moment. In this case, not just the function W must be annulled identically, but also the six partial derivatives of W with respect to ξ, η, ζ, p, q, r for the values ξ_0, η_0, ζ_0, p_0, q_0, r_0, resp., of these symbols. Assume, moreover, that W is developable in a neighborhood of $\xi = \xi_0$, $\eta = \eta_0$, $\zeta = \zeta_0$, $p = p_0$, $q = q_0$, $r = r_0$ in positive integer powers of $\xi - \xi_0$, $\eta - \eta_0$, …, $r - r_0$. Under these conditions, one will have:

$$W = W_2 + W_3 + \ldots,$$

in which W_2, W_3, … represent homogeneous polynomials of degree 2, 3, … in the differences $\xi - \xi_0$, $\eta - \eta_0$, …, $r - r_0$.

Let the coordinates of a point M_0 of the line (M_0) in the natural state and the three parameters by means of which one expresses the direction cosines of the axes of the trihedron that is attached to that point be x_0, y_0, z_0, λ_{10}, λ_{20}, λ_{30} , respectively, and suppose that the coordinates x, y, z of the corresponding points M in the deformed state (M) and the parameters λ_1, λ_2, λ_3 that relate to the axes of the trihedron that is attached to it are functions of s_0 and h that are developable in powers of h by the formulas:

$$\begin{aligned} x &= x_0 + x_1 + \ldots + x_i + \ldots, & \lambda_1 &= \lambda_{10} + \lambda_{11} + \ldots + \lambda_{1i} + \ldots, \\ y &= y_0 + y_1 + \ldots + y_i + \ldots, & \lambda_2 &= \lambda_{20} + \lambda_{21} + \ldots + \lambda_{2i} + \ldots, \\ z &= z_0 + z_1 + \ldots + z_i + \ldots, & \lambda_3 &= \lambda_{30} + \lambda_{31} + \ldots + \lambda_{3i} + \ldots, \end{aligned}$$

in which x_i , y_i , z_i , λ_{1i} , λ_{2i} , λ_{3i} denote the terms that involve h^i as a factor. We introduce these series developments in order to abbreviate the expositions, and we assume that they pertain to the ordinary calculation procedures. Formulas (15) and (15′) permit us to calculate the developments of F, G, H, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$; X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ in powers of h. The terms that are independent of h are zero, and the terms F_1, G_1, H_1, $\mathcal{I}_1$, $\mathcal{J}_1$, $\mathcal{K}_1$; X_{01}, Y_{01}, Z_{01}, $\mathcal{L}_{01}$, $\mathcal{M}_{01}$, $\mathcal{N}_{01}$ are given by the formulas:

$$F_1 = \frac{\partial W_2}{\partial \dfrac{dx^{(1)}}{ds_0}}, \qquad G_1 = \frac{\partial W_2}{\partial \dfrac{dy^{(1)}}{ds_0}}, \qquad H_1 = \frac{\partial W_2}{\partial \dfrac{dz^{(1)}}{ds_0}},$$

$$\mathcal{I}_1 = \frac{\partial W_2}{\partial \dfrac{d\lambda_1^{(1)}}{ds_0}}, \qquad \mathcal{J}_1 = \frac{\partial W_2}{\partial \dfrac{d\lambda_2^{(1)}}{ds_0}}, \qquad \mathcal{K}_1 = \frac{\partial W_2}{\partial \dfrac{d\lambda_3^{(1)}}{ds_0}},$$

$$X_{01} = \frac{d}{ds_0}\frac{\partial W_2}{\partial \dfrac{dx^{(1)}}{ds_0}}, \qquad Y_{01} = \frac{d}{ds_0}\frac{\partial W_2}{\partial \dfrac{dy^{(1)}}{ds_0}}, \qquad Z_{01} = \frac{d}{ds_0}\frac{\partial W_2}{\partial \dfrac{dz^{(1)}}{ds_0}},$$

$$\mathcal{L}_{01} = \frac{d}{ds_0}\frac{\partial W_2}{\partial \dfrac{d\lambda_1^{(1)}}{ds_0}} - \frac{\partial W_2}{\partial \lambda_1^{(1)}}, \quad \mathcal{M}_{01} = \frac{d}{ds_0}\frac{\partial W_2}{\partial \dfrac{d\lambda_2^{(1)}}{ds_0}} - \frac{\partial W_2}{\partial \lambda_2^{(1)}}, \quad \mathcal{N}_{01} = \frac{d}{ds_0}\frac{\partial W_2}{\partial \dfrac{d\lambda_3^{(1)}}{ds_0}} - \frac{\partial W_2}{\partial \lambda_3^{(1)}},$$

in which one sets:

$$x^{(1)} = x_0 + x_1, \qquad y^{(1)} = y_0 + y_1, \qquad z^{(1)} = z_0 + z_1,$$
$$\lambda_1^{(1)} = \lambda_{10} + \lambda_{11}, \quad \lambda_2^{(1)} = \lambda_{20} + \lambda_{21}, \quad \lambda_3^{(1)} = \lambda_{30} + \lambda_{31}.$$

We consider – under the name of a *state of deformation that is infinitely close to the natural state* – the state (M) where the point M has the coordinates $x^{(1)}$, $y^{(1)}$, $z^{(1)}$, and where the parameters that relate to the trihedron that is attached to it have the values $\lambda_1^{(1)}$, $\lambda_2^{(1)}$, $\lambda_3^{(1)}$. On the other hand, if we let the terms *effort, moment of deformation, external force, and external moment* refer to the vectors (F_1, G_1, H_1), ($\mathcal{I}_1$, $\mathcal{J}_1$, $\mathcal{K}_1$), (X_{01}, Y_{01}, Z_{01}), (L_{01}, M_{01}, N_{01}), where L_{01}, M_{01}, N_{01} are calculated by means of λ_{10}, λ_{20}, λ_{30}, $\mathcal{L}_{01}$, $\mathcal{M}_{01}$, $\mathcal{N}_{01}$ in the same fashion as L_0, M_0, N_0 are calculated by means of λ_1, λ_2, λ_3, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$, then we will arrive at the hypotheses that were generally made by the classical authors, where the first two vectors are linear functions of the elements that characterize the deformed state considered. As a consequence, we recover what one calls the *generalized Hooke law*, but when one limits it, as is convenient, by *the condition that it must respect the principle of the conservation of energy*. In the classical method, in order to satisfy that condition one must repeat the path that we just followed in our exposition, but in the opposite sense.

The coefficients in the linear functions that express Hooke's law are the *moduli of deformation* of the deformable line in its state that is infinitely close to the natural state; they are *invariable* at a given point of the line. This notion of moduli can be generalized by envisioning the first and second derivatives of the function W; aside from the case in which the general moduli are defined and continuous, one can consider the one in which they have critical values.

The preceding considerations can be easily repeated for the various particular deformable lines.

10. Dynamics of the trihedron. – The dynamics of the trihedron is related to the foregoing in a completely direct manner. It suffices to regard the arc s_0 as *time t*, and the deformable line as a *trajectory*. That simple assertion immediately explains the analogies have been known for a long time between the classical dynamics of the point and invariable body and the statics of the deformable line. In a previous work ([1]), to which we shall not return here, we have envisioned dynamics from the viewpoint that we just pointed out.

We recall only that classical dynamics appears to be the study of *states of motion that are infinitely close to the state of rest*. As Lamé has already remarked, kinetic mass presents itself as the power or coefficient of the resistance to change of the motion, a definition that is analogous to that of the coefficients of elasticity. One can, with Laplace, consider an arbitrary state of motion that is not infinitely close to the rest state, and one must then distinguish the kinetic mass from the Hamiltonian mass and the Maupertuisian mass.

Consider, to simplify, the case in which W is independent of p, q, r, and where there is no external moment. The action has the velocity v for its only argument, and the analogue of the effort of deformation is the quantity of motion in the trajectory. In classical dynamics, the work that is done by the quantity of motion is not, like that of the effort of deformation of the deformable line, combined with the work that is done by the external force, but it is united with the variation of the action in order to introduce the notion of *kinetic energy*. In the simple case that we have envisioned, one has:

$$(20) \qquad \left\{ \begin{array}{lll} F = \dfrac{1}{v}\dfrac{dW}{dv}\dfrac{dx}{dt}, & G = \dfrac{1}{v}\dfrac{dW}{dv}\dfrac{dy}{dt}, & H = \dfrac{1}{v}\dfrac{dW}{dv}\dfrac{dz}{dt}, \\ X = \dfrac{dF}{dt}, & Y = \dfrac{dG}{dt}, & Z = \dfrac{dH}{dt}, \end{array} \right.$$

and one deduces from this that:

$$(21) \qquad X\,dx + Y\,dy + Z\,dz = d\left(v\frac{dW}{dv} - W \right).$$

The quantity:

$$(22) \qquad E = v\frac{dW}{dv} - W$$

is what one calls the *kinetic energy*. In the case of the statics of the deformable line, *the energy of deformation is equal to the action of deformation W, with the opposite sign*, as we showed in no. 3. Later on, we shall return to this essential distinction between the action of deformation and energy of deformation.

([1]) E. and F. COSSERAT, "Note sur la dynamique du point et du corps invariable," in *Traité de Physique*, by O.-D. Chwolson, French ed., t. I, pp. 236-273, Paris, 1906.

II. – STATICS OF THE DEFORMABLE SURFACE AND THE DYNAMICS OF THE DEFORMABLE LINE.

11. Euclidian action of deformation on a deformable surface. External force and moment. Effort and moment of deformation. – The developments into which we have entered in regard to the deformable line will permit us to be briefer in regard to the theories of the deformable surface and the three-dimensional deformable medium, in which they are reproduced with almost no changes. We preserve the preceding notations, but now suppose that $x, y, z, \alpha, \alpha', \ldots, \gamma''$ are functions of two parameters ρ_1 and ρ_2, instead of depending upon just the parameter s_0. The trihedron $Mx'y'z'$ then describes what we call a *deformable surface,* and, with Darboux, we will have twelve kinematic arguments $\xi_i, \eta_i, \zeta_i, p_i, q_i, r_i$ ($i = 1, 2$) to consider that are given by formulas (1) and (2), in which the ordinary derivatives with respect to s_0 must be replaced with the partial derivatives with respect to ρ_i. The linear element of the surface will be given by the formula:

$$ds^2 = \mathcal{E}\, d\rho_1^2 + 2\mathcal{F}\, d\rho_1\, d\rho_2 + \mathcal{G}\, d\rho_2^2,$$

$\mathcal{E} = \xi_1^2 + \eta_1^2 + \zeta_1^2$, $\mathcal{F} = \xi_1\,\xi_2 + \eta_1\,\eta_2 + \zeta_1\,\zeta_2$, $\mathcal{G} = \xi_2^2 + \eta_2^2 + \zeta_2^2$. We will have analogous formulas in the undeformed state, which we continue to distinguish by the index zero.

If we set $\Delta_0 = \sqrt{\mathcal{E}_0\mathcal{G}_0 - \mathcal{F}_0^2}$, where $\mathcal{E}_0, \mathcal{F}_0, \mathcal{G}_0$ are the analogues of $\mathcal{E}, \mathcal{F}, \mathcal{G}$ for the natural state, and if we seek the form that the function W of two infinitely close positions of the trihedron $Mx'y'z'$ must take in order for the integral $\iint W\,\Delta_0\, d\rho_1\, d\rho_2$, when taken over an arbitrary portion of the surface (M_0), to have zero variation under a likewise arbitrary infinitesimal transformation from the group of Euclidian displacements then we will be led to the following *remarkable form:*

$$W(\rho_1, \rho_2;\ \xi_1, \eta_1, \zeta_1;\ p_1, q_1, r_1;\ \xi_2, \eta_2, \zeta_2;\ p_2, q_2, r_2).$$

The argument is identical to the one that we gave in no. 2.

Let Δ denote the quantity that is analogous to Δ_0 and is defined by the formula $\Delta = \sqrt{\mathcal{E}\,\mathcal{G} - \mathcal{F}^2}$. If we multiply W by the element of area:

$$d\sigma_0 = \Delta_0\, d\rho_1\, d\rho_2$$

of the surface (M_0) then the product $W\,\Delta_0\, d\rho_1\, d\rho_2$ will be an invariant under the group of Euclidian displacements that is analogous to the area element of the surface (M). Similarly, when the integral $\iint_{C_0} \frac{\Delta}{\Delta_0}\Delta_0\, d\rho_1 d\rho_2 = \iint_C \Delta\, d\rho_1 d\rho_2$ is taken over the interior of a contour C_0 on the surface (M_0), or the corresponding contour C on the surface (M), this will determine the *area* of the domain on (M) that is delimited by C. Likewise, in the same spirit, by associating the action for the passage from the natural state (M_0) to the deformed state (M), we attach the function W to the defining elements of the deformable surface, and we say that the integral $\iint_{C_0} W\Delta_0 \, d\rho_1 d\rho_2$ is the *action of deformation* for the interior of the contour C on the deformed surface. On the other hand, we say that W is the *density* of the action of deformation at a point of the deformed surface when referred to the unit of area of the undeformed surface; $W \Delta_0 / \Delta$ will be that density at a point when referred to the unit of area of the deformed surface.

Consider an *arbitrary* variation of the action of deformation in the interior of a contour C on the surface (M), namely, $\delta\iint_{C_0} W\Delta_0 \, d\rho_1 d\rho_2$. By virtue of formulas (7) and (8), when extended to the case of two independent parameters ρ_1 and ρ_2 , and after applying Green's formula to the terms that refer explicitly to a derivative with respect to ρ_1 or ρ_2 , we will have, upon letting ds_0 denote the absolute value of the element of arc length of the contour C_0 that is traced on the surface (M_0):

$$\begin{aligned}
&\delta\iint_{C_0} W\Delta_0 \, d\rho_1 d\rho_2 \\
&\qquad = \int_{C_0} (F_0'\,\delta'x + G_0'\,\delta'y + H_0'\,\delta'z + I_0'\,\delta i' + J_0'\,\delta j' + K_0'\,\delta k')\, ds_0 \\
&\qquad - \iint_{C_0} (X_0'\,\delta'x + Y_0'\,\delta'y + Z_0'\,\delta'z + L_0'\,\delta i' + M_0'\,\delta j' + N_0'\,\delta k')\,\Delta_0 d\rho_1 d\rho_2 ,
\end{aligned}$$

in which we have set:

$$\begin{aligned}
F_0' &= \Delta_0 \left(\frac{\partial W}{\partial \xi_1}\frac{d\rho_2}{ds_0} - \frac{\partial W}{\partial \xi_2}\frac{d\rho_1}{ds_0} \right), \\
G_0' &= \Delta_0 \left(\frac{\partial W}{\partial \eta_1}\frac{d\rho_2}{ds_0} - \frac{\partial W}{\partial \eta_2}\frac{d\rho_1}{ds_0} \right), \\
H_0' &= \Delta_0 \left(\frac{\partial W}{\partial \zeta_1}\frac{d\rho_2}{ds_0} - \frac{\partial W}{\partial \zeta_2}\frac{d\rho_1}{ds_0} \right), \\
I_0' &= \Delta_0 \left(\frac{\partial W}{\partial p_1}\frac{d\rho_2}{ds_0} - \frac{\partial W}{\partial p_2}\frac{d\rho_1}{ds_0} \right), \\
J_0' &= \Delta_0 \left(\frac{\partial W}{\partial q_1}\frac{d\rho_2}{ds_0} - \frac{\partial W}{\partial q_2}\frac{d\rho_1}{ds_0} \right), \\
K_0' &= \Delta_0 \left(\frac{\partial W}{\partial r_1}\frac{d\rho_2}{ds_0} - \frac{\partial W}{\partial r_2}\frac{d\rho_1}{ds_0} \right),
\end{aligned}$$

in which the signs of $d\rho_1$ and $d\rho_2$ are specified by the sense of positive traversal of the curvilinear integral, and, in addition:

$$X_0' = \sum_i \left[\frac{1}{\Delta_0}\frac{\partial}{\partial \rho_i}\left(\Delta_0 \frac{\partial W}{\partial \xi_i} \right) + q_i \frac{\partial W}{\partial \zeta_i} - r_i \frac{\partial W}{\partial \eta_i} \right],$$

$$Y_0' = \sum_i \left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial\eta_i}\right) + r_i\frac{\partial W}{\partial\xi_i} - p_i\frac{\partial W}{\partial\zeta_i}\right],$$

$$Z_0' = \sum_i \left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial\zeta_i}\right) + p_i\frac{\partial W}{\partial\eta_i} - q_i\frac{\partial W}{\partial\xi_i}\right]$$

and

$$L_0' = \sum_i \left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial p_i}\right) + q_i\frac{\partial W}{\partial r_i} - r_i\frac{\partial W}{\partial q_i} + \eta_i\frac{\partial W}{\partial\zeta_i} - \zeta_i\frac{\partial W}{\partial\eta_i}\right],$$

$$M_0' = \sum_i \left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial q_i}\right) + r_i\frac{\partial W}{\partial p_i} - p_i\frac{\partial W}{\partial r_i} + \zeta_i\frac{\partial W}{\partial\xi_i} - \xi_i\frac{\partial W}{\partial\zeta_i}\right],$$

$$N_0' = \sum_i \left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial r_i}\right) + p_i\frac{\partial W}{\partial q_i} - q_i\frac{\partial W}{\partial p_i} + \xi_i\frac{\partial W}{\partial\eta_i} - \eta_i\frac{\partial W}{\partial\xi_i}\right].$$

Upon first regarding the double integral that figures in the expression for $\delta\iint_{C_0} W\Delta_0\, d\rho_1 d\rho_2$, we shall call the line segments that have their origins at M and whose projections onto the axes Mx', My', Mz' are X_0', Y_0', Z_0' and L_0', M_0', N_0', respectively, the *external force and external moment at the point M, when referred to the unit of area of the undeformed surface*. Upon then regarding the curvilinear integral, we then call the line segments that issue from M and whose projections onto the axes Mx', My', Mz' are $-F_0'$, $-G_0'$, $-H_0'$ and $-I_0'$, $-J_0'$, $-K_0'$, respectively, the *external effort and external moment of deformation at the point M on the contour C* of the deformed surface, *when referred to the unit of length on the contour* C_0. As is easy to see, at a well-defined point M of C, these last six quantities *depend upon only the direction of the exterior normal* to the curve C_0, when it is drawn through the point M_0 in the plane that is tangent to (M_0). It remains invariable if the direction of the exterior normal does not change when the region of (M_0) considered varies, and it changes sign if that direction is replaced with the opposite direction.

Suppose that one traces out a line Σ in the interior of the deformed surface that is bounded by the contour C that circumscribes a subset (A) of the surface, either by itself or along with a portion of the contour C, and let (B) denote what remains of the surface outside of the subset (A). Let Σ_0 be the curve on (M_0) that corresponds to the curve Σ of (M), and let (A_0) and (B_0) be the regions of (M_0) that correspond to the regions (A) and (B) of (M), resp. Mentally separate the two subsets (A) and (B). One can regard the two line segments ($-F_0'$, $-G_0'$, $-H_0'$) and ($-I_0'$, $-J_0'$, $-K_0'$) that are determined for the point M and for the direction of the normal that is drawn through Σ_0 in the plane that is tangent to (M_0) and to the exterior of (A_0) as the external effort and moment of deformation, resp., at the point M of the contour Σ to the region (A). One can likewise regard the two line segments ($+F_0'$, $+G_0'$, $+H_0'$) and ($+I_0'$, $+J_0'$, $+K_0'$) as the external effort and moment of deformation, resp., at the point M of the contour Σ to the region (B). By reason of that remark, we say that $-F_0'$, $-G_0'$, $-H_0'$ and $-I_0'$, $-J_0'$, $-K_0'$ are the components along the axes Mx', My', Mz' of *the effort and moment of deformation, resp., that are exerted at M on the portion* (A) *of the surface* (M), and that $+F_0'$, $+G_0'$, $+H_0'$ and $+I_0'$, $+J_0'$, $+K_0'$ are the components along the axes Mx', My', Mz' of *the effort and moment of deformation, resp., that are exerted at M on the portion* (B) *of the surface* (M).

12. Diverse specifications of the effort and moment of deformation. Notions of the energy of deformation and the natural state. – Set:

$$A_i' = \Delta_0 \frac{\partial W}{\partial \xi_i}, \qquad B_i' = \Delta_0 \frac{\partial W}{\partial \eta_i}, \qquad C_i' = \Delta_0 \frac{\partial W}{\partial \zeta_i},$$
$$P_i' = \Delta_0 \frac{\partial W}{\partial p_i}, \qquad Q_i' = \Delta_0 \frac{\partial W}{\partial q_i}, \qquad R_i' = \Delta_0 \frac{\partial W}{\partial r_i}.$$

$\frac{1}{\sqrt{\mathcal{G}_0}} A_1'$, $\frac{1}{\sqrt{\mathcal{G}_0}} B_1', \frac{1}{\sqrt{\mathcal{G}_0}} C_1'$ and $\frac{1}{\sqrt{\mathcal{G}_0}} P_1', \frac{1}{\sqrt{\mathcal{G}_0}} Q_1', \frac{1}{\sqrt{\mathcal{G}_0}} R_1'$ represent the projections onto Mx', My', Mz' of the effort and the moment of deformation, resp., that are exerted at the point M on a curve that admits the same tangent as ρ_1 = const. That effort and moment of deformation *are referred to the unit of length of the undeformed contour.* In regard to ρ_1 = const., the effort and moment of deformation have the projections $\frac{1}{\sqrt{\mathcal{G}_0}} A_2'$, $\frac{1}{\sqrt{\mathcal{G}_0}} B_2', \frac{1}{\sqrt{\mathcal{G}_0}} C_2'$ and $\frac{1}{\sqrt{\mathcal{G}_0}} P_2', \frac{1}{\sqrt{\mathcal{G}_0}} Q_2', \frac{1}{\sqrt{\mathcal{G}_0}} R_2'$, respectively. The new efforts and momenta of deformations that we just defined are coupled to the elements that were introduced in no. 11 by the following relations:

$$\Delta_0 X_0' = \sum_i \left(\frac{\partial A_i'}{\partial \rho_i} + q_i C_i' - r_i B_i' \right), \qquad F_0' = A_1' \frac{d\rho_2}{ds_0} - A_2' \frac{d\rho_1}{ds_0},$$
$$\Delta_0 Y_0' = \sum_i \left(\frac{\partial B_i'}{\partial \rho_i} + r_i A_i' - p_i C_i' \right), \qquad G_0' = B_1' \frac{d\rho_2}{ds_0} - B_2' \frac{d\rho_1}{ds_0},$$
$$\Delta_0 Z_0' = \sum_i \left(\frac{\partial C_i'}{\partial \rho_i} + p_i B_i' - q_i A_i' \right), \qquad H_0' = C_1' \frac{d\rho_2}{ds_0} - C_2' \frac{d\rho_1}{ds_0},$$

$$\Delta_0 L_0' = \sum_i \left(\frac{\partial P_i'}{\partial \rho_i} + q_i R_i' - r_i Q_i' + \eta_i C_i' - \zeta_i B_i' \right), \qquad I_0' = P_1' \frac{d\rho_2}{ds_0} - P_2' \frac{d\rho_1}{ds_0},$$
$$\Delta_0 M_0' = \sum_i \left(\frac{\partial Q_i'}{\partial \rho_i} + r_i P_i' - p_i R_i' + \zeta_i A_i' - \xi_i C_i' \right), \qquad J_0' = Q_1' \frac{d\rho_2}{ds_0} - Q_2' \frac{d\rho_1}{ds_0},$$
$$\Delta_0 N_0' = \sum_i \left(\frac{\partial R_i'}{\partial \rho_i} + p_i Q_i' - q_i P_i' + \xi_i B_i' - \eta_i A_i' \right), \qquad K_0' = R_1' \frac{d\rho_2}{ds_0} - R_2' \frac{d\rho_1}{ds_0}.$$

One can propose, as in no. 3, to transform the relations that we just wrote down *independently of the values of the quantities that figure in them that are calculated by means of W*. Indeed, instead of defining the line segments that we have attached to the point M by their projections onto Mx', My', Mz', we can just as well define them by their projections onto other axes.

First, consider the fixed axes Ox, Oy, Oz. Let X_0, Y_0, Z_0 and L_0, M_0, N_0 denote the projections onto these axes of the external force and external moment, respectively, at an arbitrary point M of the deformed medium. Let F_0, G_0, H_0 and I_0, J_0, K_0 denote the projections of the effort and moment of deformation, resp., that relate to the direction $(d\rho_1, d\rho_2)$ of the tangent to a curve C, which are referred to the unit of length of the undeformed curve C_0, and which were defined previously; let A_i, B_i, C_i and P_i, Q_i, R_i be the projections of the effort (A_i', B_i', C_i') and the moment of deformation (P_i', Q_i', R_i'), resp. The transforms of the preceding relations are obviously:

$$\Delta_0 X_0 = \frac{\partial A_1}{\partial \rho_1} + \frac{\partial A_2}{\partial \rho_2}, \qquad F_0 = A_1 \frac{d\rho_2}{ds_0} - A_2 \frac{d\rho_1}{ds_0},$$

$$\Delta_0 Y_0 = \frac{\partial B_1}{\partial \rho_1} + \frac{\partial B_2}{\partial \rho_2}, \qquad G_0 = B_1 \frac{d\rho_2}{ds_0} - B_2 \frac{d\rho_1}{ds_0},$$

$$\Delta_0 Z_0 = \frac{\partial C_1}{\partial \rho_1} + \frac{\partial C_2}{\partial \rho_2}, \qquad H_0 = C_1 \frac{d\rho_2}{ds_0} - C_2 \frac{d\rho_1}{ds_0},$$

$$\Delta_0 L_0 = \frac{\partial P_1}{\partial \rho_1} + \frac{\partial P_2}{\partial \rho_2} + C_1 \frac{\partial y}{\partial \rho_1} + C_2 \frac{\partial y}{\partial \rho_2} - B_1 \frac{\partial z}{\partial \rho_1} - B_2 \frac{\partial z}{\partial \rho_2}, \qquad I_0 = P_1 \frac{d\rho_2}{ds_0} - P_2 \frac{d\rho_1}{ds_0},$$

$$\Delta_0 M_0 = \frac{\partial Q_1}{\partial \rho_1} + \frac{\partial Q_2}{\partial \rho_2} + A_1 \frac{\partial z}{\partial \rho_1} + A_2 \frac{\partial z}{\partial \rho_2} - C_1 \frac{\partial x}{\partial \rho_1} - C_2 \frac{\partial x}{\partial \rho_2}, \qquad J_0 = Q_1 \frac{d\rho_2}{ds_0} - Q_2 \frac{d\rho_1}{ds_0},$$

$$\Delta_0 N_0 = \frac{\partial R_1}{\partial \rho_1} + \frac{\partial R_2}{\partial \rho_2} + B_1 \frac{\partial x}{\partial \rho_1} + B_2 \frac{\partial x}{\partial \rho_2} - A_1 \frac{\partial y}{\partial \rho_1} - A_2 \frac{\partial y}{\partial \rho_2}, \qquad K_0 = R_1 \frac{d\rho_2}{ds_0} - R_2 \frac{d\rho_1}{ds_0}.$$

$\frac{d\rho_1}{ds_0}$ and $\frac{d\rho_2}{ds_0}$ can be replaced by:

$$-\frac{1}{\Delta_0}\left(\lambda_0 \frac{\partial x_0}{\partial \rho_2} + \mu_0 \frac{\partial y_0}{\partial \rho_2} + \nu_0 \frac{\partial z_0}{\partial \rho_2}\right), \qquad -\frac{1}{\Delta_0}\left(\lambda_0 \frac{\partial x_0}{\partial \rho_1} + \mu_0 \frac{\partial y_0}{\partial \rho_1} + \nu_0 \frac{\partial z_0}{\partial \rho_1}\right),$$

respectively, and $\frac{d\rho_1}{ds}$ and $\frac{d\rho_2}{ds}$, by:

$$-\frac{1}{\Delta}\left(\lambda \frac{\partial x}{\partial \rho_2} + \mu \frac{\partial y}{\partial \rho_2} + \nu \frac{\partial z}{\partial \rho_2}\right), \qquad -\frac{1}{\Delta}\left(\lambda \frac{\partial x}{\partial \rho_1} + \mu \frac{\partial y}{\partial \rho_1} + \nu \frac{\partial z}{\partial \rho_1}\right),$$

respectively, if one denotes the direction cosines of the exterior normal at C_0 with respect to the fixed axes by λ_0, μ_0, ν_0, and lets λ, μ, ν be those of the exterior normal at C. These equations give, in particular, the equations that relate to the *infinitely small deformation* of a planar surface that were used by Lord Kelvin and Tait.

Instead of referring the elements that relate to the point M to the fixed axes $Oxyz$, imagine a tri-rectangular trihedron $Mx_1'y_1'z_1'$ whose axis Mz_1' is normal to the surface (M) at M, and M_1 is referred to the trihedron $Mx'y'z'$. Let l, l', l'' be the direction cosines of Mx_1' relative to the axes of the latter, let m, m', m'' be those of My_1', and let n, n', n'' be those of Mz_1'. We define the cosines n, n', n'' precisely by the formulas:

$$n = \frac{1}{\Delta}(\eta_1\,\zeta_1 - \eta_2\,\zeta_2), \qquad n' = \frac{1}{\Delta}(\zeta_1\,\xi_1 - \zeta_2\,\xi_2), \qquad n'' = \frac{1}{\Delta}(\xi_1\,\eta_1 - \xi_2\,\eta_2).$$

We assume that the trihedron $Mx_1'y_1'z_1'$ has the same disposition as the other ones, and, for the moment, we do not make any particular hypotheses on the other cosines. We then let $\xi_i^{(1)}$, $\eta_i^{(1)}$, $\zeta_i^{(1)}$ denote the components of the velocity of the origin M of the axes Mx_1', My_1', Mz_1' along those axes when only ρ_i varies and plays the role of time. We likewise let $p_i^{(1)}$, $q_i^{(1)}$, $r_i^{(1)}$ be the projections onto these same axes of the instantaneous rotation of the trihedron $Mx_1'y_1'z_1'$ relative to the parameter ρ_i. In the latter definitions, the trihedron $Mx_1'y_1'z_1'$ is naturally referred to the fixed trihedron $Oxyz$. We will have:

$$\begin{aligned}
\xi_i^{(1)} &= l\,\xi_i + l'\,\eta_i + l''\,\zeta_i\,,\\
\eta_i^{(1)} &= m\,\xi_i + m'\,\eta_i + m''\,\zeta_i\,,\\
\zeta_i^{(1)} &= n\,\xi_i + n'\,\eta_i + n''\,\zeta_i = 0,
\end{aligned}$$

and three formulas such as the following one:

$$p_i^{(1)} = l\,p_i + l'\,q_i + l''\,r_i + \sum n\frac{\partial m}{\partial \rho_i}\,,$$

in which the trihedra considered are supposed to have the same disposition.

Let X_0'', Y_0'', Z_0'' and L_0'', M_0'', N_0'' denote the projections onto Mx_1', My_1', Mz_1' of the external force and external moment, respectively, at an arbitrary point M of the deformed surface, and which are referred to the unit of area of the undeformed surface. Let F_0'', G_0'', H_0'' and I_0'', J_0'', K_0'' be the projections onto the same axes of the effort (F_0, G_0, H_0) and moment (I_0, J_0, K_0), resp., and let A_i'', B_i'', C_i'' and P_i'', Q_i'', R_i'' be the projections of the effort (A_i', B_i', C_i') and moment (P_i', Q_i', R_i'), resp., that were defined previously. The transforms of the preceding relations or the original relations are obviously:

$$\Delta_0 X_0'' = \sum_i \left(\frac{\partial A_i''}{\partial \rho_i} + q_i^{(1)} C_i'' - r_i^{(1)} B_i''^{(1)} \right), \qquad F_0'' = A_1'' \frac{d\rho_2}{ds_0} - A_2'' \frac{d\rho_1}{ds_0},$$
$$\Delta_0 Y_0'' = \sum_i \left(\frac{\partial B_i''}{\partial \rho_i} + r_i^{(1)} A_i'' - p_i^{(1)} C_i^{(1)} \right), \qquad G_0'' = B_1'' \frac{d\rho_2}{ds_0} - B_2'' \frac{d\rho_1}{ds_0},$$
$$\Delta_0 Z_0'' = \sum_i \left(\frac{\partial C_i''}{\partial \rho_i} + p_i^{(1)} B_i'' - q_i^{(1)} A_i'' \right), \qquad H_0'' = C_1'' \frac{d\rho_2}{ds_0} - C_2'' \frac{d\rho_1}{ds_0},$$

$$\Delta_0 L_0'' = \sum_i \left(\frac{\partial P_i''}{\partial \rho_i} + q_i^{(1)} R_i'' - r_i^{(1)} Q_i'' + \eta_i^{(1)} C_i'' \right), \qquad I_0'' = P_1'' \frac{d\rho_2}{ds_0} - P_2'' \frac{d\rho_1}{ds_0},$$
$$\Delta_0 M_0'' = \sum_i \left(\frac{\partial Q_i''}{\partial \rho_i} + r_i^{(1)} P_i'' - p_i^{(1)} R_i'' + \xi_i^{(1)} C_i'' \right), \qquad J_0'' = Q_1'' \frac{d\rho_2}{ds_0} - Q_2'' \frac{d\rho_1}{ds_0},$$
$$\Delta_0 N_0'' = \sum_i \left(\frac{\partial R_i''}{\partial \rho_i} + p_i^{(1)} Q_i'' - q_i^{(1)} P_i'' + \xi_i^{(1)} B_i'' - \eta_i^{(1)} A_i'' \right), \qquad K_0'' = R_1'' \frac{d\rho_2}{ds_0} - R_2'' \frac{d\rho_1}{ds_0}.$$

One can replace $\frac{d\rho_1}{ds}$ and $\frac{d\rho_2}{ds}$ with $-\frac{1}{\Delta}(\lambda'' \xi_2^{(1)} + \mu'' \eta_2^{(1)})$ and $-\frac{1}{\Delta}(\lambda'' \xi_1^{(1)} + \mu'' \eta_1^{(1)})$, resp., here, if λ'', μ'', 0 denote the direction cosines of the exterior normal to the contour C with respect to the trihedron $Mx_1' y_1' z_1'$. One then obtains:

$$F_0'' \frac{ds_0}{ds} = \lambda'' \frac{\xi_1^{(1)} A_1'' + \xi_2^{(1)} A_2''}{\Delta} + \mu'' \frac{\eta_1^{(1)} A_1'' + \eta_2^{(1)} A_2''}{\Delta},$$
$$I_0'' \frac{ds_0}{ds} = \lambda'' \frac{\xi_1^{(1)} P_1 + \xi_2^{(1)} P_2}{\Delta} + \mu'' \frac{\eta_1^{(1)} P_1 + \eta_2^{(1)} P_2}{\Delta},$$

and two systems of analogous formulas.

These formulas lead one to replace the twelve auxiliary functions A_i'', B_i'', C_i'', P_i'', Q_i'', R_i'' with twelve new auxiliary functions that will be the coefficients of λ'' and μ'' in the preceding expressions for the efforts and moments when referred to the unit of length on C, or will be related to these coefficients in a simple fashion. We set:

$$\frac{1}{\Delta}(\xi_1^{(1)} A_1'' + \xi_2^{(1)} A_2'') = N_1, \qquad \frac{1}{\Delta}(\eta_1^{(1)} A_1'' + \eta_2^{(1)} A_2'') = T - S_3,$$
$$\frac{1}{\Delta}(\xi_1^{(1)} B_1'' + \xi_2^{(1)} B_2'') = T + S_3, \qquad \frac{1}{\Delta}(\eta_1^{(1)} B_1'' + \eta_2^{(1)} B_2'') = N_2,$$
$$\frac{1}{\Delta}(\xi_1^{(1)} C_1'' + \xi_2^{(1)} C_2'') = S_2, \qquad \frac{1}{\Delta}(\eta_1^{(1)} C_1'' + \eta_2^{(1)} C_2'') = S_1,$$

upon introducing six primary auxiliary functions N_1, N_2, T, S_1, S_2, S_3, and similarly:

$$\frac{1}{\Delta}(\xi_1^{(1)}P_1''+\xi_2^{(1)}P_2'') = \mathcal{N}_1, \qquad \frac{1}{\Delta}(\eta_1^{(1)}P_1''+\eta_2^{(1)}P_2'') = \mathcal{T}-\mathcal{S}_3,$$
$$\frac{1}{\Delta}(\xi_1^{(1)}Q_1''+\xi_2^{(1)}Q_2'') = \mathcal{T}+\mathcal{S}_3, \qquad \frac{1}{\Delta}(\eta_1^{(1)}Q_1''+\eta_2^{(1)}Q_2'') = \mathcal{N}_2,$$
$$\frac{1}{\Delta}(\xi_1^{(1)}R_1''+\xi_2^{(1)}R_2'') = \mathcal{S}_2, \qquad \frac{1}{\Delta}(\eta_1^{(1)}R_1''+\eta_2^{(1)}R_2'') = \mathcal{S}_1,$$

upon introducing six other auxiliary functions. The twelve equations that just wrote down can be solved immediately with respect the original auxiliary functions A_i'', B_i'', C_i'', P_i'', Q_i'', R_i''. Upon remarking that $\xi_1^{(1)}\eta_2^{(1)}-\xi_2^{(1)}\eta_1^{(1)} = \Delta$, one gets:

$$A_1'' = \eta_2^{(1)}N_1-\xi_2^{(1)}(T-S_3), \qquad A_2'' = \xi_1^{(1)}(T-S_3)-\eta_1^{(1)}N_1,$$
$$B_1'' = \eta_2^{(1)}(T+S_3)-\xi_2^{(1)}N_2, \qquad B_2'' = \xi_1^{(1)}N_2-\eta_1^{(1)}(T+S_3),$$
$$C_1'' = \eta_2^{(1)}S_2-\xi_2^{(1)}S_1, \qquad C_2'' = \xi_1^{(1)}S_1-\eta_1^{(1)}S_2,$$

and six analogous formulas for P_i'', Q_i'', R_i'', with italic symbols in the right-hand sides. One can obviously give notations to the components of the effort and moment of deformation that we just introduced that are analogous to the ones that are in use for the deformable line. Therefore, one can call N_1, N_2 the *efforts of tension*, while the components $T - S_3$, $T + S_3$ are the *shearing efforts* in the plane that is tangent to the deformed surface; the components S_1, S_2 are the *shearing efforts* that are normal to the deformed surface. Likewise, the components $\mathcal{N}_1$, $\mathcal{N}_2$ are the *moments of torsion*, while the components $\mathcal{T}-\mathcal{S}_3$, $\mathcal{T}+\mathcal{S}_3$ have the character of *moments of flexure*; the components $\mathcal{S}_1$, $\mathcal{S}_2$ can be called the *moments of geodesic flexure*.

The notions of energy of deformation and natural present themselves here exactly as they do for the deformable line.

13. The flexible and inextensible surface of Poisson and Lamé. The fluid membrane, referred to as a particular case of the surface that was envisioned by Lagrange, Poisson, and Duhem. The flexible and inextensible surface of the geometers. – The reader can extend for himself the general remarks that we made at the end of no. 3 in regard to the deformable line, and at the beginning of no. 5 on the subject of the notions of *hidden trihedron* and *hidden action*.

First, suppose that W depends upon only ρ_1, ρ_2, ξ_1, η_1, ζ_1, ξ_2, η_2, ζ_2. In this case, the equations of no. 12 reduce to the following ones:

$$\Delta_0 X_0 = \frac{\partial}{\partial\rho_1}\frac{\partial(W\Delta_0)}{\partial\dfrac{\partial x}{\partial\rho_1}}+\frac{\partial}{\partial\rho_2}\frac{\partial(W\Delta_0)}{\partial\dfrac{\partial x}{\partial\rho_2}}, \qquad \Delta_0\,\mathcal{L}_0 = -\frac{\partial W}{\partial\lambda_1},$$

$$\Delta_0\, Y_0 = \frac{\partial}{\partial\rho_1}\frac{\partial(W\,\Delta_0)}{\partial\dfrac{\partial y}{\partial\rho_1}} + \frac{\partial}{\partial\rho_2}\frac{\partial(W\,\Delta_0)}{\partial\dfrac{\partial y}{\partial\rho_2}}, \qquad \Delta_0\,\mathcal{M}_0 = -\frac{\partial W}{\partial\lambda_2},$$

$$\Delta_0\, Z_0 = \frac{\partial}{\partial\rho_1}\frac{\partial(W\,\Delta_0)}{\partial\dfrac{\partial z}{\partial\rho_1}} + \frac{\partial}{\partial\rho_2}\frac{\partial(W\,\Delta_0)}{\partial\dfrac{\partial z}{\partial\rho_2}}, \qquad \Delta_0\,\mathcal{N}_0 = -\frac{\partial W}{\partial\lambda_3},$$

in which λ_1, λ_2, λ_3 are three parameters, by means of which one expresses α, α', …, γ'', and W is a function of only ρ_1, ρ_2, $\frac{\partial x}{\partial\rho_1}$, …, $\frac{\partial z}{\partial\rho_2}$, λ_1, λ_2, λ_3 ; $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ here that has the same significance that it did in no. 4.

Imagine the case in which the functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are zero. The equations $\frac{\partial W}{\partial\lambda_1} = 0$, $\frac{\partial W}{\partial\lambda_2} = 0$, $\frac{\partial W}{\partial\lambda_3} = 0$ amount to either:

$$\begin{aligned}
\frac{\partial y}{\partial\rho_1}C_1 - \frac{\partial z}{\partial\rho_1}B_1 + \frac{\partial y}{\partial\rho_2}C_2 - \frac{\partial z}{\partial\rho_2}B_2 &= 0,\\
\frac{\partial z}{\partial\rho_1}A_1 - \frac{\partial x}{\partial\rho_1}C_1 + \frac{\partial z}{\partial\rho_2}A_2 - \frac{\partial x}{\partial\rho_2}C_2 &= 0,\\
\frac{\partial x}{\partial\rho_1}B_1 - \frac{\partial y}{\partial\rho_1}A_1 + \frac{\partial x}{\partial\rho_2}B_2 - \frac{\partial y}{\partial\rho_2}A_2 &= 0,
\end{aligned}$$

or to $S_1 = S_2 = S_3 = 0$, in such a way that the effort at a point of an arbitrary curve is in the plane that is tangent to the deformed surface and the shearing efforts that are exerted on two rectangular directions are equal.

Let the two states of the surface, (M_0) and (M), be given. When the functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are zero, this can present itself accidentally or for any deformed surface (M) as a consequence of the form of W. The function W will depend simply upon ρ_1, ρ_2, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, and one will have:

$$\begin{aligned}
N_1 &= 2\frac{\Delta_0}{\Delta}\left(\frac{\partial W}{\partial\mathcal{E}}\xi_1^{(1)2} + \frac{\partial W}{\partial\mathcal{F}}\xi_1^{(1)}\xi_2^{(1)} + \frac{\partial W}{\partial\mathcal{G}}\xi_2^{(1)2}\right),\\
T &= \frac{\Delta_0}{\Delta}\left(2\frac{\partial W}{\partial\mathcal{E}}\xi_1^{(1)}\eta_1^{(1)} + \frac{\partial W}{\partial\mathcal{F}}(\xi_2^{(1)}\eta_2^{(1)} + \xi_1^{(1)}\eta_2^{(1)}) + 2\frac{\partial W}{\partial\mathcal{G}}\xi_2^{(1)}\eta_2^{(1)}\right),\\
N_2 &= 2\frac{\Delta_0}{\Delta}\left(\frac{\partial W}{\partial\mathcal{E}}\eta_1^{(1)2} + \frac{\partial W}{\partial\mathcal{F}}\eta_1^{(1)}\eta_2^{(1)} + \frac{\partial W}{\partial\mathcal{G}}\eta_2^{(1)2}\right).
\end{aligned}$$

The trihedron is completely hidden, and we can also regard the surface as simply point-like.

The consideration of the infinitely small deformation, when applied to the preceding surface, permits us to recover the surface – or *membrane* – that was studied by Poisson and Lamé.

A particularly interesting case that we call the *fluid membrane* is obtained by supposing that one has $T = 0$, $N_1 = N_2$ in regard to the three functions thus defined. It is easy to see that W then depends upon $\mathcal{E}, \mathcal{F}, \mathcal{G}$ only by the intermediary of $\Delta = \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}$ and that it is, in turn, a function of ρ_1, ρ_2, and $\mu = \Delta / \Delta_0 - 1$. Upon continuing to let W denote the expression for W in terms of ρ_1, ρ_2, μ, one will have:

$$N_1 = N_2 = \frac{\partial W}{\partial \mu} = N, \qquad T = 0.$$

If we suppose, in addition, that W depends upon only μ then we will find ourselves in the presence of the surface that was considered by Lagrange and studied by Poisson and Duhem. If one introduces the variables x, y then one is led to the system:

$$\frac{\partial N}{\partial x} = \frac{\Delta_0}{\Delta}\left(X_0 + Z_0 \frac{\partial z}{\partial x}\right), \qquad \frac{\partial N}{\partial y} = \frac{\Delta_0}{\Delta}\left(Y_0 + Z_0 \frac{\partial z}{\partial y}\right),$$

$$N\left(\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2}\right) = \frac{\Delta_0}{\Delta}(X_0\, n + Y_0\, n' + Z_0\, n''),$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the principal radii of curvature of the deformed surface (M).

Return to the general case in which W is an arbitrary function of ρ_1, ρ_2, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$. We imagine that we direct our attention to just the deformations of the surface for which one has $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$. It will suffice to introduce these hypotheses in the definitions of the forces. The usual problems that correspond to the given of the function W and to the case where $\mathcal{E} - \mathcal{E}_0$, $\mathcal{F} - \mathcal{F}_0$, $\mathcal{G} - \mathcal{G}_0$ are not generally zero can then be posed only for special givens.

If we suppose that *only* the function W_0 that is obtained by setting $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$ in $W(\rho_1, \rho_2, \mathcal{E}, \mathcal{F}, \mathcal{G})$ is given, and that one does not know the values of the derivatives of W with respect to $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ for $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$, so *the action is hidden*, then we see that N_1, T, N_2 become three auxiliary functions that one can attach to x, y, z, in such a way that in the case where the forces that act upon the elements of the surface are given we will have six partial differential equations in six unknown functions; one will then have a determinate problem only if one adds some accessory conditions. If the deformed figure is assigned *a priori* then one will have three equations in the unknown functions N_1, T, N_2.

The equations to which we have thus arrived are the ones that define the *flexible and inextensible surface* of the geometers. The considerations that related to the flexible and inextensible line are repeated in regard to the latter surface, which can thus be defined *a priori*.

14. Deformable surface where the axis *Mz′* is normal to (*M*) at *M*. Surface of Sophie Germain and Poisson. Surface of Lord Kelvin and Tait. – We propose to introduce the condition that *Mz′* be normal to the surface (*M*). One can do this either by starting with the previously-defined, general, deformable surface and studying *only* the deformations that verify the conditions that $\zeta_1 = \zeta_2 = 0$, or by defining a new deformable surface for which one develops the theory by analogy with that of the former, but while observing the *conditions* that $\zeta_1 = \zeta_2 = 0$.

If we place ourselves at the first viewpoint then it will suffice to append the hypotheses $\zeta_1 = \zeta_2 = 0$ to the formulas that serve to define the force and analogous elements. One sees that if the function W that serves as the point of departure is *given* then one cannot give the external forces and moments arbitrarily, since one appends the two equations $\zeta_1 = \zeta_2 = 0$ to the six equations that determine them.

If one wishes to pursue the idea of specializing the general surface then one must not suppose that the function W is given, but introduce the notion of hidden action, which takes on an entirely special aspect here.

By the fact of the conditions $\zeta_1 = \zeta_2 = 0$, the trihedron, instead of depending upon the six parameters $x, y, z, \lambda_1, \lambda_2, \lambda_3$, now depends upon only four parameters – for example, x, y, z, m, where the last one is the angle that Mx' makes with the curve (ρ_2) at M. The translations can be calculated by means of the system:

$$\frac{\eta_1}{\xi_1} = \tan m, \qquad \xi_1^2 + \eta_1^2 = \mathcal{E}, \quad \xi_1\,\xi_2 + \eta_1\,\eta_2 = \mathcal{F}, \qquad \xi_2^2 + \eta_2^2 = \mathcal{G}.$$

As for the rotations, if one introduces the quantities that Darboux denoted by D, D', D'' and the two Christoffel symbols:

$$\Sigma_1 = \frac{1}{2\Delta^2}\left(-\mathcal{E}\frac{\partial \mathcal{E}}{\partial \rho_2} + 2\mathcal{E}\frac{\partial \mathcal{E}}{\partial \rho_1} - \mathcal{F}\frac{\partial \mathcal{E}}{\partial \rho_1}\right),$$

$$\Sigma_2 = \frac{1}{2\Delta^2}\left(\mathcal{E}\frac{\partial \mathcal{G}}{\partial \rho_1} - \mathcal{F}\frac{\partial \mathcal{E}}{\partial \rho_2}\right)$$

then they will have the expressions:

$$p_1 = \frac{1}{\Delta^2}(\xi_1\,D' - \xi_2\,D), \qquad p_2 = \frac{1}{\Delta^2}(\xi_1\,D'' - \xi_2\,D'),$$

$$q_1 = \frac{1}{\Delta^2}(\eta_1\,D' - \eta_2\,D), \qquad q_2 = \frac{1}{\Delta^2}(\eta_1\,D'' - \eta_2\,D'),$$

$$r_1 = -\frac{\partial m}{\partial \rho_1} + \frac{\Sigma_1 \Delta}{\mathcal{E}}, \qquad r_2 = -\frac{\partial m}{\partial \rho_2} + \frac{\Sigma_2 \Delta}{\mathcal{E}}.$$

If one substitutes these values into the function that is obtained setting $\zeta_1 = \zeta_2 = 0$ in W then one will obtain a function W_0 that depends upon ρ_1, ρ_2, m, $\frac{\partial m}{\partial \rho_1}$, $\frac{\partial m}{\partial \rho_2}$, and x, y, z, and their first and second derivatives by the intermediary of *nine independent expressions* m, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, ρ_1, ρ_2, D, D', D'', whereas W refers to *ten* arguments apart from ρ_1, ρ_2, ζ_1, ζ_2. That reduction comes about from one of the equations to which Darboux gave the name of *Codazzi equations*, which is written $p_1\,\eta_2 - q_1\,\xi_2 - p_2\,\eta_1 + q_2\,\xi_1 = 0$, here. *When just the function* W_0 *is known,* we will thus have, by definition, three auxiliary unknowns. Nevertheless, these three auxiliary functions can all be eliminated, even if they lead to a total number of functions that is greater by one unit than the number of equations when they are combined with the other elements that are defined by W_0.

One recovers the same remarkable result that we already pointed out for the deformable line, and which we must also confine ourselves to simply stating. The equations of statics of the deformable surface that are presently considered can be summarized in the following relation:

$$\iint [\delta(W_0\,\Delta_0) + \Delta_0\,(\mathcal{X}_0\,\delta x + \mathcal{Y}_0\,\delta y + \mathcal{Z}_0\,\delta z - \delta m)]\,d\rho_1\,d\rho_2 = 0,$$

where:

$$\Delta_0\,\mathcal{X}_0 = \Delta_0\,X_0 + \frac{\partial}{\partial \rho_1}\left[\gamma\frac{\Delta_0}{\Delta}\left(L_0\frac{\partial x}{\partial \rho_2} + M_0\frac{\partial y}{\partial \rho_2} + N_0\frac{\partial z}{\partial \rho_2}\right) - \frac{\mathcal{E}\dfrac{\partial x}{\partial \rho_2} - \mathcal{F}\dfrac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}}\Delta_0 N_0'\right]$$

$$- \frac{\partial}{\partial \rho_2}\left[\gamma\frac{\Delta_0}{\Delta}\left(L_0\frac{\partial x}{\partial \rho_1} + M_0\frac{\partial y}{\partial \rho_1} + N_0\frac{\partial z}{\partial \rho_1}\right)\right],$$

with analogous formulas, in which γ, γ', γ'' are the direction cosines of the normal Mz' to (M). A particularly interesting case is the one where $W_0\,\Delta_0 / \Delta$ does not depend upon r_1, r_2, and whose only arguments are ρ_1, ρ_2, and the two expressions:

$$\frac{1}{\mathcal{R}_1\mathcal{R}_2} = \frac{DD'' - D'^2}{\Delta^4}, \qquad \frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2} = \frac{\mathcal{G}D' + \mathcal{E}D'' - 2\mathcal{F}D'}{\Delta^2},$$

in which $\mathcal{R}_1$ and $\mathcal{R}_2$ denote the principal radii of curvature of the surface. If one takes x, y to be variables then one is led to a generalization of the equations that were envisioned by Sophie Germain, Lagrange, and Poisson.

One can further adopt the specification of the effort and moment of deformation by means of the twelve auxiliary functions N_1, T, N_2, S_1, S_2, S_3 ; $\mathcal{N}_1$, $\mathcal{T}$, $\mathcal{N}_2$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$. The fact that C_1', C_2' become unknowns for $\zeta_1 = \zeta_2 = 0$ when W is *hidden* then translates into saying that the two auxiliary functions S_1 and S_2 are unknowns. Now, suppose that W_0

does not contain the arguments ρ_1, ρ_2 and depends upon p_1, q_1, p_2, q_2 only by the intermediary of the expressions:

$$p_1\xi_1 + q_1\eta_1, \qquad p_1\xi_2 + q_1\eta_2 + p_2\xi_1 + q_2\eta_1, \qquad p_2\xi_2 + q_2\eta_2\ ;$$

one has the relations $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}_3 = 0$. The three preceding expressions are *presently* (i.e., $\zeta_1 = \zeta_2 = 0$) the coefficients of $d\rho_1^2$, $d\rho_1\, d\rho_2$, $d\rho_2^2$ in the differential equation for the lines of curvature of (M).

The preceding permits one to pass easily to the infinitely small deformation, which was *the only one* that was considered by Kelvin and Tait, and to recover the case in which their theory gives the results that were first stated by Sophie Germain and Poisson.

15. Dynamics of the deformable line. – The dynamics of the deformable line is associated with the preceding exposition, for which it will suffice to regard one of the parameters – ρ_2, for example – as the time t. One then has an action of deformation and motion simultaneously. Under the influence of the trihedron, the velocity of a point of the deformable line enters into W by way of the three arguments ξ_2, η_2, ζ_2, and one finds oneself in the presence of the notion of *kinetic anisotropy* that was envisioned by Rankine, and which was subsequently introduced in several theories of physics; for example, in the theories of double refraction and the rotational polarization of light. Even if W is independent of rotations and leads to zero external moments, the argument of pure deformation $\xi_1^2 + \eta_1^2 + \zeta_1^2$ and the purely kinetic argument $\xi_2^2 + \eta_2^2 + \zeta_2^2$ are generally accompanied by the mixed argument $\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2$. An argument of this type is no longer new in mechanics, and appears notably, as we will show, in the theory of the action of forces at a distance. When W does not contain the mixed argument $\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2$, one must, in general, consider the state of deformation and motion that is infinitely close to the natural state in order to find oneself in the case of classical mechanics, where the *action of deformation is completely separate from the kinetic action*. One then obtains d'Alembert's principle by supposing that the external force and external moment are zero; i.e., by expressing the idea that the deformable line is subject to no action from the outside world and by introducing, in turn, the fundamental notion of the *isolated system*, which Duhem has found to be necessary for the rational construction of mechanics.

III. – STATICS AND DYNAMICS OF THE DEFORMABLE MEDIUM.

16. Euclidian action of deformation for a deformable medium. External force and moment. Effort and moment of deformation. – The theories of the deformable line and deformable surface that we just presented lead us to imagine a deformable, three-dimensional medium that is more general than the one that has been the usual object of the theory of elasticity. Consider a space (M_0) that is described by a point M_0 whose coordinates are x_0, y_0, z_0 with respect to three fixed rectangular axes Ox, Oy, Oz. We can regard these coordinates as functions of three parameters ρ_1, ρ_2, ρ_3 that are chosen in an arbitrary manner. However, to simplify, we suppose that $x_0 = \rho_1$, $y_0 = \rho_2$, $z_0 = \rho_3$, and according to what is convenient, we sometimes employ, in turn, the notation x_0, y_0, z_0, and sometimes the notation ρ_1, ρ_2, ρ_3 . While preserving our usual notations, we suppose that $x, y, z, \alpha, \alpha', \ldots, \gamma''$ are functions of x_0, y_0, z_0 . The trihedron $Mx'y'z'$ will then describe what we call a *deformable medium*, and we will have eighteen kinematical arguments $\xi_i, \eta_i, \zeta_i, p_i, q_i, r_i$ $(i = 1, 2, 3)$ that are given by formulas (1) and (2), where the ordinary derivatives with respect to s_0 must be replaced with partial derivatives with respect to x_0, y_0, z_0, or, if one prefers, with respect to ρ_i . The linear element of the deformed medium (M), when referred to the independent variables x_0, y_0, z_0, will be defined by the formula:

$$\begin{aligned} ds^2 = {} & (1 + 2\varepsilon_1)\, dx_0^2 + (1 + 2\varepsilon_2)\, dy_0^2 + (1 + 2\varepsilon_3)\, dz_0^2 \\ & + 2\gamma_1\, dy_0\, dz_0 + 2\gamma_2\, dz_0\, dx_0 + 2\gamma_3\, dx_0\, dy_0 , \end{aligned}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_1, \gamma_2, \gamma_3$ are calculated by the following formulas:

$$\begin{aligned} \varepsilon_1 &= \tfrac{1}{2}(\xi_1^2 + \eta_1^2 + \zeta_1^2 - 1), & \gamma_1 &= \xi_2\, \xi_3 + \eta_2\, \eta_3 + \zeta_2\, \zeta_3 , \\ \varepsilon_2 &= \tfrac{1}{2}(\xi_2^2 + \eta_2^2 + \zeta_2^2 - 1), & \gamma_2 &= \xi_3\, \xi_1 + \eta_3\, \eta_1 + \zeta_3\, \zeta_1 , \\ \varepsilon_3 &= \tfrac{1}{2}(\xi_3^2 + \eta_3^2 + \zeta_3^2 - 1), & \gamma_3 &= \xi_1\, \xi_2 + \eta_1\, \eta_2 + \zeta_1\, \zeta_2 . \end{aligned}$$

We will have analogous formulas for the undeformed state (M_0), which we continue to distinguish by the index zero. We also introduce the known quantity Δ that is defined by the formula $\Delta = \dfrac{D(x, y, z)}{D(x_0, y_0, z_0)}$, whose square is expressed as a function of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_1, \gamma_2, \gamma_3$ in the following manner:

$$\Delta^2 = \begin{vmatrix} 1+2\varepsilon_1 & \gamma_3 & \gamma_2 \\ \gamma_3 & 1+2\varepsilon_2 & \gamma_1 \\ \gamma_2 & \gamma_1 & 1+2\varepsilon_3 \end{vmatrix} .$$

If we seek what the function W of two infinitely close positions of the trihedron $Mx'y'z'$ must be in order for the integral $\iiint W\, dx_0\, dy_0\, dz_0$, which is taken over an arbitrary portion of the space (M_0), to have a zero variation under an *infinitesimal transformation* *of the group of Euclidian displacements* then, as before, we will be led to the following *remarkable form:*

$$W(x_0, y_0, z_0, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i).$$

The argument will always remain identical to the one that we made in no. 2.

If we multiply W by the volume element $dx_0\, dy_0\, dz_0$ of the space (M_0) then the product $W\, dx_0\, dy_0\, dz_0$ that is obtained is an invariant under the group of Euclidian displacements that is analogous to the volume element of the medium (M). Just as the common value of the integrals $\iiint_{S_0} |\Delta|\, dx_0\, dy_0\, dz_0$, $\iiint_S dx\, dy\, dz$, which are taken over the interior of a surface S_0 in the medium (M_0) and the interior of the corresponding surface S of the medium (M), respectively, determine the *volume* of the domain that is bounded by the surface S, similarly, upon associating, in that spirit, the notion of action for the passage from the natural state (M_0) to the deformed state (M), we attach the function W to the defining elements of the deformable medium, and we say that the integral $\iiint_{S_0} W\, dx_0\, dy_0\, dz_0$ is the *action of deformation* for the interior of the surface S on the deformed medium. On the other hand, we say that W is the *density* of the action of deformation *at a point* of the deformed medium, when referred to the unit of volume of the undeformed medium and that $W / |\Delta|$ is the density of that action at a point, when referred to the unit of deformed volume.

Consider an *arbitrary* variation of the action of deformation for the interior of a surface (S) on the medium (M), namely:

$$\delta \iiint_{S_0} W\, dx_0\, dy_0\, dz_0 .$$

By virtue of formulas (7) and (8), when extended to the case of three independent parameters x_0, y_0, z_0 or ρ_i ($i = 1, 2, 3$), and after applying Green's formula to the terms that refer to a derivative with respect to one of the variables ρ_i explicitly, it becomes, upon letting l_0, m_0, n_0 denote the direction cosines with respect to the fixed axes Ox, Oy, Oz of the exterior normal to the surface S_0, which bounds the medium before deformation, and letting $d\sigma_0$ denote the area element of that surface:

$$\begin{aligned}
&\delta \iiint_{S_0} W\, dx_0\, dy_0\, dz_0 \\
&\quad = \iint_{S_0} (F_0'\,\delta' x + G_0'\,\delta' y + H_0'\,\delta' z + I_0'\,\delta i' + J_0'\,\delta j' + K_0'\,\delta k')\, d\sigma_0 \\
&\quad - \iiint_{S_0} (X_0'\,\delta' x + Y_0'\,\delta' y + Z_0'\,\delta' z + L_0'\,\delta i' + M_0'\,\delta j' + N_0'\,\delta k')\, dx_0 dy_0 dz_0 ,
\end{aligned}$$

where one has set:

$$(23)\quad \begin{cases} F_0' = l_0 \dfrac{\partial W}{\partial \xi_1} + m_0 \dfrac{\partial W}{\partial \xi_2} + n_0 \dfrac{\partial W}{\partial \xi_3}, & I_0' = l_0 \dfrac{\partial W}{\partial p_1} + m_0 \dfrac{\partial W}{\partial p_2} + n_0 \dfrac{\partial W}{\partial p_3}, \\ G_0' = l_0 \dfrac{\partial W}{\partial \eta_1} + m_0 \dfrac{\partial W}{\partial \eta_2} + n_0 \dfrac{\partial W}{\partial \eta_3}, & J_0' = l_0 \dfrac{\partial W}{\partial q_1} + m_0 \dfrac{\partial W}{\partial q_2} + n_0 \dfrac{\partial W}{\partial q_3}, \\ H_0' = l_0 \dfrac{\partial W}{\partial \zeta_1} + m_0 \dfrac{\partial W}{\partial \zeta_2} + n_0 \dfrac{\partial W}{\partial \zeta_3}, & K_0' = l_0 \dfrac{\partial W}{\partial r_1} + m_0 \dfrac{\partial W}{\partial r_2} + n_0 \dfrac{\partial W}{\partial r_3}, \end{cases}$$

$$(24)\quad \begin{cases} X_0' = \sum_i \left(\dfrac{\partial}{\partial \rho_i} \dfrac{\partial W}{\partial \xi_i} + q_i \dfrac{\partial W}{\partial \zeta_i} - r_i \dfrac{\partial W}{\partial \eta_i} \right), \\ Y_0' = \sum_i \left(\dfrac{\partial}{\partial \rho_i} \dfrac{\partial W}{\partial \eta_i} + r_i \dfrac{\partial W}{\partial \xi_i} - p_i \dfrac{\partial W}{\partial \zeta_i} \right), \\ Z_0' = \sum_i \left(\dfrac{\partial}{\partial \rho_i} \dfrac{\partial W}{\partial \zeta_i} + p_i \dfrac{\partial W}{\partial \eta_i} - q_i \dfrac{\partial W}{\partial \xi_i} \right), \\ \\ L_0' = \sum_i \left(\dfrac{\partial}{\partial \rho_i} \dfrac{\partial W}{\partial p_i} + q_i \dfrac{\partial W}{\partial r_i} - r_i \dfrac{\partial W}{\partial q_i} + \eta_i \dfrac{\partial W}{\partial \zeta_i} - \zeta_i \dfrac{\partial W}{\partial \eta_i} \right), \\ M_0' = \sum_i \left(\dfrac{\partial}{\partial \rho_i} \dfrac{\partial W}{\partial q_i} + r_i \dfrac{\partial W}{\partial p_i} - r_i \dfrac{\partial W}{\partial r_i} + \zeta_i \dfrac{\partial W}{\partial \xi_i} - \xi_i \dfrac{\partial W}{\partial \zeta_i} \right), \\ N_0' = \sum_i \left(\dfrac{\partial}{\partial \rho_i} \dfrac{\partial W}{\partial r_i} + p_i \dfrac{\partial W}{\partial q_i} - r_i \dfrac{\partial W}{\partial p_i} + \xi_i \dfrac{\partial W}{\partial \eta_i} - \eta_i \dfrac{\partial W}{\partial \xi_i} \right). \end{cases}$$

If we first regard the triple integral that figures in the expression for $\delta \iiint_{S_0} W \; dx_0 \, dy_0 \, dz_0$ then we will call the line segments that have their origins at M and whose projections onto the axes Mx', My', Mz' are X_0', Y_0', Z_0' and L_0', M_0', N_0' the *external force and external moment at the point M*, respectively, *when referred to the volume element of the undeformed medium*.

Upon then regarding the surface integral that figures in $\delta \iiint_{S_0} W \; dx_0 \, dy_0 \, dz_0$, we then call the line segments that issue from the point M whose projections onto the axes Mx', My', Mz' are $-F_0'$, $-G_0'$, $-H_0'$ and $-I_0'$, $-J_0'$, $-K_0'$ the *external effort and external moment of deformation*, respectively, *at the point M of the surface S*, which bounds the deformed medium, *when referred to the unit of area of the surface* S_0. At a well-defined point M of (S), these last six quantities depend upon only the direction of the exterior normal to the surface (S). They remain invariable if the direction of the exterior normal does not change when the region considered of (M) varies, and they change sign if the direction is replaced with the opposite direction. Suppose that one traces out a surface (Σ) in the interior of the deformed medium that is bounded by (S), which circumscribes a subset (A) of the medium, either by itself or along with a portion of the surface (S), and let (B)

denote what remains of the medium outside of the subset (A). Let (Σ_0) be the surface of (M_0) that corresponds to the surface (Σ) of (M), and let (A_0) and (B_0) be the regions of (M_0) that correspond to the regions (A) and (B) of (M). Mentally separate the two subsets (A) and (B). One can regard the two segments $(-F_0', -G_0', -H_0')$ and $(-I_0', -J_0', -K_0')$ that are determined for M and for the direction of the normal that is drawn through (Σ_0) towards the exterior of (A_0) as the external effort and moment of deformation, resp., at the point M of the frontier (Σ) of the region (A). One can likewise regard the two line segments $(+F_0', +G_0', +H_0')$ and $(+I_0', +J_0', +K_0')$ as the external effort and moment of deformation, resp., at the point M of the frontier (Σ) of the region (B). By reason of that remark, we say that $-F_0', -G_0', -H_0'$ and $-I_0', -J_0', -K_0'$ are the components along the axes Mx', My', Mz' of *the effort and moment of deformation, resp., that are exerted at M on the portion* (A) *of the medium* (M), and that $+F_0', +G_0', +H_0'$ and $+I_0', +J_0', +K_0'$ are the components along the axes Mx', My', Mz' of *the effort and moment of deformation, resp., that are exerted at M on the portion* (B) *of the medium* (M).

17. Various specifications of the effort and moment of deformation. Notions of energy of deformation and the natural state. Clapeyron's theorem. – Set:

$$A_i' = \frac{\partial W}{\partial \xi_i}, \qquad B_i' = \frac{\partial W}{\partial \eta_i}, \qquad C_i' = \frac{\partial W}{\partial \zeta_i}, \qquad P_i' = \frac{\partial W}{\partial p_i}, \qquad Q_i' = \frac{\partial W}{\partial q_i}, \qquad R_i' = \frac{\partial W}{\partial r_i}.$$

A_i', B_i', C_i', P_i', Q_i', R_i' represent the projections onto Mx', My', Mz' of the effort and moment of deformation, respectively, that are exerted at the point M of a surface that has, before deformation, an interior normal at the point M_0 that is parallel to the coordinate axis Ox, Oy, Oz that corresponds to the index i. That effort and moment of deformation are referred to the unit of area of the undeformed surface. The new efforts and moments of deformations that were just defined are coupled to the elements that were introduced in no. 16 by the following formulas:

$$(25)\quad \begin{cases} X'_0 = \sum_i \left(\dfrac{\partial A'_i}{\partial \rho_i} + q_i C'_i - r_i B'_i \right), \\ Y'_0 = \sum_i \left(\dfrac{\partial B'_i}{\partial \rho_i} + r_i A'_i - p_i C'_i \right), \\ Z'_0 = \sum_i \left(\dfrac{\partial C'_i}{\partial \rho_i} + p_i B'_i - q_i A'_i \right), \\ \\ L'_0 = \sum_i \left(\dfrac{\partial P'_i}{\partial \rho_i} + q_i R'_i - r_i Q'_i + \eta_i C'_i - \zeta_i B'_i \right), \\ M'_0 = \sum_i \left(\dfrac{\partial Q'_i}{\partial \rho_i} + r_i P'_i - p_i R'_i + \zeta_i A'_i - \xi_i C'_i \right), \\ N'_0 = \sum_i \left(\dfrac{\partial R'_i}{\partial \rho_i} + p_i Q'_i - q_i P'_i + \xi_i B'_i - \eta_i A'_i \right), \end{cases}$$

$$(26)\quad \begin{cases} F'_0 = l_0 A'_1 + m_0 A'_2 + n_0 A'_3, \\ G'_0 = l_0 B'_1 + m_0 B'_2 + n_0 B'_3, \\ H'_0 = l_0 C'_1 + m_0 C'_2 + n_0 C'_3, \\ \\ I'_0 = l_0 P'_1 + m_0 P'_2 + n_0 P'_3, \\ J'_0 = l_0 Q'_1 + m_0 Q'_2 + n_0 Q'_3, \\ K'_0 = l_0 R'_1 + m_0 R'_2 + n_0 R'_3. \end{cases}$$

As in nos. 3 and 12, one can propose to transform the relations that we just wrote down *independently of the values of the quantities that figure in them that are calculated by means of W*. Indeed, instead of defining the line segments that we have attached to the point M by their projections onto Mx', My', Mz', we can just as well define them by their projections onto other axes.

We confine ourselves to the consideration of fixed axes Ox, Oy, Oz. Let X_0, Y_0, Z_0 and L_0, M_0, N_0 denote the projections onto these axes of the external force and external moment at an arbitrary point M of the deformed medium, and let F_0, G_0, H_0 and I_0, J_0, K_0 denote the projections of the effort and moment of deformation, respectively, onto a surface whose interior normal had the direction cosines l_0, m_0, n_0 before deformation. Let A_i, B_i, C_i and P_i, Q_i, R_i denote the projections of the effort (A'_i, B'_i, C'_i) and moment (P'_i, Q'_i, R'_i) of deformation , respectively. The transforms of the preceding relations are obviously:

$$(27)\quad \begin{cases} X_0 = \dfrac{\partial A_1}{\partial x_0} + \dfrac{\partial A_2}{\partial y_0} + \dfrac{\partial A_3}{\partial z_0}, \\ Y_0 = \dfrac{\partial B_1}{\partial x_0} + \dfrac{\partial B_2}{\partial y_0} + \dfrac{\partial B_3}{\partial z_0}, \\ Z_0 = \dfrac{\partial C_1}{\partial x_0} + \dfrac{\partial C_2}{\partial y_0} + \dfrac{\partial C_3}{\partial z_0}, \\ \\ L_0 = \dfrac{\partial P_1}{\partial x_0} + \dfrac{\partial P_2}{\partial y_0} + \dfrac{\partial P_3}{\partial z_0} + C_1 \dfrac{\partial y}{\partial x_0} + C_2 \dfrac{\partial y}{\partial y_0} + C_3 \dfrac{\partial y}{\partial z_0} - B_1 \dfrac{\partial z}{\partial x_0} - B_2 \dfrac{\partial z}{\partial y_0} - B_3 \dfrac{\partial z}{\partial z_0}, \\ M_0 = \dfrac{\partial Q_1}{\partial x_0} + \dfrac{\partial Q_2}{\partial y_0} + \dfrac{\partial Q_3}{\partial z_0} + A_1 \dfrac{\partial z}{\partial x_0} + A_2 \dfrac{\partial z}{\partial y_0} + A_3 \dfrac{\partial z}{\partial z_0} - C_1 \dfrac{\partial x}{\partial x_0} - C_2 \dfrac{\partial x}{\partial y_0} - C_3 \dfrac{\partial x}{\partial z_0}, \\ N_0 = \dfrac{\partial R_1}{\partial x_0} + \dfrac{\partial R_2}{\partial y_0} + \dfrac{\partial R_3}{\partial z_0} + B_1 \dfrac{\partial x}{\partial x_0} + B_2 \dfrac{\partial x}{\partial y_0} + B_3 \dfrac{\partial x}{\partial z_0} - A_1 \dfrac{\partial y}{\partial x_0} - A_2 \dfrac{\partial y}{\partial y_0} - A_3 \dfrac{\partial y}{\partial z_0}, \end{cases}$$

$$(28)\quad \begin{cases} F_0 = l_0 A_1 + m_0 A_2 + m_0 A_3, \\ G_0 = l_0 B_1 + m_0 B_2 + m_0 B_3, \\ H_0 = l_0 C_1 + m_0 C_2 + m_0 C_3, \\ \\ I_0 = l_0 P_1 + m_0 P_2 + m_0 P_3, \\ J_0 = l_0 Q_1 + m_0 Q_2 + m_0 Q_3, \\ K_0 = l_0 R_1 + m_0 R_2 + m_0 R_3, \end{cases}$$

which are relations that are the three-dimensional generalizations of the equations of Lord Kelvin and Tait for one and two dimensions, and of the ones that we developed in a previous work. We can transform them in such a fashion as to obtain the generalizations of the well-known equations of the theory of elasticity that relate to effort. Moreover, if the surface of the medium (M) that corresponds to the surface S_0 of (M_0) is always indicated by S then we let X, Y, Z, L, M, N denote the projections onto the Ox, Oy, Oz of the external force and moment, resp., that are applied to the point M and referred to the unit of volume of the deformed medium (M), and let F, G, H, I, J, K denote the projections onto Ox, Oy, Oz of the effort and moment of deformation, resp., that are exerted on S at the point M, when referred to the unit of area of S. In addition, we introduce eighteen new auxiliary functions p_{xx}, …, q_{xx}, … by the formulas:

$$\Delta\, p_{xx} = A_1 \frac{\partial x}{\partial x_0} + A_2 \frac{\partial x}{\partial y_0} + A_3 \frac{\partial x}{\partial z_0}, \qquad \Delta\, q_{xx} = P_1 \frac{\partial x}{\partial x_0} + P_2 \frac{\partial x}{\partial y_0} + P_3 \frac{\partial x}{\partial z_0},$$

$$\Delta\, p_{yx} = A_1 \frac{\partial y}{\partial x_0} + A_2 \frac{\partial y}{\partial y_0} + A_3 \frac{\partial y}{\partial z_0}, \qquad \Delta\, q_{yx} = P_1 \frac{\partial y}{\partial x_0} + P_2 \frac{\partial y}{\partial y_0} + P_3 \frac{\partial y}{\partial z_0},$$

$$\Delta\, p_{zx} = A_1 \frac{\partial z}{\partial x_0} + A_2 \frac{\partial z}{\partial y_0} + A_3 \frac{\partial z}{\partial z_0}, \qquad \Delta\, q_{zx} = P_1 \frac{\partial z}{\partial x_0} + P_2 \frac{\partial z}{\partial y_0} + P_3 \frac{\partial z}{\partial z_0},$$

and analogous formulas that are obtained by replacing A_1, A_2, A_3, p_{xx}, p_{yx}, p_{zx}, P_1, P_2, P_3, q_{xx}, q_{yx}, q_{zx} with B_1, B_2, B_3, p_{xy}, p_{yy}, p_{zy}, Q_1, Q_2, Q_3, q_{xy}, q_{yy}, q_{zy}, and then with C_1, C_2, C_3, p_{xz}, p_{yz}, p_{zz}, R_1, R_2, R_3, q_{xz}, q_{yz}, q_{zz}, respectively. It is easy to see that one obtains the transformed equations:

$$(29)\qquad \left\{\begin{aligned}
X &= \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}, \\
Y &= \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z}, \\
Z &= \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z}, \\
\\
L &= \frac{\partial q_{xx}}{\partial x} + \frac{\partial q_{yx}}{\partial y} + \frac{\partial q_{zx}}{\partial z} + p_{yz} - p_{zy}, \\
M &= \frac{\partial q_{xy}}{\partial x} + \frac{\partial q_{yy}}{\partial y} + \frac{\partial q_{zy}}{\partial z} + p_{zx} - p_{xz}, \\
N &= \frac{\partial q_{xz}}{\partial x} + \frac{\partial q_{yz}}{\partial y} + \frac{\partial q_{zz}}{\partial z} + p_{xy} - p_{yx},
\end{aligned}\right.$$

$$(30)\qquad \left\{\begin{aligned}
F &= l p_{xx} + m p_{yx} + n p_{zx}, \\
G &= l p_{xy} + m p_{yy} + n p_{zy}, \\
H &= l p_{xz} + m p_{yz} + n p_{zz}, \\
\\
I &= l q_{xx} + m q_{yx} + n q_{zx}, \\
J &= l q_{xy} + m q_{yy} + n q_{zy}, \\
K &= l q_{xz} + m q_{yz} + n q_{zz}.
\end{aligned}\right.$$

The significance of the eighteen new auxiliary functions p_{xx}, …, q_{xx}, … results immediately from the relations that just found. Indeed, it is clear that the coefficients p_{xx}, p_{xy}, p_{xz} of l in the expressions for F, G, H represent the projections onto Ox, Oy, Oz of the effort that is exerted at the point M on a surface whose interior normal is parallel to Ox, and that the coefficients q_{xx}, q_{xy}, q_{xz} of l in the expressions for I, J, K are the projections onto Ox, Oy, Oz of the moment of deformation at M relative to the same surface. The coefficients of m and n give rise to an analogous interpretation in regard to the surfaces whose interior normals are parallel to Oy and Oz.

The notions of energy of deformation and natural state again present themselves here exactly as they did for the line and surface. By considering the infinitely small deformation, that leads immediately to Clapeyron's theorem.

18. The continuous medium of the usual theory of elasticity. The invariable body. – The general remarks that we made at the end of no. 3 in regard to the deformable line and at the beginning of no. 5 in the context of the notions of *hidden trihedron* and *hidden action* extend to not only the deformable surface, but – as we said in no. 13 – also to the deformable medium that we presently consider. Suppose that W depends upon only the quantities x_0, y_0, z_0, ξ_i, η_i, ζ_i, and not upon p_i, q_i, r_i . In this case, the equations of no. 17 reduce to the following ones:

$$X_0 = \frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial x}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial x}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial x}{\partial z_0}}, \qquad \mathcal{L}_0 = -\frac{\partial W}{\partial \lambda_1},$$

$$Y_0 = \frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial y}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial y}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial y}{\partial z_0}}, \qquad \mathcal{M}_0 = -\frac{\partial W}{\partial \lambda_2},$$

$$Z_0 = \frac{\partial}{\partial x_0}\frac{\partial W}{\partial \frac{\partial z}{\partial x_0}} + \frac{\partial}{\partial y_0}\frac{\partial W}{\partial \frac{\partial z}{\partial y_0}} + \frac{\partial}{\partial z_0}\frac{\partial W}{\partial \frac{\partial z}{\partial z_0}}, \qquad \mathcal{N}_0 = -\frac{\partial W}{\partial \lambda_3},$$

in which λ_1, λ_2, λ_3 are three parameters, by means of which one expresses α, α', …, γ'', and W is a function of only x_0, y_0, z_0, $\frac{\partial x}{\partial x_0}$, …, $\frac{\partial z}{\partial z_0}$, λ_1, λ_2, λ_3, here; $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ have the same significance as they did in no. 4.

Imagine the case in which the functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are zero. The equations $\frac{\partial W}{\partial \lambda_1} = 0$, $\frac{\partial W}{\partial \lambda_2} = 0$, $\frac{\partial W}{\partial \lambda_3} = 0$ amount to:

$$C_1\frac{\partial y}{\partial x_0} + C_2\frac{\partial y}{\partial y_0} + C_3\frac{\partial y}{\partial z_0} - B_1\frac{\partial z}{\partial x_0} - B_2\frac{\partial z}{\partial y_0} - B_3\frac{\partial z}{\partial z_0} = 0,$$

$$A_1\frac{\partial z}{\partial x_0} + A_2\frac{\partial z}{\partial y_0} + A_3\frac{\partial z}{\partial z_0} - C_1\frac{\partial x}{\partial x_0} - C_2\frac{\partial x}{\partial y_0} - C_3\frac{\partial x}{\partial z_0} = 0,$$

$$B_1\frac{\partial x}{\partial x_0} + B_2\frac{\partial x}{\partial y_0} + B_3\frac{\partial x}{\partial z_0} - A_1\frac{\partial y}{\partial x_0} - A_2\frac{\partial y}{\partial y_0} - A_3\frac{\partial y}{\partial z_0} = 0;$$

i.e., $p_{yz} = p_{zy}$, $p_{zx} = p_{xz}$, $p_{xy} = p_{yx}$, which are relations whose interpretation is immediate.

Having said that, observe, as before, that if one starts with two positions (M_0) and (M) that are assumed to be *given* then it can happen that this result presents itself accidentally in the case where the three functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are zero; i.e., for a certain set of special deformed media. However, it can happen that it presents itself for any sort of deformed medium (M) and results from the form of W.

Imagine the latter case, which is particularly interesting. The trihedron is completely hidden, and we can also regard the medium as simply point-like. W is then a simple function of ρ_1, ρ_2, ρ_3, and the six expressions ε_i, γ_i that are defined by the formulas of no. 16, and the equations of no. 17 reduce to either:

$$
\begin{aligned}
X_0' &= \sum_i \left(\frac{\partial A_i'}{\partial \rho_i} + q_i C_i' - r_i B_i' \right), & F_0' &= l_0 A_1' + m_0 A_2' + n_0 A_3', \\
Y_0' &= \sum_i \left(\frac{\partial B_i'}{\partial \rho_i} + r_i A_i' - p_i C_i' \right), & G_0' &= l_0 B_1' + m_0 B_2' + n_0 B_3', \\
Z_0' &= \sum_i \left(\frac{\partial C_i'}{\partial \rho_i} + p_i B_i' - q_i A_i' \right), & H_0' &= l_0 C_1' + m_0 C_2' + n_0 C_3',
\end{aligned}
$$

in which one has:

$$
\left.
\begin{aligned}
A_i' &= \Delta\left(\xi_i \frac{\partial W}{\partial \varepsilon_i} + \xi_k \frac{\partial W}{\partial \gamma_j} + \xi_j \frac{\partial W}{\partial \gamma_k} \right), \\
B_i' &= \Delta\left(\eta_i \frac{\partial W}{\partial \varepsilon_i} + \eta_k \frac{\partial W}{\partial \gamma_j} + \eta_j \frac{\partial W}{\partial \gamma_k} \right), \\
C_i' &= \Delta\left(\zeta_i \frac{\partial W}{\partial \varepsilon_i} + \zeta_k \frac{\partial W}{\partial \gamma_j} + \zeta_j \frac{\partial W}{\partial \gamma_k} \right),
\end{aligned}
\right\} \qquad (i, j, k = 1, 2, 3),
$$

or to:

$$
\begin{aligned}
X_0 &= \frac{\partial A_1}{\partial x_0} + \frac{\partial A_2}{\partial y_0} + \frac{\partial A_3}{\partial z_0}, & F_0 &= l_0 A_1 + m_0 A_2 + n_0 A_3, \\
Y_0 &= \frac{\partial B_1}{\partial x_0} + \frac{\partial B_2}{\partial y_0} + \frac{\partial B_3}{\partial z_0}, & G_0 &= l_0 B_1 + m_0 B_2 + n_0 B_3, \\
Z_0 &= \frac{\partial C_1}{\partial x_0} + \frac{\partial C_2}{\partial y_0} + \frac{\partial C_3}{\partial z_0}, & H_0 &= l_0 C_1 + m_0 C_2 + n_0 C_3,
\end{aligned}
$$

in which one has:

$$
\begin{aligned}
A_1 &= \Omega_1 \frac{\partial x}{\partial x_0} + \Xi_3 \frac{\partial x}{\partial y_0} + \Xi_2 \frac{\partial x}{\partial z_0} \\
A_2 &= \Xi_3 \frac{\partial x}{\partial x_0} + \Omega_3 \frac{\partial x}{\partial y_0} + \Xi_1 \frac{\partial x}{\partial z_0}
\end{aligned}
$$

$$A_3 = \Xi_2 \frac{\partial x}{\partial x_0} + \Xi_1 \frac{\partial x}{\partial y_0} + \Omega_3 \frac{\partial x}{\partial z_0},$$

and

$$B_1 = \Omega_1 \frac{\partial y}{\partial x_0} + \Xi_3 \frac{\partial y}{\partial y_0} + \Xi_2 \frac{\partial y}{\partial z_0}$$

$$B_2 = \Xi_3 \frac{\partial y}{\partial x_0} + \Omega_3 \frac{\partial y}{\partial y_0} + \Xi_1 \frac{\partial y}{\partial z_0}$$

$$B_3 = \Xi_2 \frac{\partial y}{\partial x_0} + \Xi_1 \frac{\partial y}{\partial y_0} + \Omega_3 \frac{\partial y}{\partial z_0},$$

$$C_1 = \Omega_1 \frac{\partial z}{\partial x_0} + \Xi_3 \frac{\partial z}{\partial y_0} + \Xi_2 \frac{\partial z}{\partial z_0}$$

$$C_2 = \Xi_3 \frac{\partial z}{\partial x_0} + \Omega_3 \frac{\partial z}{\partial y_0} + \Xi_1 \frac{\partial z}{\partial z_0}$$

$$C_3 = \Xi_2 \frac{\partial z}{\partial x_0} + \Xi_1 \frac{\partial z}{\partial y_0} + \Omega_3 \frac{\partial z}{\partial z_0},$$

upon setting $\Omega_i = \dfrac{\partial W}{\partial \varepsilon_i}$, $\Xi_i = \dfrac{\partial W}{\partial \gamma_i}$, to abbreviate the notation. Furthermore, let:

$$X = \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}, \qquad F = l\,p_{xx} + m\,p_{yx} + n\,p_{zx},$$

$$Y = \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z}, \qquad G = l\,p_{xy} + m\,p_{yy} + n\,p_{zy},$$

$$Z = \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z}, \qquad H = l\,p_{xz} + m\,p_{yz} + n\,p_{zz},$$

in which one has:

$$p_{xx} = \frac{1}{\Delta}\left[\Omega_1 \left(\frac{\partial x}{\partial x_0}\right)^2 + \Omega_2 \left(\frac{\partial x}{\partial y_0}\right)^2 + \Omega_3 \left(\frac{\partial x}{\partial y_0}\right)^2 + 2\Xi_1 \frac{\partial x}{\partial y_0}\frac{\partial x}{\partial z_0} + 2\Xi_2 \frac{\partial x}{\partial z_0}\frac{\partial x}{\partial x_0} + 2\Xi_3 \frac{\partial x}{\partial x_0}\frac{\partial x}{\partial x_0}\right],$$

and analogous formulas for p_{yz}, … As one sees, one recovers the deformable continuous medium that the usual theory of elasticity is concerned with.

A particularly interesting case is obtained by looking for the form that W must take if one is to have $p_{yz} = 0$, $p_{zz} = 0$, $p_{xy} = 0$ for any $\dfrac{\partial x}{\partial x_0}$, … One finds that W must be a simple function of x_0, y_0, z_0, and the expression Δ that was defined in no. 15, and one has:

$$p_{xz} = p_{yy} = p_{zz} = \frac{\partial W}{\partial \Delta}.$$

If one supposes that W depends upon only Δ, and if one sets $p = \frac{\partial W}{\partial \Delta}$, and if X, Y, Z are given as functions of x, y, z then the equations will reduce to the following ones:

$$X = \frac{\partial p}{\partial x}, \qquad Y = \frac{\partial p}{\partial y}, \qquad Z = \frac{\partial p}{\partial z}, \qquad F = l\,p, \qquad G = m\,p, \qquad H = n\,p,$$

which serve as the basis for hydrostatics. If the function W is *hidden* then p is an auxiliary function whose significance is well-known.

We start with a medium in which W is a function of x_0, y_0, z_0, ε_i, γ_i, since the function W is *hidden*. We can conceive that one directs one's attention to just those transformations of the medium for which one has $\varepsilon_i = \gamma_i = 0$. It suffices to introduce these hypotheses into the definitions of forces, etc., and if the forces are given, to introduce these six conditions. In the latter case, the usual problems, which correspond to being given the function W and the general case in which ε_i, γ_i are not zero, can be posed only if the givens are special. If we suppose that *only* the function W_0, which is obtained by setting $\varepsilon_i = \gamma_i = 0$ in W, is given, and that one does not know the values of the derivatives of W with respect to ε_i, γ_i for $\varepsilon_i = \gamma_i = 0$ – so W is *hidden* – then we see that p_{xx}, …, p_{zz}, for example, become six auxiliary functions that one must append to x, y, z in such a way that in the case where the forces that act on the volume elements are given, we will have nine partial differential equations in nine unknowns. The integral of the system $\varepsilon_i = \gamma_i = 0$ corresponds to a collective displacement of the medium that is assumed to be deformed in a continuous manner. It therefore remains for us to determine the six integration constants and the six auxiliary functions p_{xx}, …, p_{zz}. One sees that by leaving aside the latter, one recovers the usual problems of the mechanics of the *invariable body*.

We can also conceive that one might seek to define a medium *sui generis* whose definition already takes the conditions $\varepsilon_i = \gamma_i = 0$ into account. Start with the defining identity:

$$\iiint_{S_0} \delta W\, dx_0\, dy_0\, dz_0 = \iint_{S_0} (F_0'\,\delta' x + \cdots + K_0'\,\delta k')\, d\sigma_0 - \iiint_{S_0} (X_0'\,\delta' x + \ldots + N_0'\,\delta' k')\, dx_0 dy_0 dz_0\,.$$

This identity must no longer be true when $\varepsilon_i = \gamma_i = 0$, and we will be led to append the expression $\mu_1\,\delta\varepsilon_1 + \mu_2\,\delta\varepsilon_2 + \mu_3\,\delta\varepsilon_3 + \mu_4\,\delta\gamma_1 + \mu_5\,\delta\gamma_2 + \mu_6\,\delta\gamma_3$ to δW in the integral of the left-hand side, which contains six auxiliary function μ_1, …, μ_6 of x_0, y_0, z_0. We thus fall back upon the theory of the medium that corresponds to the function $W_1 = W + \mu_1\,\delta\varepsilon_1 + \mu_2\,\delta\varepsilon_2 + \mu_3\,\delta\varepsilon_3 + \mu_4\,\delta\gamma_1 + \mu_5\,\delta\gamma_2 + \mu_6\,\delta\gamma_3$ when we confine ourselves to studying the deformations that relate to $\varepsilon_i = \gamma_i = 0$. If, by a change of auxiliary functions, we take the latter conditions into account in W *a priori* then we must simply apply the theory to the function $\mu_1\,\delta\varepsilon_1 + \ldots + \mu_6\,\delta\gamma_3$, and upon supposing that μ_1, …, μ_6 are *unknown*, we are reduced to a theory of the invariable body of the kind that one can create from the ideas of Lagrange.

We finally point out a third method of constituting an invariable medium that follows Lord Kelvin and Tait, which is always subject to the same equations and which will be a limiting case of the original medium. *Moreover, like that of Lagrange, it also applies to the various cases of the deformable line and surfaces.* Imagine that the function W that serves as the definition of the original medium is variable. In order to fix ideas, suppose that for $\varepsilon_1, \ldots, \gamma_3$ that are close to zero W is developable in a MacLaurin series by means of the formula:

$$W = W_1 + W_2 + \ldots + W_i + \ldots,$$

where W_i represents the set of terms of i^{th} degree, and assume that the coefficients of W_2 (which can depend upon x_0, y_0, z_0) can increase indefinitely under their variation. *If we desire that W should preserve a finite value* then we must suppose that ε_i, γ_i tend to zero. In other words, we can then consider only the deformations that satisfy $\varepsilon_i = \gamma_i = 0$, and the body that we will arrive at in the limit can take on only collective displacements. We make the preceding more precise by regarding the coefficients in $W_1, W_2, \ldots$ as functions of a parameter h such that when h tends to zero the coefficients in W_2 increase indefinitely; for example, they might be functions that are linear with respect to $1 / h$. On the other hand, we suppose x, y, z vary with h in such a manner that when ε_i, γ_i are developed in positive powers of h, the first terms in the development will be the ones in h. Under these conditions, W will tend to zero, while $\frac{\partial W}{\partial \varepsilon_i}, \frac{\partial W}{\partial \gamma_i}$ will tend to certain limits (which can be functions of x_0, y_0, z_0). The equations of no. 16 that serve to define the external force and moment finally lead us to some formulas *in which the notion of the function W will have disappeared*, and in which six auxiliary functions $F_0', G_0', H_0', I_0', J_0', K_0'$ will figure.

In order to not leave the scope of this note – i.e., in order to remain within the domain of mechanics, properly speaking – we must confine ourselves to pointing out that the case in which the functions $\mathcal{L}_0, \mathcal{M}_0, \mathcal{N}_0$ are not zero leads to the consideration of media such as the *contractible ether* of Lord Kelvin, for example. We also mention that the most general case, in which traces of the derivatives of the action W with respect to the rotations p_i, q_i, r_i remain in the expressions for the external moment, leads in the most natural manner to the notion of *magnetic induction* that was introduced by Maxwell.

19. Euclidian action of deformation and motion for a continuous medium in motion. The notion of Euclidian energy of deformation and motion. – What we said in no. 15 about the dynamics of the deformable line extends with no difficulty to the derivatives of the deformable surface. Since we shall enter into the statics of media that depend upon more than three geometric parameters here, we shall directly present the theory of the notion of a deformable, three-dimensional medium. The functions $x, y, z, \alpha, \alpha', \ldots, \gamma''$ then depend upon x_0, y_0, z_0, t, where the coordinates x_0, y_0, z_0 define the position at the instant t_0. The continuous, three-dimensional set of trihedra $Mx'y'z'$ for a given value of t will be what we call *the deformed state* of the deformable medium considered at the instant t. The continuous, four-dimensional set of trihedra $Mx'y'z'$ that is obtained by making t vary will be the *trajectory of the deformed state* of the deformable medium. In its original state at the instant t, the medium will be said to be in *the natural state*, and its trajectory, when one then makes t vary, will be the *trajectory of the natural state*, or also *the natural state of motion of the medium*.

To the kinematical arguments ξ_i, η_i, ζ_i, p_i, q_i, r_i of no. 16, one appends the six new arguments:

$$\text{(30 } bis\text{)} \qquad \left\{\begin{aligned} \xi &= \alpha\frac{dx}{dt} + \alpha'\frac{dy}{dt} + \alpha''\frac{dz}{dt}, & p &= \sum \gamma\frac{d\beta}{dt}, \\ \eta &= \beta\frac{dx}{dt} + \beta'\frac{dy}{dt} + \beta''\frac{dz}{dt}, & q &= \sum \alpha\frac{d\gamma}{dt}, \\ \zeta &= \gamma\frac{dx}{dt} + \gamma'\frac{dy}{dt} + \gamma''\frac{dz}{dt}, & r &= \sum \beta\frac{d\alpha}{dt}. \end{aligned}\right.$$

We further seek to determine a function W of two infinitely close positions of trihedra $Mx'y'z'$ such that the quadruple integral $\iiint\int W\,dx_0dy_0dz_0dt$, when taken over an arbitrary portion of the space (M_0) and the time interval that is found between the instants t_1 and t_2, will have a zero variation when one submits the set of all trihedra on what we have called the *trajectory of the deformed state* to the same arbitrary infinitesimal transformation of the group of Euclidian displacements. We are always led to the remarkable form $W(x_0, y_0, z_0, t, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i, \xi, \eta, \zeta, p, q, r)$. We say that the integral:

$$\text{(31)} \qquad \int_{t_1}^{t_2}\iiint_{S_0} W dx_0 dy_0 dz_0 dt$$

is the *action of deformation and motion* in the interior of the surface S on a deformable medium and in the time interval that is comprised between the instant t_1 and t_2 . On the other hand, we say that W is the *density* of the action of deformation and motion *at a point* of the deformed medium that is taken *at a given instant*, and referred to the unit of volume of the undeformed medium and to the unit of time. Upon giving Δ the same significance as it had in no. 16, $W / |\Delta|$ will be the density of that action at a point and a given instant, when referred to the unit of volume of the deformed medium and the unit of time.

Consider an *arbitrary* variation of the action of deformation and motion (31). By a calculation that is completely similar to the one in no. 16, we will be led to formulas such as (23), where one must give W its *present significance* and to the following six new formulas:

$$\text{(32)} \quad A' = \frac{\partial W}{\partial \xi}, \quad B' = \frac{\partial W}{\partial \eta}, \quad C' = \frac{\partial W}{\partial \zeta}, \quad P' = \frac{\partial W}{\partial p}, \quad Q' = \frac{\partial W}{\partial q}, \quad R' = \frac{\partial W}{\partial r}.$$

Moreover, one must add the terms:

$$(33)\quad \begin{cases} \dfrac{d}{dt}\dfrac{\partial W}{\partial \xi}+q\dfrac{\partial W}{\partial \zeta}-r\dfrac{\partial W}{\partial \eta}, & \dfrac{d}{dt}\dfrac{\partial W}{\partial p}+q\dfrac{\partial W}{\partial r}-r\dfrac{\partial W}{\partial q}+\eta\dfrac{\partial W}{\partial \zeta}-\zeta\dfrac{\partial W}{\partial \eta}, \\ \dfrac{d}{dt}\dfrac{\partial W}{\partial \eta}+r\dfrac{\partial W}{\partial \zeta}-p\dfrac{\partial W}{\partial \zeta}, & \dfrac{d}{dt}\dfrac{\partial W}{\partial q}+r\dfrac{\partial W}{\partial p}-p\dfrac{\partial W}{\partial r}+\zeta\dfrac{\partial W}{\partial \xi}-\xi\dfrac{\partial W}{\partial \zeta}, \\ \dfrac{d}{dt}\dfrac{\partial W}{\partial \zeta}+p\dfrac{\partial W}{\partial \zeta}-q\dfrac{\partial W}{\partial \xi}, & \dfrac{d}{dt}\dfrac{\partial W}{\partial r}+p\dfrac{\partial W}{\partial q}-q\dfrac{\partial W}{\partial p}+\xi\dfrac{\partial W}{\partial \eta}-\eta\dfrac{\partial W}{\partial \xi}, \end{cases}$$

respectively, to the formulas (24), and we will have:

$$(34)\quad \begin{cases} \delta\displaystyle\int_{t_1}^{t_2}\iiint_{S_0} W\,dx_0\,dy_0\,dz_0\,dt \\ \qquad =\left[\displaystyle\iiint_{S_0}(A'\,\delta'x+B'\,\delta'y+C'\,\delta'z+P'\,\delta i'+Q'\,\delta j'+R'\,\delta k')\,dx_0\,dy_0\,dz_0\right]_{t_1}^{t_2} \\ \qquad +\displaystyle\int_{t_1}^{t_2}\iint_{S_0}(F_0'\,\delta'x+G_0'\,\delta'y+H_0'\,\delta'z+I_0'\,\delta i'+J_0'\,\delta j'+K_0'\,\delta k')\,d\sigma_0\,dt \\ \qquad -\displaystyle\int_{t_1}^{t_2}\iiint_{S_0}(X_0'\,\delta'x+Y_0'\,\delta'y+Z_0'\,\delta'z+L_0'\,\delta i'+M_0'\,\delta j'+N_0'\,\delta k')\,dx_0\,dy_0\,dz_0\,dt. \end{cases}$$

We call the line segments that have origins at M and whose projections onto the axes Mx', My', Mz' are A', B', C' and P', Q', R' the *quantity of motion and moment of the quantity of motion, respectively, at the point M of the deformed medium* (M) *at the instant t*. We say that the first terms of the external force X_0', Y_0', Z_0' and external moment L_0', M_0', N_0', which are given by the right-hand sides of formulas (24), where W must take the significance:

$$W(x_0, y_0, z_0, t, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i, \xi, \eta, \zeta, p, q, r)$$

represent the *static part* of that external force and moment. The additional terms (33) will be the *dynamical part.* Moreover, as in no. 17, we can introduce some various way of specifying the effort and moment of deformations, as well as the quantity of motion and moment of the quantity of motion. The right-hand sides of formulas (25) will constitute the static part of the external force and moment. The dynamical part that one must add to it will be given by the expressions:

$$\begin{aligned} &\frac{dA'}{dt}+qC'-rB', &\qquad &\frac{dP'}{dt}+qR'-rQ'+\eta C'-\zeta B', \\ &\frac{dB'}{dt}+rA'-pC', &\qquad &\frac{dQ'}{dt}+rP'-pR'+\zeta A'-\xi C', \\ &\frac{dA'}{dt}+pB'-qA', &\qquad &\frac{dR'}{dt}+qQ'-qP'+\xi B'-\eta A'. \end{aligned}$$

The static part of the external force and moment with respect to the axes Ox, Oy, Oz will be given by the right-hand sides of formulas (27), and if one lets A, B, C and P, Q, R

denote the projections of the quantity of motion and the moment of the quantity of motion, respectively, onto the axes *Ox*, *Oy*, *Oz* then the dynamical part will be given by the expressions:

$$\begin{array}{ll} \dfrac{dA}{dt}, & \dfrac{dP}{dt} + C\dfrac{dy}{dt} - B\dfrac{dz}{dt}, \\ \dfrac{dB}{dt}, & \dfrac{dQ}{dt} + A\dfrac{dz}{dt} - C\dfrac{dx}{dt}, \\ \dfrac{dC}{dt}, & \dfrac{dR}{dt} + B\dfrac{dx}{dt} - A\dfrac{dy}{dt}. \end{array}$$

Likewise, to the static part that is represented by the right-hand sides of formulas (29), one will add the dynamical part that is defined by the following expressions:

$$\begin{array}{ll} \dfrac{1}{\Delta}\dfrac{dA}{dt}, & \dfrac{1}{\Delta}\dfrac{dP}{dt} + \dfrac{C}{\Delta}\dfrac{dy}{dt} - \dfrac{B}{\Delta}\dfrac{dz}{dt}, \\ \dfrac{1}{\Delta}\dfrac{dB}{dt}, & \dfrac{1}{\Delta}\dfrac{dQ}{dt} + \dfrac{A}{\Delta}\dfrac{dz}{dt} - \dfrac{C}{\Delta}\dfrac{dx}{dt}, \\ \dfrac{1}{\Delta}\dfrac{dC}{dt}, & \dfrac{1}{\Delta}\dfrac{dR}{dt} + \dfrac{B}{\Delta}\dfrac{dx}{dt} - \dfrac{A}{\Delta}\dfrac{dy}{dt}. \end{array}$$

If we write the equality (34) in the form:

$$\delta \int_{t_1}^{t_2} \iiint_{S_0} W\, dx_0 dy_0 dz_0 dt = -\, \delta \mathcal{T}_e \tag{35}$$

then $\delta\mathcal{T}_e$ will be the external virtual work and can be given various expressions according to the specifications that were adopted for the effort and moment of deformation, and for the quantity of motion and moment of quantity of motion. Since δW must be identically zero for an arbitrary Euclidian virtual displacement, by virtue of the invariance of W under the group of Euclidian displacements, one will have the relations:

$$\left\{ \begin{array}{l} \left[\iiint_{S_0} A\, dx_0 dy_0 dz_0 \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \iint_{S_0} F_0\, d\sigma_0 dt - \int_{t_1}^{t_2} \iiint_{S_0} X_0 dx_0 dy_0 dz_0 dt = 0, \\ \\ \left[\iiint_{S_0} (P + Cy - Bz)\, dx_0 dy_0 dz_0 \right]_{t_1}^{t_2} \\ \quad + \int_{t_1}^{t_2} \iint_{S_0} (I_0 + H_0 y - G_0 z) d\sigma_0 dt - \int_{t_1}^{t_2} \iiint_{S_0} (L_0 + Z_0 y - X_0 z)\, dx_0 dy_0 dz_0 dt = 0, \end{array} \right. \tag{36}$$

and some analogous equalities. One thus obtains the generalization of Saint-Guilhem's theorem, and upon introducing the notion of *impulsion*, the generalization of the classical theory of *impulsions*.

We remark that the present exposition contains the statics of deformable bodies as a special case. Indeed, it suffices to consider a *reversible virtual modification of the action*, in the sense of Duhem, instead of imagining, as we just did, a *realizable virtual modification.*

This observation will lead us to the notion of energy of deformation and motion. We propose to determine the work that is done by the external forces and moments and the efforts and external moments of deformation during an arbitrary time interval for a *real modification.* For this, it will suffice for us to calculate the elementary work that relates to the time dt. The latter is:

$$\left[\iiint_{S_0}(\xi X_0'+\eta Y_0'+\cdots)\,dx_0\,dy_0\,dz_0-\iint_{S_0}(\xi F_0'+\eta G_0'+\cdots)\,d\sigma\right]dt\,.$$

If one replaces X_0', Y_0', …, F_0', G_0', … with their expressions as functions of the action and if one performs a calculation that is inverse to the one that led to their definition then one will obtain immediately, by virtue of the Codazzi equations:

$$\left[\iiint_{S_0}\left(\frac{dE}{dt}+\frac{\partial W}{\partial t}\right)dx_0 dy_0 dz_0\right]dt\,,$$

upon setting:

$$E = \xi\frac{\partial W}{\partial \xi}+\eta\frac{\partial W}{\partial \eta}+\zeta\frac{\partial W}{\partial \zeta}+p\frac{\partial W}{\partial p}+q\frac{\partial W}{\partial q}+r\frac{\partial W}{\partial r}-W. \tag{37}$$

If one considers, in particular, the case in which W does not contain t explicitly, in such a way that $\dfrac{\partial W}{\partial t}$ is zero, then the preceding value becomes the derivatives with respect to time of the expression:

$$\iiint_{S_0} E\;dx_0\,dy_0\,dz_0\,,$$

which can be called the *energy of deformation and motion at the instant t.*

At the point to which we have arrived in our exposition, we can make some important general remarks that will once more find their application in the theory of the Euclidian action in what follows.

The only notion of Euclidian action of deformation and motion that *suffices* for us provides – in a very extended case – a *constructive* definition of the quantity of motion and moment of the quantity of motion, the effort and the moment of deformation, and of the external force and moment. The distinction that we have made between the dynamical part and the static part of the external force and moment, which amount to grouping, on the one hand, the terms that contain only dynamical acceleration, and on the other hand, the terms that contain only what one can call the *kinematical acceleration*, obviously express an extension of *d'Alembert's principle*. Likewise, the reasoning that we made in order to arrive at the notion of energy of deformation and motion shows that there is a sort of separation of that energy into two parts – the one dynamical, and the other, kinematical. If we suppose that the external work done is zero then the energy of deformation and motion will be constant, and consequently, the total dynamical and kinematical energy will remain constant in time. We thus obtain the notion of *conservation of energy*, which simply translates into the hypothesis that the medium is *isolated* from the external world. We recover, in turn, all of the fundamental ideas of classical mechanics, and it is obvious that the particular form that it takes in the latter case leads one to envision only the state of motion and deformation that is *infinitely close to the natural state*, where one supposes that the action W and its derivatives are zero.

We can further remark that the deductive path that we have followed sidesteps the objection that Carnot already raised a century ago against the reverse progress in classical mechanics. In it, the force and analogous quantities are *a priori* notions. True, the dynamical force can receive a definition in such a way that it expresses the second law of motion that was posed by Newton in his *Principia*. Likewise, if one prefers, static force can be regarded as defined by Hooke's law when the deformation is infinitely small, as Reech has proposed. Finally, the force that relates to the state of deformation and motion can be expressed by d'Alembert's principle. However, one must further submit to definitions that possess a certain arbitrary and contingent character, such as saying that it must verify the principle of the conservation of energy *a priori* in due course, and one can only postpone the difficulty by introducing energy as a metaphysical notion. The same thing is not true for the Euclidian action from which we have derived everything, since it analogous to the distance between two infinitely close points, and consequently translates into simply the idea of measure in the world of phenomena, and in a manner that has been consecrated by all past experiments, moreover.

Finally, it appears that the generality of the form of the action that we adopted in our exposition, and which corresponds to an arbitrary state of motion and deformation, is found to be justified, not only by the consideration of the *critical phenomena* of motion and deformation, but also by the fact that it introduces the regular and uniform method that one must follow by definition, even when one confines oneself to a state that is infinitely close to the natural state in order to establish or verify the conservation of energy.

IV. – EUCLIDIAN ACTION AT A DISTANCE. THE ACTION OF CONSTRAINT AND DISSIPATIVE ACTION.

20. Euclidian action of deformation and motion for a discontinuous medium. – Consider a discrete system of n trihedra, in which each trihedron is distinguished by an index i that consequently takes the values 1, 2, …, n. Let $M_i x'_i y'_i z'_i$ be the trihedron whose index is i and whose summit M_i will have the coordinates x_i, y_i, z_i, and the axes $M_i x'_i$, $M_i y'_i$, $M_i z'_i$ will have the direction cosines α_i, α'_i, α''_i, β_i, β'_i, β''_i, γ_i, γ'_i, γ''_i with respect to the three fixed rectangular axes Ox, Oy, Oz. We suppose that the quantities x_i, y_i, z_i, α_i, …, γ''_i are functions of time t, and we introduce the six arguments ξ_i, η_i, ζ_i, p_i, q_i, r_i that are defined by the formulas (30) with the index i.

Imagine a function W of two infinitely close positions of the system of trihedra $M_i x'_i y'_i z'_i$; i.e., a function of t, x_i, y_i, z_i, α_i, …, γ''_i, and their first derivatives with respect to t (when i takes the values 1, 2, …, n). We propose to determine the form that W must take in order for that function to remain invariant under all infinitesimal transformations of the group of Euclidian displacements such as (9). Observe that the relations (30) permit one to express – by means of well-known formulas – the first derivatives of the nine cosines α_i, α'_i, …, γ''_i with respect to t in terms of these cosines and p_i, q_i, r_i, and on the other hand, to express the nine cosines α_i, α'_i, …, γ''_i in terms of ξ_i, η_i, ζ_i, and the first derivatives of x_i, y_i, z_i with respect to t. We can thus finally express the function W that we seek as a function of t, x_i, y_i, z_i, and their first derivatives, and finally of ξ_i, η_i, ζ_i, p_i, q_i, r_i, which we indicate by writing:

$$W = W\left(t, x_i, y_i, z_i, \frac{dx_i}{dt}, \frac{dy_i}{dt}, \frac{dz_i}{dt}, \xi_i, \eta_i, \zeta_i, p_i, q_i, r_i\right).$$

Since the variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$, δp_i, δq_i, δr_i are zero in the present case, as this would result from the well-known theory of the moving trihedron, we must write the new form of W, by virtue of formulas (9), when taken with the index i, and for any a_1, a_2, a_3, ω_1, ω_2, ω_3:

$$\sum_i \left(\frac{\partial W}{\partial x_i}\delta x_i + \frac{\partial W}{\partial y_i}\delta y_i + \frac{\partial W}{\partial z_i}\delta z_i + \frac{\partial W}{\partial \frac{dx_i}{dt}}\delta\frac{dx_i}{dt} + \frac{\partial W}{\partial \frac{dy_i}{dt}}\delta\frac{dy_i}{dt} + \frac{\partial W}{\partial \frac{dz_i}{dt}}\delta\frac{dz_i}{dt} \right) = 0.$$

Replace δx_i, δy_i, δz_i with their values in (9) and $\delta\frac{dx_i}{dt}$, $\delta\frac{dy_i}{dt}$, $\delta\frac{dz_i}{dt}$ with the values that one deduces by differentiation, and equate the coefficients of a_1, a_2, a_3, ω_1, ω_2, ω_3 to zero. One then gets the following conditions:

(38) $$\sum_i \frac{\partial W}{\partial x_i} = 0, \quad \sum_i \frac{\partial W}{\partial y_i} = 0, \quad \sum_i \frac{\partial W}{\partial z_i} = 0,$$

and

$$\sum_i \left(y_i \frac{\partial W}{\partial z_i} - z_i \frac{\partial W}{\partial y_i} + \frac{dy_i}{dt} \frac{\partial W}{\partial \frac{dz_i}{dt}} - \frac{dz_i}{dt} \frac{\partial W}{\partial \frac{dy_i}{dt}} \right) = 0, \tag{39}$$

with two analogous relations.

If we suppose that *the points* (x_i, y_i, z_i) *can describe all possible trajectories* then we will arrive at some identities that are verified by the function W of $6n$ arguments x_i, y_i, z_i, $\frac{dx_i}{dt}$, $\frac{dy_i}{dt}$, $\frac{dz_i}{dt}$, and some further arguments ξ_i, η_i, ζ_i, p_i, q_i, r_i, which we can leave aside for the moment. We seek to exhibit the form that results for W.

We commence by treating the case of the system of three equations:

$$\left\{ \begin{aligned} &\sum_{i=1}^{p} \left(y_i \frac{\partial W}{\partial z_i} - z_i \frac{\partial W}{\partial y_i} \right) = 0, \\ &\sum_{i=1}^{p} \left(z_i \frac{\partial W}{\partial x_i} - x_i \frac{\partial W}{\partial z_i} \right) = 0, \\ &\sum_{i=1}^{p} \left(x_i \frac{\partial W}{\partial y_i} - y_i \frac{\partial W}{\partial x_i} \right) = 0, \end{aligned} \right. \tag{40}$$

which determine a function W of $3n$ arguments x_i, y_i, z_i . We have already encountered this system in the context of the dynamics of a point and the statics of a line, and of the surface and continuous, three-dimensional medium in the cases where $p = 1$, $p = 2$, $p = 3$, respectively. We leave aside the case of $p = 1$, where the three equations reduce to two. For $p = 2$ and $p = 3$, we have three equations that define a complete system. For $p = 2$, we have three equations, six variables, and three independent solutions:

$$x_i^2 + y_i^2 + z_i^2 \quad (i = 1, 2), \qquad x_1\, x_2 + y_1\, y_2 + z_1\, z_2 .$$

For $p = 3$, we have three equations, nine variables, and six independent solutions:

$$x_i^2 + y_i^2 + z_i^2 \quad (i = 1, 2, 3), \quad x_i\, x_j + y_i\, y_j + z_i\, z_j \qquad (i, j = 1, 2, 3).$$

For $p > 3$, the system is still complete. In order to prove this, it suffices to show that it admits $3p - 3$ independent solutions, since the number of equations is 3, and the number of variables is $3p$. Now, we first have essentially the p solutions:

$$x_i^2 + y_i^2 + z_i^2 \quad (i = 1, 2, \ldots, p),$$

and then the solution:

$$x_1\, x_2 + y_1\, y_2 + z_1\, z_2,$$

and finally the $2(p - 2)$ solutions:

$$x_1\,x_i + y_1\,y_i + z_1\,z_i\,,\qquad x_2\,x_i + y_2\,y_i + z_2\,z_i\qquad (i = 3, 4, 5, \ldots, p),$$

which are all independent. W is then a function of the $3(p - 1)$ independent arguments that we just enumerated.

We now return to the proposed system that was defined the conditions (38) and (39). The conditions (38) prove that W depends upon $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n$ only by the intermediary of the expressions:

$$\begin{array}{llll} X_2 = x_2 - x_1, & X_3 = x_3 - x_1, & \ldots, & X_n = x_n - x_1, \\ Y_2 = y_2 - y_1, & Y_3 = y_3 - y_1, & \ldots, & Y_n = y_n - y_1, \\ Z_2 = z_2 - z_1, & Z_3 = z_3 - z_1, & \ldots, & Z_n = z_n - z_1\,. \end{array}$$

On the other hand, set:

$$\frac{dx_i}{dt} = X_{n+i}\,,\qquad \frac{dy_i}{dt} = Y_{n+i}\,,\qquad \frac{dz_i}{dt} = Z_{n+i}\,,$$

and write that equations (39) are verified by the function W of the arguments $X_2, X_3, \ldots, X_{3n}\,;\ Y_2, Y_3, \ldots, Y_{3n}\,;\ Z_2, Z_3, \ldots, Z_{3n}$. For example, consider the first of equations (39). It becomes:

$$-\,y_1\left(\frac{\partial W}{\partial Z_2}+\frac{\partial W}{\partial Z_3}+\cdots+\frac{\partial W}{\partial Z_n}\right)+z_1\left(\frac{\partial W}{\partial Y_2}+\frac{\partial W}{\partial Y_3}+\cdots+\frac{\partial W}{\partial Y_n}\right)$$
$$+\,(y_1+Y_2)\,\frac{\partial W}{\partial Z_2}-(z_1+Z_2)\,\frac{\partial W}{\partial Y_2}+\ldots=0.$$

y_1 and z_1 disappear, and what remains is the first of the equations:

$$\sum_{i=1}^{2n}\left(Y_i\,\frac{\partial W}{\partial Z_i}-Z_i\,\frac{\partial W}{\partial Y_i}\right)=0,$$
$$\sum_{i=1}^{2n}\left(Z_i\,\frac{\partial W}{\partial X_i}-X_i\,\frac{\partial W}{\partial Z_i}\right)=0,$$
$$\sum_{i=1}^{2n}\left(X_i\,\frac{\partial W}{\partial Y_i}-Y_i\,\frac{\partial W}{\partial X_i}\right)=0.$$

We thus fall back upon the system (40), where x_i, y_i, z_i are replaced with $X_{i+1}, Y_{i+1}, Z_{i+1}$, and p is replaced with $2n - 1$.

If we first suppose that $n = 2$ then we will see that W is – abstracting from the arguments $\xi_i, \eta_i, \zeta_i, p_i, q_i, r_i$ – a function of the independent expressions:

$$X_2^2+Y_2^2+Z_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$

$$X_3^2 + Y_3^2 + Z_3^2 = \left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dy_1}{dt}\right)^2 + \left(\frac{dz_1}{dt}\right)^2 = \xi_1^2 + \eta_1^2 + \zeta_1^2,$$

$$X_4^2 + Y_4^2 + Z_4^2 = \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dy_2}{dt}\right)^2 + \left(\frac{dz_2}{dt}\right)^2 = \xi_2^2 + \eta_2^2 + \zeta_2^2,$$

$$X_2 X_3 + Y_2 Y_3 + Z_2 Z_3 = (x_2 - x_1)\frac{dx_1}{dt} + (y_2 - y_1)\frac{dy_1}{dt} + (z_2 - z_1)\frac{dz_1}{dt},$$

$$X_2 X_4 + Y_2 Y_4 + Z_2 Z_4 = (x_2 - x_1)\frac{dx_2}{dt} + (y_2 - y_1)\frac{dy_2}{dt} + (z_2 - z_1)\frac{dz_2}{dt},$$

$$X_3 X_4 + Y_3 Y_4 + Z_3 Z_4 = \frac{dx_1}{dt}\frac{dx_2}{dt} + \frac{dy_1}{dt}\frac{dy_2}{dt} + \frac{dz_1}{dt}\frac{dz_2}{dt}.$$

Therefore, we finally have that W is a function of t, ξ_i, η_i, ζ_i, p_i, q_i, r_i, and the *four arguments:*

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$

$$(x_2 - x_1)\frac{dx_1}{dt} + (y_2 - y_1)\frac{dy_1}{dt} + (z_2 - z_1)\frac{dz_1}{dt},$$

$$(x_2 - x_1)\frac{dx_2}{dt} + (y_2 - y_1)\frac{dy_2}{dt} + (z_2 - z_1)\frac{dz_2}{dt},$$

$$\frac{dx_1}{dt}\frac{dx_2}{dt} + \frac{dy_1}{dt}\frac{dy_2}{dt} + \frac{dz_1}{dt}\frac{dz_2}{dt}.$$

If we suppose that $n > 2$ then we will see that W is – abstracting from the arguments ξ_i, η_i, ζ_i, p_i, q_i, r_i – a function of $6(n - 1)$ independent expressions:

$$X_i^2 + Y_i^2 + Z_i^2 = \begin{cases} \text{either } (x_i - x_1)^2 + (y_i - y_1)^2 + (z_i - z_1)^2 \quad (i = 1, 2, \dots, n), \\ \text{or } \left(\frac{dx_k}{dt}\right)^2 + \left(\frac{dy_k}{dt}\right)^2 + \left(\frac{dz_k}{dt}\right)^2 = \xi_k^2 + \eta_k^2 + \zeta_k^2, \end{cases}$$

$$X_2 X_3 + Y_2 Y_3 + Z_2 Z_3 = (x_2 - x_1)(x_3 - x_1) + (y_2 - y_1)(y_3 - y_1) + (z_2 - z_1)(z_3 - z_1),$$

$$X_2 X_i + Y_2 Y_i + Z_2 Z_i = \begin{cases} \text{either } (x_2 - x_1)(x_i - x_1) + (y_2 - y_1)(y_i - y_1) + (z_2 - z_1)(z_i - z_1), \\ \text{or } (x_2 - x_1)\frac{dx_k}{dt} + (y_2 - y_1)\frac{dy_k}{dt} + (z_2 - z_1)\frac{dz_k}{dt}, \end{cases}$$

$$X_3 X_i + Y_3 Y_i + Z_3 Z_i = \begin{cases} \text{either } (x_3 - x_1)(x_i - x_1) + (y_3 - y_1)(y_i - y_1) + (z_3 - z_1)(z_i - z_1), \\ \text{or } (x_3 - x_1)\frac{dx_k}{dt} + (y_3 - y_1)\frac{dy_k}{dt} + (z_3 - z_1)\frac{dz_k}{dt}. \end{cases}$$

We remark that one has:

$$(x_i - x_j)(x_i - x_k) + (y_i - y_j)(y_i - y_k) + (z_i - z_j)(z_i - z_k) = \tfrac{1}{2}(r_{ij}^2 + r_{ik}^2 - r_{kj}^2),$$

where r is the distance between two points of the system. By reason of symmetry, one might have to make arguments that are *not independent* figure in W, and one can take then the following arguments independently of $\xi_i, \eta_i, \zeta_i, p_i, q_i, r_i$:

$$r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2,$$

$$\psi_{ij} = \frac{dx_i}{dt}\frac{dx_j}{dt} + \frac{dy_i}{dt}\frac{dy_j}{dt} + \frac{dz_i}{dt}\frac{dz_j}{dt},$$

$$\lambda_{ijk} = (x_i - x_j)\frac{dx_k}{dt} + (y_i - y_j)\frac{dy_k}{dt} + (z_i - z_j)\frac{dz_k}{dt}.$$

The last ones consist of arguments λ_{iji} with two indices and arguments λ_{ijk} with three indices. The latter figure only when there are more then two points, and one sees that in this case the action on two points is influenced by all of the other points. It is easy to establish the very complex relations that exist between these non-independent arguments. They are analogous to the known relations between the distances r_{ij} when the number of points is ≥ 5.

If we know the expression for the Euclidian action W on the system of trihedra considered then we can easily, by a calculation that repeats what we have done previously, find the expression for the external force and moment on an arbitrary trihedron. Since the action W is a function of x_i, y_i, z_i, $\frac{dx_i}{dt}, \frac{dy_i}{dt}, \frac{dz_i}{dt}$ by the intermediary of $r_{ij}, \psi_{ij}, \lambda_{ijk}$, it is convenient to first regard W as a function of x_i, y_i, z_i, $\frac{dx_i}{dt}, \frac{dy_i}{dt}, \frac{dz_i}{dt}$ and of $\xi_i, \eta_i, \zeta_i, p_i, q_i, r_i$. We will have:

$$\delta\int_{t_1}^{t_2} W\,dt$$

$$= \left[\sum_i (A_i\delta x_i + B_i\delta y_i + C_i\delta z_i + P_i\delta i_i + Q_i\delta j_i + R_i\delta k_i\right]_{t_1}^{t_2}$$

$$- \int_{t_1}^{t_2}\sum_i (X_i\delta x_i + Y_i\delta y_i + Z_i\delta z_i + L_i\delta i_i + M_i\delta j_i + N_i\delta k_i)\,dt,$$

upon setting:

$$A_i = \alpha_i\frac{\partial W}{\partial \xi_i} + \beta_i\frac{\partial W}{\partial \eta_i} + \gamma_i\frac{\partial W}{\partial \zeta_i}, \qquad P_i = \alpha_i\frac{\partial W}{\partial p_i} + \beta_i\frac{\partial W}{\partial q_i} + \gamma_i\frac{\partial W}{\partial r_i},$$

$$B_i = \alpha_i'\frac{\partial W}{\partial \xi_i} + \beta_i'\frac{\partial W}{\partial \eta_i} + \gamma_i'\frac{\partial W}{\partial \zeta_i}, \qquad Q_i = \alpha_i'\frac{\partial W}{\partial p_i} + \beta_i'\frac{\partial W}{\partial q_i} + \gamma_i'\frac{\partial W}{\partial r_i},$$

$$C_i = \alpha_i''\frac{\partial W}{\partial \xi_i} + \beta_i''\frac{\partial W}{\partial \eta_i} + \gamma_i''\frac{\partial W}{\partial \zeta_i}, \qquad R_i = \alpha_i''\frac{\partial W}{\partial p_i} + \beta_i''\frac{\partial W}{\partial q_i} + \gamma_i''\frac{\partial W}{\partial r_i},$$

where (A_i, B_i, C_i) and (P_i, Q_i, R_i) are the quantity of motion and the moment of the quantity of motion, respectively, of the trihedron with index i, and:

$$X_i = \frac{dA_i}{dt} + \frac{d}{dt}\left(\frac{\partial W}{\partial \frac{dx_i}{dt}}\right) - \frac{\partial W}{\partial x_i}, \qquad L_i = \frac{dP_i}{dt} + C_i \frac{dy_i}{dt} - B_i \frac{dz_i}{dt},$$

$$Y_i = \frac{dB_i}{dt} + \frac{d}{dt}\left(\frac{\partial W}{\partial \frac{dy_i}{dt}}\right) - \frac{\partial W}{\partial y_i}, \qquad M_i = \frac{dQ_i}{dt} + A_i \frac{dz_i}{dt} - C_i \frac{dx_i}{dt},$$

$$Z_i = \frac{dC_i}{dt} + \frac{d}{dt}\left(\frac{\partial W}{\partial \frac{dz_i}{dt}}\right) - \frac{\partial W}{\partial z_i}, \qquad N_i = \frac{dR_i}{dt} + B_i \frac{dx_i}{dt} - A_i \frac{dy_i}{dt},$$

where (X_i, Y_i, Z_i) and (L_i, M_i, N_i) are the external force and external moment, respectively, of the trihedron with index i. In these calculations, one sees that it is easy to exhibit the arguments r_{ij}, ψ_{ij}, λ_{ijk} .

We remark that the expression for the external force is found to decompose into two parts: The first one, which depends upon the line segments (A_i, B_i, C_i) and (P_i, Q_i, R_i), and their derivatives, is the properly dynamical part, while the second one, which results from the presence of the arguments r_{ij}, ψ_{ij}, λ_{ijk} in W, corresponds to the force that the trihedron with index i is subjected to under the influence of all the other trihedra of the system. Consider the expression:

$$\sum_i \Big[X_i \frac{dx_i}{dt} + Y_i \frac{dy_i}{dt} + Z_i \frac{dz_i}{dt}$$
$$+ L_i(\alpha_i p_i + \beta_i q_i + \gamma_i r_i) + M_i(\alpha'_i p_i + \beta'_i q_i + \gamma'_i r_i) + N_i(\alpha''_i p_i + \beta''_i q_i + \gamma''_i r_i)\Big] dt,$$

which represents the sum of the elementary works that are done by the forces that are applied to the various trihedra. If we calculate them by replacing X_i, Y_i, Z_i, L_i, M_i, N_i with the preceding values then we will find the following expression for the elementary work that relates to the dynamical part of the external force and external moment:

$$\sum \Big[\frac{d}{dt}\left(\xi_i \frac{\partial W}{\partial \xi_i} + \eta_i \frac{\partial W}{\partial \eta_i} + \zeta_i \frac{\partial W}{\partial \zeta_i} + p_i \frac{\partial W}{\partial p_i} + q_i \frac{\partial W}{\partial q_i} + r_i \frac{\partial W}{\partial r_i} \right)$$
$$- \left(\frac{\partial W}{\partial \xi_i}\frac{d\xi_i}{dt} + \frac{\partial W}{\partial \eta_i}\frac{d\eta_i}{dt} + \cdots + \frac{\partial W}{\partial r_i}\frac{dr_i}{dt} \right) \Big] dt,$$

which is analogous to what we have already obtained in no. 19, and the elementary work that is done by the forces that are exerted between the trihedra of the system:

$$\sum_i \left[\frac{d}{dt}\left(\frac{dx_i}{dt}\frac{\partial W}{\partial \frac{dx_i}{dt}} + \frac{dy_i}{dt}\frac{\partial W}{\partial \frac{dy_i}{dt}} + \frac{dz_i}{dt}\frac{\partial W}{\partial \frac{dz_i}{dt}} \right) \right.$$
$$\left. - \left(\frac{\partial W}{\partial \frac{dx_i}{dt}}\frac{d^2 x_i}{dt^2} + \frac{\partial W}{\partial \frac{dy_i}{dt}}\frac{d^2 y_i}{dt^2} + \frac{\partial W}{\partial \frac{dz_i}{dt}}\frac{d^2 z_i}{dt^2} + \frac{\partial W}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial W}{\partial y_i}\frac{dy_i}{dt} + \frac{\partial W}{\partial z_i}\frac{dz_i}{dt} \right) \right] dt .$$

If we add these two expressions, and if we set:

$$E = \sum_i \left(\xi_i \frac{\partial W}{\partial \xi_i} + \eta_i \frac{\partial W}{\partial \eta_i} + \zeta_i \frac{\partial W}{\partial \zeta_i} + p_i \frac{\partial W}{\partial p_i} + q_i \frac{\partial W}{\partial q_i} + r_i \frac{\partial W}{\partial r_i} + \frac{dx_i}{dt}\frac{\partial W}{\partial \frac{dx_i}{dt}} + \frac{dy_i}{dt}\frac{\partial W}{\partial \frac{dy_i}{dt}} + \frac{dz_i}{dt}\frac{\partial W}{\partial \frac{dz_i}{dt}} - W \right)$$

then we will see that the sum of the elementary works is:

$$dE + \frac{\partial W}{\partial t}\, dt.$$

Upon supposing that W is independent of t and giving E the name of *energy of motion and position* for the system in question, we will obtain a proposition that is entirely analogous to the one in no. 19.

It is easy to deduce a dynamical law for systems from the foregoing that is established on the same plane as in classical mechanics, without having to restrict oneself to envisioning central forces, as one does in that theory. Moreover, the present exposition has the advantage of giving the true origins to the various laws of force and distance that were studied by Gauss, Riemann, Weber, and Clausius, which all introduce just the arguments r_{ij}, ψ_{ij}, λ_{ijk} . We shall not insist upon this point, which leaves the traditional scope of mechanics, and which is mainly of interest to theoretical physics.

21. The Euclidian action of constraint and the dissipative Euclidian action. – The considerations that we just developed in regard to the Euclidian action at a distance lead to the notion of *constraint* in a most natural manner, which is due to Gauss and was, as one knows, applied by Hertz to the study of the foundations of mechanics by following a path that was already traversed by Beltrami, R. Lipschitz, and G. Darboux.

To simplify, let there be given a point that describes a trajectory that is defined by three functions x_0, y_0, z_0 of time t when its motion is *free*. On the other hand, let x, y, z denote the functions of time t that define the trajectory when it is subject to constraints. We can envision the two points (X, Y, Z), (X_0, Y_0, Z_0) whose coordinates are obtained, for example, from the formulas:

$$X = x + \frac{dx}{dt}dt + \frac{1}{2}\frac{d^2x}{dt^2}dt^2, \qquad X_0 = x_0 + \frac{dx_0}{dt}dt + \frac{1}{2}\frac{d^2x_0}{dt^2}dt^2,$$
$$Y = y + \frac{dy}{dt}dt + \frac{1}{2}\frac{d^2y}{dt^2}dt^2, \qquad Y_0 = y_0 + \frac{dy_0}{dt}dt + \frac{1}{2}\frac{d^2y_0}{dt^2}dt^2,$$
$$Z = z + \frac{dz}{dt}dt + \frac{1}{2}\frac{d^2z}{dt^2}dt^2, \qquad Z_0 = y_0 + \frac{dz_0}{dt}dt + \frac{1}{2}\frac{d^2z_0}{dt^2}dt^2,$$

which come from the Taylor development, when it is limited to the first three terms. Upon assuming that the constraints are *frictionless*, one can write that at the instant t considered, one will have:

$$x = x_0, \quad x = x_0, \quad x = x_0, \quad \frac{dx}{dt} = \frac{dx_0}{dt}, \quad \frac{dy}{dt} = \frac{dy_0}{dt}, \quad \frac{dz}{dt} = \frac{dz_0}{dt}.$$

Having said that, after having considered the *Euclidian action at a distance* $U_1(r)$ for the two points (X, Y, Z) and (X_0, Y_0, Z_0), whose separation we denote by r, the introduction of the notion of constraint that is due to Gauss amounts to replacing r with its value, in such a way that one is led to the function U of the argument γ that is defined by the formula:

$$\dot{\gamma}^2 = \left(\frac{d^2x}{dt^2} - \frac{d^2x_0}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2} - \frac{d^2y_0}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2} - \frac{d^2z_0}{dt^2}\right)^2.$$

If we then apply the method of variable action then we will get:

$$\delta U = X\left(\delta\frac{d^2x}{dt^2} - \delta\frac{d^2x_0}{dt^2}\right) + Y\left(\delta\frac{d^2y}{dt^2} - \delta\frac{d^2y_0}{dt^2}\right) + Z\left(\delta\frac{d^2z}{dt^2} - \delta\frac{d^2z_0}{dt^2}\right),$$

by setting:

$$X = \frac{1}{\gamma}\frac{dU}{d\gamma}\left(\frac{d^2x}{dt^2} - \frac{d^2x_0}{dt^2}\right), \qquad Y = \frac{1}{\gamma}\frac{dU}{d\gamma}\left(\frac{d^2y}{dt^2} - \frac{d^2y_0}{dt^2}\right), \qquad Z = \frac{1}{\gamma}\frac{dU}{d\gamma}\left(\frac{d^2z}{dt^2} - \frac{d^2z_0}{dt^2}\right).$$

If, with Gauss, one calls the argument γ the *constraint* then the force X, Y, Z can be called the *force of constraint* that is applied to the point (x, y, z), and can be regarded as having the effect of preventing the motion of the point from being *free*. On the contrary, the force $-X$, $-Y$, $-Z$ is applied to the point (x_0, y_0, z_0) in order to bring about the transformation from free motion into constrained motion.

The essential difference between the present conception of force and the one that results from Newton's laws of motion is the following: In the latter, one considers the action that relates to two infinitely close positions – the one, present, the other, future – *on the same trajectory*. In the way that Gauss and Hertz looked at things, the action is referred to two future positions, one of which is on the trajectory that is called the *free* one, while the other is on the trajectory that is called the *constrained* one. In the two cases, one obviously has a theory that permits one to *predict* a future motion, which is the objective of the dynamics of a point. However, in addition – and this is the point that we would like to demonstrate especially – the action is *Euclidian*.

On this subject, it is interesting to remark that Gauss has explicitly established an agreement between the action of constraint and the *law of errors*, which does indeed have the same form. One thus sees that the fundamental character of the law of errors is the *Euclidian invariance* of that law, and that the new branch of mechanics that was created by Maxwell, Boltzmann, and W. Gibbs under the name of *statistical mechanics* can likewise take on the deductive form that we have tried to give to ordinary mechanics here.

We can further observe that the forces of constraint can just as well translate into the mechanics that one deduces from the ideas of Newton as in the mechanics that one can deduce from the Gauss's notion of constraint, which is an *indeterminacy* that is produced in the definition of force, and which leads one to introduce the Lagrange multipliers.

Gauss's idea can also be applied to friction by imagining a Euclidian action on the two points:

$$X = x + \frac{dx}{dt}dt, \qquad X_0 = x_0 + \frac{dx_0}{dt}dt,$$
$$Y = y + \frac{dy}{dt}dt, \qquad Y_0 = y_0 + \frac{dy_0}{dt}dt,$$
$$Z = z + \frac{dz}{dt}dt, \qquad Z_0 = z_0 + \frac{dz_0}{dt}dt,$$

where the point x_0, y_0, z_0 is referred to a free trajectory and the point x, y, z, to a trajectory that is traversed with friction. As it amounts to a sliding friction here, one must set $x = x_0$, $y = y_0$, $z = z_0$, $\frac{dx}{dt} = \mu\,\frac{dx_0}{dt}, \frac{dy}{dt} = \mu\,\frac{dy_0}{dt}, \frac{dz}{dt} = \mu\,\frac{dz_0}{dt}$. One is then led to an action that is a function of the velocity $v_0 = \sqrt{\left(\frac{dx_0}{dt}\right)^2 + \left(\frac{dy_0}{dt}\right)^2 + \left(\frac{dz_0}{dt}\right)^2}$ that is affected with the factor $1 - \mu$, which describes precisely the notion of *dissipation of the free action at the point* x_0, y_0, z_0.

The arguments r_{ij}, ψ_{ij}, λ_{ijk} that we have considered in no. 20 translates definitively an analogous idea in regard to a trihedron that is assumed to be isolated in the system of n trihedra that is envisioned. One can, if one prefers, distinguish these arguments and say that r_{ij} is a *potential* argument, while ψ_{ij}, λ_{ijk} are *dissipative*. The central-force hypothesis thus amounts to considering only the dynamics of systems without *friction at a distance* in mechanics. On the other hand, one can derive the special argument dr_{ij} / dt of Weber from the arguments r_{ij}, ψ_{ij}, λ_{ijk}, and if one passes from the discontinuous medium to a *continuous* medium, the conception of which is based upon considering the ds^2 of space, then one is then led to introduce the *viscosity arguments* $\frac{d\varepsilon_1}{dt}, \frac{d\varepsilon_2}{dt}, \frac{d\varepsilon_3}{dt}, \frac{d\gamma_1}{dt}, \frac{d\gamma_2}{dt}, \frac{d\gamma_3}{dt}$ into the action W. Aside from such arguments, which were envisioned for the first time by Navier and Poisson, one must obviously also present arguments such as the mixed argument $\xi_1\ \xi_2 + \eta_1\ \eta_2 + \zeta_1\ \zeta_2$ that was at issue in no. 15. We confine ourselves to these summary observations about viscosity, which has not been studied in a sufficiently systematic manner up to now, moreover.

V. – THE EUCLIDIAN ACTION FROM THE EULERIAN VIEWPOINT.

22. The action of deformation and motion of a continuous medium from the Eulerian viewpoint. The notion of the radiation of energy. – In the statics and dynamics of deformable continuous media, we took the independent variables to be x_0, y_0, z_0, and x_0, y_0, z_0, t, respectively. In the case of statics, x_0, y_0, z_0 were the coordinates of the point M_0 of the natural state (M_0). In the case of dynamics, x_0, y_0, z_0 were the coordinates at the instant t_0 of the point M_0 that became the point M at time t. The independent variables that we thus considered were the *Lagrange variables.*

One can now imagine that one performs a change of variables on the independent variables. In particular, by analogy with what one does in hydrodynamics, one can take x, y, z or x, y, z, t to be the new independent variables, and with this particular choice one has what one calls the *Euler variables*. $x_0, y_0, z_0, \alpha, \alpha', \ldots, \gamma''$ then become functions of x, y, z or of x, y, z, t according to whether one is dealing with statics or dynamics, respectively.

Along with the Lagrange variables, we have considered the Euclidian arguments ξ_i, η_i, ζ_i, p_i, q_i, r_i, ξ, η, ζ, p, q, r. Along with the Euler variables, we envision the new arguments (ξ_i), (η_i), (ζ_i), (p_i), (q_i), (r_i); (ξ), (η), (ζ), (p), (q), (r). We shall define them and show that they are, like the former, Euclidian invariants. Upon recalling that $x_0 = \rho_1$, $y_0 = \rho_2$, $z_0 = \rho_3$, set:

$$[\xi_i] = \frac{\partial \rho_i}{\partial x}, \qquad [\eta_i] = \frac{\partial \rho_i}{\partial y}, \qquad [\zeta_i] = \frac{\partial \rho_i}{\partial z},$$
$$[p_1] = \sum \gamma \frac{\partial \beta}{\partial x}, \quad [q_1] = \sum \gamma \frac{\partial \beta}{\partial y}, \quad [r_1] = \sum \gamma \frac{\partial \beta}{\partial z},$$

with analogous formulas for $[p_2]$, $[q_2]$, $[r_2]$ and $[p_3]$, $[q_3]$, $[r_3]$ that are obtained by first changing γ, β into α, β and then into β, α. The arguments (ξ_i), (η_i), (ζ_i) will be the projection onto the axes Mx', My', Mz' of the vector whose projections onto the axes Ox, Oy, Oz are $[\xi_i]$, $[\eta_i]$, $[\zeta_i]$. Similarly, (p_i), (q_i), (r_i) will be the projections onto the axes Mx', My', Mz' of the vector whose projections onto the axes Ox, Oy, Oz are $[p_i]$, $[q_i]$, $[r_i]$. In addition, in the case where there is motion, we take:

$$(\xi) = \frac{\partial \rho_1}{\partial t}, \qquad (\eta) = \frac{\partial \rho_2}{\partial t}, \qquad (\eta) = \frac{\partial \rho_3}{\partial t},$$
$$(p) = \sum \gamma \frac{\partial \beta}{\partial x}, \quad (q) = \sum \gamma \frac{\partial \beta}{\partial y}, \quad (r) = \sum \gamma \frac{\partial \beta}{\partial z}.$$

It is easy to see that one has:

$$\xi_i (\xi_i) + \eta_i (\eta_i) + \zeta_i (\zeta_i) = 1, \quad \xi_j (\xi_k) + \eta_j (\eta_k) + \zeta_j (\zeta_k) = 0 \quad (j \neq k),$$

$$(\xi) + \xi(\xi_1) + \eta (\eta_1) + \zeta (\zeta_1) = 0$$
$$(\eta) + \xi(\xi_2) + \eta (\eta_2) + \zeta (\zeta_2) = 0$$

$$(\zeta) + \xi(\xi_3) + \eta\,(\eta_3) + \zeta\,(\zeta_3) = 0,$$

and then:

$$(p_1) = \sum_i p_i(\xi_i)\,, \qquad (q_1) = \sum_i p_i(\eta_i)\,, \qquad (r_1) = \sum_i p_i(\zeta_i)\,,$$

with analogous formulas for (p_2), (q_2), (r_2) and (p_3), (q_3), (r_3) that are obtained by first changing p_i into q_i, and then into r_i . Finally:

$$(p) = p_1(\xi) + p_2(\eta) + p_3(\zeta) + p,$$

with analogous formulas for (q) and (r) that are obtained by changing p, p_i into q, q_i, and then into r, r_i . One sees that the Eulerian arguments, being functions of *only* the Lagrangian arguments, are indeed also Euclidian invariants.

Suppose that W, which we call the *Lagrangian action density*, is expressed by means of the Eulerian arguments (ξ_i), (η_i), (ζ_i), (p_i), (q_i), (r_i), (ξ), (η), (ζ), (p), (q), (r), and set:

$$W = \Omega\,\Delta.$$

We call Ω, which will have the remarkable form:

$$\Omega[x_0, y_0, z_0, t, (\xi_i), (\eta_i), (\zeta_i), (p_i), (q_i), (r_i), (\xi), (\eta), (\zeta), (p), (q), (r)],$$

the *Eulerian action density*. The action will take the form:

$$\int_{t_1}^{t_2}\iiint \Omega\; dx\,dy\,dz\,dt.$$

When the integration over x, y, z is taken, as before, over the volume that is bounded by the surface S of the deformed medium – i.e., a *domain that varies with time* – we will get the Lagrangian action. On the contrary, if the integration is taken over a *fixed domain that is independent of t* then we will get the Eulerian action.

In order to apply the calculus of variations to an action that is taken in one or the other of the forms that we just pointed out, it is convenient – following the example of Poincaré – to establish the following distinction between the variations that a function V of x, y, z, t can receive. From the Eulerian viewpoint, the function V experiences a variation that we will denote by (δV) and which is due to a *change of the function*. From the Lagrangian viewpoint, it experiences the variation:

$$\delta V = (\delta V) + \frac{\partial V}{\partial x}\delta x + \frac{\partial V}{\partial y}\delta y + \frac{\partial V}{\partial z}\delta z\,,$$

that one can call the *total* – or Lagrangian – *variation*. The *special role* that is played in the present theory by the functions x_0, y_0, z_0 of x, y, z, t translates into writing that their Lagrangian variations are *zero,* in such a way that one will have three formulas such as the following one:

$$0 = (\delta x_0) + \frac{\partial x_0}{\partial x}\delta x + \frac{\partial x_0}{\partial y}\delta y + \frac{\partial x_0}{\partial z}\delta z .$$

A distinction that is analogous to the preceding one must be made between the derivatives with respect to time of the function V. The Eulerian derivative is what one usually distinguishes by the symbol $\partial V / \partial t$. As for the total – or Lagrangian – derivative, it is expressed by the formula:

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} .$$

The sign δ commutes with $\frac{\partial}{\partial x_0}$, $\frac{\partial}{\partial y_0}$, $\frac{\partial}{\partial z_0}$, $\frac{d}{dt}$, and the sign (δ) commutes with $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial t}$. Similarly, in a Lagrangian integral whose domain varies with time, one cannot invert the integration with respect to t and the system of integrations that relate to x, y, z. One can make that inversion only when one is dealing with the variables x_0, y_0, z_0, and the integrations by parts must be true for the Lagrangian derivatives d / dt. For an Eulerian integral, one can invert the integration with respect to t and the integration with respect to the field of variables x, y, z, and when that inversion has been performed, the integration with respect to time must be done by imagining that x, y, z are constant. The integrations by parts must refer to the derivatives $\partial / \partial t$, and not the derivatives d / dt.

First, consider the Lagrangian action. Its variation will be:

$$\int_{t_1}^{t_2}\iiint (\Delta\,\delta\Omega + \Omega\,\delta\Delta)\,dx_0\,dy_0\,dz_0\,dt = \int_{t_1}^{t_2}\iiint \left(\delta\Omega + \Omega\frac{\delta\Delta}{\Delta}\right) dx\,dy\,dz\,dt,$$

or furthermore, upon remarking that $\frac{\delta\Delta}{\Delta} = \frac{\partial\,\delta x}{\partial x} + \frac{\partial\,\delta y}{\partial y} + \frac{\partial\,\delta z}{\partial z}$:

$$\int_{t_1}^{t_2}\iiint \left[\delta\Omega + \Omega\left(\frac{\partial\,\delta x}{\partial x} + \frac{\partial\,\delta y}{\partial y} + \frac{\partial\,\delta z}{\partial z}\right)\right] dx\,dy\,dz\,dt.$$

If one carries out the calculations along the same lines as in no. 19, and takes into account the remarks that we just made a moment ago then one will recover the formulas that we already know:

$$X = \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} + \frac{1}{\Delta}\frac{dA}{dt},$$

$$Y = \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} + \frac{1}{\Delta}\frac{dB}{dt},$$

$$Z = \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} + \frac{1}{\Delta}\frac{dC}{dt},$$

$$L = \frac{\partial q_{xx}}{\partial x} + \frac{\partial q_{yx}}{\partial y} + \frac{\partial q_{zx}}{\partial z} + p_{yz} - p_{zy} + \frac{1}{\Delta}\frac{dP}{dt} + \frac{C}{\Delta}\frac{dy}{dt} - \frac{B}{\Delta}\frac{dz}{dt},$$

$$M = \frac{\partial q_{xy}}{\partial x} + \frac{\partial q_{yy}}{\partial y} + \frac{\partial q_{zy}}{\partial z} + p_{zx} - p_{xz} + \frac{1}{\Delta}\frac{dQ}{dt} + \frac{A}{\Delta}\frac{dz}{dt} - \frac{C}{\Delta}\frac{dx}{dt},$$

$$N = \frac{\partial q_{xz}}{\partial x} + \frac{\partial q_{yz}}{\partial y} + \frac{\partial q_{zz}}{\partial z} + p_{xy} - p_{yx} + \frac{1}{\Delta}\frac{dR}{dt} + \frac{B}{\Delta}\frac{dx}{dt} - \frac{A}{\Delta}\frac{dy}{dt},$$

$$F = l\,p_{xx} + n\,p_{yx} + n\,p_{zx}\,,$$
$$G = l\,p_{xy} + n\,p_{yy} + n\,p_{zy}\,,$$
$$H = l\,p_{xz} + n\,p_{yz} + n\,p_{zz}\,,$$

$$I = l\,q_{xx} + n\,q_{yx} + n\,q_{zx}\,,$$
$$J = l\,q_{xy} + n\,q_{yy} + n\,q_{zy}\,,$$
$$K = l\,q_{xz} + n\,q_{yz} + n\,q_{zz}\,.$$

However, if one sets, by analogy with the notations of no. 16:

$$(A_i') = \frac{\partial \Omega}{\partial(\xi_i)}, \qquad (B_i') = \frac{\partial \Omega}{\partial(\eta_i)}, \qquad (C_i') = \frac{\partial \Omega}{\partial(\zeta_i)},$$
$$(A') = \frac{\partial \Omega}{\partial(\xi)}, \qquad (B') = \frac{\partial \Omega}{\partial(\eta)}, \qquad (C') = \frac{\partial \Omega}{\partial(\zeta)},$$

$$(P_i') = \frac{\partial \Omega}{\partial(p_i)}, \qquad (Q_i') = \frac{\partial \Omega}{\partial(q_i)}, \qquad (R_i') = \frac{\partial \Omega}{\partial(r_i)},$$
$$(P') = \frac{\partial \Omega}{\partial(p)}, \qquad (Q') = \frac{\partial \Omega}{\partial(q)}, \qquad (R') = \frac{\partial \Omega}{\partial(r)},$$

where (A_i'), (B_i'), (C_i'), (P_i'), (Q_i'), (R_i'), (A'), (B'), (C'), (P'), (Q'), (R') define four vectors, respectively, that are referred to the axes Mx', My', Mz', where we denote the components with respect to Ox, Oy, Oz by $[A_i]$, $[B_i]$, $[C_i]$, $[P_i]$, $[Q_i]$, $[R_i]$, $[A]$, $[B]$, $[C]$, $[P]$, $[Q]$, $[R]$, respectively, then one will find that:

$$\frac{A}{\Delta} = -(A')[\xi_1] - (B')[\xi_2] - (A')[\xi_3] - (P')[p_1] - (Q')[p_2] - (R')[p_3]\,,$$

$$p_{xx} = \Omega - [A_1]\,[\xi_1] - [A_2]\,[\xi_2] - [A_3]\,[\xi_3] - [P_1]\,[p_1] - [P_2]\,[p_2] - [P_3]\,[p_3] - \frac{A}{\Delta}\frac{dx}{dt},$$

$$p_{yx} = \Omega - [B_1]\,[\xi_1] - [B_2]\,[\xi_2] - [B_3]\,[\xi_3] - [Q_1]\,[Q_1] - [Q_2]\,[p_2] - [Q_3]\,[p_3] - \frac{A}{\Delta}\frac{dy}{dt},$$
$$p_{zx} = \Omega - [C_1]\,[\xi_1] - [C_2]\,[\xi_2] - [C_3]\,[\xi_3] - [R_1]\,[p_1] - [R_2]\,[p_2] - [R_3]\,[p_3] - \frac{A}{\Delta}\frac{dz}{dt},$$

with analogous formulas for B, C, and p_{xy}, p_{yy}, p_{zy}, p_{xz}, p_{yz}, p_{zz} . In addition:

$$q_{xx} = \alpha[P_1] + \beta[P_2] + \gamma[P_3] - \frac{P}{\Delta}\frac{dx}{dt},$$
$$q_{yx} = \alpha[Q_1] + \beta[Q_2] + \gamma[Q_3] - \frac{P}{\Delta}\frac{dy}{dt}, \qquad \frac{1}{\Delta}P = [P],$$
$$q_{zx} = \alpha[R_1] + \beta[R_2] + \gamma[R_3] - \frac{P}{\Delta}\frac{dz}{dt},$$

with analogous formulas for Q, R, and q_{xy} , q_{yy} , q_{zy} , q_{xz} , q_{yz} , q_{zz} . These results are also obtained by directly transforming the formulas of no. 19 by means of the relations between the Lagrangian arguments and the Eulerian arguments that were discussed above.

We can likewise obtain the density of energy that corresponds to the notion of Lagrangian action by one or another of the paths that we just pointed out. We have seen that *when it is referred to the space of* x_0, y_0, z_0 this density is:

$$\xi\frac{\partial W}{\partial \xi} + \eta\frac{\partial W}{\partial \eta} + \zeta\frac{\partial W}{\partial \zeta} + p\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial q} + r\frac{\partial W}{\partial r} - W.$$

This same density, *when it is referred to the space of* x, y, z and expressed by means of the function Ω of the Eulerian arguments (ξ_i), (η_i), (ζ_i), (p_i), (q_i), (r_i); (ξ), (η), (ζ), (p), (q), (r), is:

$$(\xi)\frac{\partial \Omega}{\partial(\xi)} + (\eta)\frac{\partial \Omega}{\partial(\eta)} + (\zeta)\frac{\partial \Omega}{\partial(\zeta)} + (p)\frac{\partial \Omega}{\partial(p)} + (q)\frac{\partial \Omega}{\partial(q)} + (r)\frac{\partial \Omega}{\partial(r)} - \Omega.$$

In no. 19, we found that the elementary work that was done by the external forces and external moments, as well as the external efforts and moments of deformations that were exercised on a portion (M) of the medium that occupied the portion (M_0) of the natural state at the instant t had the expression:

$$\left(\iiint_{S_0} \frac{dE}{dt}\,dx_0\,dy_0\,dz_0\right)dt,$$

in which W is supposed to be independent of t. The same result persists if one considers a *fixed region*(M) of space. Therefore, if we observe that we have the following identity:

$$\frac{1}{\Delta}\frac{dE}{dt} = \frac{\partial}{\partial t}\left(\frac{E}{\Delta}\right) + \frac{\partial}{\partial x}\left(\frac{E}{\Delta}\frac{dx}{dt}\right) + \frac{\partial}{\partial y}\left(\frac{E}{\Delta}\frac{dy}{dt}\right) + \frac{\partial}{\partial z}\left(\frac{E}{\Delta}\frac{dz}{dt}\right),$$

which was employed by Poincaré and which is applied to an arbitrary function, then we will arrive at the following expression for the new elementary work:

$$\left\{\frac{\partial}{\partial t}\iiint_S \frac{E}{\Delta}\,dx\,dy\,dz \;+\; \iiint_S \left[\frac{\partial}{\partial x}\left(\frac{E}{\Delta}\frac{dx}{dt}\right) + \frac{\partial}{\partial y}\left(\frac{E}{\Delta}\frac{dy}{dt}\right) + \frac{\partial}{\partial z}\left(\frac{E}{\Delta}\frac{dz}{dt}\right)\right] dx\,dy\,dz\right\} dt\,,$$

or

$$\left[\frac{\partial}{\partial t}\iiint_S \frac{E}{\Delta}\,dx\,dy\,dz + \iint_S \frac{E}{\Delta}\left(l\frac{dx}{dt} + m\frac{dy}{dt} + n\frac{dz}{dt}\right) d\sigma\right] dt\,.$$

The double integral that figures in the coefficient of dt corresponds to what one can call the *energy flux of deformation and motion* that traverses the fixed surface S in the deformed body.

Now, consider the action from the Eulerian viewpoint. It is, first of all, interesting to confirm that the values of the external forces and external moments remain the same, but that the following terms disappear in the expressions for the efforts p_{xx}, p_{yx}, p_{zx} :

$$\pi_{xx} = \Omega - \frac{A}{\Delta}\frac{dx}{dt}, \qquad \pi_{yx} = \Omega - \frac{B}{\Delta}\frac{dy}{dt}, \qquad \pi_{zx} = \Omega - \frac{C}{\Delta}\frac{dz}{dt},$$

and the following terms in the expressions for the moments of deformation q_{xx}, q_{yx}, q_{zx} :

$$\chi_{xx} = -\frac{P}{\Delta}\frac{dx}{dt}, \qquad \chi_{yx} = -\frac{Q}{\Delta}\frac{dy}{dt}, \qquad \chi_{zx} = -\frac{R}{\Delta}\frac{dz}{dt},$$

with analogous expressions for the quantities π_{xy}, π_{yy}, …, χ_{xy}, χ_{yy}, … It results from this that the elementary work that is obtained in the preceding case must be augmented with a new surface integral, which has the expression:

$$\left\{\iint_S \left[\Omega\left(l\frac{dx}{dt} + m\frac{dy}{dt} + n\frac{dz}{dt}\right) - \frac{1}{\Delta}(\xi^2 + \eta^2 + \zeta^2)(lA + mB + nC) \right.\right.$$
$$\left.\left. - \frac{1}{\Delta}(p\xi + q\eta + r\zeta)(lP + mQ + nR)\right] d\sigma\right\} dt\,.$$

One can call this new integral the *radiant energy flux that traverses the frontier S of the deformed body.*

The argument that was made in no. 19 and was founded upon the Euclidian invariance of the action density will no longer lead to the same conclusions (36) in regard to the external forces and moments, as well as in regard to the *new* external efforts and moments of deformation. One can express this by saying that the new efforts and moments of deformation no longer satisfy what Poincaré called the *principle of reaction.*

As one knows, this latter conclusion is likewise reached in the electrical theories of Lorentz. However, the existence of the radiation that we just exhibited permits us to reconcile the efforts and moments of deformation π_{xx}, π_{yx}, ,,,, χ_{xx}, χ_{yx}, … with those of Maxwell using considerations that are inferred from the electromagnetic theory of light, and which Bartoli, in the context of thermodynamics, has called the *pressure of radiant energy*, so that one can therefore once more respect the *principle of reaction*.