THEORY

OF

DEFORMABLE BODIES

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FOREWORD

This volume contains the development of a summary note on the *Théorie de l'action euclidienne* that APPELL has seen fit to introduce in the 2nd edition of his *Traité de mécanique rationelle.* The reproduction of an appendix to the French edition of the *Traité de physique* of CHWOLSON, explains several peculiarities of the editing and the reference that we make to a prior work on the dynamics of the point and rigid body, which is likewise combined with the work of the Russian savant. We profited from that new printing by correcting several mistakes in our text.

We do not seek to actually deduce all of the consequences of the general results that we will arrive at; throughout, we strive only to rediscover and clarify the classical doctrines. In order for this sort of verification of the theory of the Euclidian action to appear more complete in each of the parts of our exposition we will have to establish the form that the equations of deformable bodies take when one is limited to the consideration of *infinitely close states;* however, this is a point that we have already addressed, with all of the necessary details, in our *Premiere mémoire sur la Théorie de l'élasticité* that we wrote in 1896 (*Annales de la Faculté des Sciences de Toulouse,* Tome X). We suppose, moreover, that the masterful lessons of G. DARBOUX on the *Théorie générale des surfaces* are completely familiar to the reader.

Our researches will make sense only when have shown how one may envision the theories of heat and electricity by following the path that we follow. We dedicate two notes in tomes III and IV of the treatise of CHWOLSON to this subject. The *subdivision,* to use the language of pragmatism, appears to be a scientific necessity; nevertheless, one must not lose sight of the fact that it solves grave questions. We have attempted to give an idea of these difficulties in our note on the *Théorie of corps minces,* published in 1908 in the Comptes Rendus de l'Académie des Sciences and whose substance was also indicated by APPELL in his treatise.

E. & F. COSSERAT

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THEORY

OF

DEFORMABLE BODIES

By Messrs **E. and F. COSSERAT**

I. - GENERAL CONSIDERATIONS

1. Development of the idea of a continuous medium. - The notion of a deformable body has played an important role in the development of theoretical physics in the last century, and FRESNEL ([1]) has to be regarded as the equal of NAVIER, POISSON, and CAUCHY ([2]) as one of the precursors to the present theory of elasticity. At the time of these savants, under the influence of Newtonian ideas, one considered only discrete systems of points. With the memorable research of G. GREEN ([3]), continuous systems of points appeared. One has since attempted to enlarge the ideas of GREEN, which are insufficient to give the theory of luminous waves all that it requires. In particular, LORD KELVIN ([4]) is associated with defining a continuous medium in which a moment may be exerted at any point. The same tendency has been attributed to the school of HELMHOLTZ ([5]), and the contradiction, due to J. BERTRAND ([6]) in regard to the theory of electromagnetism, is quite characteristic. One may return to the origin of this

[1] FRESNEL. - *Oeuvres complétes,* Paris, 1886; see the introduction by I. VERDET.

[2] See ISAAC TODHUNTER and KARL PEARSON. - *A History of the Theory of Elasticity and the Strength of Materials, from* GALILEI *to the present time,* Vol. I, GALILEI to SAINT-VENANT, 1886; Vol. II, Part I and II, SAINT-VENANT to LORD KELVIN, 1893. This remarkable work contains a very complete and very precise analysis of the work of the founders of the theory of elasticity.

[3] G. GREEN. - *Math. Papers,* edited by N.M. FERRERS, facsimile reprint, Paris, A. Hermann, 1903.

[4] LORD KELVIN. - *Math. and phys. Papers,* volume I, 1882; vol. II, 1884; vol. III, 1890; *Reprint of Papers on Electrostatics and Magnetism*, 2nd ed. 1884; *Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light*, 1904; W. THOMSON and P.G. TAIT, *Treatise on Natural Philosophy*, 1st ed. Oxford 1867; 2nd ed. Cambridge 1879-1883.

[5] HELMHOLTZ. - *Vorles. über die Dynamik diskreter Massenpunkte,* Berlin 1897; *Vorles. über die electromagnetische Theorie des Lichtes*, Leipzig 1897; *Wiss. Abhandl.,* 3 vol. Leipzig, 1892-1895.

[6] J. BERTRAND. - *C.R.* **73**, pp. 965; **75**, pp. 860; **77**, pp. 1049; see also H. POINCARI, *Electricité et Optique*, II, *Les théories de* HELMHOLTZ *et les experiences de* HERTZ, Paris, 1891, pp. 51; 2nd ed. 1901, pp. 275.

evolution, which was, on the one hand, the concepts that were introduced in the theory of the resistance of materials by BERNOULLI and EULER ([7]), and, on the other hand, POINSOT's theory of couples ([8]). One is therefore naturally led to unite the various concepts of deformable bodies that one considers today in natural philosophy into a single geometric definition. *A deformable line is a continuous one-parameter set of triads, a deformable surface is a two-parameter set, and a deformable medium is a three-parameter set* (ρ_i); *when there is motion, one must add time t to these geometric parameters* ρ_i . As one knows, the mathematical continuity that one supposes in such a definition leaves the trace of an invariant solid unchanged at every point. As a result, one may anticipate that the well-known moments that have been studied in line and surface elasticity since EULER and BERNOULLI, and which LORD KELVIN and HELMHOLTZ have sought to find in three-dimensional media, will appear in the mechanical viewpoint.

2. Difficulties presented by the inductive method in mechanics. - The primary form of mechanics is inductive; this is what one neatly perceives in the theory of deformable bodies. This theory imprinted propositions that relate to the notion of static force on the mechanics of invariable bodies, which one applies by the principle of solidification; next, the relation between effort and deformation was established hypothetically (generalized Hooke's law), and one sought, *a posteriori,* the conditions under which energy is conserved (GREEN). A century ago, CARNOT ([9]) pointed out the problem with that method: that one constantly appeals to *a priori* notions and that the path that one follows is not always certain. Indeed, the static force has no constructive definition in our classical form for mechanics, and the importance of the revision that REECH ([10]) has proposed in regards to that in 1852 has remained largely unrecognized

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[7] See TODHUNTER and PEARSON. - *Op. cit.*

[8] AUGUST COMPTE. - *Cours de Philosophie positive.* - 5th ed. Paris, 1907, Tome I, page 338: "No matter what the fundamental qualities of the conception of POINSOT that relate to statics may be in reality, one must nevertheless recognize, it seems to me, that it is, above all, essentially destined, by its nature, to represent the quintessence of dynamics; moreover, in regard to that, one may be assured that this conception has not exerted its ultimate influence up to this point in time."

[9] CARNOT, in his 1783 *Essai sur les machines en général,* who foresaw in 1803, *les Principes fondamentaux de l'équilibre et du mouvement,* sought to reduce mechanics to precise definitions and principles that were completely devoid of any metaphysical character and vague terms that the philosophers dispute to no avail. This reaction took CARNOT a little too far, since it led him to contest the legitimacy of the notion of force, a notion that was obscure according to him, and for which he would like to substitute the idea of motion exclusively. By the same reasoning, he would not accept as rigorous any of the known proofs of the force parallelogram rule: "the very existence of the word force in the stated proposition renders this proof impossible by the very nature of things." (Cf. COMBES, PHILLIPS, and COLLIGNON, eds., *Exposé de la situation de la mécanique appliquée,* Paris 1867).

[10] F. REECH. - *Cours de Mécanique, d'après la nature généralement flexible et élastique des corps,* Paris 1852. This work was written by the illustrious marine engineer in order to revise the teaching of mechanics at l'Ecole Polytechnique. His ideas have been discussed further by J. ANDRADE, *Leçons de mécanique physique,* Paris, 1898, and by marine engineer in chief, MARBEC, in his elementary course in mechanics

up to our present time. Perhaps this is due to the considerable uncertainty that elasticians have about making Hooke's law one of the rational foundations. Analogous reservations are, moreover, manifest in almost the same form in all of the other domains of physics ([11]).

To avoid these difficulties, HELMHOLTZ has attempted to construct what one calls *energetics*, which rests on the least action principle and on the same idea of energy; force, whatever its origin, then becomes a secondary notion of deductive origin. However, the principle of a minimum in natural phenomena ([12]) and the concept of energy ([13]) itself are things we replace on account of the defects of the inductive method. Why a minimum, and what definition can be given to energy if one would have not merely a physical theory, but a truly mechanical theory? HELMHOLTZ does not appear to have responded to these questions. Nonetheless, he has contributed more completely than anyone before him to establishing the distinction between two notions that appear to agree in classical dynamics: energy and action. We believe it is the latter that we must begin with in order to describe the viewpoint of HELMHOLTZ with full precision, and to give mechanics, or, more generally, theoretical physics, a perfectly deductive form.

3. Theory of the Euclidian action. - When one is concerned with the motion of a point, the essential element that enters into the definition of the action is the Euclidian distance between two infinitely close positions of the moving point. We have previously shown ([14]) that one can deduce all of the fundamental definitions of classical mechanics from this notion alone, such as those of the quantity of motion, of force and of energy.

We actually propose to establish that one may follow an identical path in the study of static or dynamic deformations of discrete systems of points and of continuous bodies and that one thus arrives at the construction of a *general theory of action on the extension*

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at l'Ecole de Maistrance de Toulon (1906). See also J. Perrin, *Traité de Chimie physique, les Principes,* Paris 1903.

[11] The remarks of LORD KELVIN, in his *Baltimore Lectures* pp. 131, on the work of BLANCHET, is particularly interesting in this regard; he points out that POISSON, CORIOLIS, and STURM (*C.R.* **7**, pp. 1143), as well as CAUCHY, LIOUVILLE and DUHAMEL (1841) have accepted the 36 coefficients that BLANCHET introduced into the generalized Hooke law without objection. LORD KELVIN has also argued against WEBER's law of force at a distance from the same viewpoint in the 1st edition of *Natural Philosophy*. More recently, the application of the static adiabatic law to the study of waves of finite amplitude was criticized by LORD RAYLEIGH for the same reasons, and one knows that HUGONIOT has proposed a dynamic adiabatic law.

[12] MAUPERTUIS himself has warned of the danger of the principle that he introduced into mechanics when he wrote in 1744: "We do not know very well what the *objective* of Nature is, and we may misunderstand the quantity that we will regard as its cost in the production of its effects." LAGRANGE first had the intention of making the least action principle the basis for his *analytical mechanics*, but, much later, he recognized the superiority of the method that consisted of considering the virtual works.

[13] HERTZ, *Die Prinzipien der Mechanik, etc.*, 1894; see the introduction, in particular.

[14] *Note sur la dynamique du point et du corps invariable,* Tome I, page 236.

and the motion, which embraces all that is directly subject to the laws of mechanics in theoretical physics.

Here, the action will likewise be a function of two elements that are infinitely close elements, both in time and in the space of the medium considered. Upon introducing the condition of invariance into the groups of Euclidian displacements and defining the medium that we indicated in section **1** *the action density at a point will have the same remarkable form as the one that we have already encountered in the dynamics of the point and the invariable body.* With the notations of the *Leçons* of DARBOUX, let (ξ_i, η_i, ζ_i), (p_i, q_i, r_i) be the geometric velocities of translation and rotation of the elementary triad, and let (ξ, η, ζ), (p, q, r), be the analogous velocities relative to the motion of the triad. The action will be the integral:

$$\int_{t}^{t_2}\int\cdots\int W(\rho_i, t; \xi_i, \eta_i, \varsigma_i, p_i, q_i, r_i; \xi, \eta, \varsigma, p, q, r)\, d\rho_1, \cdots, d\rho i, \cdots, dt.$$

It will suffice to consider the variation of that action if we are to be led to the definition of the quantity of motion and to those of the effort and the moment of deformation, of force and external moment, and finally, to those of the energy of deformation and motion, by the intermediary of the notion of work.

In that theory, statics becomes entirely autonomous, which conforms to the views of CARNOT and REECH. For this, one will have to take only an action density W that is independent of the velocities (ξ, η, ζ) and (p, q, r), i.e., to consider a body *without inertia,* or again, a body endowed with an inertia, but on the condition that we regard the deformation as a *reversible transformation* in the sense of DUHEM. On the other hand, upon appealing to the notion of *hidden arguments* one will recover all of the concepts of mechanical origin that are employed in physics. For example, those of flexible and inextensible line, flexible and inextensible surface, and of invariable body, as well as the less particular definitions that have been proposed for the deformable line from D. BERNOULLI and EULER up to THOMSON and TAIT, for the deformable surface from SOPHIE GERMAIN and LAGRANGE up to LORD RAYLEIGH, and for the deformable medium from NAVIER and GREEN up to LORD KELVIN and W. VOIGT.

Upon envisioning deformation and motion at the same time one will arrive at the idea that contains d'Alembert's principle in a purely deductive manner, a principle that relates only to the case where *the action of deformation is completely separate from the kinetic action.* Finally, if one suppose that the deformable body is not subject to any action from the exterior world, and if one introduces, in turn, the fundamental notion of *isolated system,* of which DUHEM ([15]), and subsequently LE ROY ([16]) have seen the necessity in the rational construction of theoretical physics, one will be naturally led to the idea of a minimum that HELMHOLTZ took for his point of departure, at the same time as the appearance of the principle of the conservation of energy, which is at the basis for our present scientific system.

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[15] P. DUHEM. - *Commentaire aux principes de la Thermodynamique,* 1892; *la Théorie physique, its objet et sa structure,* 1906.

[16] E. LE ROY. - *La Science positive et les philosophies de la liberté, Congrès int. de Philosophie,* T. I, 1900.

Apparently, one will thus ultimately avoid all of the difficulties, as well as the trial and error of inductive research, as we have previously said.

4. A critique of the principles of mechanics. - In the form that we just sketched out, the theory of Euclidian action makes a primary contribution to the critique of the principles of mechanics.

Its generality permits us to foresee that there are singular phenomena for the action of the motion, as well as in the deformation of the extension; for example, the speed of solids in the plastic state or when close to a rupture, and that of fluids under great efforts ([17]). Under ordinary circumstances, this generality may be reduced by the consideration of states that are infinitely close to the natural state; this is a point that we discussed in our preceding note.

However, one may also suppose that one or more dimensions of the deformable body becomes infinitely small and envision what one might call a *slender body* ([18]). This notion was developed in 1828 by POISSON and also, a little later, by CAUCHY; their objective, as of all of the elasticians that were occupied with that arduous question later on, was to establish a passage between the distinct theories of bodies of one, two, and three dimensions. One knows that one very important part of the work of BARRI de SAINT-VENANT and of KIRCHHOFF is attached to the discussion of the research of POISSON and CAUCHY. Nevertheless, these savants, and later, their disciples, have not extricated themselves from the veritable difficulty of the question. This difficulty consists in the fact that *generally the zero value of the parameter that was introduced is not an ordinary point, as was assumed by* POISSON *and* CAUCHY, *nor even a pole, but an essential singular point. This important fact justifies the separate study* of the line, the surface and the medium that is found in the present work ([19]).

In concluding these preliminary observations we remark that the theory of the Euclidian action rests on the notion of *differential invariant*, taken in its simplest form. If one enlarges this notion in such a manner as to understand the idea of a *differential parameter* then modern theoretical physics appears as an immediate prolongation of mechanics, properly speaking, *to the Eulerian viewpoint*, and one is naturally led to the principles of the theory of heat and to present electric doctrines. This new field of research, in which we commence to enter into the deduction of the idea of the radiation of energy from the consideration of deformable bodies, will be explored more completely in an ultimate work. We may thus introduce a new precision into the views of H.

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[17] E. and F. COSSERAT. **-** *Sur la mécanique générale, C.R.* **145**, pp. 1139, 1907.

[18] E. and F. COSSERAT. **-** *Sur la théorie des corps minces, C.R.* **146**, pp 169, 1908.

[19] It is true that the interest and the importance of the theories of the deformable line and surface are poorly appreciated nowadays; there is no place for them in the *Encyclopédie des Sciences mathématiques pures et appliquées,* which is presently published in Germany. W. THOMSON and TAIT are guarded about omitting them from their *Natural Philosophy,* and they are presented *before* the theory of the elastic body in three dimensions; similarly for P. DUHEM, *Hydrodynamique, Elasticité, Acoustique,* Paris, 1891.

LORENTZ ([20]) and H. POINCARÉ ([21]) on the subject of what one calls the *principle of reaction* in mechanics.

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[20] H. LORENTZ. - *Versuch einer Theorie der electrischen und optischen Ersheinungen in Bewegten Körpern*, Leiden 1895; reprinted in Leipzig in 1906. *Abhandl. gber theoretische Physik*, 1907; *Encyklop. Der Math. Wissenschaften*, V_2, *Elektronen theorie*, 1903.

[21] H. POINCARÉ. - *Electricité et Optique*, 2nd ed., 1901, pp. 448.

II. - STATICS OF THE DEFORMABLE LINE

5. Deformable line. Natural state and deformed state. - Consider a curve (M_0) that is described by a point M_0 whose coordinates x_0, y_0, z_0 with respect to the three fixed rectangular axes *Ox*, *Oy*, *Oz* are functions of the same parameter, which we suppose in the sequel to be the arc length s_0 of the curve, measured from a definite origin in some definite sense. Add to each point M_0 of the curve (M_0) a tri-rectangular triad whose axes $M_0x_0', M_0y_0', M_0z_0'$ have the direction cosines $\alpha_0, \alpha_0', \alpha_0'', \beta_0, \beta_0', \beta_0'', \gamma_0, \gamma_0', \gamma_0''$, respectively, with respect to the axes *Ox*, *Oy*, *Oz*, and which are functions of the same parameter s_0.

The continuous one-dimensional set of such triads $M_0x_0'y_0'z_0'$ will be what we call a *deformable line.*

Give a displacement M_0M to the point M_0. Let *x*, *y*, *z* be the coordinates of a point *M* with respect to the fixed axes *Ox*, *Oy*, *Oz*. In addition, endow the triad $M_0x_0'y_0'z_0'$ with a rotation that will ultimately make these axes agree with those of a triad $Mx'y'z'$ that we affix to the point *M*. We define this rotation upon giving the axes Mx', My', Mz' the direction cosines $\alpha, \alpha', \alpha'', \beta, \beta', \beta'', \gamma, \gamma', \gamma''$ with respect to the fixed axes *Ox*, *Oy*, *Oz*.

The continuous one-dimensional set of triads $Mx'y'z'$ will be what we call the *deformed state* of the deformable line, which, when considered in its primitive state, will be called the *natural state.*

6. Kinematical elements that relate to the states of the deformable line. - Suppose that s_0 varies and that, for the moment, we make it play the role of time. Upon employing the notations of DARBOUX ([22]), we denote the projections of the velocity of the origin M_0 onto the axes $M_0x_0', M_0y_0', M_0z_0'$ by ξ_0, η_0, ζ_0, and the projections of the velocity of instantaneous rotation of the triad $M_0x_0'y_0'z_0'$ onto the same axes by p_0, q_0, r_0. We denote the analogous quantities for the triad $Mx'y'z'$ when one refers it, like the triad $M_0x_0'y_0'z_0'$, to the fixed triad *Oxyz* by ξ, η, ζ, and *p*, *q*, *r*.

The elements that we introduced are calculated in the habitual fashion; in particular, one has:

[22] G. DARBOUX. - *Leçons sur la théorie générale des surfaces,* T. I., Paris, 1887.

(1) $$\begin{cases} \xi = \alpha \dfrac{dx}{ds_0} + \alpha' \dfrac{dy}{ds_0} + \alpha'' \dfrac{dz}{ds_0}, \\ \eta = \beta \dfrac{dx}{ds_0} + \beta' \dfrac{dy}{ds_0} + \beta'' \dfrac{dz}{ds_0}, \\ \varsigma = \gamma \dfrac{dx}{ds_0} + \gamma' \dfrac{dy}{ds_0} + \gamma'' \dfrac{dz}{ds_0}, \end{cases}$$ (2) $$\begin{cases} p = \sum \gamma \dfrac{d\beta}{ds_0} = -\sum \beta \dfrac{d\gamma}{ds_0}, \\ q = \sum \alpha \dfrac{d\gamma}{ds_0} = -\sum \gamma \dfrac{d\alpha}{ds_0}, \\ r = \sum \beta \dfrac{d\alpha}{ds_0} = -\sum \alpha \dfrac{d\beta}{ds_0}. \end{cases}$$

With these quantities, the linear element *ds* of the curve described by the point *M* is defined by the formula:

$$ds^2 = (\xi^2 + \eta^2 + \varsigma^2) ds_0^2.$$

Denote the projections of the segment *OM* onto the axes Mx', My', Mz' by x', y', z', in such a way that the coordinates of the *fixed point O* with respect to these axes are $-x', -y', -z'$. We have the well-known formulas:

$$\xi - \frac{dx'}{ds_0} - qz' + ry' = 0, \qquad \eta - \frac{dy'}{ds_0} - rx' + pz' = 0, \qquad \varsigma - \frac{dz'}{ds_0} - py' + qx' = 0,$$

which give the new expressions for ξ, η, ζ.

7. Expressions for the variations of the velocities of translation and rotation of the triad relative to the deformed state. - Suppose that one endows each of the triads of the deformed state with an infinitely small displacement that may vary in a continuous fashion with these triads. Denote the variations of $x, y, z; x', y', z'; \alpha, \alpha', \cdots, \gamma''$, by δx, δy, δz, $\delta x', \delta y', \delta z'$, $\delta\alpha, \delta\alpha', \ldots, \delta\gamma''$, respectively. The variations $\delta\alpha, \delta\alpha', \cdots, \delta\gamma''$ are expressed by formulas such as the following:

$$\delta\alpha = \beta \delta K' - \gamma \delta J',$$

by means of the three auxiliary variables $\delta I', \delta J', \delta K'$, which are the components of the well-known instantaneous rotation attached to the infinitely small displacement under consideration, relative to Mx', My', Mz'. The variations *dx*, *dy*, *dz* are the projections of the infinitely small displacement experienced by *M* onto *Ox*, *Oy*, *Oz*; the projections $\delta' x, \delta' y, \delta' z$ of this displacement onto Mx', My', Mz' are deduced immediately, and have the values:

(6) $$\delta' x = \delta x' + z' \delta I' - y' \delta K', \qquad \delta' z = \delta y' + x' \delta K' - z' \delta I', \qquad \delta' z = \delta z' + y' \delta I' - x' \delta K'.$$

We propose to determine the variations $\delta\xi$, $\delta\eta$, $\delta\zeta$, δp, δq, δr that are experienced by ξ, η, ζ, p, q, r. From formulas (2), we have:

$$\delta p = \sum\left(\frac{d\beta}{ds_0}\delta\gamma + \gamma\frac{d\delta\beta}{ds_0}\right),$$
$$\delta q = \sum\left(\frac{d\gamma}{ds_0}\delta\alpha + \alpha\frac{d\delta\gamma}{ds_0}\right),$$
$$\delta r = \sum\left(\frac{d\alpha}{ds_0}\delta\beta + \beta\frac{d\delta\alpha}{ds_0}\right).$$

If we replace $\delta\alpha$ by its value $\beta\delta K' - \gamma\delta J'$, and $\delta\alpha', \cdots, \delta\gamma''$, by their analogous values, then we get

(7) $$\delta p = \frac{d\delta I'}{ds_0} + q\delta K' - r\delta J', \qquad \delta q = \frac{d\delta J'}{ds_0} + r\delta I' - p\delta K',$$
$$\delta r = \frac{d\delta K'}{ds_0} + p\delta J' - q\delta I',$$

Similarly, formulas (4) give us three formulas, where the first one is:

$$\delta\xi = \frac{d\delta x'}{ds_0} + q\delta z' - r\delta y' + z'\delta q - y'\delta r.$$

If we replace δp, δq, δr, by their values as given by formulas (7) then we obtain:

(14) $$\begin{cases} \delta\xi = \eta\delta K' - \varsigma\delta J' + \dfrac{d\delta' x}{ds_0} + q\delta' x - r\delta' y, \\ \delta\eta = \varsigma\delta I' - \xi\delta K' + \dfrac{d\delta' y}{ds_0} + r\delta' y - p\delta' z, \\ \delta\varsigma = \xi\delta J' - \eta\delta I' + \dfrac{d\delta' z}{ds_0} + p\delta' z - q\delta' x, \end{cases}$$

where we have introduced the three symbols, $\delta' x, \delta', \delta' z$, which are defined by formulas (6), to abbreviate the notation.

8. Euclidian action of deformation on a deformable line. - Consider a function *W* of *two infinitely close positions* of the triad $Mx'y'z'$, i.e., a function of s_0, of x, y, z, α, $\alpha', \cdots, \gamma''$, and of their first derivatives with respect to s_0. We propose to determine what the form of *W* must be in order for the integral:

$$\int W\, ds_0,$$

when taken over an arbitrary portion of the line (M_0), to have a null variation when one subjects the set of all the triads of the deformable line, taken in its deformed state, *to the same arbitrary infinitesimal transformation from the group of Euclidean displacements.*

By definition, this amounts to determining W in such a fashion that one has:

$$\delta W = 0$$

when, on the one hand, the origin M of the triad $Mx'y'z'$ is subject to an infinitely small displacement whose projections δx, δy, δz on the axes Ox, Oy, Oz are:

$$(15)\qquad \begin{cases} \delta x = (a_1 + \omega_2 z - \omega_3 y)\delta t, \\ \delta y = (a_2 + \omega_3 x - \omega_1 z)\delta t, \\ \delta z = (a_3 + \omega_1 y - \omega_2 x)\delta t, \end{cases}$$

where a_1, a_2, a_3, ω_1, ω_2, ω_3 are six arbitrary constant and δt is an infinitely small quantity that is independent of s_0, and where, on the other hand, the triad $Mx'y'z'$ is subjected to an infinitely small rotation whose components along the axes Ox, Oy, Oz are:

$$\omega_1 \delta t,\ \omega_2 \delta t,\ \omega_3 \delta t.$$

Observe that, in the present case, the variations $\delta\xi$, $\delta\eta$, $\delta\zeta$, δp, δq, δr of the six expressions ξ, η, ζ, p, q, r are null, since this results from the well-known theory of moving triads, and as we have, moreover, verified immediately by means of formulas (7) and (8), upon replacing $\delta' x, \delta' I$ by their present values:

$$(9')\qquad \begin{cases} \delta' x = \alpha(a_1 + \omega_2 z - \omega_3 y)\delta t + \alpha'(a_2 + \omega_3 x - \omega_1 z)\delta t + \alpha''(a_3 + \omega_1 y - \omega_2 x)\delta t \\ \delta I' = (\alpha\omega_1 + \alpha'\omega_2 + \alpha''\omega_3)\delta t, \end{cases}$$

and $\delta' y, \delta' z, \delta J', \delta K'$ with their analogous present values. It results from this we have obtained a solution to the question, upon taking an arbitrary function of s_0 and the six expressions ξ, η, ζ, p, q, r for W; we shall now show that we thus obtain the general solution ([23]) to the problem that we have posed.

To that effect, observe that by means of well-known formulas relations (2) permit us to express the first derivatives of the nine cosines $\alpha, \alpha', \cdots, \gamma''$ with respect to s_0 by means of the cosines of p, q, r. On the other hand, we remark that formulas (1) permit us to conceive that one expresses the nine cosines $\alpha, \alpha', \cdots, \gamma''$ by means of the ξ, η, ζ, and the first derivatives of x, y, z with respect to s_0. Therefore, we may finally express the desired function W as a function of s_0, and x, y, z, and their first derivatives, and ultimately of ξ, η, ζ, p, q, r, which we indicate upon writing:

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[23] We suppose, in what follows, that the deformable line *is susceptible to all possible deformations*, and, as a result, that *the deformed state may be taken to be absolutely arbitrary;* this is what one may express upon saying that the deformable line is *free.*

$$W = W\left(s_0, x, y, z, \frac{dx}{ds_0}, \frac{dy}{ds_0}, \frac{dz}{ds_0}, \xi, \eta, \varsigma, p, q, r\right).$$

Since the variations $\delta\xi$, $\delta\eta$, $\delta\zeta$, δp, δq, δr are null in the present case, as we have remarked that there is such an instant, we finally have to write the new form of W that one obtains, by virtue of formulas (9), and for any a_1, a_2, a_3, ω_1, ω_2, ω_3:

$$\frac{\partial W}{\partial x}\delta x + \frac{\partial W}{\partial y}\delta y + \frac{\partial W}{\partial z}\delta z + \frac{\partial W}{\partial \frac{dx}{ds_0}}\delta\frac{dx}{ds_0} + \frac{\partial W}{\partial \frac{dy}{ds_0}}\delta\frac{dy}{ds_0} + \frac{\partial W}{\partial \frac{dz}{ds_0}}\delta\frac{dz}{ds_0} = 0.$$

We replace dx, dy, dz by their values (9) and $\delta\frac{dx}{ds_0}, \delta\frac{dy}{ds_0}, \delta\frac{dz}{ds_0}$ by the values that one deduces upon differentiating; equating the coefficients of a_1, a_2, a_3, ω_1, ω_2, ω_3 to zero; we obtain the following six conditions:

$$\frac{\partial W}{\partial x} = 0, \qquad \frac{\partial W}{\partial y} = 0, \qquad \frac{\partial W}{\partial z} = 0,$$

$$\frac{\partial W}{\partial \frac{dy}{ds_0}}\frac{dz}{ds_0} - \frac{\partial W}{\partial \frac{dz}{ds_0}}\frac{dy}{ds_0} = 0, \quad \frac{\partial W}{\partial \frac{dz}{ds_0}}\frac{dx}{ds_0} - \frac{\partial W}{\partial \frac{dx}{ds_0}}\frac{dz}{ds_0} = 0, \quad \frac{\partial W}{\partial \frac{dx}{ds_0}}\frac{dy}{ds_0} - \frac{\partial W}{\partial \frac{dy}{ds_0}}\frac{dx}{ds_0} = 0,$$

The first three show, as we may easily foresee, that W is independent of x, y, z; the last three express that W depends on $\frac{dx}{ds_0}, \frac{dy}{ds_0}, \frac{dz}{ds_0}$ only by the intermediary of the quantity:

$$\left(\frac{dx}{ds_0}\right)^2 + \left(\frac{dy}{ds_0}\right)^2 + \left(\frac{dz}{ds_0}\right)^2,$$

and since the latter is, from formula (3), equal to $\xi^2 + \eta^2 + \zeta^2$ we finally see that *the desired function W has the remarkable form:*

$$W(s_0, \xi, \eta, \zeta, p, q, r).$$

If we multiply W by ds_0 then the product $W ds_0$ that we obtain is an invariant of the group of Euclidean displacements that is analogous to the one that, under the name of *linear element*, provides the distance between two infinitely close points of the curve (M) that is described by the point M.

Similarly, the common value of the integrals:

$$\int_{A_0}^{B_0}\frac{ds}{ds_0}ds_0, \qquad \int_A^B ds,$$

when taken between two points A_0 and B_0 of the curve (M_0) and the corresponding points A and B on the curve (M), determines the *length* of the arc AB of that curve (M); in the same spirit, upon associating the notion of *action* to the passage from that natural state (M_0) to the deformed state (M) we add the function W to the elements of the definition of the deformable line, and we say that the integral:

$$\int_{A_0}^{B_0} W\, ds_0$$

is the *action of deformation* on the deformed line between two points A and B, which correspond to the points A_0 and B_0 of (M_0). In this definition and in what follows, we suppose that the arcs s_0 and s, are regarded in the sense of A_0 going to B_0 and A going to B, or conversely, that the notations A_0, B_0, A, B denote the extremities of the line in the natural state and the deformed state, corresponding to that convention.

We also say that W is the *density* of the action of deformation *at a point* of the deformed line relative to the unit of length of the undeformed line; $W\frac{ds_0}{ds}$ will be the action density at a point relative to the unit of length of the deformed line.

9. Force and external moment. Effort and the moment of external deformation. Effort and the moment of deformation at a point of the deformed line. - Consider an *arbitrary* variation of the action of deformation between two points A and B of the line (M), namely:

$$\delta\int_{A_0}^{B_0} W\, ds_0 = \int_{A_0}^{B_0}\left(\frac{\partial W}{\partial \xi}\delta\xi + \frac{\partial W}{\partial \eta}\delta\eta + \frac{\partial W}{\partial \varsigma}\delta\varsigma + \frac{\partial W}{\partial p}\delta p + \frac{\partial W}{\partial q}\delta q + \frac{\partial W}{\partial r}\delta r\right)ds_0 .$$

By virtue of formulas (7) and (8) of sec. **7**, we may write this as:

(?)

$$\begin{aligned}
&+\frac{\partial W}{\partial \eta}\left(\varsigma\delta I' - \xi\delta K' + \frac{d\delta' y}{ds_0} + r\delta' x - p\delta' z\right)\\
&+\frac{\partial W}{\partial \varsigma}\left(\xi\delta J' - \eta\delta I' + \frac{d\delta' z}{ds_0} + p\delta' y - q\delta' x\right)\\
&+\frac{\partial W}{\partial p}\left(\frac{d\delta I'}{ds_0} + q\delta K' - r\delta J'\right) + \frac{\partial W}{\partial q}\left(\frac{d\delta J'}{ds_0} + r\delta I' - q\delta K'\right)\\
&+\frac{\partial W}{\partial r}\left(\frac{d\delta K'}{ds_0} + p\delta J' - q\delta I'\right)\Bigg] ds_0 .
\end{aligned}$$

We integrate the six terms that refer explicitly to the derivatives with respect to s_0 by parts and obtain:

$$\begin{aligned}\delta\int_{A_0}^{B_0} W\,ds_0 = &\left[\frac{\partial W}{\partial\xi}\delta' x+\frac{\partial W}{\partial\eta}\delta' y+\frac{\partial W}{\partial\varsigma}\delta' z+\frac{\partial W}{\partial p}\delta I'+\frac{\partial W}{\partial q}\delta J'+\frac{\partial W}{\partial r}\delta K'\right]_{A_0}^{B_0}\\
&-\int_{A_0}^{B_0}\left[\left(\frac{d}{ds_0}\frac{\partial W}{\partial\xi}+q\frac{\partial W}{\partial\varsigma}-r\frac{\partial W}{\partial\eta}\right)\delta' x+\left(\frac{d}{ds_0}\frac{\partial W}{\partial\eta}+r\frac{\partial W}{\partial\xi}-p\frac{\partial W}{\partial\varsigma}\right)\delta' y\right.\\
&+\left(\frac{d}{ds_0}\frac{\partial W}{\partial\varsigma}+p\frac{\partial W}{\partial\eta}-q\frac{\partial W}{\partial\xi}\right)\delta' z+\left(\frac{d}{ds_0}\frac{\partial W}{\partial p}+q\frac{\partial W}{\partial r}-r\frac{\partial W}{\partial q}-\eta\frac{\partial W}{\partial\varsigma}-\varsigma\frac{\partial W}{\partial\eta}\right)\delta I'\\
&+\left(\frac{d}{ds_0}\frac{\partial W}{\partial q}+r\frac{\partial W}{\partial p}-p\frac{\partial W}{\partial r}+\eta\frac{\partial W}{\partial\xi}-\xi\frac{\partial W}{\partial\varsigma}\right)\delta J'\\
&\left.+\left(\frac{d}{ds_0}\frac{\partial W}{\partial r}+p\frac{\partial W}{\partial q}-q\frac{\partial W}{\partial p}-\xi\frac{\partial W}{\partial\eta}-\eta\frac{\partial W}{\partial\xi}\right)\delta K'\right]ds_0.\end{aligned}$$

Set:

$$\begin{cases}F'=\dfrac{\partial W}{\partial\xi},G'=\dfrac{\partial W}{\partial\eta},H'=\dfrac{\partial W}{\partial\varsigma},I'=\dfrac{\partial W}{\partial p},J'=\dfrac{\partial W}{\partial q},K'=\dfrac{\partial W}{\partial r},\\
X_0'=\dfrac{d}{ds_0}\dfrac{\partial W}{\partial\xi}+q\dfrac{\partial W}{\partial\varsigma}-r\dfrac{\partial W}{\partial\eta},\\
Y_0'=\dfrac{d}{ds_0}\dfrac{\partial W}{\partial\eta}+r\dfrac{\partial W}{\partial\xi}-p\dfrac{\partial W}{\partial\varsigma},\\
Z_0'=\dfrac{d}{ds_0}\dfrac{\partial W}{\partial\varsigma}+p\dfrac{\partial W}{\partial\eta}-q\dfrac{\partial W}{\partial\xi},\\
L_0'=\dfrac{d}{ds_0}\dfrac{\partial W}{\partial p}+q\dfrac{\partial W}{\partial r}-r\dfrac{\partial W}{\partial q}+\eta\dfrac{\partial W}{\partial\varsigma}-\varsigma\dfrac{\partial W}{\partial\eta},\\
M_0'=\dfrac{d}{ds_0}\dfrac{\partial W}{\partial r}+r\dfrac{\partial W}{\partial p}-p\dfrac{\partial W}{\partial r}+\varsigma\dfrac{\partial W}{\partial\xi}-\xi\dfrac{\partial W}{\partial\varsigma},\\
N_0'=\dfrac{d}{ds_0}\dfrac{\partial W}{\partial q}+p\dfrac{\partial W}{\partial q}-q\dfrac{\partial W}{\partial p}+\xi\dfrac{\partial W}{\partial\eta}-\eta\dfrac{\partial W}{\partial\xi},\end{cases}$$

We have:

$$\begin{aligned}\delta\int_{A_0}^{B_0} W\,ds_0 = &[F'\delta' x+G'\delta' y+H'\delta' z+I'\delta I'+J'\delta J'+K'\delta K']_{A_0}^{B_0}\\
&-\int_{A_0}^{B_0}(X_0'\delta' x+Y_0'\delta' y+Z_0'\delta' z+L_0'\delta I'+M_0'\delta J'+N_0'\delta K')ds_0.\end{aligned}$$

Upon first considering the integral that figures in the expression of $\delta\int_{A_0}^{B_0} W\,ds_0$, we call the segments that issue from M whose projections on the axes Mx', My', Mz' are X_0', Y_0', Z_0' and L_0', M_0', N_0' the *external force and external moment at the point M relative to the unit of length of the undeformed line,* respectively. Upon regarding the completely integrated part of $\delta\int_{A_0}^{B_0} W\,ds_0$, we call the segments that issue from *B* whose projections on the axes Mx', My', Mz' have the values $-F_{B_0}', -G_{B_0}', -H_{B_0}'$ and $-I_{B_0}', -J_{B_0}', -K_{B_0}'$ that the expressions $-F', -G', -H'$ and $-I', -J', -K'$ take at the point B_0 the *external effort and external moment of deformation at the point B*, respectively. We call the analogous segments that are formed from the values $-F_{A_0}', -G_{A_0}', -H_{A_0}'$ and $-I_{A_0}', -J_{A_0}', -K_{A_0}'$ that the expressions $-F', -G', -H'$ and $-I', -J', -K'$ take at the point A_0 the *external effort and external moment of deformation at the point A*, respectively.

The points *A* and *B* are not presented in the same fashion here, which conforms to the convention that distinguishes them and the convention that was made regarding the sense of the arc s_0.

Suppose that one cuts the deformed line *AB* at the point M, and that one separates the two parts *AM* and *MB*; one may regard the two segments $(-F', -G', -H')$ and $(-I', -J', -K')$ that are determined by the point *M* as the effort and the external moment of deformation of the part *AM* at the point *M*, and the two segments (F', G', H') and (I', J', K') as the effort and the external moment of the part *MB* at the point *M*. It amounts to the same thing if, instead of considering *AM* and *MB* one imagines two portions of the deformable line that belong to *AM* and *MB*, respectively, and have an extremity at *M*. By reason of these remarks, we say that $-F', -G', -H'$ and $-I', -J', -K'$ are the components of *the effort and the moment of deformation exerted on AM and on any portion of AM ending at M at the point M* along the axes Mx', My', Mz', and that F', G', H' and I', J', K' are the components of *the effort and moment of deformation exerted on MB and any portion of MB ending at M at the point M* along the axes Mx', My', Mz'.

We observe that if one replaces the triad $Mx'y'z'$ by a triad that is invariably related then one is led to conclusions that are identical to the ones that we have previously indicated ([24]).

10. Relations between the elements defined in the preceding section; diverse transformations of these relations. - The different elements that were introduced in the preceding section are coupled by the following relations, which result immediately from comparing the formulas that serve to define them:

[24] *Note sur la dynamique du point et du corps invariable,* Tome I, pages 260 and 269.

(11)
$$\begin{cases} \dfrac{dF'}{ds_0} + qH' - rG' - X_0' = 0, & \dfrac{dI'}{ds_0} + qK' - rJ' + \eta H' - \varsigma G' - L_0' = 0, \\ \dfrac{dG'}{ds_0} + rF' - pH' - Y_0' = 0, & \dfrac{dJ'}{ds_0} + rI' - pK' + \varsigma HF'\xi H' - M_0' = 0, \\ \dfrac{dH'}{ds_0} + pG' - qF' - Z_0' = 0, & \dfrac{dK'}{ds_0} + pJ' - qI' + \xi G' - \eta F' - N_0' = 0. \end{cases}$$

One may propose to transform the relations that we proceed to write, *independently of the values of the quantities that figure in them that are calculated by means of W*. Indeed, these relations apply between the segments that are attached to the point *M*, and which we have given names to. Instead of defining these segments by their projections on Mx', My', Mz', we can just as well define them by their projections on other axes. These latter projections will be coupled by relations that are transforms of the preceding ones.

The transformed relations are obtained immediately if one remarks that the primitive formulas have a simple and immediate interpretation by the addition of axes that are parallel translated from the point *O* to the moving axes.

1. First consider fixed axes *Ox*, *Oy*, *Oz*. Denote the projections of the force and external moment at an arbitrary point of the deformed line onto these axes by X_0, Y_0, Z_0 and L_0, M_0, N_0, and the projections of the effort and the moment of deformation on the same axes by *F*, *G*, *H* and *I*, *J*, *K*, so the projections of the above on the Mx', My', Mz' axes will be F', G', H' and I', J', K'. Evidently, the transforms of the preceding relations are:

$$\frac{dF}{ds_0} - X_0 = 0,$$

$$\frac{dI}{ds_0} + H\frac{dy}{ds_0} - G\frac{dz}{ds_0} - L_0 = 0,$$

$$\frac{dG}{ds_0} - Y_0 = 0,$$

$$\frac{dJ}{ds_0} + F\frac{dz}{ds_0} - H\frac{dx}{ds_0} - M_0 = 0,$$

$$\frac{dH}{ds_0} - Z_0 = 0,$$

$$\frac{dK}{ds_0} + G\frac{dx}{ds_0} - F\frac{dy}{ds_0} - N_0 = 0.$$

We may regard the force X_0', Y_0', Z_0' and the moment L_0', M_0', N_0', or, if one prefers, the force X_0, Y_0, Z_0 and the moment L_0, M_0, N_0 as distributed in a continuous manner along the line; this force and moment will be referred to the unit of length of the undeformed line. In order to have the force and moment referred to the unit of length of the deformed line, it suffices to multiply $X'_0, Y'_0, Z'_0, L'_0, M'_0, N'_0$, or X_0, Y_0, Z_0, L_0, M_0, N_0 by $\dfrac{ds_0}{ds}$, where ds is the linear element of the deformed line that corresponds to the linear element ds_0 of the undeformed line. We introduce the projections of the force and external moment on the fixed axes *Ox*, *Oy*, *Oz*, namely, *X*, *Y*, *Z*, *L*, *M*, *N*, which are referred to the unit of length of the deformed line; we obtain the relations:

$$(12)\qquad \begin{cases} \dfrac{dF}{ds} - X = 0, & \dfrac{dI}{ds} + H\dfrac{dy}{ds} - G\dfrac{dz}{ds} - L = 0, \\ \dfrac{dG}{ds} - Y = 0, & \dfrac{dJ}{ds} + F\dfrac{dz}{ds} - H\dfrac{dx}{ds} - M = 0, \\ \dfrac{dH}{ds} - Z = 0, & \dfrac{dK}{ds} + G\dfrac{dx}{ds} - F\dfrac{dy}{ds} - N = 0, \end{cases}$$

which are identical with those considered by several authors, and, in particular, by LORD KELVIN and TAIT ([25]). However, the latter are obtained upon applying what one calls, in classical mechanics, the principle of solidification, and upon starting with the notions of forces and couples, *a priori*, which are thus expressed as a function of the deformations, *a posteriori*, by virtue of the hypotheses. Under these hypotheses, we have imagined only infinitely small deformations up till now, whereas now we presently place ourselves in the most general case.

2. One may give a new form to the equations relative to the fixed axes *Ox*, *Oy*, *Oz*. We may express the nine cosines $\alpha, \alpha', \alpha'', \dots, \gamma''$ by means of three auxiliary variables; let λ_1, λ_2, λ_3 be these three auxiliary variables. Set:

$$\begin{aligned} \sum \gamma d\beta &= -\sum \beta d\gamma = \varpi'_1 d\lambda_1 + \varpi'_2 d\lambda_2 + \varpi'_3 d\lambda_3, \\ \sum \alpha d\gamma &= -\sum \gamma d\alpha = \chi'_1 d\lambda_1 + \chi'_2 d\lambda_2 + \chi'_3 d\lambda_3, \\ \sum \beta d\alpha &= -\sum \alpha d\beta = \sigma'_1 d\lambda_1 + \sigma'_2 d\lambda_2 + \sigma'_3 d\lambda_3 . \end{aligned}$$

The functions $\varpi'_i, \chi'_i, \sigma'_i$ of λ_1, λ_2, λ_3 so defined satisfy the relations:

$$\begin{aligned} &\frac{\partial \varpi'_j}{\partial \lambda_i} - \frac{\partial \varpi'_i}{\partial \lambda_j} + \chi'_i \sigma'_j - \chi'_j \sigma'_i = 0, \\ &\frac{\partial \chi'_j}{\partial \lambda_i} - \frac{\partial \chi'_i}{\partial \lambda_j} + \sigma'_i \varpi'_j - \sigma'_j \varpi'_i = 0, \qquad (i, j = 1, 2, 3) \\ &\frac{\partial \sigma'_j}{\partial \lambda_i} - \frac{\partial \sigma'_i}{\partial \lambda_j} + \varpi'_i \chi'_j - \varpi'_j \chi'_i = 0, \end{aligned}$$

and one has:

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[25] LORD KELVIN AND TAIT. - *Natural Philosophy*, Part. II, sec. **614**.

$$p = \varpi_1' \frac{d\lambda_1}{ds_0} + \varpi_2' \frac{d\lambda_2}{ds_0} + \varpi_3' \frac{d\lambda_3}{ds_0},$$
$$q = \chi_1' \frac{d\lambda_1}{ds_0} + \chi_2' \frac{d\lambda_2}{ds_0} + \chi_3' \frac{d\lambda_3}{ds_0},$$
$$r = \sigma_1' \frac{d\lambda_1}{ds_0} + \sigma_2' \frac{d\lambda_2}{ds_0} + \sigma_3' \frac{d\lambda_3}{ds_0}.$$

When we denote the projections on the fixed *Ox*, *Oy*, *Oz* axes of the segment whose projections on the Mx', My', Mz' axes are $\varpi_i', \chi_i', \sigma_i'$ by ϖ_i, χ_i, σ_i we have:

$$\begin{aligned}
\alpha' d\alpha'' + \beta' d\beta'' + \gamma' d\gamma'' &= -(\alpha'' d\alpha' + \beta'' d\beta' + \gamma'' d\gamma') &= \sum \varpi_i d\lambda_i,\\
\alpha'' d\alpha + \beta'' d\beta + \gamma'' d\gamma &= -(\alpha d\alpha'' + \beta d\beta'' + \gamma d\gamma'') &= \sum \chi_i d\lambda_i,\\
\alpha d\alpha' + \beta d\beta' + \gamma d\gamma' &= -(\alpha' d\alpha + \beta' d\beta + \gamma' d\gamma) &= \sum \sigma_i d\lambda_i,
\end{aligned}$$

by virtue of which ([26]), the new functions ϖ_i, χ_i, σ_i of λ_1, λ_2, λ_3 satisfy the relations:

$$\begin{aligned}
\frac{\partial \varpi_j}{\partial \lambda_i} - \frac{\partial \varpi_i}{\partial \lambda_j} &= \chi_i \sigma_j - \chi_j \sigma_i \\
\frac{\partial \chi_j}{\partial \lambda_i} - \frac{\partial \chi_i}{\partial \lambda_j} &= \sigma_i \varpi_j - \sigma_j \varpi_i \qquad (i, j = 1, 2, 3), \\
\frac{\partial \sigma_j}{\partial \lambda_i} - \frac{\partial \sigma_i}{\partial \lambda_j} &= \varpi_i \chi_j - \varpi_j \chi.
\end{aligned}$$

We again make the remark, which will be of use later on, that if one denotes the variations of λ_1, λ_2, λ_3 that correspond to the variations $\delta\alpha, \delta\alpha', \cdots, \delta\gamma''$ of $\alpha, \alpha', \cdots, \gamma''$ by $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$ then one will have:

$$\begin{aligned}
\delta I' &= \varpi_1' \delta\lambda_1 + \varpi_2' \delta\lambda_2 + \varpi_3' \delta\lambda_3,\\
\delta J' &= \chi_1' \delta\lambda_1 + \chi_2' \delta\lambda_2 + \chi_3' \delta\lambda_3,\\
\delta K' &= \sigma_1' \delta\lambda_1 + \sigma_2' \delta\lambda_2 + \sigma_3' \delta\lambda_3,\\
\delta I &= \alpha \delta I' + \beta \delta J' + \gamma \delta K' = \varpi_1 L_0 + \chi_1 M_0 + \sigma_1 N_0,
\end{aligned}$$

[26] These formulas may serve to define the functions ϖ_i, χ_i, σ_i directly, and may be substituted for:

$$\begin{aligned}
\varpi_i &= \alpha \varpi_i' + \beta \chi_i' + \gamma \sigma_i',\\
\chi_i &= \alpha' \varpi_i' + \beta' \chi_i' + \gamma' \sigma_i', \qquad (i = 1, 2, 3).\\
\sigma_i &= \alpha'' \varpi_i' + \beta'' \chi_i' + \gamma'' \sigma_i'.
\end{aligned}$$

$$\delta J = \alpha'\delta I' + \beta'\delta J' + \gamma'\delta K' = \varpi_2 L_0 + \chi_2 M_0 + \sigma_2 N_0,$$
$$\delta K = \alpha''\delta I' + \beta''\delta J' + \gamma''\delta K' = \varpi_3 L_0 + \chi_3 M_0 + \sigma_3 N_0,$$

where δI, δJ, δK are the projections onto the fixed axes of the segment whose projections onto Mx', My', Mz' are $\delta I', \delta J', \delta K'$.

Now set:

$$\begin{aligned}
\mathcal{I} &= \varpi_1' I' + \chi_1' J' + \sigma_1' K' = \varpi_1 I + \chi_1 J + \sigma_1 K\\
\mathcal{J} &= \varpi_2' I' + \chi_2' J' + \sigma_2' K' = \varpi_2 I + \chi_2 J + \sigma_2 K\\
\mathcal{K} &= \varpi_3' I' + \chi_3' J' + \sigma_3' K' = \varpi_3 I + \chi_3 J + \sigma_3 K\\
\mathcal{L}_0 &= \varpi_1' L_0' + \chi_1' M_0' + \sigma_1' N_0' = \varpi_1 L_0 + \chi_1 M_0 + \sigma_1 N_0\\
\mathcal{M}_0 &= \varpi_2' L_0' + \chi_2' M_0' + \sigma_2' N_0' = \varpi_2 L_0 + \chi_2 M_0 + \sigma_2 N_0\\
\mathcal{N}_0 &= \varpi_3' L_0' + \chi_3' M_0' + \sigma_3' N_0' = \varpi_3 L_0 + \chi_3 M_0 + \sigma_3 N_0,
\end{aligned}$$

and we will have the equation:

$$\begin{aligned}
\frac{d\mathcal{I}}{ds_0} - I'\left(\frac{d\varpi_1'}{ds_0} + q\sigma_1' - r\chi_1'\right) - J'\left(\frac{d\chi_1'}{ds_0} + r\varpi_1' - p\sigma_1'\right) - K'\left(\frac{d\sigma_1'}{ds_0} + p\chi_1' - q\varpi_1'\right)&\\
+ F'(\chi_1'\varsigma - \sigma_1'\eta) + G'(\sigma_1'\xi - \varpi_1'\varsigma) + H'(\varpi_1'\eta - \chi_1'\xi) - \mathcal{L}_0 = 0,&
\end{aligned}$$

with two analogous equations. If one remarks that the functions ξ, η, ζ, p, q, r of λ_1, λ_2, λ_3, $\frac{d\lambda_1}{ds_0}, \frac{d\lambda_2}{ds_0}, \frac{d\lambda_3}{ds_0}$ give rise to the formulas:

$$\begin{aligned}
&\frac{\partial \xi}{\partial \lambda_i} + \chi_i'\varsigma - \sigma_i'\eta = 0, &\qquad &\frac{\partial p}{\partial \lambda_i} = \frac{d\varpi_i'}{ds_0} + q\sigma_i' - r\chi_i',\\
&\frac{\partial \eta}{\partial \lambda_i} + \sigma_i'\xi - \varpi_i'\varsigma = 0, & &\frac{\partial q}{\partial \lambda_i} = \frac{d\chi_i'}{ds_0} + r\varpi_i' - p\sigma_i',\\
&\frac{\partial \varsigma}{\partial \lambda_i} + \varpi_i'\eta - \chi_i'\xi = 0, & &\frac{\partial r}{\partial \lambda_i} = \frac{d\sigma_i'}{ds_0} + p\chi_i' - q\varpi_i',
\end{aligned}$$

which result from the defining relations for the functions $\varpi_i', \chi_i', \sigma_i'$, and the nine identities that they verify, then one may give a new form to the preceding equation:

$$\frac{d\mathcal{I}}{ds_0} - F'\frac{\partial \xi}{\partial \lambda_1} - G'\frac{\partial \eta}{\partial \lambda_1} - H'\frac{\partial \varsigma}{\partial \lambda_1} - I'\frac{\partial p}{\partial \lambda_1} - J'\frac{\partial q}{\partial \lambda_1} - K'\frac{\partial r}{\partial \lambda_1} - \mathcal{L}_0 = 0,$$

with two analogous equations.

Upon setting:

$$\mathcal{I}' = \varpi_1'(I' + y'H' - z'G') + \chi_1'(J' + z'F' - x'H') + \sigma_1'(K' + x'G' - y'F'),$$

$$\mathcal{L}' = \varpi_1'(L_0' + y'Z_0' - z'Y_0') + \chi_1'(M_0' + z'X_0' - x'Z_0') + \sigma_1'(N_0' + x'Y_0' - y'X_0'),$$

with analogous formulas for $\mathcal{J}', \mathcal{K}', \mathcal{M}_0', \mathcal{N}_0'$ one similarly finds the form of the equation:

$$\frac{d\mathcal{I}'}{ds_0} - (I' + y'H' - z'G')\frac{\partial p}{\partial \lambda_1} - (J' + z'F' - x'H')\frac{\partial q}{\partial \lambda_1} - (K' + x'G' - y'F')\frac{\partial r}{\partial \lambda_1} - \mathcal{L}_0' = 0,$$

with two analogous expressions.

We will soon apply the transformations that we just indicated; for the moment, we limit ourselves to making the remark that *the expressions* $\delta I', \delta J', \delta K'$, and δI, δJ, δK *are not exact differentials.*

3. Instead of referring the elements that relate to the point *M* to the fixed axes *Oxyz*, imagine that in order to define these elements, a trirectangular triad $Mx'y'z'$ moving with *M*, whose axis Mx_1' is subject to being directed along the tangent to the curve (*M*) given the sense of the increasing arc length. To define this triad $Mx_1'y_1'z_1'$ refer it to the triad $Mx'y'z'$, and let l, l', l'' be the direction cosines of Mx_1' with respect to the latter triad, m, m', m'', those of My_1', and n, n', n'', those of Mz_1'. The cosines l, l', l'' will be defined by the formulas:

$$l = \xi\frac{ds_0}{ds}, \qquad l' = \eta\frac{ds_0}{ds}, \qquad l'' = \varsigma\frac{ds_0}{ds},$$

i.e., by the following:

$$l = \frac{\xi}{\varepsilon}, \qquad l' = \frac{\eta}{\varepsilon}, \qquad l'' = \frac{\varsigma}{\varepsilon},$$

upon setting:

$$\varepsilon = \sqrt{\xi^2 + \eta^2 + \varsigma^2}\,.$$

We assume that the triad $Mx_1'y_1'z_1'$ has the same disposition as the others. We make no other particular hypotheses on the other cosines; from their definition, they will be simply subject to verifying the relations:

$$m\xi + m'\eta + m''\varsigma = 0,$$
$$n\xi + n'\eta + n''\varsigma = 0.$$

Suppose that s_0 varies and that, for an instant, one makes it play the role of time. Moreover, refer the triad $Mx_1'y_1'z_1'$ to the fixed triad *Oxyz* and denote the respective projections of the instantaneous rotation of the triad $Mx_1'y_1'z_1'$ onto the axes Mx_1', My_1', Mz_1' by p_1, q_1, r_1 in such a way that one will have three formulas such as the following:

$$p_1 = lp + l'q + l''r + \sum n \frac{dm}{ds_0},$$

upon admitting the same disposition for the triads.

Finally, denote the projections of the force and external moment at an arbitrary point M of the deformed line Mx_1', My_1', Mz_1' onto (?) by (?) and referred to the unit of length of the undeformed line, and the projections of the effort and the moment of deformation by F_1', G_1', H_1' and I_1', J_1', K_1'. The transforms of the equations of the preceding section are obviously:

$$(13) \quad \begin{cases} \dfrac{dF_1'}{ds_0} + q_1 H_1' - r_1 G_1' - X_1' = 0 & \dfrac{dI_1'}{ds_0} + q_1 K_1' - r_1 J_1' \qquad - L_1' = 0 \\ \dfrac{dG_1'}{ds_0} + r_1 F_1' - p_1 H_1' - Y_1' = 0 & \dfrac{dJ_1'}{ds_0} + r_1 I_1' - p_1 K_1' - \varepsilon H_1' - M_1 = 0 \\ \dfrac{dH_1'}{ds_0} + p_1 G_1' - q_1 F_1' - Z_1' = 0 & \dfrac{dK_1'}{ds_0} + p_1 J_1' - q_1 I_1' - \varepsilon G_1' - N_1' = 0 \end{cases}$$

In the strength of materials, one calls F_1' the *effort of tension;* the components G_1', H_1' are the *shear efforts* in the plane normal to the deformed line. Similarly, the component I_1' of the moment of deformation is a *moment of torsion;* the components J_1', K_1' are called the *moments of flexion.*

If, in the fourth equation (13), one has $L_1' = 0$ and $q_1 = 0$, then it follows that:

$$\frac{dI_1'}{ds_0} - r_1 J_1' = 0,$$

from which results the proposition, which was established by POISSON ([27]) for the case where $L_1' = 0$, $M_1' = 0$, $N_1' = 0$, $q_1 = 0$, that *if* $J_1' = 0$ *then one has* $I_1' = const.$

11. External virtual work. Varignon's theorem. Remarks on the auxiliary variables introduced in the preceding section. - For the deformed line AB, given an arbitrary virtual deformation, we give the name of *external work* to the expression:

$$\delta \mathcal{T}_e = -[F'\delta' x + G'\delta' y + H'\delta' z + I'\delta I' + J'\delta J' + K'\delta K']_{A_0}^{B_0}$$

[27] POISSON. - *Sur les lignes élastiques à double courbure*, Correspondance sur l'Ecole Polytechnique, T. III, no. 3, pp. 355-360, January 1816. POISSON'S proposition is independent of the formulas that define the effort and the moment of deformation by means of W; POISSON established them by writing the equations of equilibrium of a portion of the line by the principle of solidification; BERTRAND gave them a proof in a note in the *Mécanique analytique* of LAGRANGE, which we will review.

$$+\int_{A_0}^{B_0}(X'_0\delta' x + Y'_0\delta' y + Z'_0\delta' z + L'_0\delta I' + M'_0\delta J' + N'_0\delta K')ds_0.$$

From the preceding section, upon setting $\mathcal{L} = \mathcal{L}_0\dfrac{ds_0}{ds}$, ..., $\mathcal{L}' = \mathcal{L}'_0\dfrac{ds_0}{ds}$, ..., one may give the following forms to that expression:

$$\delta\mathcal{T}_e = -[F\delta x + G\delta y + H\delta z + I\delta I + J\delta J + K\delta K]_A^B$$
$$+\int_A^B(X\delta x + Y\delta y + Z\delta z + L\delta I + M\delta J + N\delta K)ds,$$
$$\delta\mathcal{T}_e = -[F\delta x + G\delta y + H\delta z + \mathcal{I}\delta\lambda_1 + \mathcal{J}\delta\lambda_2 + \mathcal{K}\delta\lambda_3]_A^B$$
$$+\int_A^B(X\delta x + Y\delta y + Z\delta z + \mathcal{L}\delta\lambda_1 + \mathcal{M}\delta\lambda_2 + \mathcal{N}\delta\lambda_3)ds,$$
$$\delta\mathcal{T}_e = -[F'\delta' x + G'\delta' y + H'\delta' z + \mathcal{I}\delta\lambda_1 + \mathcal{J}\delta\lambda_2 + \mathcal{K}\delta\lambda_3]_A^B$$
$$+\int_A^B(X'_0\delta' x + Y'_0\delta' y + Z'_0\delta' z + \mathcal{L}\delta\lambda_1 + \mathcal{M}\delta\lambda_2 + \mathcal{N}\delta\lambda_3)ds.$$

We will apply the last two later on. As for the first two, we shall deduce a fundamental proposition of statics here, where the idea, though not its present form, is due to VARIGNON, and which we have encountered already in the interpretation given by SAINT-GUILHEM of the relations that couple the external forces and quantities of motion in dynamics. Identifying the effort and the moment of deformation at a point M of the line M with the resultant and the resultant moment of a system of vectors relative to the point M; let $P\nu$, $P\sigma$ be the general resultant and the resultant moment relative to a point P of space. Similarly, identify the force and the external moment at a point M referred to the unit of length of (M), with the resultant and the resultant moment of a system of vectors relative to the point M; let PN and PS be the resultant and the resultant moment relative to a point P of space; one has this proposition:

When arc length is identified with time, the velocities of the geometric points ν and σ are equal and parallel to the segments PN and PS, respectively.

This proposition is obviously the translation of equations (12), which one may write:

$$(12')\qquad \begin{cases} \dfrac{dF}{ds} - X = 0, & \dfrac{d}{ds}(I + Hy - Gz) - (L + Zy - Yz) = 0, \\ \dfrac{dG}{ds} - Y = 0, & \dfrac{d}{ds}(J + Fz - Hx) - (M + Xz - Zx) = 0, \\ \dfrac{dH}{ds} - Z = 0, & \dfrac{d}{ds}(K + Gx - Fy) - (N + Yx - Xy) = 0. \end{cases}$$

We may also arrive at this result in the following manner. Start with:

$$\int_A^B \delta W\, ds_0 = -\delta\mathcal{T}_e,$$

where δT_e is taken between A and B. Since δW may be identically null, by virtue of the invariance of W under the group of Euclidean displacements, when the expressions $\delta' x, \ldots, \delta I'$, … are given by the formulas (9′) or, what amounts to the same thing, when δx, δy, δz are given by formulas (9), and $\delta' I = \omega_1 \delta t$, $\delta' J = \omega_2 \delta t$, $\delta' K = \omega_3 \delta t$, and this is true for any value of the constants a_1, a_2, a_3, ω_1, ω_2, ω_3, from which we conclude that one has:

$$[F]_A^B - \int_A^M X\,ds = 0, \qquad [G]_A^B - \int_A^M Y\,ds = 0, \qquad [H]_A^B - \int_A^M Z\,ds = 0,$$
$$[I + yH - zG]_A^M - \int_A^M (L + yZ - zY)\,ds = 0,$$

and two analogous formulas; *in these relations, one may regard M as variable,* and they are also equivalent to equations (12′). One will remark that these formulas are easily deduced from the ones that one ordinarily write by means of the principle of solidification; we will return to this point later on in the context of the reasoning made by POISSON and reprised by BERTRAND in regard to the deformable line considered by BINET.

Along with the expressions F', G', H', I', J', K' that were first introduced, we have imagined other expressions that one may propose to calculate. On the other hand, in these calculations, one may make functions appear explicitly that one introduces according to the nature of the problem, which will be, for example, x, y, z or x', y', z', and three parameters λ_1, λ_2, λ_3, by means of which, one expresses $\alpha, \alpha', \cdots, \gamma''$ ([28]).

If one introduces x, y, z and λ_1, λ_2, λ_3 then one will have:

$$F = \frac{\partial W}{\partial \frac{dx}{ds_0}}, \qquad G = \frac{\partial W}{\partial \frac{dy}{ds_0}}, \qquad H = \frac{\partial W}{\partial \frac{dz}{ds_0}},$$
$$\mathcal{I} = \frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}}, \qquad \mathcal{J} = \frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}}, \qquad \mathcal{K} = \frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}}.$$

If one introduces x', y', z' and three parameters λ_1, λ_2, λ_3 then one will have:

$$F = \frac{\partial W}{\partial \frac{dx'}{ds_0}}, \qquad G = \frac{\partial W}{\partial \frac{dy'}{ds_0}}, \qquad H = \frac{\partial W}{\partial \frac{dz'}{ds_0}},$$
$$\mathcal{I}' = \frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}}, \qquad \mathcal{J}' = \frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}}, \qquad \mathcal{K}' = \frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}}.$$

[28] For the auxiliary variables λ_1, λ_2, λ_3 one may take, for example, the components of rotation that make the fixed axes Ox, Oy, Oz parallel to Mx', My', Mz'.

12. Notion of the energy of deformation. - Imagine the two states (M_0) or A_0B_0 and (M) or AB of a deformable line, and consider an arbitrary sequence of states that start from (M_0) and end at (M). To that effect, it suffices to consider functions x, y, z; $\alpha, \alpha', \cdots, \gamma''$ of s_0 and one variable h, which reduce to x_0, y_0, z_0; $\alpha_0, \alpha'_0, \cdots, \gamma''_0$, respectively, for the value zero of h, and to the values x, y, z; $\alpha, \alpha', \cdots, \gamma''$, respectively, for the value h of h relative to (M).

Upon making the parameter h vary from h to 0 in a continuous fashion, we obtain a continuous deformation that permits us to pass from the state A_0B_0 to the state AB. For this continuous deformation, imagine the *total work* performed by the forces and external moments of deformation that are applied to the extremities of the line. To obtain the total work, it suffices to integrate the differential so obtained from 0 to h, upon starting with one of the expressions for $\delta\mathcal{T}_e$ that were defined in the preceding section, and substituting the partial differentials that correspond to increasing h by δh for the variations of x, y, z; $\alpha, \alpha', \cdots, \gamma''$. The formula:

$$\delta\mathcal{T}_e = -\int_{A_0}^{B_0} \delta W \, ds_0$$

gives the expression $-\int_{A_0}^{B_0} \frac{\partial W}{\partial h} dh \, ds_0$ for the present value of $\delta\mathcal{T}_e$, and we obtain:

$$-\int_0^h \left(\int_{A_0}^{B_0} \frac{\partial W}{\partial h} ds_0 \right) dh =$$

$$-\int_{A_0}^{B_0} [W(s_0, \xi, \eta, \varsigma, p, q, r) - W(s_0, \xi_0, \eta_0, \varsigma_0, p_0, q_0, r_0)] ds_0$$

for the total work.

The work considered is independent of the intermediary states and depends only on the extreme states (M_0) and (M).

This leads us to introduce the notion of the *energy of deformation,* which must be distinguished from the preceding action we described; we say that $-W$ is the *deformation energy density*, referred to the unit of length of the deformed line.

13. Natural state of the deformable line. General indications of the problems that the consideration of that line leads to. In the foregoing, we started with a state of the deformable line that we called *natural*, and we were given a state that we called *deformed;* we have indicated the formulas that permit us to calculate the external force and the elements that are analogous to the ones that are adjoined to the function, W, that represents the action of deformation at a point for the deformable line.

Let us pause for a moment on the notion of *natural state.* The latter is, in the preceding, a state that has not been subjected to any deformation. Regard the functions $x, y, z, \ldots$ as determining the deformed state, which depends upon one parameter such that one recovers the natural state for a particular value of this parameter; the latter will thus appear as a particular case of the deformed state, and we are led to attempt to apply the notions relating to the latter.

One may, without changing the values of the elements defined by formulas (10), replace the function W by that function augmented by an arbitrary *definite* function of s_0, and if one was left inspired by the idea of *action* that we associated to the passage from the natural state (M_0) to the deformed state (M) one may, if one prefers, suppose that the function of s_0 that is defined by the expression:

$$W(s_0, \xi_0, \eta_0, \zeta_0, p_0, q_0, r_0)$$

is identically null; however, the values obtained for the external force and the analogous elements in regard to the natural state will not be necessarily null; we say that they define the external force and the analogous elements relative to the natural state ([29]).

In what we just discussed, the natural state presented itself as the initial state of a sequence of deformed states, as a state with which to begin our study of the deformation. As a result, one is led to demand that it is not possible for it to play the role of one of the deformed states, since the role that we have made the natural state play, and likewise the elements that were defined in section 9, (external force, external effort, ...), that were calculated for the other deformed states, have the same value if one refers the first of these elements to the unit of length of the deformed line. This question receives a response only if one introduces and clarifies the notion of action corresponding to the passage from a deformed state to another deformed state.

The simplest hypothesis consists of assuming that this latter action is obtained by subtracting the action that corresponds to the passage from the natural state (M_0) to the first deformed state ($M_{(0)}$) from the action that corresponds to the passage from the natural state (M_0) to the second deformed state (M). If we denote the arc length of ($M_{(0)}$) by $s_{(0)}$, and the quantities that are analogous to $\xi, \eta, \zeta, p, q, r$ by $\xi_{(0)}, \eta_{(0)}, \zeta_{(0)}, p_{(0)}, q_{(0)}, r_{(0)}$ then one is led to adopt the expression:

$$\int_{A_0}^{B_0} [W(s_0, \xi, \eta, \varsigma, p, q, r) - W(s_0, \xi_{(0)}, \eta_{(0)}, \varsigma_{(0)}, p_{(0)}, q_{(0)}, r_{(0)})]\, ds_0 . \tag{14}$$

Introduce $s_{(0)}$ for the independent variable instead of s_0, and denote the variables that become $\xi, \eta, \zeta, p, q, r,$ when one makes $s_{(0)}$ play the role that was played by s_0 by $\xi^{(0)}, \eta^{(0)}, \zeta^{(0)}, p^{(0)}, q^{(0)}, r^{(0)}$; one will have relations such as the following:

$$\xi = \xi_0 \frac{ds_{(0)}}{ds_0},$$

and, upon denoting the points of ($M_{(0)}$) that correspond to the points A_0, B_0 of (M_0) by $A_{(0)}$, $B_{(0)}$ expression (14) becomes:

$$\int_{A_{(0)}}^{B_{(0)}} W_0^{(0)}(s_{(0)}, \xi^{(0)}, \eta^{(0)}, \varsigma^{(0)}, p^{(0)}, q^{(0)}, r^{(0)})\, ds_{(0)}, \tag{15}$$

upon denoting the expression:

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[29] We may then speak of the external force and moment, the effort and moment of deformation, because we regard the natural state as the limit of a sequence of states for which we know the external force and moments, the effort and the moment of deformation; this is because the external force and moment, the effort and moment of deformation, are defined, up till now, only when there is a deformation that makes it possible to manifest and measure them.

$$[W(s_0,\xi^{(0)}\frac{ds_{(0)}}{ds_0},\eta^{(0)}\frac{ds_{(0)}}{ds_0},\cdots,r^{(0)}\frac{ds_{(0)}}{ds_0})-W(s_0,\xi_{(0)},\eta_{(0)},\cdots,)]\frac{ds_0}{ds_{(0)}},$$

by $W_0^{(0)}(s_{(0)}, \xi^{(0)}, \eta^{(0)}, \zeta^{(0)}, p^{(0)}, q^{(0)}, r^{(0)})$, in which s_0 is replaced as a function of $s_{(0)}$.

Furthermore, from the remark made at the beginning of this section, one may, if one prefers, substitute the following expression:

$$(15') \qquad \int_{A_{(0)}}^{B_{(0)}} W^{(0)}(s_{(0)},\xi^{(0)},\eta^{(0)},\varsigma^{(0)},p^{(0)},q^{(0)},r^{(0)})ds_{(0)}$$

for (15), where the function $W^{(0)}(s_{(0)}, \xi^{(0)}, \eta^{(0)}, \zeta^{(0)}, p^{(0)}, q^{(0)}, r^{(0)})$, is the expression:

$$W(s_0,\xi^{(0)}\frac{ds_{(0)}}{ds_0},\cdots,r^{(0)}\frac{ds_{(0)}}{ds_0}),$$

in which s_0, $\frac{ds_0}{ds_{(0)}},\frac{ds_{(0)}}{ds_0}$ are expressed as functions of $s_{(0)}$.

One immediately confirms that the application of the formulas of section 9 to expression (15) or expression (15′) gives, upon starting with $(M_{(0)})$ as the natural state, the same values for the external force and moment relative to the state (M), referred to the unit of length of (M), as well as the same values for the effort and the moment of deformation.

Therefore we may consider (M) as a deformed state when $(M_{(0)})$ is the natural state, provided that the function W that is associated to the state (M) is presently $W_{(0)}$ and $W_0^{(0)}$ ([30]).

We now give several general indications about the problems that may lead to the consideration of the deformable line.

In the preceding, as well as in what we already did, we gave formulas that determined the external force and the analogous elements when one supposed that the functions x, y, z, ... of s_0 that define the deformed state were known.

We immediately remark that if one starts with the givens of x, y, z, ..., and if one calculates X_0', Y_0', Z_0' – to fix ideas – then, after doing all the calculations, one obtains definite functions of s_0. However, by virtue of the formulas that define x, y, z, ... as functions of s_0, one may obviously express X_0', Y_0', Z_0' by means of s_0, x, y, z, ..., and their derivatives up to whatever order one desires. Upon imagining a problem in which X_0', Y_0', Z_0', for example, figure among the givens, we may imagine that these expressions are given as functions of s_0, but we may just as well suppose that they refer x, y, z, ..., and the derivatives of the latter with respect to s_0.

[30] As we said at the beginning of this section, this permits us to generalize the notion of natural state that we first introduced. Instead of simply making the idea of a particular state correspond to that word, we may, in a more general fashion, make it correspond to the idea of an arbitrary state that we start with to study the deformation.

Consider a problem in which the projections of the external force and moment, either on the fixed axes *Ox*, *Oy*, *Oz* or on the axes Mx', My', Mz', figure among the givens, and suppose, to fix ideas, that these projections are given functions of s_0, x, y, z, $\alpha, \alpha', \cdots, \gamma''$, and their first and second order derivatives. In addition, suppose that the external force and moment are referred to the unit of length of (M_0) and that x_0, y_0, z_0, are given functions of s_0. It is clear that under these conditions the formulas of section 9 that serve to define $X_0', Y_0', Z_0', L_0', M_0', N_0'$ become six differential equations between the unknowns x, y, z, λ_1, λ_2, λ_3 the last three being three auxiliary functions, by means of which one may express the nine cosines $\alpha, \alpha', \cdots, \gamma''$. These differential equations, with the hypothesis that one proceeds to make on the external force and moment, do not involve derivatives of order higher than two.

To complete the search for the unknowns, if the problem we posed is well-defined, or at least if it does not involve an indeterminacy as great as the one that results in only the differential equations that we will eventually discuss, then one will have to take the complementary givens into account. The latter may be limit conditions, i.e., conditions that are satisfied by the unknowns at the extremities A_0 and B_0; for example, one may give the values at A_0 and B_0 of a certain number of expressions x, y, z, λ_1, λ_2, λ_3, and expressions such as $F_0', G_0', H_0', I_0', J_0', K_0'$ that relate to the effort and the moment of deformation, or similarly to functions – more often than not, linear – of x, y, z, λ_1, λ_2, λ_3 and $F_0', G_0', H_0', I_0', J_0', K_0'$.

We shall show, by particular examples, with particular hypotheses, how differential equations and complementary conditions may correspond to various problems; however, one may vary the questions.

If the arc length *s* figures explicitly in the givens then one will consider *s* as a supplementary variable, and one may adjoin the relation:

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$$

It often happens that one may devote most of one's attention to the deformed line (M) with the line (M_0) remaining in the background, so to speak. If we suppose that the expression of *W* as a function of s_0, x, h, z, p, q, r is given and does not necessitate being given (M_0) for its determination then the function *W* will finally be a function of s_0, the first derivatives of x, y, z, of λ_1, λ_2, λ_3, and the first derivatives of λ_1, λ_2, λ_3 . If the external force and moment are also given explicitly by means of s_0, x, y, z, λ_1, λ_2, λ_3 and their derivatives then it is clear that the problem may be considered as comprising, on the one hand, the determination of the state (M) by means of a variable relating to that state – *s*, for example – or one of the letters x, y, z, and, on the other hand, the determination of the relation that couples s_0 and s. With the hypotheses that we just made, s_0 may figure explicitly, and, in addition([31]), its differential ds_0 may figure, or, if one prefers, the

[31] If one gives the external force and moment referred to the unit of length of (M), and, more generally, if one gives these elements as functions of s_0, s, x, ..., and the first derivatives with respect to one of these letters.

expression $\frac{ds_0}{ds}$ or its inverse $\frac{ds}{ds_0}$. We remark that the notion of the quotient, which gives the derivative of s with respect to s_0, corresponds to the linear dilatation felt by the line element ds_0 that issues from the point M_0 of (M_0), and which becomes the element ds that issues from the point M of (M) that corresponds to the point M_0. We return to the dilatation that LAMÉ specifically imagined for the particular deformable line that he studied ([32]).

Another type of problem will be developed later on when we seek to attach some very special lines that were considered by geometers who used to be occupied with the present subject, to the deformable line that was defined up till now, i.e., the *free line* ([33]), which is susceptible to all possible deformations, upon imagining the study of the former as the study of particular deformations of the free line.

14. Normal form for the equations of the deformable line when the external force and moment are given as simple functions of s_0 and elements that fix the position of the triad $Mx'y'z'$. Castigliano's minimum work principle. – Conforming to the indications of the preceding section, suppose that the external force and moment are given by means of simple functions of s_0 and elements that fix the position of the triad $Mx'y'z'$. Suppose, moreover, that the natural state is given. We may consider the equations of sec. 9 as differential equations in the unknowns x, y, z and the three parameters λ_1, λ_2, λ_3 by means of which one expresses $\alpha, \alpha', \cdots, \gamma''$, or again, in the unknowns x', y', z' and the three parameters λ_1, λ_2, λ_3, which corresponds to a change of variables. These two viewpoints are the ones that most naturally present themselves. In the first case, the expressions ξ, η, ζ, p, q, r are functions of $\frac{dx}{ds_0}, \frac{dy}{ds_0}, \frac{dz}{ds_0}$, λ_1, λ_2, λ_3, $\frac{d\lambda_1}{ds_0}, \frac{d\lambda_2}{ds_0}, \frac{d\lambda_3}{ds_0}$ that one may calculate by means of formulas (1) and (2). In the second case, these will be functions of x', y', z', $\frac{dx'}{ds_0}, \ldots$, λ_1, ..., $\frac{d\lambda_1}{ds_0}, \ldots$ that one may calculate by means of formulas (2) and (4).

The first case is the most interesting, by reason of the analogy that exists between the present question and dynamics of points, and between triads and rigid bodies. We examine it first.

1. Assume that $X'_0, Y'_0, Z'_0, L'_0, M'_0, N'_0$, or, what amounts to the same thing, X_0, Y_0, Z_0, L_0, M_0, N_0 are given functions of s_0, x, y, z, λ_1, λ_2, λ_3. The expression W is, after

[32] LAMÉ. - *Leçons sur la théorie mathématique de l'élasticité des corps solides*, 2nd ed., pp. 98-99 (8th lesson, sec. **41**, entitled *Dilatation du fil*).

[33] Here, the expression "*free*" signifies that the theory starts with the function W that depends on elements that result from considering only that line, and which are susceptible to all possible variations.

substituting values for ξ, η, ζ, p, q, r that are related by formulas (1) and (2) to definite functions of s_0, $\frac{dx}{ds_0}, \frac{dy}{ds_0}, \frac{dz}{ds_0}$, λ_1, λ_2, λ_3, $\frac{d\lambda_1}{ds_0}, \frac{d\lambda_2}{ds_0}, \frac{d\lambda_3}{ds_0}$, which we continue to denote by W, and the equations of the problem may be written:

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dx}{ds_0}} - X_0 = 0, \qquad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}} - \frac{\partial W}{\partial \lambda_1} - \mathcal{L}_0 = 0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dy}{ds_0}} - Y_0 = 0, \qquad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}} - \frac{\partial W}{\partial \lambda_2} - \mathcal{M}_0 = 0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dz}{ds_0}} - Z_0 = 0, \qquad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}} - \frac{\partial W}{\partial \lambda_3} - \mathcal{N}_0 = 0,$$

$\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$, are functions of s_0, x, y, z, λ_1, λ_2, λ_3 that result in the functions of sec. **10**.

This results immediately either from the formulas of the preceding sections or, in a more immediate fashion, from the formulas of the definition of X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$, F, G, H, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ may be summarized in the relation:

$$\delta\int_{A_0}^{B_0} W\, ds_0 + \delta\mathcal{T}_e = 0,$$

i.e., in:

$$\delta\int_{A_0}^{B_0} W\, ds_0 = [F\delta x + G\delta y + H\delta z + \mathcal{I}\delta\lambda_1 + \mathcal{J}\delta\lambda_2 + \mathcal{K}\delta\lambda_3]_{A_0}^{B_0}$$
$$-\int_{A_0}^{B_0} (X_0\delta x + Y_0\delta y + Z_0\delta z + \mathcal{L}_0\delta\lambda_1 + \mathcal{M}_0\delta\lambda_2 + \mathcal{N}_0\delta\lambda_3)\, ds_0 .$$

We may replace the preceding system by a system of first order equations upon introducing six unknown auxiliary variables for which, instead of first order derivatives of x, y, z, λ_1, λ_2, λ_3, we choose the six expressions that we just considered:

$$F = \frac{\partial W}{\partial \frac{dx}{ds_0}}, \qquad G = \frac{\partial W}{\partial \frac{dy}{ds_0}}, \qquad H = \frac{\partial W}{\partial \frac{dz}{ds_0}},$$

$$\mathcal{I} = \frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}}, \qquad \mathcal{J} = \frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}}, \qquad \mathcal{K} = \frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}}.$$

Upon supposing that the Hessian of W with respect to $\frac{dx}{ds_0}, \frac{dy}{ds_0}, \frac{dz}{ds_0}, \frac{d\lambda_1}{ds_0}, \frac{d\lambda_2}{ds_0}, \frac{d\lambda_3}{ds_0}$ is non-null (which amounts to supposing that the Hessian of the function W is non-null when it is expressed in terms of ξ, η, ζ, p, q, r), we may derive values for the last six derivatives $\frac{dx}{ds_0}, \ldots, \frac{d\lambda_3}{ds_0}$ as functions of F, G, H, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$. We substitute these values in the expression:

$$\mathcal{E} = \frac{dx}{ds_0}\frac{\partial W}{\partial \frac{dx}{ds_0}} + \frac{dy}{ds_0}\frac{\partial W}{\partial \frac{dy}{ds_0}} + \frac{dz}{ds_0}\frac{\partial W}{\partial \frac{dz}{ds_0}} + \sum \frac{d\lambda_i}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_i}{ds_0}} - W,$$

which is none other than the expression of:

$$\xi\frac{\partial W}{\partial \xi} + \eta\frac{\partial W}{\partial \eta} + \varsigma\frac{\partial W}{\partial \varsigma} + p\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial q} + r\frac{\partial W}{\partial r} - W,$$

as a function of $s_0, \frac{dx}{ds_0}, \frac{dy}{ds_0}, \frac{dz}{ds_0}, \lambda_1, \cdots, \frac{d\lambda_{13}}{ds_0}, \cdots$ After substitution, we obtain a function of s_0, λ_1, λ_2, λ_3, F, G, H, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, which we continue to denote by the letter $\mathcal{E}$. Now, the total differential of the latter functions is obviously:

$$\frac{dx}{ds_0} d\frac{\partial W}{\partial \frac{dx}{ds_0}} + \cdots + \frac{d\lambda_1}{ds_0} d\frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}} + \cdots - \frac{\partial W}{\partial s_0} ds_0 - \sum \frac{\partial W}{\partial \lambda_i} d\lambda_i,$$

or

$$\frac{dx}{ds_0} dF + \frac{dy}{ds_0} dG + \frac{dz}{ds_0} dH + \frac{d\lambda_1}{ds_0} d\mathcal{I} + \frac{d\lambda_2}{ds_0} d\mathcal{J} + \frac{d\lambda_3}{ds_0} d\mathcal{K} - \frac{\partial W}{\partial s_0} ds_0 - \sum \frac{\partial W}{\partial \lambda_i} d\lambda_i,$$

and as a result one has the following form for the system that defines x, y, z, λ_1, λ_2, λ_3, F, G, H, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$:

$$\frac{dx}{ds_0} = \frac{\partial \mathcal{E}}{\partial F}, \quad \frac{dy}{ds_0} = \frac{\partial \mathcal{E}}{\partial G}, \quad \frac{dz}{ds_0} = \frac{\partial \mathcal{E}}{\partial H}, \quad \frac{d\lambda_1}{ds_0} = \frac{\partial \mathcal{E}}{\partial \mathcal{I}}, \quad \frac{d\lambda_2}{ds_0} = \frac{\partial \mathcal{E}}{\partial \mathcal{J}}, \quad \frac{d\lambda_3}{ds_0} = \frac{\partial \mathcal{E}}{\partial \mathcal{K}},$$

$$\frac{dF}{ds_0} - X_0 = 0, \qquad \frac{dG}{ds_0} - Y_0 = 0, \qquad \frac{dH}{ds_0} - Z_0 = 0,$$

$$\frac{d\mathcal{I}}{ds_0} + \frac{\partial \mathcal{E}}{\partial \lambda_1} - \mathcal{L}_0 = 0, \qquad \frac{d\mathcal{J}}{ds_0} + \frac{\partial \mathcal{E}}{\partial \lambda_2} - \mathcal{M}_0 = 0, \quad \frac{d\mathcal{K}}{ds_0} + \frac{\partial \mathcal{E}}{\partial \lambda_3} - \mathcal{N}_0 = 0.$$

We have supposed that, by virtue of the formulas that define x, y, z, λ_1, λ_2, λ_3 as functions of s_0, one can express X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ as a function of s_0, x, y, z, λ_1, λ_2, λ_3; this is possible in an infinitude of ways, and one may choose the new forms for X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ in such a way that the partial derivatives $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}, \frac{\partial U}{\partial \lambda_1}, \frac{\partial U}{\partial \lambda_2}, \frac{\partial U}{\partial \lambda_3}$, respectively, change the sign of the same function of $\mathcal{U}$, which is or is not independent of s_0. Suppose that this is the case and let $\mathcal{V}$ denote the function of x, y, z, λ_1, λ_2, λ_3 (and maybe s_0) that is defined by the formula:

$$\mathcal{V} = \mathcal{E} + \mathcal{U};$$

the preceding system takes the form:

$$\frac{dx}{ds_0} = \frac{\partial \mathcal{V}}{\partial F}, \quad \frac{dy}{ds_0} = \frac{\partial \mathcal{V}}{\partial G}, \quad \frac{dz}{ds_0} = \frac{\partial \mathcal{V}}{\partial H}, \quad \frac{d\lambda_1}{ds_0} = \frac{\partial \mathcal{V}}{\partial \mathcal{I}}, \quad \frac{d\lambda_2}{ds_0} = \frac{\partial \mathcal{V}}{\partial \mathcal{J}}, \quad \frac{d\lambda_3}{ds_0} = \frac{\partial \mathcal{V}}{\partial \mathcal{K}},$$

$$\frac{dF}{ds_0} = -\frac{\partial \mathcal{V}}{\partial x}, \quad \frac{dG}{ds_0} = -\frac{\partial \mathcal{V}}{\partial y}, \quad \frac{dH}{ds_0} = -\frac{\partial \mathcal{V}}{\partial z},$$

$$\frac{d\mathcal{I}}{ds_0} = -\frac{\partial \mathcal{V}}{\partial \lambda_1}, \quad \frac{d\mathcal{J}}{ds_0} = -\frac{\partial \mathcal{V}}{\partial \lambda_2}, \quad \frac{d\mathcal{K}}{ds_0} = -\frac{\partial \mathcal{V}}{\partial \lambda_3}.$$

Here we have equations that are presented in the form of HAMILTON'S equations from dynamics. In particular, if we suppose that the new forms of X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are chosen, as is always possible, in such a fashion that s_0 does not figure and that they are partial derivatives of a function $-\mathcal{U}$ of x, y, z, λ_1, λ_2, λ_3, and if, in addition, we suppose that $W(s_0, \xi, \eta, \zeta, p, q, r)$ does not depend on s_0 ([34]), then we have, more particularly, a canonical system of equations.

2. Now look at the functions x', y', z', and suppose furthermore that the functions $\alpha, \alpha', \cdots, \gamma''$ are expressed by means of three auxiliary functions λ_1, λ_2, λ_3. Assume that $X'_0, Y'_0, Z'_0, L'_0, M'_0, N'_0$ are given functions of s_0, x', y', z', λ_1, λ_2, λ_3. The expression W is, after substituting the values for ξ, η, ζ, p, q, r that are derived from formulas (2) and (4), a well-defined function of s_0, x', y', z', λ_1, λ_2, λ_3 that we continue to denote by W, and the equations of the problem may be written:

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dx'}{ds_0}} - X'_0 = 0, \quad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}} - \mathcal{L}'_0 = 0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dy'}{ds_0}} - Y'_0 = 0, \quad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}} - \mathcal{M}'_0 = 0,$$

[34] To express this hypothesis one may say that in this case - and by definition - the line is *homogenous.*

$$\frac{d}{ds_0}\frac{\partial W}{\partial \dfrac{dz'}{ds_0}} - Z_0' = 0\,, \quad \frac{d}{ds_0}\frac{\partial W}{\partial \dfrac{d\lambda_3}{ds_0}} - \mathcal{N}_0' = 0\,,$$

where $\mathcal{L}_0', \mathcal{M}_0', \mathcal{N}_0'$ are the functions of s_0, x', y', z', λ_1, λ_2, λ_3 that result from sec. **10**.

We may replace the preceding system by a system of first order equations upon introducing six auxiliary unknowns for which, instead of first order derivatives of $x', y', z', \lambda_1, \lambda_2, \lambda_3$, we choose the six preceding expressions that we already envisioned:

$$F' = \frac{\partial W}{\partial \dfrac{dx'}{ds_0}}, \quad G' = \frac{\partial W}{\partial \dfrac{dy'}{ds_0}}, \quad H' = \frac{\partial W}{\partial \dfrac{dz'}{ds_0}},$$

$$\mathcal{I}' = \frac{\partial W}{\partial \dfrac{d\lambda_1}{ds_0}}, \quad \mathcal{J}' = \frac{\partial W}{\partial \dfrac{d\lambda_2}{ds_0}}, \quad \mathcal{K}' = \frac{\partial W}{\partial \dfrac{d\lambda_3}{ds_0}}.$$

Upon supposing that the Hessian of W with respect to $\dfrac{dx'}{ds_0}, \dfrac{dy'}{ds_0}, \dfrac{dz'}{ds_0}, \dfrac{d\lambda_1}{ds_0}, \dfrac{d\lambda_2}{ds_0}, \dfrac{d\lambda_3}{ds_0}$, is non-null, we may derive values for these latter six derivatives as functions of F', G', H', $\mathcal{I}'$, $\mathcal{J}'$, $\mathcal{K}'$ from these six relations; we transport these values into the expression:

$$\mathcal{E}' = \frac{dx'}{ds_0}\frac{\partial W}{\partial \dfrac{dx'}{ds_0}} + \frac{dy'}{ds_0}\frac{\partial W}{\partial \dfrac{dy'}{ds_0}} + \frac{dz'}{ds_0}\frac{\partial W}{\partial \dfrac{dz'}{ds_0}} + \sum \frac{d\lambda_i}{ds_0}\frac{\partial W}{\partial \dfrac{d\lambda_i}{ds_0}} - W,$$

we obtain, after substitution, a function of s_0, x', y', z', λ_1, λ_2, λ_3, F', G', H', $\mathcal{I}'$, $\mathcal{J}'$, $\mathcal{K}'$ that we continue to denote by the letter $\mathcal{E}'$. Now, the total differential of this latter function is obviously:

$$\begin{aligned}&\frac{dx'}{ds_0}dF' + \frac{dy'}{ds_0}dG' + \frac{dz'}{ds_0}dH' + \frac{d\lambda_1}{ds_0}d\mathcal{I}' + \frac{d\lambda_2}{ds_0}d\mathcal{J}' + \frac{d\lambda_3}{ds_0}d\mathcal{K}' \\ &\quad - \frac{\partial W}{\partial s_0}ds_0 - \frac{\partial W}{\partial x'}dx' - \frac{\partial W}{\partial y'}dy' - \frac{\partial W}{\partial z'}dz' - \sum \frac{\partial W}{\partial \lambda_i}d\lambda_i,\end{aligned}$$

and, as a result, one has the following form for the system that defines x', y', z', λ_1, λ_2, λ_3, F', G', H', $\mathcal{I}'$, $\mathcal{J}'$, $\mathcal{K}'$:

$$\frac{dx'}{ds_0} = \frac{\partial \mathcal{E}'}{\partial F'}, \quad \frac{dy'}{ds_0} = \frac{\partial \mathcal{E}'}{\partial G'}, \quad \frac{dz'}{ds_0} = \frac{\partial \mathcal{E}'}{\partial H'}, \quad \frac{d\lambda_1}{ds_0} = \frac{\partial \mathcal{E}'}{\partial \mathcal{I}'}, \quad \frac{d\lambda_2}{ds_0} = \frac{\partial \mathcal{E}'}{\partial \mathcal{J}'}, \quad \frac{d\lambda_3}{ds_0} = \frac{\partial \mathcal{E}'}{\partial \mathcal{K}'},$$

$$\frac{dF'}{ds_0} + \frac{\partial \mathcal{E}'}{\partial x'} - X_0' = 0, \qquad \frac{dG'}{ds_0} + \frac{\partial \mathcal{E}'}{\partial y'} - Y_0' = 0, \qquad \frac{dH'}{ds_0} + \frac{\partial \mathcal{E}'}{\partial z'} - Z_0' = 0,$$

$$\frac{d\mathcal{I}'}{ds_0}+\frac{\partial\mathcal{E}'}{\partial\lambda_1}-\mathcal{L}_0'=0\,,\qquad \frac{d\mathcal{J}'}{ds_0}+\frac{\partial\mathcal{E}'}{\partial\lambda_2}-\mathcal{M}_0'=0\qquad \frac{d\mathcal{E}'}{ds_0}+\frac{\partial\mathcal{K}'}{\partial\lambda_3}-\mathcal{N}_0'=0\,.$$

By virtue of the formulas that define x', y', z', λ_1, λ_2, λ_3 as functions of s_0, we have supposed that one can express them as functions of s_0, x', y', z', $\lambda_1, \lambda_2, \lambda_3$. This is possible in an infinitude of ways and one may choose the new forms for them in such a way that they are the partial derivatives, up to sign, of the same functions $\mathcal{U}'$, which may or may not be independent of s_0. Suppose that this is true and introduce the function of x', y', z', λ_1, λ_2, λ_3, (and maybe s_0) that is defined by the formula:

$$\mathcal{V}' = \mathcal{E}' + \mathcal{U}';$$

the preceding system then takes the form:

$$\frac{dx'}{ds_0}=\frac{\partial\mathcal{V}'}{\partial F'},\quad \frac{dy'}{ds_0}=\frac{\partial\mathcal{V}'}{\partial G'},\quad \frac{dz'}{ds_0}=\frac{\partial\mathcal{V}'}{\partial H'},\quad \frac{d\lambda_1}{ds_0}=\frac{\partial\mathcal{E}'}{\partial\mathcal{I}'},\quad \frac{d\lambda_2}{ds_0}=\frac{\partial\mathcal{V}'}{\partial\mathcal{J}'},\quad \frac{d\lambda_3}{ds_0}=\frac{\partial\mathcal{V}'}{\partial\mathcal{K}'},$$
$$\frac{dF'}{ds_0}=-\frac{\partial\mathcal{V}'}{\partial x'},\qquad \frac{dG'}{ds_0}=-\frac{\partial\mathcal{V}'}{\partial y'},\qquad \frac{dH'}{ds_0}=-\frac{\partial\mathcal{V}'}{\partial z'},$$
$$\frac{d\mathcal{I}'}{ds_0}=-\frac{\partial\mathcal{V}'}{\partial\lambda_1},\qquad \frac{d\mathcal{J}'}{ds_0}=-\frac{\partial\mathcal{V}'}{\partial\lambda_2},\qquad \frac{d\mathcal{K}'}{ds_0}=-\frac{\partial\mathcal{V}'}{\partial\lambda_3}.$$

In the case where the forces and external moments are zero, the equation:

$$\delta\int W\,ds_0+\delta\mathcal{T}_e=0$$

corresponds to Castigliano's *principle of minimum work* ([35]), which was already considered by VINE, COURNOT, MENABREA, and others.

Consider the equations in the normal form:

$$\frac{dx}{ds_0}=\frac{\partial\mathcal{E}}{\partial F},\cdots,\frac{dF}{ds_0}-X_0=0,\cdots$$

Upon integrating from A_0 to B_0, they become:

$$x_{B_0}-x_{A_0}=\int_{A_0}^{B_0}\frac{\partial\mathcal{E}}{\partial F}ds_0\,,\ldots,\qquad F_{B_0}-F_{A_0}=\int_{A_0}^{B_0}X_0ds_0,\ldots$$

[35] CASTIGLIANO. - *Théorie de l'équilibre des systèmes élastiques et ses applications*, Turin 1879. See also MÜLLER-BRESLAU, *Die neueren Methoden der Festigkeitslehre*, 3rd ed., Leipzig, 1904.

For example, if one supposes that X_0, Y_0, Z_0 are null then one has $F = const. = F_{B_0} = F_{A_0} = G = const., H = const.$ In the three formulas such as:

$$x_{B_0} - x_{A_0} = \int_{A_0}^{B_0} \frac{\partial \mathcal{E}}{\partial F} ds_0 ,$$

F, G, H are independent of s_0, and one may write:

$$x_{B_0} - x_{A_0} = \frac{\partial}{\partial F} \int_{A_0}^{B_0} \mathcal{E} \, ds_0 .$$

If $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are null, and if $\frac{\partial \mathcal{E}}{\partial \lambda_1} = \frac{\partial \mathcal{E}}{\partial \lambda_2} = \frac{\partial \mathcal{E}}{\partial \lambda_3} = 0$ then one obtains analogous theorems that relate to λ_1, λ_2, λ_3. One is therefore led, in a very direct and natural manner, to what one calls the theorems of CASTIGLIANO in the strength of materials. One therefore generally imagines the simple case of an infinitely small deformation; W is a quadratic form, and the same things are true for $\mathcal{E}$ as those we deduced for W as its adjoint form.

15. Notions of hidden triad and hidden ***W*****.** - In the study of the deformable line, it is natural to direct one's attention to the curve described by the line, in particular, This amounts to starting with x, y, z and considering $\alpha, \alpha', \cdots, \gamma''$ as simple auxiliary variables. This is what we may likewise express by imagining that one ignores the existence of the triads that determine the deformable line, and that one knows only the vertices of these triads. Upon taking this viewpoint, in order to envision the differential equations that one is led to in this case, we may introduce the notion of *hidden triad,* and we are led to a resulting classification of the diverse circumstances that may present themselves in the elimination of $\alpha, \alpha', \cdots, \gamma''$.

A first question that presents itself is therefore that of the reductions that may be produced in the elimination of the $\alpha, \alpha', \cdots, \gamma''$. In the corresponding particular case where our attention is directed almost exclusively upon the curve described by the deformed line (M) one may occasionally make an abstraction from (M_0), and, as a result, from the deformation that permits us to pass from (M_0) to (M). It is from this latter viewpoint that we may recover the line that is called flexible and inextensible in rational mechanics.

The triad may be considered in another fashion. We may make several particular hypotheses on it, and similarly on the curve (M), which amounts to envisioning particular deformations of the free deformable line. If the relations that we impose are simple relations between ξ, η, ζ, p, q, r, as will be the case in the applications that we have to study, we may account for these relations in the calculations of W and derive more particular functions from W. The interesting question that this poses will be to simply introduce these particular forms, and to consider the general function W that will serve as point of departure as hidden, in a way. We will therefore have a *theory that will be special to the particular forms suggested by the given relations between* ξ, η, ζ, *p*, *q*, *r*.

We verify that one may thus, by means of the theory of the free deformable line, assemble the equations that are the result of special problems that one encounters in the habitual exposition of rational mechanics and in the classical theory of elasticity under the title of particular cases with a common origin.

In the latter theory, one often places oneself in the appropriate circumstances so as to avoid the consideration of deformations; in reality, they need to be completed. In practical applications this is what one may do when imagining the infinitely small deformation.

Take the case where the force and the external moment refer only to the first derivatives of the unknowns *x*, *y*, *z* and $\lambda_1, \lambda_2, \lambda_3$. The second derivatives of these unknowns will be introduced into the differential equations only by way of *W*. Now, the derivatives of *x*, *y*, *z* figure only in ξ, η, ζ and those of $\lambda_1, \lambda_2, \lambda_3$ present themselves only in *p*, *q*, *r*. One therefore sees that if *W* depends only on ξ, η, ζ or only on *p*, *q*, *r* then there will be a reduction in the orders of the derivatives that enter into the system of differential equations, and, as a result, there will also be a reduction in the system that is obtained by the elimination of *p*, *q*, *r*. We commence to examine the first two cases.

16. Case where ***W*** **depends only on** s_0, ξ, η, ζ. **How one recovers the equations of Lagrange's theory of the flexible and inextensible line.** - Suppose that *W* depends only on s_0, ξ, η, ζ. The equations of sec. **14** then reduce to the following:

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dx}{ds_0}} - X_0 = 0, \qquad \frac{\partial W}{\partial \lambda_1} + \mathcal{L}_0 = 0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dy}{ds_0}} - Y_0 = 0, \qquad \frac{\partial W}{\partial \lambda_2} + \mathcal{M}_0 = 0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dz}{ds_0}} - Z_0 = 0, \qquad \frac{\partial W}{\partial \lambda_3} + \mathcal{M}_0 = 0,$$

in which *W* depends only on $s_0, \frac{dx}{ds_0}, \frac{dy}{ds_0}, \frac{dz}{ds_0}, \lambda_1, \lambda_2, \lambda_3$. We show that if we take the simple case where X_0, Y_0, Z_0, $\lambda_1, \lambda_2, \lambda_3$ are given functions ([36]) of s_0, *x*, *y*, *z*, $\frac{dx}{ds_0}, \frac{dy}{ds_0}, \frac{dz}{ds_0}, \lambda_1, \lambda_2, \lambda_3$ then the three equations on the right may be solved for $\lambda_1, \lambda_2, \lambda_3$, and one finally obtains three differential equations that involve only s_0, *x*, *y*, *z*, and the first and second derivatives.

[36] In order to simplify the exposition, and to indicate more conveniently the things to which we are alluding, we suppose that X_0, Y_0, Z_0, L_0, M_0, N_0 do not refer to the derivatives of $\lambda_1, \lambda_2, \lambda_3$.

First, imagine the particular case where the given functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are null; the same will be true for the corresponding values of the functions of any of the systems: (L', M', N'), (L_0, M_0, N_0), (L, M, N). From this, it results that the following equations:

$$\frac{\partial W}{\partial \lambda_1} = 0, \qquad \frac{\partial W}{\partial \lambda_2} = 0, \qquad \frac{\partial W}{\partial \lambda_3} = 0,$$

amount to:

$$\frac{F}{\frac{dx}{ds}} = \frac{G}{\frac{dy}{ds}} = \frac{H}{\frac{dz}{ds}},$$

and, upon denoting the common value of these ratios by $-T$, the equations (?), in which it is necessary to carry λ_1, λ_2, λ_3, may be written:

$$\frac{d}{ds_0}\left(T\frac{dx}{ds}\right) + X_0 = 0, \qquad \frac{d}{ds_0}\left(T\frac{dy}{ds}\right) + Y_0 = 0, \qquad \frac{d}{ds_0}\left(T\frac{dz}{ds}\right) + Z_0 = 0,$$

or, if one prefers:

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + X = 0, \qquad \frac{d}{ds}\left(T\frac{dy}{ds}\right) + Y = 0, \qquad \frac{d}{ds}\left(T\frac{dz}{ds}\right) + Z = 0,$$

The effort actually reduces to an *effort of tension T*.

Having said this, observe that if one starts with two positions (M_0) and (M), which are assumed *given*, and one deduces the functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ from them, as in sec. **9** and **10**, then in the case where the three functions are null one may arrive at the conclusion that this result presents itself accidentally, i.e., only for a certain set of particular deformations. However, one may also arrive at the conclusion that it presents itself for *any* deformed (M), since it is a consequence of the nature of (M), i.e., the form of W.

Imagine the latter case, which is particularly interesting: W is then a simple function of s_0 and $\xi^2 + \eta^2 + \zeta^2$, or, from (37), what amounts to the same thing, of s_0 and $\frac{ds_0}{ds}$. The equations $\frac{\partial W}{\partial \lambda_i} = 0$, (i = 1, 2, 3) reduce to identities (38) and if one supposes that W is expressed by means of s_0 and $\mu = \frac{ds_0}{ds} - 1$ (where μ represents the linear dilatation at the point), then all that remains are the equations:

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(37) One may also say that W is a function of s_0 and the linear dilatation $\mu = \frac{ds_0}{ds} - 1$ at the point M, as was considered by LAMÉ in his *Leçons sur la théorie mathématique de l'élasticité des corps solides*, pp. 98, 99, in the 2nd edition.

(38) The *triad* is completely hidden; we may also understand that we have a *pointlike* line.

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+X=0, \qquad \frac{d}{ds}\left(T\frac{dy}{ds}\right)+Y=0, \qquad \frac{d}{ds}\left(T\frac{dz}{ds}\right)+Z=0,$$

where one has:

$$T=-\frac{\partial W}{\partial \mu}.$$

If we suppose that the function W is known, then that gives us X, Y, Z or X_0, Y_0, Z_0 as functions of s_0, s, x, y, z, and the fourth derivatives of the latter (39) with respect to one of the others; the preceding equations, combined with:

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1,$$

provide four differential equations that define four of the variables s_0, s, x, y, z by means of the fifth.

If s does not figure explicitly then one may eliminate ds by means of the relation that one derives, and what remains are three differential equations that define the three unknowns x, y, z as functions of s_0.

If we imagine the particular case in which W depends on only μ and s_0 does not figure explicitly then we find ourselves in the presence of the equations that were proposed by LAGRANGE (40) for the study of the line that he qualified as a "flexible and, at the same time, extensible and contractible filament." We must remark that explanations given by LAGRANGE, in the second of the sections that he dedicated to the question (sec. 43) must be revised in the following fashion: if we regard W as a given function of μ then the same is also true for T (which corresponds to the assertion of LAGRANGE that expresses – with these notations – the fact that F is a given function of $\frac{ds}{d\sigma}$). We may substitute the unknown T for the unknown μ since the knowledge of one of them as a function of s implies the same for the other, and finally one is led to the study of four functions of s: T, x, y, z by means of the four preceding equations (and supplementary conditions if they are given). One observes, in addition, that if, as LAGRANGE seems to have supposed, the given expressions of X, Y, Z do not refer to s explicitly then one is limited to the consideration of the first three equations and the three variables x, y, z, where the differential of s was eliminated by means of the fourth equation.

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[39] One may suppose that derivatives of order higher than the first have been introduced.

[40] LAGRANGE. – *Mécanique analytique,* 1st part, Section V, par. 11, nos. 42-43, 4th edition, pp. 156-158. The same question has been raised by LAMÉ, in his *Leçons sur la théorie mathématique de l'élasticité des corps solides,* 2nd edition, 8th lesson, and then by DUHEM, in Tome II of his work, *Hydrodynamique, Elasticité, Acoustique,* pp. 1 and following. The exposition of LAMÉ, as well as the remarks of TODHUNTER and PEARSON on page 235 of Tome I of their *History,* etc., is the reproduction of the one that was given by POISSON, on pages 422 and following, of his *Mémoire sur le mouvement des corps élastiques,* printed in 1829 in Tome VIII of the *Mémoires de l'Institut de France.*

In the first of the sections that we cited (no. 43), LAGRANGE remarked that he was led to the same equations for the filament that he had already considered in his exposition under the name of flexible and inextensible filament, and in no. 44 he returned to tension. It seems to us that there is some confusion in the exposition of LAGRANGE on the subject of the notion of force (a confusion that was already pointed out by J. BERTRAND from the viewpoint of dynamics alone in the note he appended to no. 44). Indeed, it is clear that the viewpoint of LAGRANGE is that of dynamics, and that the word *equilibrium* is equivalent to the word *rest* in his exposition. Upon introducing, at the beginning of no. 44, "the force F by which every element ds of the filament curve tends to be contracted," LAGRANGE introduced a notion of force that no longer conforms to the definition posed at the beginning of his work (page 1), which is not a kinetic force, but a force that we may qualify as a *static force,* which is measured by means of the deformations.

17. **The flexible and inextensible filament. –** How, while remaining in the domain of the section on statics, where one measures forces by means of deformation, may one conceive and introduce the notion of *flexible and inextensible filament?* To give a definition of flexible and inextensible filament, it will suffice for us to follow – *but in the opposite sense* – the path that is habitually adopted, i.e., what one is often inspired to call *the solidification principle* ([41]).

In a general manner, imagine the deformable line of sec. **5**, with its natural state (M_0) and its deformed state (M). Suppose that for the deformations of the line, which are defined as in sec. **5**, i.e., by a *correspondence* between the points of (M_0) and those of the deformation (M), we impose the condition ([42]) that an arbitrary portion of (M) has the same length as the *corresponding* portion, which amounts to saying that one subjects x, y, z to the condition,

$$ds = ds_0 ,$$

upon supposing, as we did before, that ds and ds_0 have the same sign. One must assume that for such a line one would like to define the elements: exterior force, … We imagine a deformable line of the type considered up till now, and, instead of considering an arbitrary deformation (M) of the natural state (M_0), we direct our attention towards the deformations (M) for which one has $ds = ds_0$. As far as the position of the points and the associated triads are concerned, these deformations coincide with the deformations of the given inextensible line. For the definition of external force, …, acting on the latter, we assume the preceding formulas that we adopted with regard to any deformable line, which one applies to the positions of that line that coincide with those of the given inextensible line.

[41] APPELL. – 1st edition, T. I, no. 132, pp. 165; in the 2nd edition, T. I, no. 120, pp. 161, the expression *solidification principle* is omitted; the same is true for THOMSON and TAIT, *Treatise on Natural Philosophy*, vol. I, Part II, sec. **564**, pp. 110.

[42] We shall repeat this assumption in different analogous circumstances where one is led to adjoin what we shall later call later the *internal constraints* of the system that we previously studied.

In particular, if we imagine a flexible and inextensible line then we deduce the definition of external forces, relative to that line, that act on the line considered before, and for which W is a simple function of s and μ, *by considering the deformations of the latter for which the function μ reduces to zero.* Retaining only the letters s, X, Y, Z (since $s = s_0$, $X = X_0$, $Y = Y_0$, $Z = Z_0$), one is led to the system:

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+X=0, \qquad \frac{d}{ds}\left(T\frac{dy}{ds}\right)+Y=0, \qquad \frac{d}{ds}\left(T\frac{dz}{ds}\right)+Z=0,$$

in which $\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1$, and where T represents the function of s that is defined by the formula: $T=-\left(\frac{\partial W}{\partial \mu}\right)_{\mu=0}$.

It will not be necessary for us to suppose that the function T is known in order to obtain a *well-defined* problem; it will suffice to adjoin suitable limits to the conditions.

18. Case where W depends only on s_0, ξ, η, ζ, and where $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are non-null. – Now imagine the general case, where $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are not all three of them null. Upon introducing the auxiliary functions F, G, H the equations:

$$\frac{\partial W}{\partial \lambda_1}+\mathcal{L}_0=0, \qquad \frac{\partial W}{\partial \lambda_2}+\mathcal{M}_0=0, \qquad \frac{\partial W}{\partial \lambda_3}+\mathcal{N}_0=0,$$

amount to the relations:

$$\begin{aligned} H\frac{dy}{ds}-C\frac{dz}{ds}-L&=0,\\ F\frac{dz}{ds}-H\frac{dx}{ds}-M&=0,\\ G\frac{dx}{ds}-F\frac{dy}{ds}-N&=0,\end{aligned}$$

in such a way that in the present case the component of the effort that is tangent to the line, which one may call the *effort of tension,* the component of the effort that is normal to the line, which one may call the *transverse effort,* as is it is called in the strength of materials, and finally, the vector (L, M, N) determine a tri-rectangular triad.

Again introduce the effort of tension:

$$T=-\left(F\frac{dx}{ds}+G\frac{dy}{ds}+H\frac{dz}{ds}\right),$$

as an auxiliary, and we obtain:

$$-F=T\frac{dx}{ds}+N\frac{dy}{ds}-M\frac{dz}{ds},$$

$$-G = T\frac{dy}{ds} + L\frac{dz}{ds} - N\frac{dx}{ds},$$
$$-H = T\frac{dz}{ds} + M\frac{dx}{ds} - L\frac{dy}{ds},$$
$$L\frac{dx}{ds} + M\frac{dy}{ds} + N\frac{dz}{ds} = 0.$$

As a result, if X, Y, Z, L, M, N are given as functions of s, x, y, z and their first derivatives then one comes upon three equations such as the following:

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + X + \frac{d}{ds}\left(N\frac{dy}{ds} - M\frac{dz}{ds}\right) = 0,$$

to which we may adjoin:

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1, \qquad L\frac{dx}{ds} + M\frac{dy}{ds} + N\frac{dz}{ds} = 0,$$

in such a way that *for the last problem we posed* we have *five* differential equations that refer to *four* unknowns, namely, x, y, z, and the auxiliary unknown T.

19. Case where W depends only on s_0, p, q, r. – Suppose that W depends only on s_0, p, q, r. The equations of sec. **14**, which reduce to the following:

$$X_0 = 0, \qquad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}} - \frac{\partial W}{\partial \lambda_1} - \mathcal{L}_0 = 0,$$
$$Y_0 = 0, \qquad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}} - \frac{\partial W}{\partial \lambda_2} - \mathcal{M}_0 = 0$$
$$Z_0 = 0, \qquad \frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}} - \frac{\partial W}{\partial \lambda_3} - \mathcal{N}_0 = 0,$$

in which W depends only on s_0, λ_1, λ_2, λ_3, $\frac{d\lambda_1}{ds_0}, \frac{d\lambda_2}{ds_0}, \frac{d\lambda_3}{ds_0}$, then show us that if we take the simple case where X_0, Y_0, Z_0 do not refer to the derivatives of x, y, z then one may obtain x, y, z from the equations on the left and substitute their values into the equations on the right, i.e., into $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$. If these latter three do not refer to the derivatives of order higher than the first of x, y, z then, when X_0, Y_0, Z_0 refer only to s_0, x, y, z, λ_i, and

$\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ refer only to s_0, x, y, z, $\frac{dx}{ds_0}$, $\frac{dy}{ds_0}$, $\frac{dz}{ds_0}$, λ_i, $\frac{d\lambda_i}{ds_0}$, $\frac{d^2\lambda_i}{ds_0^2}$, one then comes down to three second order equations that determine λ_1, λ_2, λ_3.

The particular case in which the given functions X_0, Y_0, Z_0 are identically null is particularly interesting. One has simply the three equations on the right which, if $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ depend only on λ_1, λ_2, λ_3, and their derivatives, constitute three differentials equations that determine λ_1, λ_2, λ_3.

20. Case where W **is a function of** s_0, ξ, η, ζ, p, q, r **that depends on** ξ, η, ζ **only by the intermediary of** $\xi^2 + \eta^2 + \zeta^2$**, or, what amounts to the same thing, by the intermediary of** $\mu = \frac{ds}{ds_0} - 1$. - Consider the effort at a point of the deformed line and suppose that *for any deformation* it reduces to a tension effort. This amounts to saying that the function W of s_0, ξ, η, ζ, p, q, r verifies the identities:

$$\frac{\frac{\partial W}{\partial \xi}}{\xi} = \frac{\frac{\partial W}{\partial \eta}}{\eta} = \frac{\frac{\partial W}{\partial \varsigma}}{\varsigma},$$

i.e., they depend on ξ, η, ζ only by the intermediary of the quantity $\xi^2 + \eta^2 + \zeta^2$, or, what amounts to the same thing, the quantity $\mu = \frac{ds}{ds_0} - 1$.

Once again, we presently have:

$$\frac{F}{\frac{dx}{ds}} = \frac{G}{\frac{dy}{ds}} = \frac{H}{\frac{dz}{ds}},$$

and, upon introducing the common value $-T$ of these ratios, which is defined by the formula:

$$T = -\frac{\partial W}{\partial \mu},$$

we may give the system the following form:

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + X = 0, \qquad (1+\mu)\frac{d}{ds}\frac{\partial W}{\partial \frac{d\lambda_1}{ds_0}} - \frac{\partial W}{\partial \lambda_1} - \mathcal{L}_0 = 0,$$

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) + Y = 0, \qquad (1+\mu)\frac{d}{ds}\frac{\partial W}{\partial \frac{d\lambda_2}{ds_0}} - \frac{\partial W}{\partial \lambda_2} - \mathcal{M}_0 = 0,$$

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right)+Z=0, \qquad (1+\mu)\frac{d}{ds}\frac{\partial W}{\partial \frac{d\lambda_3}{ds_0}}-\frac{\partial W}{\partial \lambda_3}-\mathcal{N}_0=0,$$

by which x, y, z, λ_1, λ_2, λ_3, and s_0 are defined as functions of s (here, μ denotes $\frac{1}{\frac{ds}{ds_0}}-1$).

If we envision – to fix ideas – the case in which X, Y, Z are given functions of only the letters s, x, y, z then one sees that one may separately determine x, y, z, and the auxiliary T by means of the system of differential equations:

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+X=0, \quad \frac{d}{ds}\left(T\frac{dy}{ds}\right)+Y=0, \quad \frac{d}{ds}\left(T\frac{dz}{ds}\right)+Z=0,$$
$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1.$$

Once again, we recover the system that was presented in the context of LAGRANGE'S flexible and inextensible filament, and in the context of the flexible inextensible filament.

21. The deformable line that is obtained by supposing that Mx' is the tangent to (M) **at** M. – We may repeat what we said about the passage from the flexible inextensible filament of LAGRANGE to the flexible inextensible filament of rational mechanics in regard to the general case and that of *arbitrary particular deformations.* We shall consider the following case, which is important in the theory of the strength of materials, and will lead us later on to the deformable line as was studied by LORD KELVIN and TAIT, in particular, but only, as we have already observed, from the standpoint of infinitely small deformations ([43]).

We refer back to the deformable line of sec. **5**, and suppose that we have defined the external force, etc., as in sec. **9**. Now imagine that we direct our attention exclusively to the deformation (M) of (M_0), where the axis Mx' is tangent to the curve (M) at each point, and suppose, moreover, and in such a way that these deformations form a continuous sequence starting with (M_0), that the latter is constructed such that $M_0x'_0$ is a tangent to M_0. By a convenient choice of the sense in which one understands s_0 and s this amounts to supposing that one has:

$$(14) \qquad \begin{cases} \alpha_0=\dfrac{dx_0}{ds_0}, & \alpha'_0=\dfrac{dy_0}{ds_0}, & \alpha''_0=\dfrac{dz_0}{ds_0}, \\ \alpha=\dfrac{dx}{ds} & \alpha'=\dfrac{dy}{ds}, & \alpha''=\dfrac{dz}{ds}, \end{cases}$$

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[43] W. THOMSON and TAIT. – *Treatise on Natural Philosophy,* vol. I, Part II, 1883 edition, sec. **588** ff., pp. 130 ff.

or that:

$$(15) \qquad \eta = \eta_0 = 0, \qquad \zeta = \zeta_0 = 0, \qquad \xi_0 = 1, \qquad \xi = \frac{ds}{ds_0}.$$

The application of these definitions gives us definite expressions for the external force, etc.. We may say that the study of these expressions and the problems they lead to by the repetition of all that has been said constitutes the object of the study of the line that is subject to the conditions defined by formulas (14) and (15).

Limiting the deformations of (M_0) to those deformations (M) that verify conditions (14) or (15) or admitting the new conception of a line that is susceptible only to deformations that verify the preceding conditions are regarded as identical here from the standpoint of calculations that define elements such as external force, etc. This way of thinking is absolutely consistent with the principle called *solidification,* which is introduced by the authors in *the opposite order,* in a sense, as we have said.

Before considering the form that the formulas of sec. **9** take here, we establish several formulas that relate to the triad $Mx'y'z'$, either under particular conditions or as they presently present themselves. Suppose that we take the principal normal *Mn* and the binormal *Mb* to the curve (M) at M. If they, along with Mx', form a triad $Mx'nb$ with the same disposition as the triad $Mx'y'z'$ then we may designate the direction cosines of *Mn* and *Mb* with respect to the axes Mx', My', Mz', respectively, by 0, cos ω, sin ω, and 0, – sin ω, cos ω, which amounts to saying that we have, moreover:

$$(16) \qquad \begin{cases} \beta = \beta_1 \cos\omega - \gamma_1 \sin\omega, & \gamma = \beta_1 \sin\omega - \gamma_1 \cos\omega, \\ \beta' = \beta_1' \cos\omega - \gamma_1' \sin\omega & \gamma' = \beta_1' \sin\omega - \gamma_1' \cos\omega, \\ \beta'' = \beta_1'' \cos\omega - \gamma_1'' \sin\omega & \gamma'' = \beta_1'' \sin\omega - \gamma_1'' \cos\omega, \end{cases}$$

upon denoting the direction cosines of *Mn* with respect to the fixed axes *Ox*, *Oy*, *Oz* by $\beta_1, \beta_1', \beta_1''$, and those of *Mb* with respect to the same axes by $\gamma_1, \gamma_1', \gamma_1''$, and upon introducing an auxiliary variable ω as well, which is the angle My' makes with *Mn*, taken in a convenient sense.

We may then determine ω by means of the expressions that we already introduced. The principal normal is the tangent to the indicatrix of P. SERRET, considered to be the point whose coordinates are 1, 0, 0, with respect to this triad, for which the vertex O is fixed and the axes are parallel to those of $Mx'y'z'$. The projections of the displacement of this point onto the axes of the moving triad, or onto those of $Mx'y'z'$, are:

$$0, \qquad r\,ds_0, \quad -q\,ds_0,$$

and one has:

$$\frac{\cos\omega}{r} = -\frac{\sin\omega}{r}.$$

One may obtain more complete formulas upon replacing the cosines $\beta, \beta', \ldots, \gamma''$ in the formulas (2) of sec. **6** with their expression (16); they become:

$$p\frac{ds_0}{ds} = \sum \gamma \frac{d\beta}{ds} = \sum \gamma_1 \frac{d\beta_1}{ds} - \frac{d\omega}{ds},$$
$$q\frac{ds_0}{ds} = \sum \alpha \frac{d\gamma}{ds} = \cos\omega \sum \alpha \frac{d\gamma_1}{ds} + \sin\omega \sum \frac{d\beta_1}{ds},$$
$$r\frac{ds_0}{ds} = \sum \beta \frac{d\alpha}{ds} = \cos\omega \sum \beta_1 \frac{d\alpha}{ds} - \sin\omega \sum \frac{d\alpha}{ds},$$

i.e.,

$$(17)\qquad \begin{cases} p\dfrac{ds_0}{ds} = \dfrac{1}{\tau} - \dfrac{d\omega}{ds}, \\ r\dfrac{ds_0}{ds} = -\dfrac{\sin\omega}{\rho}, \\ r\dfrac{ds_0}{ds} = \dfrac{\cos\omega}{\rho}, \end{cases}$$

upon setting

$$\frac{1}{\rho} = \sum \beta_1 \frac{d\alpha}{ds}, \qquad \frac{1}{\tau} = \sum \gamma_1 \frac{d\beta_1}{ds},$$

and recalling that $\sum \alpha \frac{d\gamma_1}{ds} = 0$. The expressions $\frac{1}{\rho}$ and $\frac{1}{\tau}$ are equal in absolute value to the curvature and torsion (the *cambrure* of BARRÉ DE SAINT-VENANT and the *tortuosity* of THOMSON and TAIT) of the curve (M) at M; the latter two formulas (17) correspond to the remarks made by THOMSON and TAIT ([44]).

We arrive at the formulas of sec. **9**. For the moment, denote the function that W becomes when one takes conditions (15) into account by W_1, i.e., set:

$$W_1 = [W(s_0, \xi, \eta, \zeta, p, q, r)]_{\eta=0, \zeta=0} = W(s_0, \xi, 0, 0, p, q, r).$$

Furthermore, upon remarking that from formulas (14):

$$\xi = \frac{ds}{ds_0} = 1 + \mu,$$

we set:

$$W_1 = W(s_0, 1 + \mu, 0, 0, p, q, r).$$

We have

$$F' = \left[\frac{\partial W}{\partial \xi}\right]_{\eta=0,\varsigma=0} = \frac{\partial W_1}{\partial \xi} = \frac{\partial W_1}{\partial \mu}, \qquad G' = \left[\frac{\partial W}{\partial \eta}\right]_{\eta=0,\varsigma=0},$$

[44] W. THOMSON and TAIT. – *Treatise on Natural Philosophy,* vol. I, Part II, 1883 edition, sec. **590**, pp. 131.

$$H' = \left[\frac{\partial W}{\partial \varsigma}\right]_{\eta=0,\varsigma=0}, \qquad I' = \left[\frac{\partial W}{\partial p}\right]_{\eta=0,\varsigma=0} = \frac{\partial W_1}{\partial p},$$

$$J' = \left[\frac{\partial W}{\partial q}\right]_{\eta=0,\varsigma=0} = \frac{\partial W_1}{\partial q}, \qquad K' = \left[\frac{\partial W}{\partial r}\right]_{\eta=0,\varsigma=0} = \frac{\partial W_1}{\partial r},$$

If we would therefore like to introduce only the function W_1, i.e., the value taken by W at $\eta = \zeta = 0$, and if we suppose that one is not given the values that are taken by the derivatives $\frac{\partial W}{\partial \eta}, \frac{\partial W}{\partial \varsigma}$ for $\eta = \zeta = 0$ then we find ourselves in the presence of six expressions, where only four of them, F', G', J', K', may be considered as given, and two of them, G', H', are left to be determined ([45]). In other words, knowledge of W_1 uniquely entails knowledge of the tension effort F' and the moment of deformation (I', J', K').

If we introduce the expressions F, G, H, I, J, K then we may say that the first three are three auxiliaries, in regard to which, one knows simply that one has ([46]):

$$F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds} = \frac{\partial W}{\partial \mu}, \tag{18}$$

and the last three may be calculated by means of one of the systems:

$$\text{(19)}\quad \begin{cases} \alpha I + \alpha' J + \alpha'' K = \dfrac{\partial W}{\partial p}, \\ \beta I + \beta' J + \beta'' K = \dfrac{\partial W}{\partial q}, \\ \gamma I + \gamma' J + \gamma'' K = \dfrac{\partial W}{\partial r}, \end{cases} \qquad \text{(19}'\text{)}\quad \begin{cases} I = \alpha\dfrac{\partial W}{\partial p} + \beta\dfrac{\partial W}{\partial q} + \gamma\dfrac{\partial W}{\partial r}, \\ J = \alpha'\dfrac{\partial W}{\partial p} + \beta'\dfrac{\partial W}{\partial q} + \gamma'\dfrac{\partial W}{\partial r}, \\ K = \alpha''\dfrac{\partial W}{\partial p} + \beta''\dfrac{\partial W}{\partial q} + \gamma''\dfrac{\partial W}{\partial r}, \end{cases}$$

where $\alpha, \alpha', \alpha'', \ldots, \gamma''$ are defined by formulas (14) and (16).

The external force and moment result from them by the formulas of sec. **9** and **10**, in the measure where they may be determined when W_1 alone is given.

Suppose that one is presently given the external force and moment. The equations:

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[45]. If we admit that we know only the function W_1 then we may suppose that we ignore the existence of the function W that has served as our point of departure, since that function is, in a sense, *hidden, along with the positions of the triad* $Mx'y'z'$ *for which* Mx' *is not tangent to the curve* (M).

46. From now on, we denote the function W_1 of s_0, μ, p, q, r by W.

$$(20)\qquad \begin{cases} \dfrac{dF}{ds} - X = 0, & \dfrac{dI}{ds} + H\dfrac{dy}{ds} - G\dfrac{dz}{ds} - L = 0, \\ \dfrac{dG}{ds} - Y = 0, & \dfrac{dJ}{ds} + F\dfrac{dz}{ds} - H\dfrac{dx}{ds} - M = 0, \\ \dfrac{dH}{ds} - Z = 0, & \dfrac{dK}{ds} + G\dfrac{dx}{ds} - F\dfrac{dy}{ds} - N = 0, \end{cases}$$

combined with equations (18) and (19), and the relation:

$$(21)\qquad \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1,$$

provide a system of eight differential equations in five of these variables (as functions of the sixth) and of F, G, H when X, Y, Z, L, M, N or X_0, Y_0, Z_0, L_0, M_0, N_0 are given functions of s_0, s, x, y, z, ω, and the derivatives of these variables with respect to each other.

If s does not figure explicitly in the given functions then one may use (21) to eliminate ds and, upon taking s_0, for example, to be the independent variable one will have a system of seven differential equations that define the seven unknowns x, y, z, ω, and F, G, H.

In the case at hand, where the function W that we started with is hidden, the expressions F, G, H are simple auxiliary functions that are defined by the differential equations of which we speak; we may propose to eliminate them. However, that elimination is easy, since they figure linearly and their derivatives are excluded from relation (18) and the three relations on the right-hand side of (20); these four relations give:

$$(22)\qquad \begin{cases} F = -T\dfrac{dx}{ds} + \left(\dfrac{dK}{ds} - N\right)\dfrac{dy}{ds} - \left(\dfrac{dJ}{ds} - M\right)\dfrac{dz}{ds}, \\ G = -T\dfrac{dy}{ds} + \left(\dfrac{dI}{ds} - L\right)\dfrac{dz}{ds} - \left(\dfrac{dK}{ds} - N\right)\dfrac{dx}{ds}, \\ H = -T\dfrac{dz}{ds} + \left(\dfrac{dJ}{ds} - M\right)\dfrac{dx}{ds} - \left(\dfrac{dI}{ds} - L\right)\dfrac{dy}{ds}, \\ \left(\dfrac{dI}{ds} - L\right)\dfrac{dx}{ds} + \left(\dfrac{dJ}{ds} - M\right)\dfrac{dy}{ds} + \left(\dfrac{dK}{ds} - N\right)\dfrac{dz}{ds} = 0. \end{cases}$$

To abbreviate the notation, we set:

$$(23)\qquad T = -\frac{\partial W}{\partial \mu},$$

from which, by elimination of F, G, H we obtain the system of four equations:

(24) $$\begin{cases} \dfrac{d}{ds}\left[-T\dfrac{dx}{ds}+\left(\dfrac{dK}{ds}-N\right)\dfrac{dy}{ds}-\left(\dfrac{dJ}{ds}-M\right)\dfrac{dz}{ds}\right]-X=0, \\ \dfrac{d}{ds}\left[-T\dfrac{dy}{ds}+\left(\dfrac{dI}{ds}-L\right)\dfrac{dz}{ds}-\left(\dfrac{dK}{ds}-N\right)\dfrac{dx}{ds}\right]-Y=0, \\ \dfrac{d}{ds}\left[-T\dfrac{dz}{ds}+\left(\dfrac{dJ}{ds}-M\right)\dfrac{dx}{ds}-\left(\dfrac{dI}{ds}-L\right)\dfrac{dy}{ds}\right]-Z=0, \end{cases}$$

(25) $$\left(\frac{dI}{ds}-L\right)\frac{dx}{ds}+\left(\frac{dJ}{ds}-M\right)\frac{dy}{ds}+\left(\frac{dK}{ds}-N\right)\frac{dz}{ds}=0,$$

in which we have replaced I, J, K, T with their values from (19′) and (23), and which, with (21), form a system of five differential equations that relate five of the variables s_0, s, x, y, z, ω, to the remaining one. If s does not figure in the given variables explicitly then one may use (21) to eliminate ds, and relations (24) and (25) provide four differential equations that define x, y, z, ω as functions of s_0.

22. Reduction of the system of the preceding section to a form that one may deduce from the calculus of variations. – In the preceding section, we finally found a function W which, by the intermediary of μ, p, q, r, depends upon $\omega, \frac{d\omega}{ds}, \frac{dx}{ds_0}, \ldots, \frac{d^3x}{ds_0^3}$, as well as on s_0.

Observe that upon taking these latter arguments into account, equation (25) may be written:

$$\frac{d}{ds_0}\left(\frac{\partial W}{\partial \frac{d\omega}{ds_0}}\right)-\frac{\partial W}{\partial\omega}+\left(L_0\frac{dx}{ds}+M_0\frac{dy}{ds}+N_0\frac{dz}{ds}\right)=0.$$

We examine whether successively combining each of equations (24) and (25) will give three equations that are susceptible to being deduced from the calculus of variations directly, i.e., equations such as the following:

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \frac{d^3x}{ds_0^3}}-\frac{d^2}{ds_0^2}\frac{\partial W}{\partial \frac{d^2x}{ds_0^2}}+\frac{d}{ds_0}\frac{\partial W}{\partial \frac{dx}{ds_0}}-X_0+\cdots=0,$$

where the terms not written depend only upon the external moments.

If we remark that the equations considered refer to derivatives that are of order at most five then one sees that one must seek to introduce the third derivatives of equations (25), which may be written:

$$\frac{d}{ds_0}\left(\frac{\partial W}{\partial p}\right)+q\frac{\partial W}{\partial r}-r\frac{\partial W}{\partial q}-\left(L_0\frac{dx}{ds}+M_0\frac{dy}{ds}+N_0\frac{dz}{ds}\right)=0,$$

or

$$V=\frac{d}{ds_0}\left(\frac{\partial W}{\partial p}\right)+q\frac{\partial W}{\partial r}-r\frac{\partial W}{\partial q}-L_0'=0,$$

with the notation of sec. **9**.

Consider the first equation of (24); it is written:

$$\frac{d}{ds_0}\left[-T\alpha+\frac{ds_0}{ds}\left(\frac{dK}{ds_0}\right)\alpha'-\frac{ds_0}{ds}\left(\frac{dJ}{ds_0}-M_0\right)\alpha''\right]-X_0=0,$$

i.e.,

$$U=\frac{d}{ds_0}\left[-T\alpha+\frac{ds_0}{ds}\left(\gamma\frac{d}{ds_0}\frac{\partial W}{\partial q}-\beta\frac{d}{ds_0}\frac{\partial W}{\partial r}+\frac{\gamma_1}{\rho}\frac{d}{ds_0}\frac{\partial W}{\partial p}\,p\beta\frac{d}{ds_0}\frac{\partial W}{\partial q}-p\gamma\frac{\partial W}{\partial r}\right)\right.$$
$$\left.-\frac{ds_0}{ds}(\alpha' N_0-\alpha'' M_0)\right]-X_0=0.$$

Upon forming the first term $\frac{d^3}{ds_0^3}\frac{\partial W}{\partial\frac{d^3x}{ds_0^3}}+\cdots$ one easily confirms, by a calculation whose details will not be given here, that the combination:

$$U_1+\frac{d^2}{ds_0^2}\left\{\frac{\gamma_1\rho}{\left(\frac{ds}{ds_0}\right)^2}V\right\}+\frac{d}{ds_0}\left\{\frac{\gamma_1\rho\frac{d^2s}{ds_0^2}}{\left(\frac{ds}{ds_0}\right)^3}V\right\}$$

reproduces the different terms of the expression in question, as well as those that go to zero with the external forces.

If we set:

$$\mathcal{X}_0=X_0+\frac{d^2}{ds_0^2}\left\{\frac{\gamma_1\rho}{\left(\frac{ds}{ds_0}\right)^2}L_0'\right\}+\frac{d}{ds_0}\left\{\frac{\gamma_1\rho\frac{d^2s}{ds_0^2}}{\left(\frac{ds}{ds_0}\right)^3}L_0'\right\}+\frac{d}{ds_0}\left[\frac{ds_0}{ds}(\alpha' N_0-\alpha'' M_0)\right],$$

and if we designate the analogous expressions that are obtained by replacing X_0, γ_1 with Y_0, γ_1', and then Z_0, γ_1'', respectively, and then making the required permutations in the last term by $\mathcal{Y}_0$, $\mathcal{Z}_0$, we obtain the system in the following form:

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \dfrac{d^3x}{ds_0^3}}-\frac{d^2}{ds_0^2}\frac{\partial W}{\partial \dfrac{d^2x}{ds_0^2}}+\frac{d}{ds_0}\frac{\partial W}{\partial \dfrac{dx}{ds_0}}-\mathcal{X}_0=0,$$

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \dfrac{d^3y}{ds_0^3}}-\frac{d^2}{ds_0^2}\frac{\partial W}{\partial \dfrac{d^2y}{ds_0^2}}+\frac{d}{ds_0}\frac{\partial W}{\partial \dfrac{dy}{ds_0}}-\mathcal{Y}_0=0,$$

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \dfrac{d^3z}{ds_0^3}}-\frac{d^2}{ds_0^2}\frac{\partial W}{\partial \dfrac{d^2z}{ds_0^2}}+\frac{d}{ds_0}\frac{\partial W}{\partial \dfrac{dz}{ds_0}}-\mathcal{Z}_0=0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \dfrac{d\omega}{ds_0}}-\frac{\partial W}{\partial \omega}+\left(L_0\frac{dx}{ds}+M_0\frac{d\gamma}{ds}+N_0\frac{dz}{ds}\right)=0,$$

which one may summarize in the formula:

$$\int_{s_0^0}^{s_0'}(\delta W+\mathcal{X}_0\delta x+\mathcal{Y}_0\delta y+\mathcal{Z}_0\delta z-L_0'\delta\omega)ds_0=0,$$

where one considers only the terms that ultimately present themselves under the integral sign (47).

This summarized form to which one is led, and which must be treated according to the rules of the calculus of variations, is particularly convenient for the purpose of effecting changes of variables.

Upon supposing that X_0, Y_0, Z_0, L_0' are of a particular form, one will have the equations for the extremals of a problem of the calculus of variations.

If we consider the case in which U denotes a function of x, y, z, $\alpha=\frac{1}{\xi}\frac{dx}{ds_0}$, $\alpha'=\frac{1}{\xi}\frac{dy}{ds_0}$, $\alpha''=\frac{1}{\xi}\frac{dz}{ds_0}$ then we have:

$$X_0=\frac{\partial U}{\partial x},\qquad Y_0=\frac{\partial U}{\partial y},\qquad Z_0=\frac{\partial U}{\partial z},$$

$$\frac{1}{\xi}\left(M_0\frac{dz}{ds}-N_0\frac{dy}{ds}\right)=\frac{\partial U}{\partial \dfrac{dx}{ds_0}}=\frac{1}{\xi}\left[\frac{\partial U}{\partial \alpha}-\frac{\alpha}{\xi}\left(\frac{\partial U}{\partial \alpha}\frac{dx}{ds_0}+\frac{\partial U}{\partial \alpha'}\frac{dy}{ds_0}+\frac{\partial U}{\partial \alpha''}\frac{dz}{ds_0}\right)\right],$$

[47] One has a form $\int_{t_0}^{t_1}(\delta T+U')dt=0$ for HAMILTON'S principle that is analogous to the one that was given by TISSERAND, pp. 4 of Tome I of his *Traité de Mécanique céleste*.

$$\frac{1}{\xi}\left(N_0\frac{dx}{ds} - L_0\frac{dx}{ds}\right) = \frac{\partial U}{\partial\dfrac{dy}{ds_0}} = \frac{1}{\xi}\left[\frac{\partial U}{\partial\alpha'} - \frac{\alpha'}{\xi}\left(\frac{\partial U}{\partial\alpha}\frac{dx}{ds_0} + \frac{\partial U}{\partial\alpha'}\frac{dy}{ds_0} + \frac{\partial U}{\partial\alpha''}\frac{dz}{ds_0}\right)\right],$$

$$\frac{1}{\xi}\left(L_0\frac{dy}{ds} - M_0\frac{dx}{ds}\right) = \frac{\partial U}{\partial\dfrac{dz}{ds_0}} = \frac{1}{\xi}\left[\frac{\partial U}{\partial\alpha''} - \frac{\alpha''}{\xi}\left(\frac{\partial U}{\partial\alpha}\frac{dx}{ds_0} + \frac{\partial U}{\partial\alpha'}\frac{dy}{ds_0} + \frac{\partial U}{\partial\alpha''}\frac{dz}{ds_0}\right)\right],$$

$$L_0\frac{dx}{ds_0} + M_0\frac{dy}{ds_0} + N_0\frac{dz}{ds_0} = 0,$$

or, what amounts to the same thing:

$$X_0 = \frac{\partial U}{\partial x}, \qquad Y_0 = \frac{\partial U}{\partial y}, \qquad Z_0 = \frac{\partial U}{\partial z},$$

$$L_0 = \left(\alpha'\frac{\partial U}{\partial\alpha''} - \alpha''\frac{\partial U}{\partial\alpha'}\right), \qquad M_0 = \left(\alpha''\frac{\partial U}{\partial\alpha} - \alpha\frac{\partial U}{\partial\alpha''}\right), \qquad N_0 = \left(\alpha\frac{\partial U}{\partial\alpha'} - \alpha'\frac{\partial U}{\partial\alpha}\right).$$

One then has:

$$\mathcal{X}_0 = \frac{\partial U}{\partial x} - \frac{d}{ds_0}\frac{\partial U}{\partial\dfrac{dx}{ds_0}}, \quad \mathcal{Y}_0 = \frac{\partial U}{\partial y} - \frac{d}{ds_0}\frac{\partial U}{\partial\dfrac{dy}{ds_0}}, \quad \mathcal{Z}_0 = \frac{\partial U}{\partial z} - \frac{d}{ds_0}\frac{\partial U}{\partial\dfrac{dz}{ds_0}}$$

as the extremal equations relative to the integral:

$$\int (W + U)\,ds_0.$$

Another particular case, which one may combine with the preceding, is the one in which W is of the form $Bp + \varphi(q^2 + r^2, \xi)$, where B is a constant. W may then be written:

$$Bp + \psi(s_0, \xi, \rho).$$

If one supposes, in addition, that $L_0' = 0$ then the four equations reduce to three, since the fourth equation reduces to an identity.

The case that we will now examine comprises, in particular, the one in which W is of the form,

$$A\frac{1}{\rho^2} + C,$$

with A and B constant. This amounts to the case considered by D. BERNOULLI, and later by EULER; it is the case that inspired SOPHIE GERMAIN and POISSON in their researches on elastic surfaces.

23. The inextensible deformable line where Mx' is the tangent to (M) at M. – Instead of simply supposing, as in the preceding case, that one has introduced conditions (14) and (15), we may suppose, in addition, that the line is inextensible, which, by virtue of (14), amounts to adjoining:

$$\xi = 1.$$

If we admit that one knows only the value of the function $S(s_0, \xi, \eta, \zeta, p, q, r)$ for $\xi = 1$, $\eta = 0$, $\zeta = 0$, or then again, starting with the line of the preceding section, to which we adjoin the condition $\mu = 0$, that we know simply the value of the function W_1 for $\mu = 0$ then we see that all three of F, G, H become indeterminate and we presently have either equations (20), where I, J, K are replaced by the values (19′), in which W denotes $W(s_0, 1, 0, 0, p, q, r)$ or $(W_1)_{\mu=0}$, and which form, with relation (21), a system of seven differential equations that define the unknowns x, y, z, F, G, H as functions of $s = s_0$, or equations (24) and (25), where I, J, K are replaced by the same values (19′), and which, with relation (21), a system of five differential equations that define the unknowns x, y, z, ω, T as functions of $s = s_0$.

However, the system so obtained coincides with the one that was introduced by THOMSON and TAIT ([48]), upon supposing that $W(s_0, 1, 0, 0, p, q, r)$ is obtained by the substitution of the values of p_0, q_0, r_0 as functions of s_0 into a quadratic form (with constant coefficients) in the expressions $p - p_0, q - q_0, r - r_0$. This is what we will arrive at if we suppose, for example, that the expression W_1 at the beginning of the preceding section is obtained by substituting the values of p_0, q_0, r_0 as functions of s_0 for these variables in a quadratic form in $p(1 + \mu) - p_0, q(1 + \mu) - q_0, r(1 + \mu) - r_0$.

Observe, in addition, that in the applications made by THOMSON and TAIT of the considerations in their sec. **614**, namely, for example, the application made in sec. **616**, they put themselves in the case of an infinitely small deformation; we therefore recover, in a completely natural way, the applications mentioned by starting with the function W in general and considering infinitely small deformations.

Here we may develop considerations that are analogous to the ones relating to the preceding line; the only difference is that one adjoins:

$$\left(\frac{dx}{ds_0}\right)^2 + \left(\frac{dy}{ds_0}\right)^2 + \left(\frac{dz}{ds_0}\right)^2 = 1.$$

One presently arrives at the formula:

$$\int_{s_0^0}^{s_0'} (\delta W + \mathcal{X}_0 \delta x + \mathcal{Y}_0 \delta y + \mathcal{Z}_0 \delta z - L_0' \delta\omega) ds_0 = 0,$$

which must happen *by virtue of the fact that:*

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[48] THOMSON and TAIT. – *Treatise on Natural Philosophy,* Vol. I, Part. II, sec. **614**, pp. 152-155.

$$\left(\frac{dx}{ds_0}\right)^2+\left(\frac{dy}{ds_0}\right)^2+\left(\frac{dz}{ds_0}\right)^2=1,$$

and where $\mathcal{X}_0$, $\mathcal{Y}_0$, $\mathcal{Z}_0$ have a significance that we shall describe.

Indeed, the equilibrium system of equations is equivalent to the following:

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \frac{d^3x}{ds_0^3}}-\frac{d^2}{ds_0^2}\frac{\partial W}{\partial \frac{d^2x}{ds_0^2}}+\frac{d}{ds_0}\left[\frac{\partial W}{\partial \frac{dx}{ds_0}}-T\frac{dx}{ds_0}\right]-\mathcal{X}_0=0,$$

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \frac{d^3y}{ds_0^3}}-\frac{d^2}{ds_0^2}\frac{\partial W}{\partial \frac{d^2y}{ds_0^2}}+\frac{d}{ds_0}\left[\frac{\partial W}{\partial \frac{dy}{ds_0}}-T\frac{dy}{ds_0}\right]-\mathcal{Y}_0=0,$$

$$\frac{d^3}{ds_0^3}\frac{\partial W}{\partial \frac{d^3z}{ds_0^3}}-\frac{d^2}{ds_0^2}\frac{\partial W}{\partial \frac{d^2z}{ds_0^2}}+\frac{d}{ds_0}\left[\frac{\partial W}{\partial \frac{dz}{ds_0}}-T\frac{dz}{ds_0}\right]-\mathcal{Z}_0=0,$$

$$\frac{d}{ds_0}\frac{\partial W}{\partial \frac{d\omega}{ds_0}}-\frac{\partial W}{\partial \omega}+L_0\frac{dx}{ds}+M_0\frac{dy}{ds}+N_0\frac{dz}{ds}=0.$$

where one must set:

$$\mathcal{X}_0=X_0+\frac{d^2}{ds_0^2}(\gamma_1\rho L_0')+\frac{d}{ds_0}(\alpha' N_0-\alpha'' M_0)$$

$$\mathcal{Y}_0=Y_0+\frac{d^2}{ds_0^2}(\gamma_1'\rho L_0')+\frac{d}{ds_0}(\alpha'' N_0-\alpha M_0),$$

$$\mathcal{Z}_0=Z_0+\frac{d^2}{ds_0^2}(\gamma_1''\rho L_0')+\frac{d}{ds_0}(\alpha N_0-\alpha' M_0).$$

24. Case where the external forces and moments are null; particular form of *W* that leads to the equations treated by Binet and Wantzel. – Instead of using equations (24) and (25), it may be more convenient to recall the equations we began with; it may also be useful to appeal to the geometric interpretation.

For example, suppose that X_0, Y_0, Z_0 are null. One concludes from this that *F*, *G*, *H* are constants equal to the values $F_{A_0}, G_{A_0}, H_{A_0}$ that they take at the one of the extremities A_0, and one has three equations:

$$\frac{dI}{ds_0} + H_{A_0}\frac{dy}{ds_0} - G_{A_0}\frac{dz}{ds_0} - L_0 = 0,$$
$$\frac{dJ}{ds_0} + F_{A_0}\frac{dz}{ds_0} - H_{A_0}\frac{dx}{ds_0} - M_0 = 0,$$
$$\frac{dK}{ds_0} + G_{A_0}\frac{dx}{ds_0} - F_{A_0}\frac{dy}{ds_0} - N_0 = 0,$$

which are the primitive equations and *actually* result from the elimination of T from (24) and (25).

If one has, in addition, that L_0, M_0, N_0 are null i.e., if the deformed (M) is subjected only to forces applied at its extremities, then we have:

$$I + H_{A_0} y - G_{A_0} z = const.,$$
$$J + F_{A_0} z - H_{A_0} x = const.,$$
$$K + G_{A_0} x - F_{A_0} y = const.,$$

relations that one also obtains from the geometric interpretation of the equations by means of formulas such as ([49]):

$$I_{M_0} - H_{M_0} y_M - G_{M_0} z_M = I_{A_0} + H_{A_0} y_A - G_{A_0} z_A - \int_{A_0}^{M_0} (Y_0 z - Z_0 y - L_0) ds_0.$$

Having made these remarks, consider the case where the function W of s_0, p, q, r is of the form ([50]):

$$\tfrac{1}{2} A(q^2 + r^2) + Bp + C,$$

where A, B, C are constants. One will have:

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[49] One will observe that the reasoning of BERTRAND *(Sur l'équilibre d'une ligne élastique*, Note III of the *Mécanique analytique* of LAGRANGE, pp. 460-464 of Tome XI of Oeuvres de LAGRANGE) amounts to the use of these formulas, or, more precisely, to equivalent ones such as:

$$I_{M_0} - I_{A_0} = G_{A_0} z_M - H_{A_0} y_M - G_{A_0} z_A + H_{A_0} y_A$$
$$- \left[\int_{A_0}^{M_0} (Y_0 z - Z_0 y - L_0) ds_0 + y_0 \int_{A_0}^{M_0} Z_0 ds_0 - z_0 \int_{A_0}^{M_0} Y_0 ds_0 \right];$$

it suffices to refer to sec. **9**, where we said that the effort and the moment of deformation at A_0 are $(F'_{A_0}, G'_{A_0}, H'_{A_0})$, $(I'_{A_0}, J'_{A_0}, K'_{A_0})$, i.e., the values of (F', G', H'), (I', J', K') at A_0.

[50] If W is obtained by replacing p_0, q_0, r_0 with their values as a function of $p - p_0$, $q - q_0$, $r - r_0$ then we suppose that $p_0 = q_0 = r_0 = 0$, in such a way that $(q_0)^2 + (r_0)^2 = 0$, and the curve (M_0) is a straight line.

$$I' = B, \qquad J' = Aq, \qquad K' = Ar;$$

the vector (I', J', K') or (I, J, K) is the resultant of a constant vector equal to B that is directed along the tangent Mx' and a vector that is directed along the binormal and has the same absolute value as $\frac{A}{\rho}$. The three equations:

$$I + H_{A_0} y - G_{A_0} z = const., \qquad J + F_{A_0} z - H_{A_0} x = const., \qquad K + G_{A_0} x - F_{A_0} y = const.,$$

are, up to notations, identical with the equations:

$$p\frac{dyd^2z - dzd^2y}{ds^3} = \theta\frac{dx}{ds} + cy - bz + a_1,$$
$$p\frac{dzd^2x - dxd^2z}{ds^3} = \theta\frac{dy}{ds} + az - cx + b_1,$$
$$p\frac{dxd^2y - dyd^2x}{ds^3} = \theta\frac{dz}{ds} + bx - ay + c_1,$$

that were considered by BINET ([51]), WANTZEL ([52]), HERMITE ([53]), in which p, θ, a, b, c, a_1, b_1, c_1 are constants.

In the previously cited note, which placed us in the realm of the analytical mechanics of LAGRANGE, and where we were said to have imitated a method discussed by POISSON in the article that was mentioned in sec. **10**, and recalled in the following section, J. BERTRAND has treated, after WANTZEL, the case where the three equations:

$$cy - bz + a_1 = 0, \qquad az - cx + b_1 = 0, \qquad bx - ay + c_1 = 0,$$

represent a straight line; if this straight line is identified by:

$$H_A(y - y_A) - G_A(z - z_A) = I_A,$$
$$F_A(z - z_A) - H_A(x - x_A) = J_A,$$
$$G_A(x - x_A) - F_A(y - y_A) = K_A$$

then the preceding hypothesis amounts to:

$$F_A I_A + G_A J_A + H_A K_A = 0,$$

[51] J. BINET. – *Mémoire sur l'intégration des équations de la courbe élastique B double courbure* (Extract), *C.R.*, **18**, pp. 1115-1119, 17 June 1844. *Réflexions sur l'intégration des formulas de la tige élastique B double courbure*, *C.R.*, **19**, pp. 1-3, 1st July 1844.

[52] WANTZEL. – *Note sur l'intégration des équations de la courbe élastique B double courbure*, *C.R.*, **18**, pp. 1197-1201, 24 June 1844.

[53] Ch. HERMITE. – *Sur quelques applications des functions elliptiques*, *C.R.*, **90**, pp. 478, 8 March 1880; see also the work of that title that appeared in 1885 (see sec. **35**).

and this amounts to supposing that the couple (I_A, J_A, K_A) and the force (F_A, G_A, H_A) reduce to a unique force.

From relation (2) on page 463 of LAGRANGE, this line, when it is of issue, does not encounter the curve (*M*); this remark was made by J. BERTRAND in the case where he defined it. What might appear strange is that a hypothesis is preserved at the top of page 462 that, from the note on page 463, entails the relation $\theta = 0$.

Upon supposing that the constant θ of BINET is null, i.e., with our notations, upon making $B = 0$, one has the particular curve considered by LAGRANGE.

Observe that in the present case the unknown that we have denoted by ω does not appear in the equations; however, the three equations:

$$\frac{dI}{ds_0} + H\frac{dy}{ds_0} - G\frac{dz}{ds_0} = 0,$$
$$\frac{dJ}{ds_0} + F\frac{dz}{ds_0} - H\frac{dx}{ds_0} = 0,$$
$$\frac{dK}{ds_0} + G\frac{dx}{ds_0} - F\frac{dy}{ds_0} = 0,$$

reduce to two because upon multiplying them by $\frac{dx}{ds_0}, \frac{dy}{ds_0}, \frac{dz}{ds_0}$ and adding them one gets zero for the particular form of *I*, *J*, *K* that was considered in the last example.

We recover the preceding line in the following section; this leads us to remark that one may present the following as it is.

We seek the case in which the effort of deformation of the line in the preceding section is perpendicular to the principal normal.

We have the condition:

$$r\frac{\partial W}{\partial q} - q\frac{\partial W}{\partial r} = 0.$$

If we suppose that this condition results from the nature of the line, i.e., from the form of its *W*, then this condition is a partial differential equation that is verified by *W*, from which *W* must depend on *q* and *r* only by the intermediary of $q^2 + r^2$. If this condition is verified then, from the remark of POISSON that we recalled in sec. **10**, the equations of the problem entail that

$$I' = const.$$

If we suppose that this conclusion results from the nature of the line, i.e., the form of its *W*, then this amounts to the condition:

$$\frac{\partial W}{\partial p} = B,$$

where *B* is a constant, and we find

$$W = Bp + \varphi,$$

where φ is a function of $q^2 + r^2 = \frac{1}{\rho^2}$; upon supposing that φ is of first degree in $q^2 + r^2$ we recover the W that served as the point of departure for this section.

25. The deformable line for which the plane $Mx'y'$ is the osculating plane of (M) **at** M; **the case in which the line is inextensible, in addition; the line considered by Lagrange and its generalization due to Binet and studied by Poisson.** – We may proceed further with the hypotheses that were made for the deformations of a deformable line. Instead of assuming simply that Mx' is tangent to the curve (M), we may suppose that the plane $Mx'y'$ is the osculating plane to the curve (M).

1. First, leave aside the hypothesis of inextensibility. Assume that one still has relations (14) or (15), and, in addition:

$$q = q_0 = 0.$$

If, for the moment, we let W_2 denote the function that is obtained by setting $\eta = \zeta = q = 0$ in W, or $q = 0$ in W_1 then we have:

$$F' = \frac{\partial W_2}{\partial \mu}, \qquad I' = \frac{\partial W_2}{\partial p}, \qquad K' = \frac{\partial W_2}{\partial r}.$$

As for G', H', J', they may be calculated if W_2 is the only given, and may be considered as three auxiliary variables that are defined by the equations.

In the present case, equations (20) are combined with relations (18), (21), and the following:

$$(26) \qquad \begin{cases} I = \alpha \dfrac{\partial W}{\partial p} + \beta J' + \gamma \dfrac{\partial W}{\partial r}, \\ J = \alpha' \dfrac{\partial W}{\partial p} + \beta' J' + \gamma' \dfrac{\partial W}{\partial r}, \\ K = \alpha'' \dfrac{\partial W}{\partial p} + \beta'' J' + \gamma'' \dfrac{\partial W}{\partial r}, \end{cases}$$

in which W designates the expression W_2 takes when X, Y, Z, L, M, N or $X_0, Y_0, Z_0, L_0, M_0, N_0$ are given functions of s_0, s, x, y, z, and their derivatives with respect to one of them – a system of eight differential equations in four of these variables (as a function of the fifth) and F, G, H, J'.

As in the preceding, we may eliminate F, G, H, and what remains are the four equations (24) and (25), in which we have replaced I, J, K, T with their values from (26) and (23), and which, with (21), form a system of five differential equations that relate five of the variables s, s_0, x, y, z, J' to the other one.

2. In addition, introduce inextensibility by the relations:

$$\xi = \xi_0 = 1.$$

Continue to designate the function $W(s_0, 1, 0, 0, p, 0, r)$ by W and suppose that this function alone is continuous. We simply have the relations:

$$I' = \frac{\partial W}{\partial p}, \qquad K' = \frac{\partial W}{\partial r}.$$

As a result, if X, Y, Z, L, M, N or X_0, Y_0, Z_0, L_0, M_0, N_0 are given functions of s_0, s, x, y, z, and their derivatives with respect to one of them then we have the seven equations (20) and (21), where I, J, K are replaced by their values from (26), and which determines the seven unknowns x, y, z, F, G, H, J' as functions of s_0, for example. Upon eliminating F, G, H, we have the four equations (24) and (25) that define the four unknowns x, y, z, J' as functions of s_0.

It is easy to deduce the cases that were envisioned by LAGRANGE, BINET, and POISSON from the case we shall now consider.

Suppose that the given functions L, M, N are null; the three right-hand equations of (20) form a system that is equivalent to the following:

$$\begin{aligned} \frac{dI'}{ds} - rJ' &= 0, \\ \frac{dJ'}{ds} + rI' - pK' - H' &= 0, \\ \frac{dK'}{ds} + pJ' + G' &= 0, \end{aligned}$$

which the system of sec. **10** reduces to; just the same, one or two of these three equations may replace one or two of the equations on the right-hand side of (20), in general.

In particular, the relation:

$$\frac{dI'}{ds_0} - rJ' = 0 \tag{27}$$

that is obtained by adding the three equations on the right-hand side of (20), after multiplying them by $\alpha = \frac{dx}{ds}, \alpha' = \frac{dy}{ds}, \alpha'' = \frac{dz}{ds}$, may be substituted for any one of the aforementioned right-hand equations of (20), in general.

Having said this, suppose first that the function W of s_0, p, r that presently figures in relations (26) does not depend on p. We will have $I' = 0$, and relation (27) will give $J = 0$ upon supposing that $r \neq 0$. Hence, in the present case, the moment of deformation is directed along the binormal to the curve (M). In equations (20), we have replaced I, J, K by the values:

$$I = \gamma \frac{\partial W}{\partial r}, \qquad J = \gamma' \frac{\partial W}{\partial r}, \qquad K'' = \gamma'' \frac{\partial W}{\partial r}.$$

The three right-hand equations of (20) reduce to two.

We thus obtain the case envisioned by LAGRANGE in no. 46 and the following ones of sec. III, chapter III, first part, section V, of his *Mécanique analytique* (pp. 162, et seq. of Tome I of the first edition).

It might be useful to show the identity with the exposition of LAGRANGE. We may suppose:

$$I = J_1(dy\, d^2z - dz\, d^2y),$$
$$J = J_1(dz\, d^2x - dx\, d^2z),$$
$$K = J_1(dx\, d^2y - dy\, d^2x),$$

since the vector I, J, K is perpendicular to the osculating plane of (M).

The right-hand equations of (20), which may really be written ($L = M = N = 0$):

$$dy\, d(J_1\, d^2z) - dz\, d(J_1\, d^2y) = - H\, dy + G\, dz,$$
$$dz\, d(J_1\, d^2x) - dx\, d(J_1\, d^2z) = - F\, dz + G\, dx,$$
$$dx\, d(J_1\, d^2y) - dy\, d(J_1\, d^2x) = - G\, dx + F\, dy,$$

or

$$\frac{d(J_1 d^2 x) + F}{dx} = \frac{d(J_1 d^2 y) + G}{dy} = \frac{d(J_1 d^2 z) + H}{dz},$$

which permits us to set:

$$F = \lambda \frac{dx}{ds} - d(J_1 d^2 x),$$
$$G = \lambda \frac{dy}{ds} - d(J_1 d^2 y),$$
$$H = \lambda \frac{dz}{ds} - d(J_1 d^2 z),$$

after introducing an auxiliary variable λ.

If we transport these values into the three left-hand equations of (20) then we recover the equations that were given by LAGRANGE at the beginning of his no. 48:

$$X ds - d\,\frac{\lambda dx}{ds} + d^2(J_1 d^2 x) = 0,$$
$$Y ds - d\,\frac{\lambda dy}{ds} + d^2(J_1 d^2 y) = 0,$$
$$Z ds - d\,\frac{\lambda dz}{ds} + d^2(J_1 d^2 z) = 0.$$

In the preceding theory presented by LAGRANGE the moment of deformation is normal to the osculating plane. BINET ([54]) has proposed to consider the case where this

[54] J. BINET. – *Mémoire sur l'expression analytique de l'élasticité et de la raideur des courbes B double courbure (Bull. De la Soc. Philomatique,* 1814, pp. 159-160; *Journ. de l'Ec. Polyt.,* , Note 17, T. X, pp. 418-456, 1815).

moment of deformation is simply perpendicular to the principle normal. On the other hand, BINET supposed that the line elements were subject to external forces in a way that we shall also do in the case where $L = M = N = 0$. From (27), the hypothesis $J' = 0$ that was made by BINET entails that

$$I' = const.$$

This result, as we pointed out in sec. **10**, in the general form that is independent of W, and which is due to POISSON ([55]), may come about either because of the specification of the forces or the specification of W.

If we assume the latter case, we have:

$$W = \varphi(s_0, r) + mp,$$

where m is a constant; as a result:

$$I' = m, \qquad K' = \frac{\partial \varphi}{\partial r}.$$

With this hypothesis, one sees that if $r \neq 0$ then condition (27) amounts to saying that the unknown J' is equal to zero, and, as a result, one has to replace I, J, K in equations (20) with their values:

$$I = \alpha m + \gamma \frac{\partial \varphi}{\partial r},$$

$$J = \alpha' m + \gamma' \frac{\partial \varphi}{\partial r},$$

$$K = \alpha'' m + \gamma'' \frac{\partial \varphi}{\partial r},$$

and the three right-hand equations of (20) reduce to two. In particular, if $\frac{\partial \varphi}{\partial r}$ is derived from an expression of the form $n(r - r_0)$, where n is constant, and if one replaces r_0 as a function of s_0 then one has the hypothesis that was explicitly made by BINET and POISSON. Upon supposing, in addition, that the curve (M_0) is a straight line and that the external forces are null, in such a way that the transformation of (M_0) into (M) comes about only from forces and moments applied to the extremities, one recovers the problem treated by BINET and WANTZEL, upon which we previously stopped.

Upon supposing that $m = 0$ in all of what we proceed to discuss we revert to the case of LAGRANGE.

26. The rectilinear deformations of a deformable line. – If we suppose that (M_0) is a straight line then we must direct our attention to the deformations (M) that are likewise

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[55] POISSON. – *Sur les lignes élastiques B double courbure, Correspondance sur l'Ecole Polytechnique,* T. III, no. 3, pp. 355-360, January, 1816. This work may be considered as destined to complete what preceded it, which was due to BINET.

straight lines such that, in addition, the axis Mx' is directed along the line (M) and $M_0 x_0'$ is directed along (M_0).

1. If one first supposes that the line is extensible, then we have:

$$\eta = \eta_0 = 0, \qquad \zeta = \zeta_0 = 0, \qquad q = q_0 = 0, \qquad r = r_0 = 0.$$

Upon continuing to denote the function $W(s_0, 1 + \mu, 0, 0, p, 0, 0)$ by W, we have:

$$F' = \frac{\partial W}{\partial \mu}, \qquad I' = \frac{\partial W}{\partial p}.$$

As for G', H', J', K', they may be calculated by means of only the knowledge of the function $W(s_0, 1 + \mu, 0, 0, p, 0, 0)$. If this function is the only given one must consider G', H', J', K' as four auxiliary variables that are defined by the equations.

In the present case, when X, Y, Z, L, M, N or $X_0, Y_0, Z_0, L_0, M_0, N_0$ are given functions of s_0, s, x, y, z, and the derivatives of these variables with respect to one of the others, equations (20), combined with relations, (18), (21), and the following:

$$(28) \qquad \begin{cases} I = \alpha \dfrac{\partial W}{\partial p} + \beta J' + \gamma K', \\ J = \alpha' \dfrac{\partial W}{\partial p} + \beta' J' + \gamma' K', \\ K = \alpha'' \dfrac{\partial W}{\partial p} + \beta'' J' + \gamma'' K', \end{cases}$$

provide a system of eight differential equations in four of the above variables (as a function of the fifth) and ω, F, G, H, J', K'; in addition, one has two first degree equations (whose coefficients are to be determined) in x, y, z.

As before, one may eliminate F, G, H.

A particular case is the one where (M) coincides with (M_0) point-by-point (coincidence of the triad vertices).

2. In addition, if one introduces inextensibility by the relations:

$$\xi = \xi_0 = 1,$$

and if one continues to denote the function $W(s_0, 1, 0, 0, p, 0, 0)$ by W, one will have, upon supposing that only the this latter function is known, simply the relation:

$$I' = \frac{\partial W}{\partial p}.$$

If X, Y, Z, L, M, N or $X_0, Y_0, Z_0, L_0, M_0, N_0$ are given functions of s_0, s, x, y, z, and the derivatives of these variables with respect to one of the others then we have seven equations (20) and (21), where I, J, K are replaced by their values (28) and which, combined with two relations of first degree in x, y, z (with the coefficients to be determined by accessory conditions) determine the nine unknowns $x, y, z, \omega, F, G, H, J', K'$ as a function of s_0.

As before, one may eliminate F, G, H.

27. The deformable line obtained by adjoining the conditions $p = p_0$, $q = q_0$, $r = r_0$, **and, in particular,** $p = p_0 = 0$, $q = q_0 = 0$, $r = r_0 = 0$. – This deformable line may be studied in various fashions, either by considering the deformations (M) of the general deformable line that verify the indicated conditions, or by starting with W in general and defining a new line by the consideration of the stated conditions, or by starting with W as a function of s_0, ξ, η, ζ, and defining the line that conforms to these conditions.

Imagine the first viewpoint. For the moment, designate by W_1 what W becomes when one takes the conditions:

$$p = p_0, \qquad q = q_0, \qquad r = r_0,$$

into account; i.e., set:

$$W_1 = [W(s_0, \xi, \eta, \varsigma, p, q, r)]_{p=p_0, q=q_0, r=r_0} = W(s_0, \xi, \eta, \varsigma, p_0, q_0, r_0).$$

We have:

$$F' = \left[\frac{\partial W}{\partial \xi}\right]_{p=p_0, q=q_0, r=r_0} = \frac{\partial W_1}{\partial \xi}, \qquad I' = \left[\frac{\partial W}{\partial p}\right]_{p=p_0, q=q_0, r=r_0},$$
$$G' = \left[\frac{\partial W}{\partial \eta}\right]_{p=p_0, q=q_0, r=r_0} = \frac{\partial W_1}{\partial \eta}, \qquad J' = \left[\frac{\partial W}{\partial q}\right]_{p=p_0, q=q_0, r=r_0},$$
$$H' = \left[\frac{\partial W}{\partial \varsigma}\right]_{p=p_0, q=q_0, r=r_0} = \frac{\partial W_1}{\partial \varsigma}, \qquad K' = \left[\frac{\partial W}{\partial r}\right]_{p=p_0, q=q_0, r=r_0}.$$

Therefore, if *we would like to introduce only the function* W_1 of s_0, ξ, η, ζ, i.e., the value taken by W for $p = p_0$, $q = q_0$, $r = r_0$, and if we suppose that we are not given the values taken by the derivatives $\frac{\partial W}{\partial p}, \frac{\partial W}{\partial q}, \frac{\partial W}{\partial r}$ for $p = p_0$, $q = q_0$, $r = r_0$ then we find ourselves in the presence of six expressions, only three of which F', G', H' may be considered as given, and three of which I', J', K' are left to be determined.

The equations in question are then:

$$\frac{d}{ds_0}\left(\frac{\partial W_1}{\partial \xi}\right) + q_0 \frac{\partial W_1}{\partial \varsigma} - r_0 \frac{\partial W_1}{\partial \eta} - X_0' = 0,$$
$$\frac{d}{ds_0}\left(\frac{\partial W_1}{\partial \eta}\right) + r_0 \frac{\partial W_1}{\partial \xi} - p_0 \frac{\partial W_1}{\partial \varsigma} - Y_0' = 0,$$

$$\frac{d}{ds_0}\left(\frac{\partial W_1}{\partial \varsigma}\right) + p_0 \frac{\partial W_1}{\partial \eta} - q_0 \frac{\partial W_1}{\partial \xi} - Z_0' = 0,$$
$$\frac{dI'}{ds_0} + q_0 K' - r_0 J' + \eta \frac{\partial W_1}{\partial \varsigma} - \varsigma \frac{\partial W_1}{\partial \eta} - L_0' = 0,$$
$$\frac{dJ'}{ds_0} + r_0 I' - p_0 K' + \varsigma \frac{\partial W_1}{\partial \xi} - \xi \frac{\partial W_1}{\partial \varsigma} - M_0' = 0,$$
$$\frac{dK'}{ds_0} + p_0 J' - q_0 I' + \xi \frac{\partial W_1}{\partial \eta} - \eta \frac{\partial W_1}{\partial \xi} - N_0' = 0,$$

to which we must add $p = p_0$, $q = q_0$, $r = r_0$, and which give us, in all, nine equations in the nine unknowns x, y, z, λ_1, λ_2, λ_3, I', J', K'.

The last three formulas are similar to the ones for what MAXWELL has called the magnetic induction in the interior of a magnet.

In the particularly simple case $p = p_0 = 0$, $q = q_0 = 0$, $r = r_0 = 0$, the preceding formulas take a very simply form.

28. Deformable line subject to constraints. Canonical equations. – In all of the foregoing, we have considered a deformable line that we have qualified as *free,* i.e., the theory was developed without the intervention of external elements, and by means of a function W that is defined by the elements of the line in its natural and deformed states.

Directing our attention to certain deformations, upon adding the notion of a *hidden W* we may recover the equations that were proposed by the authors for various lines.

Instead of this exposition, we may give another in which, instead of considering the deformable line of sec. **5** and **9** for which the deformations satisfy certain definite conditions, we imagine a *sui generis* deformable line, where *the definition already accounts for* the definite conditions satisfied by the particular deformations of the preceding line.

Here is how we proceed to define the new line, while remaining in the same general neighborhood as before.

First, observe that the conditions imposed on the functions x, y, z, $\alpha, \alpha', \cdots, \gamma''$ may be of two kinds: 1. conditions between functions and their derivatives ([56]), for any s_0 . 2. conditions satisfied for certain values of s_0.

We restrict ourselves to conditions of the first type.

To fix ideas, let

$$f_1 = 0, \qquad f_2 = 0$$

be two conditions or *equations of constraint.* Instead of constructing the preceding expressions that we defined by means of the identity:

$$\int_{A_0}^{B_0} \delta W ds_0 = [F'\delta' x + G'\delta y + H'\delta' z + I'\delta I' + J'\delta J' + K'\delta K']_{A_0}^{B_0}$$

[56] Our exposition is not concerned with the distinction between holonomic and non-holonomic constraints.

$$-\int_{A_0}^{B_0}(X'\delta' x+Y'\delta' y+Z'\delta' z+L'\delta I'+M'\delta J'+N'\delta K')ds_0,$$

as functions of s_0, where we introduced $F',G',H',I',J',K';X',Y',Z',L',M',N',$ to fix ideas, we say that - by definition – the preceding identity must make sense by virtue of:

$$f_1 = 0, \qquad f_2 = 0,$$

or again that – by definition – we imagine a deformable line such that the theory results from the consideration of a function $W(s_0, \xi, \eta, \zeta, p, q, r)$ and two auxiliary functions λ_1, λ_2 of s_0, by means of the identity:

$$\int_{A_0}^{B_0}(\delta W+\lambda_1\delta f_1+\lambda_2\delta f_2)ds_0=[F'\delta' x+G'\delta y+H'\delta' z+I'\delta I'+J'\delta J'+K'\delta K']_{A_0}^{B_0}$$
$$-\int_{A_0}^{B_0}(X'\delta' x+Y'\delta' y+Z'\delta' z+L'\delta I'+M'\delta J'+N'\delta K')ds_0,$$

where, this time, all of the variations are arbitrary; we must then add

$$f_1 = 0, \qquad f_2 = 0,$$

a posteriori.

Observe, moreover, that in the case where certain of the left-hand sides f_1, f_2, …, of the equations of constraint refer to only the arguments that figure in W, one may conceive that either one proceeds in a manner as we shall describe, or that by a change of the auxiliary variables one introduces the data of these equations with particular constraints into W *a priori;* this brings us back to the notion of a *hidden W*. We stop ourselves at this point in the particular cases that follow and where the present remarks apply.

1. FLEXIBLE AND INEXTENSIBLE LINE. – Start with a function W of $\mu=\frac{ds}{ds_0}-1$ and s_0, and add the condition that $\mu = 0$. We define the functions F',G',H',X',Y',Z' by starting with:

$$\int_{A_0}^{B_0}(\delta W+\lambda\delta\mu)ds=[F'\delta' x+G'\delta' y+H'\delta' z]_{A_0}^{B_0}$$
$$-\int_{A_0}^{B_0}(X_0'\delta' x+Y_0'\delta' y+Z_0'\delta' z)ds_0.$$

This amounts to replacing W with $W_1 = W + \lambda\mu$ in the preceding, and it leads to the formulas:

$$F=\frac{\partial W_1}{\partial\frac{dx}{ds_0}},\qquad G=\frac{\partial W_1}{\partial\frac{dy}{ds_0}},\qquad H=\frac{\partial W_1}{\partial\frac{dz}{ds_0}},$$

$$\frac{dF}{ds_0} - X_0 = 0, \qquad \frac{dG}{ds_0} - Y_0 = 0, \qquad \frac{dH}{ds_0} - Z_0 = 0,$$

in which we have taken $\mu = 0$ into account, and which thus determine F, G, H, X_0, Y_0, Z_0.

As one sees, we come down to a theory of the flexible inextensible line that generalizes the theory of LAGRANGE, which corresponds to the function W_1 of s_0 and μ, and where we limit ourselves to the study of deformations that correspond to $\mu = 0$. If we take the case in which W_1 is *hidden* then we suppose that one knows simply the value $W_0(s_0)$ that W and W_1 take simultaneously for $\mu = 0$, and we therefore have the classical system of mechanics.

Observe that if, in order to construct the flexible inextensible line, we take the condition $\mu = 0$ into account in W, *a priori*, by a change of the auxiliary variables, then we are led to replace W with λ in the calculations relating to the general deformable line, and we arrive at formulas that lead furthermore to the study of the flexible extensible filament, where we limit ourselves to considering deformations that correspond to $\mu = 0$; upon supposing that λ is *unknown*, these formulas also lead us to the classical system of mechanics.

We conclude with the following remark. Suppose that, by virtue of the formulas that define the deformation, one has expressed X_0, Y_0, Z_0 as functions of s_0, x, y, z in such a way that $X_0\,dx + Y_0\,dy + Z_0\,dz$ is the total differential of a function φ of s_0, x, y, z with respect to x, y, z. Suppose, in addition, that we are dealing with the case of the hidden W_1, or in the case envisioned in the latter context, in such a way that we are reduced to the case of mechanics. From the foregoing, one recovers the remark that served as the point of departure for CLEBSCH ([57]) that the equations in question, in which X_0, Y_0, Z_0 figure, are none other than the extremal equations of the problem of the calculus of variations that consists of determining an extremum for the integral:

$$\int_{A_0}^{B_0} \varphi\, ds,$$

under the condition ([58]):

$$\left(\frac{dx}{ds_0}\right)^2 + \left(\frac{dx}{ds_0}\right)^2 + \left(\frac{dx}{ds_0}\right)^2 = 1.$$

If we set:

$$\psi_1 = -\frac{1}{2}\left[\left(\frac{dx}{ds_0}\right)^2 + \left(\frac{dy}{ds_0}\right)^2 + \left(\frac{dz}{ds_0}\right)^2 - 1\right],$$

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[57] A. CLEBSCH. – *Über die Gleichgewichtsfigur eines biegsamen Fadens, Journ. für die reine und angewandte Math.,* T. LVII, pp. 93-116 [1859], 1860.

[58] We must distinguish between the present question and the one treated by APPELL, *Traité de Mécanique rationelle,* T. I, 1st ed., sec. **158**, pp. 205 ff.; 2nd ed., sec. **146**, pp. 201 ff.

and apply the considerations developed by JORDAN ([59]), we may reduce this system to its canonical form. If we put λ_1 in place of T then the system expresses the idea that one nullifies the first variation of the integral:

$$\int_{A_0}^{B_0} F\,ds_0$$

upon setting:

$$F = -(\varphi + \lambda_1 \psi_1).$$

The equations:

$$\frac{\partial F}{\partial \dfrac{dx}{ds_0}} = p_1, \qquad \frac{\partial F}{\partial \dfrac{dy}{ds_0}} = p_2, \qquad \frac{\partial F}{\partial \dfrac{dz}{ds_0}} = p_3, \qquad \psi_1 = 0,$$

permit us to express the variables $x' = \dfrac{dx}{ds_0}$, $y' = \dfrac{dy}{ds_0}$, $z' = \dfrac{dz}{ds_0}$, λ_1 as functions of the variables x, y, z, p_1, p_2, p_3 by means of the formulas:

$$\lambda_1 = \sqrt{p_1^2 + p_2^2 + p_3^2}, \qquad x' = \frac{p_1}{\lambda_1}, \qquad y' = \frac{p_2}{\lambda_1}, \qquad z' = \frac{p_3}{\lambda_1}.$$

If we substitute these values into:

$$p_1 x' + p_2 y' + p_3 z' - F,$$

we obtain the function:

$$\mathcal{H} = \varphi(s_0, x, y, z) + \sqrt{p_1^2 + p_2^2 + p_3^2},$$

and upon denoting the coordinates x, y, z by q_1, q_2, q_3, as in APPELL ([60]), we have the equations (which are canonical if s_0 does not figure in φ):

$$\frac{dq_\nu}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_\nu}, \qquad \frac{dp_\nu}{ds_0} = -\frac{\partial \mathcal{H}}{\partial q_\nu}$$

to determine the variables x, y, z, p_1, p_2, p_3.

As one sees, we recover the results that were obtained by APPELL ([61]), in a simple form that was first given by LEGOUX ([62]), and then by MARCOLONGO ([63]), and from

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[59] JORDAN. – *Cours d'Analyse de l'Ecole Polytechnique,* T. III, 2nd edition, no. 375, pp. 501, 502.

[60] APPELL. – *Traité de mécanique rationelle,* 1st ed., T. II, Exercise 14, pp. 48-49; 2nd ed., T. I, Exercise 14, pp. 583-584.

[61] APPELL. – *Reduction à la forme canonique des équations d'un fil flexible et inextensible, C.R.,* **96**, pp. 688-691, 12 March 1883; *Traité de mécanique rationelle,* loc. cit.

which one may pass to the method of JACOBI and the results given in the first place by CLEBSCH, in the previously-cited memoir ([64]).

One may also present the preceding exposition as we did for the dynamics of a point in our first Note and for the deformable line in general.

Begin with the equations:

$$\frac{d}{ds_0}\left(T\frac{dx}{ds_0}\right)+X_0=0, \qquad \frac{d}{ds_0}\left(T\frac{dy}{ds_0}\right)+Y_0=0, \qquad \frac{d}{ds_0}\left(T\frac{dz}{ds_0}\right)+Z_0=0,$$

or rather, the system that gave rise to them:

$$F=-T\frac{dx}{ds_0}, \qquad G=-T\frac{dy}{ds_0}, \qquad H=-T\frac{dz}{ds_0},$$
$$\frac{dF}{ds_0}-X_0=0, \qquad \frac{dG}{ds_0}-Y_0=0, \qquad \frac{dH}{ds_0}-Z_0=0,$$

which may be considered as defining the six unknowns x, y, z, F, G, H. Suppose that X_0, Y_0, Z_0 are given functions of s_0, x, y, z.

If we add the three equations of the first line, after squaring, then we see that T is defined as a function of F, G, H by the relation:

$$T^2 = F^2 + G^2 + H^2,$$

from which, it results that:

$$\frac{F}{T}=\frac{\partial T}{\partial F}, \qquad \frac{G}{T}=\frac{\partial T}{\partial G}, \qquad \frac{H}{T}=\frac{\partial T}{\partial H}.$$

The normal form of the system considered is, as a result:

$$\frac{dx}{ds_0}=\frac{\partial\mathcal{H}}{\partial F}, \qquad \frac{dy}{ds_0}=\frac{\partial\mathcal{H}}{\partial G}, \qquad \frac{dz}{ds_0}=\frac{\partial\mathcal{H}}{\partial H},$$
$$\frac{dF}{ds_0}=-\frac{\partial\mathcal{H}}{\partial x}, \qquad \frac{dG}{ds_0}=-\frac{\partial\mathcal{H}}{\partial y}, \qquad \frac{dH}{ds_0}=-\frac{\partial\mathcal{H}}{\partial z}.$$

2. ELASTIC LINE OF LORD KELVIN AND TAIT. – We may repeat for this line what we did for the flexible inextensible line. Start with a function W of s_0, ξ, η, ζ, p, q,

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[62] A. LEGOUX. – *Equations canoniques, application à la recherché de l'équilibre des fils flexible et des courbes brachistrochrones, Mém. de l'Acad. des Sciences, inscriptions et belles lettres de Toulouse,* 8th Series, T. **VIII**, 2nd semester, pp. 159-184, 1885.

[63] R. MARCOLONGO. – *Sull' equilibrio di un filo flessible ed inestensibile, Rend. dell' Accad. delle scienze fisiche e matematiche (Sezione della SocietB reale di Napoli),* 2nd Series, vol. II, pp. 363-368, 1888.

64 Likewise, consult APPELL, *Sur l'équilibre d'un fil flexible et inextensible, Ann. de la Fac. Des Sc. de Toulouse,* (1), **1**, pp. B_1-B_5, 1887.

r, and add the conditions:

$$\xi = \xi_0 = 1, \qquad \eta = \eta_0 = 0, \qquad \zeta = \zeta_0 = 0 .$$

We define the functions $F', G', H', I', J', K'; X'_0, Y'_0, Z'_0, L'_0, M'_0, N'_0$ by means of the identity:

$$\int_{A_0}^{B_0} (\delta W + \mu_1 \delta\xi + \mu_2 \delta\eta + \mu_3 \delta\zeta) ds_0 = [F'\delta' x + G'\delta' y + \cdots + K'\delta K']_{B_0}^{A_0}$$
$$- \int_{A_0}^{B_0} (X'_0 \delta' x + Y'_0 \delta' y + \cdots + N'_0 \delta K') ds_0 ;$$

this amounts to replacing W with $W_1 + \mu_1(\xi - 1) + \mu_2\eta + \mu_3\zeta$ in the preceding and including the indicated formulas $\xi = \xi_0 = 1$, $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$ in these equations.

As one sees, we come down to *the theory of the deformable line that corresponds to the function* W_1 *of* s_0, ξ, η, ζ, p, q, r, *and when one limits oneself to the study of deformations that correspond to* $\xi = \xi_0 = 1$, $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$. *If we put ourselves in the case where* W_1 *is hidden* then we suppose that one knows simply the function $W(s_0, 1, 0, 0, 0, p, q, r)$ that W and W_1 simultaneously reduce to for $\xi = \xi_0 = 1$, $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$, and we recover the *theory developed by* LORD KELVIN and TAIT.

Observe that if, to construct the preceding line, we account for W *a priori* in the three conditions $\xi = \xi_0 = 1$, $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$ by a change of auxiliary variables then we are led to replace W by $W(s_0, 1, 0, 0, p, q, r) + \mu_1(\xi - 1) + \mu_2\eta + \mu_3\zeta$ in the calculations that relate to the general deformable line, and we obtain formulas that further reduce to the study of a deformable line when one is limited to imagining deformations that correspond to the three conditions $\xi = \xi_0 = 1$, $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$. Upon supposing that μ_1, μ_2, μ_3 *are not known* these formulas lead us once more to the theory of LORD KELVIN and TAIT.

Suppose that by virtue of the formulas that determine the deformation, one has expressed X_0, Y_0, Z_0, L_0, M_0, N_0 as functions of s_0, x, y, z, λ_1, λ_2, λ_3 in such a way that

$$X_0\, dx + Y_0\, dy + Z_0\, dz + \mathcal{L}_0\, d\lambda_1 + \mathcal{M}_0\, d\lambda_2 + \mathcal{N}_0\, d\lambda_3$$

is the total differential of a function U of s_0, x, y, z, λ_1, λ_2, λ_3, considered simply with respect to x, y, z, λ_1, λ_2, λ_3. In addition, suppose that we are in the case of hidden W or the case envisioned in the latter example. From the preceding, the equations in question, in which X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ figure, are none other than the extremal equations of the problem in the calculus of variations that consists of determining an extremal for the integral:

$$\int_{A_0}^{B_0} (W + U) ds_0 ,$$

where W is a given function of s_0, p, q, r, upon supposing that the six unknown functions x, y, z, λ_1, λ_2, λ_3 verify the three differential equations:

$$\xi - 1 = 0, \qquad \eta = 0, \qquad \zeta = 0.$$

If we set $\psi_1 = \xi - 1$, $\psi_2 = \eta$, $\psi_3 = \zeta$ and apply the considerations developed by JORDAN then we may reduce the system to canonical form. Upon putting F', G', H' in place of the variables λ_1, λ_2, λ_3 of JORDAN, the system expresses that one nullifies the first variation of the integral $\int_{A_0}^{B_0} \mathcal{F}\, ds_0$ upon setting:

$$\mathcal{F} = W + U + F'\psi_1 + G'\eta + H'\varsigma .$$

The equations:

$$\frac{\partial \mathcal{F}}{\partial \dfrac{dx}{ds_0}} = p_1, \quad \frac{\partial \mathcal{F}}{\partial \dfrac{dx}{ds_0}} = p_1, \quad \frac{\partial \mathcal{F}}{\partial \dfrac{dx}{ds_0}} = p_1, \quad \frac{\partial \mathcal{F}}{\partial \dfrac{d\lambda_1}{ds_0}} = p_4, \quad \frac{\partial \mathcal{F}}{\partial \dfrac{d\lambda_2}{ds_0}} = p_5, \quad \frac{\partial \mathcal{F}}{\partial \dfrac{d\lambda_3}{ds_0}} = p_6,$$

$$\psi_1 = 0, \qquad \psi_2 = 0, \qquad \psi_3 = 0$$

permit us to express the nine variables $x' = \dfrac{dx}{ds_0}$, $y' = \dfrac{dy}{ds_0}$, $z' = \dfrac{dz}{ds_0}$, $\lambda_1' = \dfrac{d\lambda_1}{ds_0}$, $\lambda_2' = \dfrac{d\lambda_2}{ds_0}$, $\lambda_3' = \dfrac{d\lambda_3}{ds_0}$, F', G', H' as functions of the twelve variables x, y, z, λ_1, λ_2, λ_3, p_1, p_2, …, p_6 by means of the formulas:

$$x' = \alpha, \qquad y' = \alpha', \qquad z' = \alpha'',$$
$$F' = \alpha p_1 + \alpha' p_2 + \alpha'' p_3, \qquad G' = \beta p_1 + \beta' p_2 + \beta'' p_3, \qquad H' = \gamma p_1 + \gamma' p_2 + \gamma'' p_3,$$

and by solving the formulas:

(29)
$$\begin{cases} p_4 = \dfrac{\partial W}{\partial p}\varpi_1' + \dfrac{\partial W}{\partial q}\chi_1' + \dfrac{\partial W}{\partial r}\sigma_1' \\ p_5 = \dfrac{\partial W}{\partial p}\varpi_2' + \dfrac{\partial W}{\partial q}\chi_2' + \dfrac{\partial W}{\partial r}\sigma_2' \\ p_6 = \dfrac{\partial W}{\partial p}\varpi_3' + \dfrac{\partial W}{\partial q}\chi_3' + \dfrac{\partial W}{\partial r}\sigma_3' \end{cases}$$

where we preserve the notations of sec. **10**, for the moment.

Substituting these values into:

$$p_1 x' + p_2 y' + p_3 z' + p_4 \lambda_1' + p_5 \lambda_2' + p_6 \lambda_3' - \mathcal{F},$$

we obtain the function $\mathcal{H}$ of s_0, x, y, z, λ_1, λ_2, λ_3, p_1, p_2, …, p_6, which is deduced from:

$$-W - U + \alpha p_1 + \alpha' p_2 + \alpha'' p_3 + p\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial q} + q\frac{\partial W}{\partial q}$$

by the substitution of the values for p, q, r as functions of s_0, λ_1, λ_2, λ_3, p_1, p_2, …, p_6 that one deduces from equations (29).

To determine the twelve variables x, y, z, λ_1, λ_2, λ_3, p_1, p_2, …, p_6, we have the equations (which are canonical if s_0 does not figure explicitly):

$$\frac{dx}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_1}, \quad \frac{dy}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_2}, \quad \frac{dz}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_3}, \quad \frac{d\lambda_1}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_4}, \quad \frac{d\lambda_2}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_5}, \quad \frac{d\lambda_3}{ds_0} = \frac{\partial \mathcal{H}}{\partial p_6},$$

$$\frac{dp_1}{ds_0} = -\frac{\partial \mathcal{H}}{\partial x}, \quad \frac{dp_2}{ds_0} = -\frac{\partial \mathcal{H}}{\partial y} \quad \frac{dp_3}{ds_0} = -\frac{\partial \mathcal{H}}{\partial z}, \quad \frac{dp_4}{ds_0} = -\frac{\partial \mathcal{H}}{\partial \lambda_1}, \quad \frac{dp_5}{ds_0} = -\frac{\partial \mathcal{H}}{\partial \lambda_2}, \quad \frac{dp_6}{ds_0} = -\frac{\partial \mathcal{H}}{\partial \lambda_3},$$

by which one may conclude the application of the method of JACOBI to the line in question.

One may also present the preceding exposition as we did for the general deformable line as well as for the dynamics of a point in our first note.

3. DEFORMABLE LINE WHERE Mx' IS TANGENT TO M AT (M). As always, start with a function W of s_0, ξ, η, ζ, p, q, r, and add the conditions that $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$. We define the functions $F', G', H', I', J', K', X'_0, Y'_0, Z'_0, L'_0, M'_0, N'_0$ by means of the identity:

$$\int_{A_0}^{B_0} (\delta W + \mu_1 \delta\eta + \mu_2 \delta\varsigma) ds = [F'\delta' x + G'\delta' y + \cdots + K'\delta K']_{A_0}^{B_0}$$
$$- \int_{A_0}^{B_0} (X'_0 \delta' x + Y'_0 \delta' y + \cdots + N'_0 \delta K') ds_0.$$

This amounts to replacing W with $W_1 = W + \mu_1\eta + \mu_2\zeta$, in the preceding, and adding the indicated conditions $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$ to the formulas.

As one sees, we recover *the theory of the deformable line that corresponds to the function* W_1 of s_0, ξ, η, ζ, p, q, r *when we limit ourselves to studying the deformations that correspond to* $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$. *If we put ourselves in the case of hidden* W_1 then we suppose that one knows simply the function $W(s_0, \xi, 0, 0, p, q, r)$ that W and W_1 simultaneously reduce to for $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$.

If, to construct the preceding line, we account for the two conditions $\eta = \eta_0 = 0$, $\zeta = \zeta_0 = 0$ in W *a priori*, by a change of the auxiliary variables, then we are led to replace W with $W(s_0, \xi, 0, 0, p, q, r) + \mu_1\eta + \mu_2\zeta$ in the calculations that relate to the general deformable line, and we arrive at formulas that once again reduce to the study of a deformable line when one is limited to studying deformations that correspond to the two conditions $\eta = \eta_0$, $\zeta = \zeta_0$.

Suppose that, by virtue of the formulas that determine the deformation, one has expressed X_0, Y_0, Z_0, L_0, M_0, N_0 as functions of s_0, x, y, z, λ_1, λ_2, λ_3, in such a way that:

$$X_0\,dx + Y_0\,dy + Z_0\,dz + \mathcal{L}_0\,d\lambda_1 + \mathcal{M}_0\,d\lambda_2 + \mathcal{N}_0\,d\lambda_3$$

is the total differential of a function U of s_0, x, y, z, λ_1, λ_2, λ_3, considered simply with respect to x, y, z, λ_1, λ_2, λ_3. Suppose, in addition, that we are dealing with the case of hidden W or in the case envisioned in the latter example. From the preceding, the equations in question, in which X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ figure, are none other than the extremal equations for the problem of the calculus of variations that consists of determining an extremum for the integral:

$$\int_{A_0}^{B_0} (W + U)\,ds_0\,,$$

where W is a given function of s_0, ξ, η, ζ, p, q, r, upon supposing that the six unknown functions x, y, z, λ_1, λ_2, λ_3 verify the two differential equations $\eta = 0$, $\zeta = 0$. The earlier considerations are thus repeated and it will be the same for all of the other particular lines that we have envisioned.

29. States infinitely close to the natural state. Hooke's modulus of deformation. Critical values of the general moduli. Concurrence with the dynamics of triads. – Return to the general deformable line. Suppose that the action is null in the natural state, as well as the effort and the moment of deformation, and similarly, the external force and moment. In this case, not only does the function W vanish identically, but also the six partial derivatives of W with respect to ξ, η, ζ, p, q, r, for the values ξ_0, η_0, ζ_0, p_0, q_0, r_0 of these variables. Suppose, moreover, that W is developable in a neighborhood of $\xi = \xi_0$, $\eta = \eta_0$, $\zeta = \zeta_0$, $p = p_0$, $q = q_0$, $r = r_0$ in positive integer powers of $\xi - \xi_0$, $\eta - \eta_0$, ..., $r - r_0$. Under these conditions, one will have:

$$W = W_2 + W_3 + \dots$$

upon representing W_2, W_3, … by homogenous polynomials of degree 2, 3, …, in the differences $\xi - \xi_0$, $\eta - \eta_0$, ..., $r - r_0$.

Suppose that the coordinates of a point M_0 of the line (M_0) in the normal state and the three parameters by means of which one expresses the direction cosines of the axes of the triad associated with that point are x_0, y_0, z_0, λ_{10}, λ_{20}, λ_{30}, respectively, and that the coordinates x, y, z of the corresponding point M in the deformed state (M), and that the parameters λ_1, λ_2, λ_3 that define the axes of the associated triad are functions of s_0 and h that are developable in powers of h by the formulas:

$$\begin{aligned} x &= x_0 + x_1 + \dots + x_i + \dots, & \lambda_1 &= \lambda_{10} + \lambda_{11} + \dots + \lambda_{1i} + \dots, \\ y &= y_0 + y_1 + \dots + y_i + \dots, & \lambda_2 &= \lambda_{20} + \lambda_{21} + \dots + \lambda_{2i} + \dots, \\ z &= z_0 + z_1 + \dots + z_i + \dots, & \lambda_3 &= \lambda_{30} + \lambda_{31} + \dots + \lambda_{3i} + \dots, \end{aligned}$$

in which x_i, y_i, z_i, λ_{1i}, λ_{2i}, λ_{3i} denote terms that refer to the h^i factor. We introduce these series developments to abbreviate the exposition and we assume that they obey the ordinary rules of calculus. The formulas of sec. **14** permit us to calculate the developments of F, G, H, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$; X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ in powers of h; the terms that are independent of h are null, and the terms F_1, G_1, H_1, $\mathcal{I}_1$, $\mathcal{J}_1$, $\mathcal{K}_1$; X_{01}, Y_{01}, Z_{01}, $\mathcal{L}_{01}$, $\mathcal{M}_{01}$, $\mathcal{N}_{01}$ are given by the formulas:

$$F_1 = \frac{\partial W_2}{\partial \frac{dx^{(1)}}{ds_0}}, \quad G_1 = \frac{\partial W_2}{\partial \frac{dy^{(1)}}{ds_0}}, \quad H_1 = \frac{\partial W_2}{\partial \frac{dz^{(1)}}{ds_0}}, \quad \mathcal{I}_1 = \frac{\partial W_2}{\partial \frac{d\lambda_1^{(1)}}{ds_0}}, \quad \mathcal{J}_1 = \frac{\partial W_2}{\partial \frac{d\lambda_2^{(1)}}{ds_0}}, \quad \mathcal{K}_1 = \frac{\partial W_2}{\partial \frac{d\lambda_3^{(1)}}{ds_0}},$$

$$X_{01} = \frac{d}{ds_0}\frac{\partial W_2}{\partial \frac{dx^{(1)}}{ds_0}}, \qquad Y_{01} = \frac{d}{ds_0}\frac{\partial W_2}{\partial \frac{dy^{(1)}}{ds_0}}, \qquad Z_{01} = \frac{d}{ds_0}\frac{\partial W_2}{\partial \frac{dz^{(1)}}{ds_0}},$$

$$\mathcal{L}_{01} = \frac{d}{ds_0}\frac{\partial W_2}{\partial \frac{d\lambda_1^{(1)}}{ds_0}} - \frac{\partial W_2}{\partial \lambda_1^{(1)}}, \qquad \mathcal{M}_{01} = \frac{d}{ds_0}\frac{\partial W_2}{\partial \frac{d\lambda_2^{(1)}}{ds_0}} - \frac{\partial W_2}{\partial \lambda_2^{(1)}}, \qquad \mathcal{N}_{01} = \frac{d}{ds_0}\frac{\partial W_2}{\partial \frac{d\lambda_3^{(1)}}{ds_0}} - \frac{\partial W_2}{\partial \lambda_3^{(1)}},$$

where we have set:

$$x^{(1)} = x_0 + x_1, \qquad y^{(1)} = y_0 + y_1, \qquad z^{(1)} = z_0 + z_1,$$
$$\lambda_1^{(1)} = \lambda_{10} + \lambda_{11}, \qquad \lambda_2^{(1)} = \lambda_{20} + \lambda_{21}, \qquad \lambda_3^{(1)} = \lambda_{30} + \lambda_{31}.$$

If we consider, under the name of *deformation state one that is infinitely close to the natural state,* then the state (M), where the point M has the coordinates $x^{(1)}$, $y^{(1)}$, $z^{(1)}$, and where the parameters that relate to the associated triad have the values $\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}$, and if, on the other hand, we call the vectors (F_1, G_1, H_1), ($\mathcal{I}_1$, $\mathcal{J}_1$, $\mathcal{K}_1$), (X_{01}, Y_{01}, Z_{01}), (L_{01}, M_{01}, N_{01}) the *effort, moment of deformation, external force, and external moment,* relative to that state, where L_{01}, M_{01}, N_{01} are calculated by means of λ_{10}, λ_{20}, λ_{30}, $\mathcal{L}_{01}$, $\mathcal{M}_{01}$, $\mathcal{N}_{01}$, in the same manner as L_0, M_0, N_0 are calculated from λ_1, λ_2, λ_3, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$, then we arrive at the general hypotheses made by the classical authors, and where the first two vectors are linear functions of the elements that characterize the deformed state in question. As a consequence, we recover what has been named the *generalized* HOOKE *law,* but limited, as is convenient, by *the condition that we respect the principle of energy conservation.* To satisfy this condition in the classical method it is necessary to retrace the path that we followed in our exposition, but in the opposite sense.

The coefficients in the linear functions that express HOOKE'S law are the *deformation moduli* of the deformable line in its state of being infinitely close to the natural state; they are *invariant* at a given point of the line. This notion of modulus may be generalized upon envisioning the first and second derivatives of the function W. Instead of the case where the general moduli are defined and continuous, one may consider the one where they have critical values.

The preceding considerations are easily repeated for different particular deformable lines; they must be reconciled with the ones that we developed in our first note. Indeed, the dynamics of triads is attached to the foregoing in a completely direct manner. It suffices to regard the arc s_0 as *time t*, and the deformable line as a *trajectory*. This simple statement immediately explains the analogies that have been recognized for quite some time between the classical dynamics of a point and the rigid body, and the statics of the deformable line.

Observe that, as in the preceding proposition that we obtained ([65]) for the case of the rigid body, with regard to the kinetic energy, there corresponds a proposition for the deformable line, from which, when W does not depend on s_0 explicitly, formulas (10) entail that the expression:

$$(\xi X_0' + \eta Y_0' + \varsigma Z_0' + pL_0' + qM_0' + rN_0')ds_0,$$

which may be put into the form:

$$X_0\,dx + Y_0\,dy + Z_0\,dz + \mathcal{L}_0\,d\lambda_1 + \mathcal{M}_0\,d\lambda_2 + \mathcal{N}_0\,d\lambda_3,$$

is equal to the differential of the quantity:

$$\xi\frac{\partial W}{\partial \xi} + \eta\frac{\partial W}{\partial \eta} + \varsigma\frac{\partial W}{\partial \varsigma} + p\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial q} + r\frac{\partial W}{\partial r} - W,$$

that was already introduced in sec. **14**.

On the other hand, observe that one may add considerations that are analogous to the ones that were developed in the present work, as far as constraints are concerned, for the deformable line to the developments that were given in our first note with regard to the rigid body.

[65] *Note sur la dynamique du point et du corps invariable*, Tome I, pp. 261.

III. – STATICS OF THE DEFORMABLE SURFACE
AND DYNAMICS OF THE DEFORMABLE LINE

30. Deformable surface. Natural state and deformed state. – As we shall see, the developments that we deduced in regard to the deformable line are reproduced, almost unchanged, in the theories of the deformable surface and deformable three-dimensional medium. This repetition shows the fecundity of the concept of Euclidian action. It suggests numerous approaches and opens up a vast field of study that the first researchers began to explore only with great difficulty, but which is now possible to begin more successfully, given the present state of the general geometric theory of surfaces and curvilinear coordinates, such as what DARBOUX has presented in his great works ([1]).

Consider a surface (M_0) that is described by a point M_0, whose coordinates x_0, y_0, z_0 with respect to three rectangular axes Ox, Oy, Oz are functions of two parameters, which we assume are chosen in a arbitrary manner and are designated by ρ_1 and ρ_2. Adjoin a trirectangular triad with axes M_0x_0', M_0y_0', Mz_0' to each point M_0 of the surface (M_0), whose direction cosines with respect to the axes Ox, Oy, Oz are $\alpha_0, \alpha_0', \alpha_0''$; $\beta_0, \beta_0', \beta_0''$; $\gamma_0, \gamma_0', \gamma_0''$, respectively, and are functions of the same parameters ρ_1 and ρ_2. The continuous two-dimensional set of all such triads $M_0x_0'y_0'z_0'$ will be what we call a *deformable surface.*

Give a displacement M_0M to the point M_0, and let x, y, z be the coordinates of the point M with respect to the fixed axes Ox, Oy, Oz. In addition, give the triad $M_0x_0'y_0'z_0'$ a rotation that ultimately brings the axes of the triad into agreement with those of a triad $Mx'y'z'$ that we adjoin to the point M; we define that rotation by giving the direction cosines $\alpha, \alpha', \alpha''$; β, β', β''; $\gamma, \gamma', \gamma''$ of the axes Mx', My', Mz' with respect to the fixed axes. The continuous two-dimensional set of all such triads $Mx'y'z'$ will be called the *deformed state* of the deformable surface under consideration, which, in its primitive state, will be called the *natural state.*

31. Kinematical elements that relate to the state of the deformable surface. – Let $\xi_i^{(0)}, \eta_i^{(0)}, \varsigma_i^{(0)}$ denote the *components of the velocity of the origin* M_0 *of the axes* M_0x_0', M_0y_0', Mz_0' *along these axes when each* ρ_i *alone varies and plays the role of time.* Likewise, let $p_i^{(0)}, q_i^{(0)}, r_i^{(0)}$ be the quantities that define the projections on those axes of the instantaneous rotation of the triad $M_0x_0'y_0'z_0'$ relative to the parameter ρ_i. We denote the analogous quantities for the triad $Mx'y'z'$ by ξ_i, η_i, ζ_i, and p_i, q_i, r_i when one refers it, like the triad $M_0x_0'y_0'z_0'$, to the fixed triad $Oxyz$.

The elements that we just introduced are calculated in the habitual fashion; one has:

[1] GASTON DARBOUX. – *Leçons sur la théorie générale des surfaces*, 4 vol., Paris, 1887-1896; *Leçons sur les systèmes orthogonaux et les coordinées curvilignes*, Tome I, Paris, 1898.

$$(30)\quad \begin{cases} \xi_i = \alpha \dfrac{\partial x}{\partial \rho_i} + \alpha' \dfrac{\partial y}{\partial \rho_i} + \alpha'' \dfrac{\partial z}{\partial \rho_i} \\ \eta_i = \beta \dfrac{\partial x}{\partial \rho_i} + \beta' \dfrac{\partial y}{\partial \rho_i} + \beta'' \dfrac{\partial z}{\partial \rho_i} \\ \varsigma_i = \gamma \dfrac{\partial x}{\partial \rho_i} + \gamma' \dfrac{\partial y}{\partial \rho_i} + \gamma'' \dfrac{\partial z}{\partial \rho_i} \end{cases} \qquad (31)\quad \begin{cases} p_i = \sum \gamma \dfrac{\partial \beta}{\partial \rho_i} = -\sum \beta \dfrac{\partial \gamma}{\partial \rho_i} \\ q_i = \sum \alpha \dfrac{\partial \gamma}{\partial \rho_i} = -\sum \gamma \dfrac{\partial \alpha}{\partial \rho_i} \\ r_i = \sum \beta \dfrac{\partial \alpha}{\partial \rho_i} = -\sum \alpha \dfrac{\partial \beta}{\partial \rho_i}. \end{cases}$$

The linear elements ds_0 and ds of the surface in its natural and deformed state will be defined by the formulas:

$$ds_0^2 = \mathcal{E}_0 d\rho_1^2 + 2\mathcal{F}_0 d\rho_1 d\rho_2 + \mathcal{G}_0 d\rho_2^2 \qquad ds^2 = \mathcal{E} d\rho_1^2 + 2\mathcal{F} d\rho_1 d\rho_2 + \mathcal{G} d\rho_2^2,$$

where $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are calculated from the following double formulas:

$$(32)\quad \begin{cases} \mathcal{E} = \left(\dfrac{\partial x}{\partial \rho_1}\right)^2 + \left(\dfrac{\partial y}{\partial \rho_1}\right)^2 + \left(\dfrac{\partial z}{\partial \rho_1}\right)^2 = \xi_1^2 + \eta_1^2 + \varsigma_1^2, \\ \mathcal{F} = \dfrac{\partial x}{\partial \rho_1}\dfrac{\partial x}{\partial \rho_2} + \dfrac{\partial y}{\partial \rho_1}\dfrac{\partial y}{\partial \rho_2} + \dfrac{\partial z}{\partial \rho_1}\dfrac{\partial z}{\partial \rho_2} = \xi_1\xi_2 + \eta_1\eta_2 + \varsigma_1\varsigma_2, \\ \mathcal{G} = \left(\dfrac{\partial x}{\partial \rho_2}\right)^2 + \left(\dfrac{\partial y}{\partial \rho_2}\right)^2 + \left(\dfrac{\partial z}{\partial \rho_2}\right)^2 = \xi_2^2 + \eta_2^2 + \varsigma_2^2, \end{cases}$$

and where $\mathcal{E}_0$, $\mathcal{F}_0$, $\mathcal{G}_0$ are calculated by analogous formulas.

Denote the projections of the segment OM onto the axes Mx', My', Mz' by x', y', z', in such a way that the coordinates of the fixed point O will be $-x', -y', -z'$ with respect to these axes. We have the following well-known formulas:

$$(33)\quad \begin{cases} \xi_i - \dfrac{\partial x'}{\partial \rho_i} - qz' + ry' = 0, \\ \eta_i - \dfrac{\partial y'}{\partial \rho_i} - rx' + pz' = 0, \\ \varsigma_i - \dfrac{\partial z'}{\partial \rho_i} - py' + qx' = 0, \end{cases}$$

which give the new expressions for ξ_i, η_i, ζ_i.

32. Expressions for the variations of the translational and rotational velocities relative to the deformed state. – Suppose that one gives an infinitely small displacement to each of the triads of the deformed states in a manner that may vary in a continuous fashion with the triads. Designate the variations of x, y, z; x', y', z'; $\alpha, \alpha', \cdots.\gamma''$ by δx, δy, δz; $\delta x', \delta y', \delta z'$; $\delta\alpha, \delta\alpha', \cdots.\delta\gamma''$, respectively. The variations $\delta\alpha, \delta\alpha', \cdots.\delta\gamma''$ are expressed by formulas such as the following:

$$\delta\alpha = \beta\delta K' - \gamma\delta J' \tag{34}$$

by means of the three auxiliary functions $\delta I', \delta J', \delta K'$, which are the components with respect to Mx', My', Mz' of the well-known instantaneous rotation that is attached to the infinitely small displacement in question. The variations δx, δy, δz are the projections on Ox, Oy, Oz of the infinitely small displacement given to the point M; the projections $\delta' x, \delta' y, \delta' z$ of this displacement on Mx', My', Mz' are deduced immediately and have the values:

$$\left\{\begin{aligned} \delta' x &= \delta x' + z'\delta J' - y'\delta K', \\ \delta' y &= \delta y' + x'\delta K' - z'\delta I', \\ \delta' z &= \delta z' + y'\delta I' - x'\delta J'. \end{aligned}\right. \tag{35}$$

We propose to the determine the variations $\delta\xi_i$, $\delta\eta_i$, $\delta\zeta_i$, δp_i , δq_i , δr_i that are implied for ξ_i , η_i , ζ_i , p_i , q_i , r_i , respectively. From the formulas (31), we have:

$$\begin{aligned} \delta p_i &= \sum\left(\frac{\partial\beta}{\partial\rho_i}\delta\gamma + \gamma\frac{\partial\delta\beta}{\partial\rho_i}\right), \\ \delta q_i &= \sum\left(\frac{\partial\gamma}{\partial\rho_i}\delta\alpha + \alpha\frac{\partial\delta\gamma}{\partial\rho_i}\right), \\ \delta r_i &= \sum\left(\frac{\partial\alpha}{\partial\rho_i}\delta\beta + \beta\frac{\partial\delta\alpha}{\partial\rho_i}\right). \end{aligned}$$

We replace $\delta\alpha$ by its value $\beta\delta K' - \gamma\delta J'$, and $\delta\alpha', \cdots.\delta\gamma''$ by their analogous values; we obtain:

$$\left\{\begin{aligned} \delta p_i &= \frac{\partial\delta I'}{\partial\rho_i} + q_i\delta K' - r_i\delta J', \\ \delta p_i &= \frac{\partial\delta J'}{\partial\rho_i} + r_i\delta I' - p_i\delta K', \\ \delta r_i &= \frac{\partial\delta K'}{\partial\rho_i} + p_i\delta J' - q_i\delta I'. \end{aligned}\right. \tag{36}$$

Likewise, formulas (35) give us three formulas, the first of which is:

$$\delta\xi_i = \frac{\partial\delta x'}{\partial\rho_i} + q_i\delta z' - r_i\delta y' - y'\delta r_i;$$

if we replace δp_i , δq_i , δr_i by the values they are given from formulas (36) then we obtain:

$$
(37)\qquad \begin{cases}
\delta\xi_i = \eta_i \delta K' - \varsigma_i \delta J' + \dfrac{\partial \delta' x}{\partial \rho_i} + q_i \delta' x - r_i \delta' y, \\
\delta\eta_i = \varsigma_i \delta I' - \xi_i \delta K' + \dfrac{\partial \delta' y}{\partial \rho_i} + r_i \delta' y - p_i \delta' z, \\
\delta\varsigma_i = \xi_i \delta J' - \eta_i \delta I' + \dfrac{\partial \delta' z}{\partial \rho_i} + p_i \delta' x - q_i \delta' x,
\end{cases}
$$

where, to abbreviate the notation, we have introduced the three symbols $\delta' x, \delta' y, \delta' z$ that are defined by formulas (35).

33. Euclidian action for the deformation of a deformable surface. - Consider a function W of *two infinitely close positions* of the triad $Mx'y'z'$, i.e., a function of ρ_1, ρ_2, x, y, $z, \alpha, \alpha', \cdots. \gamma''$, and their first derivatives with respect ρ_1 and ρ_2. If we preserve the notations of sec. 31, and set:

$$\Delta_0 = \sqrt{\mathcal{E}_0 \mathcal{F}_0 - \mathcal{G}_0^2}$$

then we propose to determine what sort of form that W must have in order for the integral:

$$\iint W \Delta_0 \, d\rho_1 d\rho_2 ,$$

to have a null variation when taken over an arbitrary portion of the surface (M_0), and when one subjects the set of all triads of the deformable surface in its deformed state to *the same arbitrary infinitesimal transformation of the group of Euclidian displacements.*

By definition, this amounts to determining W in such a fashion that one has:

$$\delta W = 0$$

when, on the one hand, the origin M of the triad $Mx'y'z'$ is subjected to an infinitely small displacement whose projection δx, δy, δz on the axes Ox, Oy, Oz are:

$$
(38)\qquad \begin{cases}
\delta x = (a_1 + \omega_2 z - \omega_3 y)\delta t \\
\delta y = (a_2 + \omega_3 x - \omega_1 z)\delta t \\
\delta z = (a_3 + \omega_1 y - \omega_2 x)\delta t,
\end{cases}
$$

where a_1, a_2, a_3, ω_1, ω_2, ω_3 are six arbitrary constants and δt is an infinitely small quantity that is independent of ρ_1, ρ_2, and when, on the other hand, this triad $Mx'y'z'$ is subjected to an infinitely small rotation whose components with respect to the axes Ox, Oy, Oz are:

$$\omega_1\,\delta t, \quad \omega_2\,\delta t, \quad \omega_3\,\delta t.$$

Observe that in the present case the variations $\delta\xi_1$, $\delta\eta_1$, $\delta\zeta_1$, δp_1, δq_1, δr_1; $\delta\xi_2$, $\delta\eta_2$, $\delta\zeta_2$, δp_2, δq_2, δr_2 of the twelve expressions ξ_1, η_1, ζ_1, p_1, q_1, r_1; ξ_2, η_2, ζ_2, p_2, q_2, r_2 are null, since this results from the well-known theory of the moving triad, and as we may, moreover, immediately verify by means of formulas (36) and (37) by replacing $\delta'x, \delta'y, \delta'z$; $\delta I', \delta J', \delta K'$ with their present values. It results from this that we may obtain a solution of the question when we let W be an arbitrary function of ρ_1, ρ_2, and the twelve expressions ξ_1, η_1, ζ_1, p_1, q_1, r_1; ξ_2, η_2, ζ_2, p_2, q_2, r_2; we shall now show that we also obtain the solution to the general problem ([1]) that we now pose.

To that effect, observe that the relations (31) permit us – by means of well-known formulas – to express the first derivatives of the nine cosines $\alpha, \alpha', \cdots, \gamma''$ with respect to ρ_1 and ρ_2 by means of the cosines and p_1, q_1, r_1; p_2, q_2, r_2. On the other hand, we remark that formulas (30) permit us to conceive that one expresses the nine cosines $\alpha, \alpha', \cdots, \gamma''$ by means of ξ_1, η_1, ζ_1, and the first derivatives of x, y, z with respect ρ_1, or by means of ξ_2, η_2, ζ_2, and the first derivatives of x, y, z with respect to ρ_2. Furthermore, in this case it is useless to make a hypothesis on the mode of solution, since it is clear that we do not obtain a more general form than the one that we are led to upon ultimately supposing that the function W that we seek is an arbitrary function of ρ_1, ρ_2, and of x, y, z, and their first derivatives with respect to ρ_1, ρ_2, and finally, of ξ_1, η_1, ζ_1, p_1, q_1, r_1; ξ_2, η_2, ζ_2, p_2, q_2, r_2, which we indicate by writing:

$$W = W\left(\rho_1, \rho_2, x, y, z, \frac{\partial x}{\partial \rho_1}, \frac{\partial y}{\partial \rho_1}, \frac{\partial z}{\partial \rho_1}, \frac{\partial x}{\partial \rho_2}, \cdots, \xi_1, \eta_1, \varsigma_1, \xi_2, \cdots, p_1, q_1, r_1, p_2, \cdots\right).$$

Since the variations $\delta\xi_i$, …, δr_i, $\delta\xi_i$, …, δr_2 are null in the present case, as they are for some instant, as we have remarked, we finally can write the new form of W that obtains from formulas (38) and for any a_1, a_2, a_3, ω_1, ω_2, ω_3:

$$\frac{\partial W}{\partial x}\delta x + \frac{\partial W}{\partial y}\delta y + \frac{\partial W}{\partial z}\delta z + \sum\left(\frac{\partial W}{\partial \dfrac{\partial x}{\partial \rho_i}}\delta\frac{\partial x}{\partial \rho_i} + \frac{\partial W}{\partial \dfrac{\partial y}{\partial \rho_i}}\delta\frac{\partial y}{\partial \rho_i} + \frac{\partial W}{\partial \dfrac{\partial z}{\partial \rho_i}}\delta\frac{\partial z}{\partial \rho_i}\right) = 0.$$

If we replace δx, δy, δz by their values in (38), and $\delta\dfrac{\partial x}{\partial \rho_i}, \delta\dfrac{\partial y}{\partial \rho_i}, \delta\dfrac{\partial z}{\partial \rho_i}$ by the values that one deduces by differentiating, and set the coefficients of a_1, a_2, a_3, ω_1, ω_2, ω_3 then we obtain the following six conditions:

[1] In what follows, we suppose that the *deformable surface is susceptible to all possible deformations*, and that, as a result, *the deformed state may be taken absolutely arbitrarily;* this is what mean when we say that *the surface is free*.

$$\frac{\partial W}{\partial x} = 0, \qquad \frac{\partial W}{\partial y} = 0, \qquad \frac{\partial W}{\partial z} = 0,$$

$$\sum_i \left(\frac{\partial W}{\partial \frac{\partial y}{\partial \rho_i}} \frac{\partial z}{\partial \rho_i} - \frac{\partial W}{\partial \frac{\partial z}{\partial \rho_i}} \frac{\partial y}{\partial \rho_i} \right) = 0, \qquad \sum_i \left(\frac{\partial W}{\partial \frac{\partial z}{\partial \rho_i}} \frac{\partial x}{\partial \rho_i} - \frac{\partial W}{\partial \frac{\partial x}{\partial \rho_i}} \frac{\partial z}{\partial \rho_i} \right) = 0,$$

$$\sum_i \left(\frac{\partial W}{\partial \frac{\partial x}{\partial \rho_i}} \frac{\partial y}{\partial \rho_i} - \frac{\partial W}{\partial \frac{\partial y}{\partial \rho_i}} \frac{\partial x}{\partial \rho_i} \right) = 0,$$

which are identities if we assume that the expressions that figure in W have been reduced to the smallest number.

The first three then show us, as one may easily foresee, that W is independent of x, y, z. The last three express that W depends on the first derivatives of x, y, z only by the intermediary of the quantities $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ that were defined by the formulas (32). We therefore finally see that *the desired function W has the remarkable form:*

$$W(\rho_1, \rho_2, \xi_1, \eta_1, \zeta_1; \xi_2, \eta_2, \zeta_2; p_1, q_1, r_1; p_2, q_2, r_2),$$

which is analogous to the one we encountered previously for the deformable line.

Let Δ denote the quantity that is analogous to Δ_0 and is defined by the formula:

$$\Delta = \sqrt{\mathcal{E}\mathcal{F} - \mathcal{G}^2}\ .$$

If we multiply W by the area element $d\sigma_0 = \Delta_0\, d\rho_1 d\rho_2$ of the surface (M_0) then the product $W\,\Delta_0\, d\rho_1 d\rho_2$ so obtained is an invariant that is analogous to the area element of the surface (M) in the group of Euclidian displacements. The same is true for the value of the integral:

$$\iint_{C_0} \frac{\Delta}{\Delta_0} \Delta_0 d\rho_1 d\rho_2 = \iint_{C_0} \Delta d\rho_1 d\rho_2$$

that is taken over the interior of a contour C_0 of the surface (M_0) or a corresponding contour C of the surface (M) that determines the *area* of the domain delimited by C on (M). Similarly, in the spirit of the notion of action for the passage from the natural state (M_0) to the deformed state (M), we adjoin the function W to the elements of the definition of the deformable surface, and we say that the integral:

$$\iint_{C_0} W\Delta_0 d\rho_1 d\rho_2,$$

is *the action of deformation* of the interior of the contour C of the deformed surface.

On the other hand, we say that W is the *density* of the action of deformation at a point of the deformed surface when referred to the unit of area for the non-deformed surface; $W\dfrac{\Delta_0}{\Delta}$ will be that density at a point when referred to the unit of area of the deformed surface.

34. External force and moment; the effort and moment of external deformation; the effort and moment of deformation at a point of the deformed surface. – Consider an *arbitrary* variation of the action of deformation of the interior of a contour C of the surface (M), namely:

$$\delta\iint_{C_0} W\Delta_0 d\rho_1 d\rho_2 = \iint_{C_0}\sum_i\left(\frac{\partial W}{\partial\xi_i}\delta\xi_i + \frac{\partial W}{\partial\eta_i}\delta\eta_i + \frac{\partial W}{\partial\varsigma_i}\delta\varsigma_i + \right.$$
$$\left. + \frac{\partial W}{\partial p_i}\delta p_i + \frac{\partial W}{\partial q_i}\delta q_i + \frac{\partial W}{\partial r_i}\delta r_i\right)\Delta_0 d\rho_1 d\rho_2 .$$

By virtue of formulas (36) and (37) of sec. **32**, we may write:

$$\delta\iint_{C_0} W\Delta_0 d\rho_1 d\rho_2 = \iint_{C_0}\sum_i\left[\frac{\partial W}{\partial\xi_i}\left(\eta_i\delta K' - \varsigma_i\delta J' + \frac{\partial\delta' x}{\partial\rho_i} + q_i\delta' z - r_i\delta' y\right)\right.$$
$$+ \frac{\partial W}{\partial\eta_i}\left(\varsigma_i\delta I' - \xi_i\delta K' + \frac{\partial\delta' y}{\partial\rho_i} + r_i\delta' x - p_i\delta' z\right)$$
$$+ \frac{\partial W}{\partial\varsigma_i}\left(\xi J' I' - \eta_i\delta I' + \frac{\partial\delta' z}{\partial\rho_i} + p_i\delta' y - x\delta' z\right)$$
$$+ \frac{\partial W}{\partial p_i}\left(\frac{\partial\delta I'}{\partial\rho_i} + q_i\delta K' - r_i\delta J'\right) + + \frac{\partial W}{\partial q_i}\left(\frac{\partial\delta J'}{\partial\rho_i} + r_i\delta I' - p_i\delta K'\right)$$
$$\left.\frac{\partial W}{\partial r_i}\left(\frac{\partial\delta K'}{\partial\rho_i} + p_i\delta J' - q_i\delta I'\right)\right]\Delta_0 d\rho_1 d\rho_2 .$$

If we apply GREEN'S formula to the terms that refer explicitly to the derivatives with respect to ρ_1 or ρ_2 then we obtain:

$$\delta\iint_{C_0} W\Delta_0 d\rho_1 d\rho_2 = \int_{C_0}\left[\left(\frac{\partial W}{\partial\xi_1}\delta' x + \frac{\partial W}{\partial\eta_1}\delta' y + \frac{\partial W}{\partial\varsigma_1}\delta' z + \frac{\partial W}{\partial p_1}\delta I' + \frac{\partial W}{\partial q_1}\delta J' + \frac{\partial W}{\partial r_1}\delta K'\right)\Delta_0 d\rho_2\right.$$
$$\left. - \left(\frac{\partial W}{\partial\xi_2}\delta' x + \frac{\partial W}{\partial\eta_2}\delta' y + \frac{\partial W}{\partial\varsigma_2}\delta' z + \frac{\partial W}{\partial p_2}\delta I' + \frac{\partial W}{\partial q_2}\delta J' + \frac{\partial W}{\partial r_2}\delta K'\right)\right]\Delta_0 d\rho_1$$

$$
\begin{aligned}
-\iint_{C_0}\Bigg\{&\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial\xi_i}\right)+q_i\frac{\partial W}{\partial\varsigma_i}-r_i\frac{\partial W}{\partial\eta_i}\right]\delta'x\\
&+\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial\eta_i}\right)+r_i\frac{\partial W}{\partial\xi_i}-p_i\frac{\partial W}{\partial\varsigma_i}\right]\delta'y\\
&+\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial\varsigma_i}\right)+p_i\frac{\partial W}{\partial\eta_i}-q_i\frac{\partial W}{\partial\xi_i}\right]\delta'z\\
&+\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial p_i}\right)+q_i\frac{\partial W}{\partial r_i}-r_i\frac{\partial W}{\partial q_i}+\eta_i\frac{\partial W}{\partial\varsigma_i}-\varsigma_i\frac{\partial W}{\partial\eta_i}\right]\delta I'\\
&+\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial q_i}\right)+r_i\frac{\partial W}{\partial p_i}-p_i\frac{\partial W}{\partial r_i}+\varsigma_i\frac{\partial W}{\partial\xi_i}-\xi_i\frac{\partial W}{\partial\varsigma_i}\right]\delta J'\\
&+\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial r_i}\right)+p_i\frac{\partial W}{\partial q_i}-q_i\frac{\partial W}{\partial p_i}+\xi_i\frac{\partial W}{\partial\eta_i}-\eta_i\frac{\partial W}{\partial\xi_i}\right]\delta K'\Bigg\}\Delta_0\,d\rho_1\,d\rho_2.
\end{aligned}
$$

The curvilinear integral that figures in the preceding formula must be clarified by specifying the sense of its traversal; as one knows, this sense is defined by means of the rotation that is given to the positive part of the curve (ρ_2), i.e., the part that corresponds to the sense in which ρ_1 varies on that augmented curve at the edge of the positive part of the curve (ρ_1). One may further specify that curvilinear integral, as in the example of BELTRAMI upon giving it the form that is provided by applying the formulas:

$$
\begin{aligned}
\iint_{C_0}\frac{\partial\varphi}{\partial\rho_1}d\rho_1 d\rho_2 &= \int_{C_0}\left(\mathcal{E}_0\frac{\partial\rho_1}{\partial n_0}+\mathcal{F}_0\frac{\partial\rho_2}{\partial n_0}\right)\frac{\varphi}{\Delta_0}ds_0,\\
\iint_{C_0}\frac{\partial\varphi}{\partial\rho_2}d\rho_1 d\rho_2 &= \int_{C_0}\left(\mathcal{F}_0\frac{\partial\rho_1}{\partial n_0}+\mathcal{G}_0\frac{\partial\rho_2}{\partial n_0}\right)\frac{\varphi}{\Delta_0}ds_0,
\end{aligned}
$$

where φ denotes a function of ρ_1, ρ_2, where ds_0 is the absolute value of the linear element of the curve (C_0), and where n_0 indicates the direction of the normal to the contour (C_0) traced in the tangent plane to the surface (M_0) and directed towards the exterior of the region delimited by that contour. To obtain the new form of the curvilinear integral, it will suffice to replace the $d\rho_1$ and $d\rho_2$ found under the integral sign in the first form that we obtained with the following values:

$$
-\left(\mathcal{F}_0\frac{\partial\rho_1}{\partial n_0}+\mathcal{G}_0\frac{\partial\rho_2}{\partial n_0}\right)\frac{ds_0}{\Delta_0},\qquad \left(\mathcal{E}_0\frac{\partial\rho_1}{\partial n_0}+\mathcal{F}_0\frac{\partial\rho_2}{\partial n_0}\right)\frac{ds_0}{\Delta_0},
$$

respectively.

If we let $\lambda_0', \mu_0', \nu_0'$ denote the direction cosines of the exterior normal to the contour C_0 in question with respect to the triad $M_0x_0'y_0'z_0'$ then one may give the following forms to the preceding two expressions that must be substituted for $d\rho_1$ and $d\rho_2$, respectively([1]):

$$(39)\qquad -(\lambda_0'\xi_2^{(0)}+\mu_0'\eta_2^{(0)}+\nu_0'\varsigma_2^{(0)})\frac{ds_0}{\Delta_0},\qquad (\lambda_0'\xi_1^{(0)}+\mu_0'\eta_1^{(0)}+\nu_0'\varsigma_1^{(0)})\frac{ds_0}{\Delta_0},$$

by virtue of the formulas:

$$\lambda_0'=\xi_1^{(0)}\frac{\partial\rho_1}{\partial n_0}+\xi_2^{(0)}\frac{\partial\rho_2}{\partial n_0},\qquad \mu_0=\eta_1^{(0)}\frac{\partial\rho_1}{\partial n_0}+\eta_2^{(0)}\frac{\partial\rho_2}{\partial n_0},\qquad \nu_0'=\varsigma_1^{(0)}\frac{\partial\rho_1}{\partial n_0}+\varsigma_2^{(0)}\frac{\partial\rho_2}{\partial n_0},$$

that determine λ_0',μ_0',ν_0'.

If ds_0 denotes the absolute value of the element of arc for the contour C_0 traced on the surface (M_0) then set:

$$F_0'=\Delta_0\left(\frac{\partial W}{\partial\xi_1}\frac{d\rho_2}{ds_0}-\frac{\partial W}{\partial\xi_2}\frac{d\rho_1}{ds_0}\right),\qquad G_0'=\Delta_0\left(\frac{\partial W}{\partial\eta_1}\frac{d\rho_2}{ds_0}-\frac{\partial W}{\partial\eta_2}\frac{d\rho_1}{ds_0}\right),$$

$$H_0'=\Delta_0\left(\frac{\partial W}{\partial\varsigma_1}\frac{d\rho_2}{ds_0}-\frac{\partial W}{\partial\varsigma_2}\frac{d\rho_1}{ds_0}\right),$$

$$I_0'=\Delta_0\left(\frac{\partial W}{\partial p_1}\frac{d\rho_2}{ds_0}-\frac{\partial W}{\partial p_2}\frac{d\rho_1}{ds_0}\right),\qquad J_0'=\Delta_0\left(\frac{\partial W}{\partial q_1}\frac{d\rho_2}{ds_0}-\frac{\partial W}{\partial q_2}\frac{d\rho_1}{ds_0}\right),$$

$$K_0'=\Delta_0\left(\frac{\partial W}{\partial r_1}\frac{d\rho_2}{ds_0}-\frac{\partial W}{\partial r_2}\frac{d\rho_1}{ds_0}\right),$$

where the signs of $d\rho_1$ and $d\rho_2$ are made precise by the sense of traversal indicated above for the curvilinear integral, or again, the values of $d\rho_1$ and $d\rho_2$ are the ones that one indicates and in which the exterior normal to the contour C_0 that is situated in the tangent plane to (M_0) figure. In addition, if we set:

$$\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial\xi_i}\right)+q_i\frac{\partial W}{\partial\varsigma_i}-r_i\frac{\partial W}{\partial\eta_i}\right]=X_0',$$

$$\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial\eta_i}\right)+r_i\frac{\partial W}{\partial\xi_i}-p_i\frac{\partial W}{\partial\varsigma_i}\right]=Y_0',$$

$$\sum_i\left[\frac{1}{\Delta_0}\frac{\partial}{\partial\rho_i}\left(\Delta_0\frac{\partial W}{\partial\varsigma_i}\right)+p_i\frac{\partial W}{\partial\eta_i}-q_i\frac{\partial W}{\partial\xi_i}\right]=Z_0',$$

[1] One naturally has analogous formulas upon introducing the direction cosines λ',μ',ν' of the exterior normal to the contour C that corresponds to C_0 with respect to the triad $Mx'y'z'$.

$$\sum_i \left[\frac{1}{\Delta_0}\frac{\partial}{\partial \rho_i}\left(\Delta_0 \frac{\partial W}{\partial p_i}\right) + q_i \frac{\partial W}{\partial r_i} - r_i \frac{\partial W}{\partial q_i} + \eta_i \frac{\partial W}{\partial \varsigma_i} - \varsigma_i \frac{\partial W}{\partial \eta_i}\right] = L'_0,$$
$$\sum_i \left[\frac{1}{\Delta_0}\frac{\partial}{\partial \rho_i}\left(\Delta_0 \frac{\partial W}{\partial q_i}\right) + r_i \frac{\partial W}{\partial p_i} - p_i \frac{\partial W}{\partial r_i} + \varsigma_i \frac{\partial W}{\partial \xi_i} - \xi_i \frac{\partial W}{\partial \varsigma_i}\right] = M'_0,$$
$$\sum_i \left[\frac{1}{\Delta_0}\frac{\partial}{\partial \rho_i}\left(\Delta_0 \frac{\partial W}{\partial r_i}\right) + p_i \frac{\partial W}{\partial q_i} - q_i \frac{\partial W}{\partial p_i} + \xi_i \frac{\partial W}{\partial \eta_i} - \eta_i \frac{\partial W}{\partial \xi_i}\right] = N'_0,$$

then we have:

$$\delta \iint_{C_0} W\Delta_0 d\rho_1 d\rho_2 = \int_{C_0} (F'_0\delta' x + G'_0\delta' y + H'_0\delta' z + I'_0\delta I' + J'_0\delta J' + K'_0\delta K')ds_0$$
$$- \iint_{C_0} (X'_0\delta' x + Y'_0\delta' y + Z'_0\delta' z + L'_0\delta I' + M'_0\delta J' + N'_0\delta K')\Delta_0 d\rho_1 d\rho_2 .$$

If we first consider the double integral that figures in the expression for $\delta \iint_{C_0} W\Delta_0 d\rho_1 d\rho_2$ then we call the segments that have their origins at *M* whose components along the axes Mx', My', Mz' are X'_0, Y'_0, Z'_0 and L'_0, M'_0, N'_0, respectively, the *external force and external moment at the point M referred to the unit of area of the non-deformed surface.* If we next consider the curvilinear integral that figures in $\delta \iint_{C_0} W\Delta_0 d\rho_1 d\rho_2$ then we call the segments that issue from the point *M*, whose projections on the axes Mx', My', Mz' are $-F'_0, -G'_0, -H'_0$ and $-I'_0, -J'_0, -K'_0$, respectively, the *external effort and external moment of deformation of the contour C of the deformed surface at the point M referred to the unit of length of the contour* C_0.

As we have seen, at a specific point *M* of *C* these last six quantities depend only on the direction of the exterior normal to the curve C_0, taken at the point M_0 in the tangent plane to (M_0). They remain invariant when the direction of the exterior normal does not change when one varies the region (M_0) in question, and they change sign if that direction is replace by the opposite direction.

Suppose that one traces a line Σ in the interior of the deformed surface that is bounded by the contour *C* in such a way that it circumscribes a subset (*A*) of the surface, either alone or with a portion of the contour *C*, and denote the rest of the surface outside of the subset (*A*) by (*B*). Let Σ_0 be the curve of (M_0) that corresponds to the curve Σ of (*M*), and let (A_0) and (B_0) be the regions of (M_0) that correspond to (*A*) and (*B*) of (*M*). Imagine that the subsets (*A*) and (*B*) are separate. One may regard the two segments $(-F'_0, -G'_0, -H'_0)$ and $(-I'_0, -J'_0, -K'_0)$ that are determined by the point *M*, the direction of the normal to Σ_0 in the tangent plane to (M_0), and the exterior to (A_0) as the external effort and the moment of deformation at the point *M* of the contour Σ of the region (*A*). Similarly, one may regard the two segments (F'_0, G'_0, H'_0) and (I'_0, J'_0, K'_0) as the external effort and moment of deformation at the point *M* of the contour Σ of the region (*B*). By reason of this remark, we say that $-F'_0, -G'_0, -H'_0$ and $-I'_0, -J'_0, -K'_0$ are the components of the *effort and moment of deformation that are exercised at M by the portion* (*A*) *of the* *surface* (M) with respect to the axes Mx', My', Mz', and that $F'_0, G'_0, H'_0, I'_0, J'_0, K'_0$ are the components of the *effort and moment of deformation that is exercised at M on the portion* (B) *of the surface* (M).

The observation made at the close of sec. **9** on the subject of replacing the triad $Mx'y'z'$ with a triad that is invariably related to it may be repeated here without modification.

35. Diverse specifications for the effort and moment of deformation. – Set:

$$A'_i = \Delta_0 \frac{\partial W}{\partial \xi_i}, \qquad B'_i = \Delta_0 \frac{\partial W}{\partial \eta_i}, \qquad C'_i = \Delta_0 \frac{\partial W}{\partial \varsigma_i},$$

$$P'_i = \Delta_0 \frac{\partial W}{\partial p_i}, \qquad Q'_i = \Delta_0 \frac{\partial W}{\partial q_i}, \qquad R'_i = \Delta_0 \frac{\partial W}{\partial r_i}$$

so that $\frac{1}{\sqrt{\mathcal{G}_0}} A'_1, \frac{1}{\sqrt{\mathcal{G}_0}} B'_1, \frac{1}{\sqrt{\mathcal{G}_0}} C'_1$ and $\frac{1}{\sqrt{\mathcal{G}_0}} P'_1, \frac{1}{\sqrt{\mathcal{G}_0}} Q'_1, \frac{1}{\sqrt{\mathcal{G}_0}} R'_1$ represent the projections on Mx', My', Mz', respectively, of the effort and moment of deformation that is exerted at the point M of the a curve that admits the same tangent as ρ_1 = const. This effort and moment of deformation are *referred to the unit of length of the non-deformed contour* As for ρ_2 = const., the effort and moment of deformation have the projections $\frac{1}{\sqrt{\mathcal{E}_0}} A'_2, \frac{1}{\sqrt{\mathcal{E}_0}} B'_2, \frac{1}{\sqrt{\mathcal{E}_0}} C'_2$ and $\frac{1}{\sqrt{\mathcal{E}_0}} P'_2, \frac{1}{\sqrt{\mathcal{E}_0}} Q'_2, \frac{1}{\sqrt{\mathcal{E}_0}} R'_2$, respectively.

The new efforts and the new moments of deformation that we shall define are related to the elements that we introduced in the preceding section by way of the following relations:

$$\sum_i \left(\frac{\partial A'_i}{\partial \rho_i} + q_i C'_i - r_i B'_i \right) = \Delta_0 X'_0, \qquad F'_0 = A'_1 \frac{d\rho_2}{ds_0} - A'_2 \frac{d\rho_1}{ds_0},$$

$$\sum_i \left(\frac{\partial B'_i}{\partial \rho_i} + r_i A'_i - p_i C'_i \right) = \Delta_0 Y'_0, \qquad G'_0 = B'_1 \frac{d\rho_2}{ds_0} - B'_2 \frac{d\rho_1}{ds_0},$$

$$\sum_i \left(\frac{\partial C_i}{\partial \rho_i} + p_i B'_i - q_i A'_i \right) = \Delta_0 Z'_0, \qquad H'_0 = C'_1 \frac{d\rho_2}{ds_0} - C'_2 \frac{d\rho_1}{ds_0},$$

$$\sum_i \left(\frac{\partial P'_i}{\partial \rho_i} + q_i R'_i - r_i Q'_i + \eta_i C'_i - \varsigma_i B'_i \right) = \Delta_0 L'_0, \qquad I'_0 = P'_1 \frac{d\rho_2}{ds_0} - P'_2 \frac{d\rho_1}{ds_0},$$

$$\sum_i \left(\frac{\partial Q'_i}{\partial \rho_i} + r_i P'_i - p_i R'_i + \eta_i A'_i - \xi_i C'_i \right) = \Delta_0 M'_0, \qquad J'_0 = Q'_1 \frac{d\rho_2}{ds_0} - Q'_2 \frac{d\rho_1}{ds_0},$$

$$\sum_i \left(\frac{\partial R'_i}{\partial \rho_i} + p_i Q'_i - q_i P'_i + \xi_i B'_i - \eta_i A'_i \right) = \Delta_0 N'_0, \qquad K'_0 = R'_1 \frac{d\rho_2}{ds_0} - R'_2 \frac{d\rho_1}{ds_0},$$

where, if one prefers, $d\rho_1$ and $d\rho_2$ are replaced by their values (39) in the equations on the right.

One may propose to transform the relations that we just wrote independently of the values of the quantities that figure in them that were calculated by means of W. Indeed, these relations apply to the segments that are attached to the point M, and that we gave names to. Instead of defining these segments by their projections on Mx', My', Mz', we may just as well define them by their projections on the other axes; the latter projections will be coupled by relations that are transforms of the preceding.

Moreover, the transformed relations are obtained immediately if one remarks that the primitive formulas have simple and immediate interpretations ([1]) by the adjunction of axes that are assumed parallel to the ones at the point O to the moving axes.

1. First consider the fixed axes Ox, Oy, Oz. Denote the projections on these axes of the external force and external moment at an arbitrary point M of the deformed medium by X_0, Y_0, Z_0 and L_0, M_0, N_0, respectively. The projections of the effort and the moment of deformation that are related to the direction ($d\rho_1$, $d\rho_2$) of the tangent to a curve C are designated by F_0, G_0, H_0 and I_0, J_0, K_0, respectively. They are referred to the unit of length of the non-deformed curve C_0, and have been previously defined. The projections of the effort (A_i', B_i', C_i'), and the moment of deformation (P_i', Q_i', R_i'), are denoted by A_i, B_i, C_i, and P_i, Q_i, R_i, respectively. The transforms of the preceding relations are obviously:

$$\frac{\partial A_1}{\partial \rho_1} + \frac{\partial A_2}{\partial \rho_2} = \Delta_0 X_0, \qquad F_0 = A_1 \frac{d\rho_2}{ds_0} - A_2 \frac{d\rho_1}{ds_0},$$

$$\frac{\partial B_1}{\partial \rho_1} + \frac{\partial B_2}{\partial \rho_2} = \Delta_0 Y_0, \qquad G_0 = B_1 \frac{d\rho_2}{ds_0} - B_2 \frac{d\rho_1}{ds_0},$$

$$\frac{\partial C_1}{\partial \rho_1} + \frac{\partial C_2}{\partial \rho_2} = \Delta_0 X_0, \qquad H_0 = C_1 \frac{d\rho_2}{ds_0} - C_2 \frac{d\rho_1}{ds_0},$$

$$\frac{\partial P_1}{\partial \rho_1} + \frac{\partial P_2}{\partial \rho_2} + C_1 \frac{\partial y}{\partial \rho_1} + C_2 \frac{\partial y}{\partial \rho_2} - B_1 \frac{\partial z}{\partial \rho_1} - B_1 \frac{\partial z}{\partial \rho_1} = \Delta_0 L_0, \qquad I_0 = P_1 \frac{d\rho_2}{ds_0} - P_2 \frac{d\rho_1}{ds_0},$$

$$\frac{\partial Q_1}{\partial \rho_1} + \frac{\partial Q_2}{\partial \rho_2} + A_1 \frac{\partial z}{\partial \rho_1} + A_2 \frac{\partial z}{\partial \rho_2} - C_1 \frac{\partial x}{\partial \rho_1} - C_1 \frac{\partial x}{\partial \rho_1} = \Delta_0 M_0, \qquad J_0 = Q_1 \frac{d\rho_2}{ds_0} - Q_2 \frac{d\rho_1}{ds_0},$$

$$\frac{\partial R_1}{\partial \rho_1} + \frac{\partial R_2}{\partial \rho_2} + B_1 \frac{\partial x}{\partial \rho_1} + B_2 \frac{\partial x}{\partial \rho_2} - A_1 \frac{\partial y}{\partial \rho_1} - A_1 \frac{\partial y}{\partial \rho_1} = \Delta_0 N_0, \qquad K_0 = R_1 \frac{d\rho_2}{ds_0} - R_2 \frac{d\rho_1}{ds_0}.$$

$\frac{d\rho_1}{ds_0}$ and $\frac{d\rho_2}{ds_0}$ must be replaced by:

[1] An interesting interpretation of note is the analogue of the one that was given by VARIGNON in the context of statics and by P. SAINT_GUILHEM in the context of dynamics.

$$-\frac{1}{\Delta_0}\left(\lambda_0\frac{\partial x_0}{\partial\rho_2}+\mu_0\frac{\partial y_0}{\partial\rho_2}+\nu_0\frac{\partial z_0}{\partial\rho_2}\right), \qquad -\frac{1}{\Delta_0}\left(\lambda_0\frac{\partial x_0}{\partial\rho_1}+\mu_0\frac{\partial y_0}{\partial\rho_1}+\nu_0\frac{\partial z_0}{\partial\rho_1}\right),$$

respectively, whereas $\frac{d\rho_1}{ds}$ and $\frac{d\rho_2}{ds}$ must be replaced by:

$$-\frac{1}{\Delta}\left(\lambda\frac{\partial x}{\partial\rho_2}+\mu\frac{\partial y}{\partial\rho_2}+\nu\frac{\partial z}{\partial\rho_2}\right), \qquad -\frac{1}{\Delta}\left(\lambda\frac{\partial x}{\partial\rho_1}+\mu\frac{\partial y}{\partial\rho_1}+\nu\frac{\partial z}{\partial\rho_1}\right),$$

respectively, where we have notated the direction cosines of the exterior normal to C_0 with respect to the fixed axes by λ_0, μ_0, ν_0, and the exterior normal to C by λ, μ, ν.

In particular, these equations give the equations of the *infinitely small deformation of a plane surface* that were used by LORD KELVIN and TAIT ([1]).

2. One may give a new form to the equations relating to the fixed axes Ox, Oy, Oz. We may express the nine cosines $\alpha, \alpha', \cdots, \gamma''$ by means of three auxiliary variables; let λ_1, λ_2, λ_3 be three such functions. Set:

$$\begin{aligned}
\sum\gamma d\beta &= -\sum\beta d\gamma = \varpi_1' d\lambda_1 + \varpi_2' d\lambda_2 + \varpi_3' d\lambda_3,\\
\sum\alpha d\gamma &= -\sum\gamma d\alpha = \chi_1' d\lambda_1 + \chi_2' d\lambda_2 + \chi_3' d\lambda_3,\\
\sum\beta d\alpha &= -\sum\alpha d\beta = \sigma_1' d\lambda_1 + \sigma_2' d\lambda_2 + \sigma_3' d\lambda_3.
\end{aligned}$$

The functions $\varpi_i', \chi_i', \sigma_i'$ of λ_1, λ_2, λ_3 that are so defined satisfy the relations:

$$\begin{aligned}
&\frac{\partial\varpi_j'}{\partial\lambda_i}-\frac{\partial\varpi_i'}{\partial\lambda_j}+\chi_i'\sigma_j'-\chi_j'\sigma_i'=0,\\
&\frac{\partial\chi_j'}{\partial\lambda_i}-\frac{\partial\chi_i'}{\partial\lambda_j}+\sigma_i'\varpi_j'-\sigma_j'\varpi_i'=0, \qquad (i, j = 1, 2, 3),\\
&\frac{\partial\sigma_j'}{\partial\lambda_i}-\frac{\partial\sigma_i'}{\partial\lambda_j}+\varpi_i'\chi_j'-\varpi_j'\chi_i'=0,
\end{aligned}$$

and one has:

$$\begin{aligned}
p_i &= \varpi_1'\frac{\partial\lambda_1}{\partial\rho_i}+\varpi_2'\frac{\partial\lambda_2}{\partial\rho_i}+\varpi_3'\frac{\partial\lambda_3}{\partial\rho_i},\\
q_i &= \chi_1'\frac{\partial\lambda_1}{\partial\rho_i}+\chi_2'\frac{\partial\lambda_2}{\partial\rho_i}+\chi_3'\frac{\partial\lambda_3}{\partial\rho_i},\\
r_i &= \sigma_1'\frac{\partial\lambda_1}{\partial\rho_i}+\sigma_2'\frac{\partial\lambda_2}{\partial\rho_i}+\sigma_3'\frac{\partial\lambda_3}{\partial\rho_i}.
\end{aligned}$$

[1] *Treatise on Natural Philosophy*, vol. I, Part II, sec. **644**, pp. 186-188.

Let ϖ_i, χ_i, σ_i denote the projections on Ox, Oy, Oz of the segment whose projections on the axes Mx', My', Mz' are $\varpi'_i, \chi'_i, \sigma'_i$. We have:

$$\sum \alpha' d\alpha'' = -\sum \alpha'' d\alpha' = \varpi_1 d\lambda_1 + \varpi_2 d\lambda_2 + \varpi_3 d\lambda_3,$$
$$\sum \alpha'' d\alpha = -\sum \alpha d\alpha'' = \chi_1 d\lambda_1 + \chi_2 d\lambda_2 + \chi_3 d\lambda_3,$$
$$\sum \alpha d\alpha' = -\sum \alpha' d\alpha = \sigma_1 d\lambda_1 + \sigma_2 d\lambda_2 + \sigma_3 d\lambda_3,$$

by virtue of which (1) the new functions ϖ_i, χ_i, σ_i of λ_1, λ_2, λ_3 satisfy the relations:

$$\begin{aligned}
&\frac{\partial \varpi_j}{\partial \lambda_i} - \frac{\partial \varpi_i}{\partial \lambda_j} + \chi_i \sigma_j - \chi_j \sigma_i = 0, \\
&\frac{\partial \chi_j}{\partial \lambda_i} - \frac{\partial \chi_i}{\partial \lambda_j} + \sigma_i \varpi_j - \sigma_j \varpi_i = 0, \qquad (\text{i, j} = 1, 2, 3), \\
&\frac{\partial \sigma_j}{\partial \lambda_i} - \frac{\partial \sigma_i}{\partial \lambda_j} + \varpi_i \chi_j - \varpi_j \chi_i = 0.
\end{aligned}$$

Again we make the remark, which will serve us later on, that if one denotes the variations of $\lambda_1, \lambda_2, \lambda_3$ by $\delta\lambda_1, \delta\lambda_2, \delta\lambda_3$, which corresponds to the variations $\delta\alpha, \delta\alpha', \cdots, \delta\gamma''$ of $\alpha, \alpha', \cdots, \gamma''$ then one will have:

$$\begin{aligned}
&\delta I' = \varpi'_1 \delta\lambda_1 + \varpi'_2 \delta\lambda_2 + \varpi'_3 \delta\lambda_3, \\
&\delta J' = \chi'_1 \delta\lambda_1 + \chi'_2 \delta\lambda_2 + \chi'_3 \delta\lambda_3, \\
&\delta K' = \sigma'_1 \delta\lambda_1 + \sigma'_2 \delta\lambda_2 + \sigma'_3 \delta\lambda_3, \\
&\delta I = \alpha \delta I' + \beta \delta J' + \gamma \delta K' = \varpi_1 \delta\lambda_1 + \varpi_1 \delta\lambda_2 + \varpi_3 \delta\lambda_3, \\
&\delta J = \alpha' \delta I' + \beta' \delta J' + \gamma' \delta K' = \chi_1 \delta\lambda_1 + \chi_1 \delta\lambda_2 + \chi_3 \delta\lambda_3, \\
&\delta K = \alpha'' \delta I' + \beta'' \delta J' + \gamma'' \delta K' = \sigma_1 \delta\lambda_1 + \sigma_1 \delta\lambda_2 + \sigma_3 \delta\lambda_3,
\end{aligned}$$

where δI, δJ, δK are the projections onto the fixed axes of the segment whose projections onto Mx', My', Mz' are $\delta I', \delta J', \delta K'$.

Now set:

$$\begin{aligned}
&\mathcal{I}_0 = \varpi'_1 I'_0 + \chi'_1 J'_0 + \sigma'_1 K'_0 = \varpi_1 I_0 + \chi_1 J_0 + \sigma_1 K_0, \\
&\mathcal{J}_0 = \varpi'_2 I'_0 + \chi'_2 J'_0 + \sigma'_2 K'_0 = \varpi_2 I_0 + \chi_2 J_0 + \sigma_2 K_0, \\
&\mathcal{K}_0 = \varpi'_3 I'_0 + \chi'_3 J'_0 + \sigma'_3 K'_0 = \varpi_3 I_0 + \chi_3 J_0 + \sigma_3 K_0,
\end{aligned}$$

---

[1] These formulas may serve to directly define the functions ϖ_i, χ_i, σ_i, and may be substituted for

$$\begin{aligned}
&\varpi_i = \alpha \varpi'_i + \beta \chi'_i + \gamma \sigma'_i \\
&\chi_i = \alpha' \varpi'_i + \beta' \chi'_i + \gamma' \sigma'_i \qquad (i, j = 1, 2, 3) \\
&\sigma_i = \alpha'' \varpi'_i + \beta'' \chi'_i + \gamma'' \sigma'.
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_0 &= \varpi_1' L_0' + \chi_1' M_0' + \sigma_1' N_0' = \varpi_1 L_0 + \chi_1 M_0 + \sigma_1 N_0 , \\
\mathcal{M}_0 &= \varpi_2' L_0' + \chi_2' M_0' + \sigma_2' N_0' = \varpi_2 L_0 + \chi_2 M_0 + \sigma_2 N_0 , \\
\mathcal{N}_0 &= \varpi_3' L_0' + \chi_3' M_0' + \sigma_3' N_0' = \varpi_3 L_0 + \chi_3 M_0 + \sigma_3 N_0 .
\end{aligned}
$$

In addition, introduce the following notation:

$$
\begin{aligned}
\Pi_i &= \varpi_1' P_i' + \chi_1' Q_i' + \sigma_1' R_i' = \varpi_1 P_i + \chi_1 Q_i + \sigma_1 R_i , \\
X_i &= \varpi_2' P_i' + \chi_2' Q_i' + \sigma_2' R_i' = \varpi_2 P_i + \chi_2 Q_i + \sigma_2 R_i , \\
\Sigma_i &= \varpi_3' P_i' + \chi_3' Q_i' + \sigma_3' R_i' = \varpi_3 P_i + \chi_3 Q_i + \sigma_3 R_i .
\end{aligned}
$$

we then have the following in place of the latter system in which either P_i', Q_i', R_i' or P_i, Q_i, R_i figure:

$$
\begin{aligned}
\mathcal{L}_0 = \sum_i \Bigg[& \frac{\partial \Pi_i}{\partial \rho_i} - A_i'(\sigma_1' \eta_i - \chi_1' \varsigma_i) - B_i'(\varpi_1' \varsigma_i - \sigma_1' \xi_i) - C_i'(\chi_i' \xi_i - \varpi_1' \sigma_i) \\
& - P_i' \left(\frac{\partial \varpi_1'}{\partial \rho_i} + q_i \sigma_1' - r_i \chi_1' \right) - Q_i' \left(\frac{\partial \chi_1'}{\partial \rho_i} + r_i \varpi_1' - p_i \sigma_1' \right) \\
& - R_i' \left(\frac{\partial \sigma_1'}{\partial \rho_i} + p_i \chi_1' - q_i \varpi_1' \right) \Bigg],
\end{aligned}
$$

with two analogous equations. If one remarks that the functions ξ_i, η_i, ζ_i, p_i, q_i, r_i of λ_1, λ_2, λ_3, $\frac{\partial \lambda_1}{\partial \rho_i}, \frac{\partial \lambda_2}{\partial \rho_i}, \frac{\partial \lambda_3}{\partial \rho_i}$, which are related by the formulas:

$$
\begin{aligned}
&\frac{\partial \xi_i}{\partial \lambda_j} + \chi_j' \varsigma_i - \sigma_j' \eta_i = 0, &\qquad& \frac{\partial p_i}{\partial \lambda_j} = \frac{\partial \varpi_j'}{\partial \rho_i} + q_i \sigma_j' - r_i \chi_j' , \\
&\frac{\partial \eta_i}{\partial \lambda_j} + \sigma_j' \xi_i - \varpi_j' \varsigma_i = 0, && \frac{\partial q_i}{\partial \lambda_j} = \frac{\partial \chi_j'}{\partial \rho_i} + r_i \varpi_j' - p_i \sigma_j' , \\
&\frac{\partial \varsigma_i}{\partial \lambda_j} + \varpi_j' \eta_i - \chi_j' \xi_i = 0, && \frac{\partial r_i}{\partial \lambda_j} = \frac{\partial \sigma_j'}{\partial \rho_i} + p_i \chi_j' - q_i \varpi_j' ,
\end{aligned}
$$

that result from the definition of the functions $\varpi_i', \chi_i', \sigma_i'$ and the nine identities that they verify, then one may give the preceding system the new form:

$$
\mathcal{L}_0 = \sum_i \left[\frac{\partial \Pi_i}{\partial \rho_i} - A_i' \frac{\partial \xi_i}{\partial \lambda_1} - B_i' \frac{\partial \eta_i}{\partial \lambda_1} - C_i' \frac{\partial \varsigma_i}{\partial \lambda_1} - P_i' \frac{\partial p_i}{\partial \lambda_1} - Q_i' \frac{\partial q_i}{\partial \lambda_1} - R_i' \frac{\partial r_i}{\partial \lambda_1} \right],
$$

with two analogous equations.

3. Instead of referring the elements that relate to the point M to the fixed axes Ox, Oy, Oz imagine that we define these elements in terms of a trirectangular triad $Mx'_1y'_1z'_1$ that is moving with M such that the axis Mz'_1 is normal to the surface (M) at M. To define this triad $Mx'_1y'_1z'_1$, we refer it to the triad $Mx'y'z'$, and let l, l', l'' be the direction cosines of Mx'_1, with m, m', m'', those of My'_1, and n, n', n'', those of Mz'_1, with respect to the latter axes.

More precisely, we define the direction cosines n, n', n'' by the formulas:

$$n = \frac{1}{\Delta}(\eta_1\varsigma_2 - \eta_2\varsigma_1), \qquad n' = \frac{1}{\Delta}(\varsigma_1\xi_2 - \varsigma_2\xi_1), \qquad n'' = \frac{1}{\Delta}(\xi_1\eta_2 - \xi_2\eta_1).$$

We assume that the triad $Mx'_1y'_1z'_1$ has the same disposition as the others and, for the moment, we make no other particular hypotheses on the other cosines.

Therefore, let $\xi_i^{(1)}, \eta_i^{(1)}, \varsigma_i^{(1)}$ denote the components of the velocity of the origin M of the axes Mx'_1, My'_1, Mz'_1 with respect to these axes when ρ_i alone varies and plays the role of time. Likewise, let $p_i^{(1)}, q_i^{(1)}, r_i^{(1)}$ be the projections of instantaneous rotation of the triad $Mx'_1y'_1z'_1$ relative to the parameter ρ_i on these same axes. In these latter definitions, the triad $Mx'_1y'_1z'_1$ is naturally referred to the fixed triad $Oxyz$. We have:

$$\xi_i^{(1)} = l\xi_i + l'\eta_i + l''\varsigma_i, \qquad \eta_i^{(1)} = m\xi_i + m'\eta_i + m''\varsigma_i, \qquad \varsigma_i^{(1)} = n\xi_i + n'\eta_i + n''\varsigma_i = 0,$$

and three formulas such as the following:

$$p_i^{(1)} = lp_i + l'q_i + l''r_i + \sum n\frac{\partial m}{\partial \rho_i},$$

in which the triads being considered have the same disposition.

Let X''_0, Y''_0, Z''_0 and L''_0, M''_0, N''_0 be the projections on the Mx'_1, My'_1, Mz'_1 of the external force and external moment, respectively, at an arbitrary point M of the deformed surface, referred to the unit of surface of the non-deformed surface. Furthermore, let F''_0, G''_0, H''_0 and I''_0, J''_0, K''_0 be the projections of the effort (F_0, G_0, H_0) and the moment (I_0, J_0, K_0), respectively, on the same axes, and let A''_i, B''_i, C''_i and P''_i, Q''_i, R''_i be the projections of the effort (A'_i, B'_i, C'_i) and the moment (P'_i, Q'_i, R'_i), respectively, as previously defined.

The transforms of the preceding relations (or the primitive relations) are obviously ([1]):

[1] It suffices to replace $\xi_i, \cdots, A'_i, \cdots$ with $\xi_i^{(1)}, \cdots, A''_i, \cdots$ and take the hypothesis $\varsigma_i^{(1)} = 0$ into account; for an *arbitrary* triad with vertex M one will have the same calculations.

$$(40)\quad \begin{cases} \sum_i \left(\frac{\partial A_i''}{\partial \rho_i} + q_i^{(1)} C_i'' - r_i^{(1)} B_i'' \right) = \Delta_0 X_0'', & F_0'' = A_1'' \frac{d\rho_2}{ds_0} - A_2'' \frac{d\rho_1}{ds_0}, \\ \sum_i \left(\frac{\partial B_i''}{\partial \rho_i} + r_i^{(1)} A_i'' - p_i^{(1)} C_i'' \right) = \Delta_0 Y_0'', & G_0'' = B_1'' \frac{d\rho_2}{ds_0} - B_2'' \frac{d\rho_1}{ds_0}, \\ \sum_i \left(\frac{\partial C_i''}{\partial \rho_i} + p_i^{(1)} B_i'' - q_i^{(1)} A_i'' \right) = \Delta_0 Z_0'', & H_0'' = C_1'' \frac{d\rho_2}{ds_0} - C_2'' \frac{d\rho_1}{ds_0}, \\ \sum_i \left(\frac{\partial P_i''}{\partial \rho_i} + q_i^{(1)} R_i'' - r_i^{(1)} Q_i'' + \eta_i^{(1)} C_i'' \right) = \Delta_0 L_0'', & I_0'' = P_1'' \frac{d\rho_2}{ds_0} - P_2'' \frac{d\rho_1}{ds_0}, \\ \sum_i \left(\frac{\partial Q_i''}{\partial \rho_i} + r_i^{(1)} P_i'' - p_i^{(1)} R_i'' + \xi_i^{(1)} C_i'' \right) = \Delta_0 M_0'', & J_0'' = Q_1'' \frac{d\rho_2}{ds_0} - Q_2'' \frac{d\rho_1}{ds_0}, \\ \sum_i \left(\frac{\partial R_i''}{\partial \rho_i} + p_i^{(1)} Q_i'' - q_i^{(1)} P_i'' + \xi_i^{(1)} B_i'' - \eta_i^{(1)} A_i'' \right) = \Delta_0 N_0'', & K_0'' = R_1'' \frac{d\rho_2}{ds_0} - R_2'' \frac{d\rho_1}{ds_0}. \end{cases}$$

Instead of replacing $d\rho_1$, $d\rho_2$ in the right-hand equations with their values in (39) or their analogues relative to (M), we may give them the following values:

$$-(\lambda'' \xi_2^{(1)} + \mu'' \eta_2^{(1)}) \frac{ds}{\Delta}, \qquad -(\lambda'' \xi_1^{(1)} + \mu'' \eta_1^{(1)}) \frac{ds}{\Delta},$$

in which we have denoted the direction cosines of the exterior normal to the contour C with respect to the triad $Mx_1'y_1'z_1'$ by $(\lambda'', \mu'', 0)$. We thus obtain:

$$(41)\quad \begin{cases} F_0'' \frac{ds_0}{ds} = \lambda'' \frac{\xi_1^{(1)} A_1'' + \xi_2^{(1)} A_2''}{\Delta} + \mu'' \frac{\eta_1^{(1)} A_1'' + \eta_2^{(1)} A_2''}{\Delta}, \\ I_0'' \frac{ds_0}{ds} = \lambda'' \frac{\xi_1^{(1)} P_1'' + \xi_2^{(1)} P_2''}{\Delta} + \mu'' \frac{\eta_1^{(1)} P_1'' + \eta_2^{(1)} P_2''}{\Delta}, \end{cases}$$

and two systems of analogous formulas.

These formulas lead us to substitute twelve new auxiliary functions for the twelve auxiliary functions $A_i'', B_i'', C_i'', P_i'', Q_i'', R_i''$, which will be the coefficients of λ'' and μ'' in the preceding expressions for the efforts and moments, when referred to the unit of length of C, or they will be related to these coefficients in a simple manner. We set:

$$\frac{\xi_1^{(1)} A_1'' + \xi_2^{(1)} A_2''}{\Delta} = N_1, \qquad \frac{\eta_1^{(1)} A_1'' + \eta_2^{(1)} A_2''}{\Delta} = T - S_3,$$
$$\frac{\xi_1^{(1)} B_1'' + \xi_2^{(1)} B_2''}{\Delta} = T + S_3, \qquad \frac{\eta_1^{(1)} B_1'' + \eta_2^{(1)} B_2''}{\Delta} = N_2,$$
$$\frac{\xi_1^{(1)} C_1'' + \xi_2^{(1)} C_2''}{\Delta} = S_2, \qquad \frac{\eta_1^{(1)} C_1'' + \eta_2^{(1)} C_2''}{\Delta} = S_1,$$

in which we have introduced the first six auxiliary functions N_1, N_2, T, S_1, S_2, S_3, and similarly:

$$\frac{\xi_1^{(1)}P_1''+\xi_2^{(1)}P_2''}{\Delta}=\mathcal{N}_1, \qquad \frac{\eta_1^{(1)}P_1''+\eta_2^{(1)}P_2''}{\Delta}=\mathcal{T}-\mathcal{S}_3,$$
$$\frac{\xi_1^{(1)}Q_1''+\xi_2^{(1)}Q_2''}{\Delta}=\mathcal{T}+\mathcal{S}_3, \qquad \frac{\eta_1^{(1)}Q_1''+\eta_2^{(1)}Q_2''}{\Delta}=\mathcal{N}_2,$$
$$\frac{\xi_1^{(1)}R_1''+\xi_2^{(1)}R_2''}{\Delta}=\mathcal{S}_2, \qquad \frac{\eta_1^{(1)}R_1''+\eta_2^{(1)}R_2''}{\Delta}=\mathcal{S}_1,$$

in which we have introduced the other six auxiliary functions $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{T}$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$.

The twelve equations that we write may be solved immediately with respect to the primitive auxiliary variables $A_i'', B_i'', C_i'', P_i'', Q_i'', R_i''$. Observe that by virtue of the hypotheses made on the common disposition of all of the triads, one has:

$$\begin{vmatrix} l & l' & l'' \\ m & m' & m'' \\ n & n' & n'' \end{vmatrix} = 1;$$

as a consequence, the formulas that define $\xi_i^{(1)}, \eta_i^{(1)}$ give:

$$\xi_1^{(1)}\eta_1^{(1)} - \xi_1^{(1)}\eta_1^{(1)} = \Delta.$$

As a result, we obtain:

$$A_1'' = N_1\eta^{(1)} - (T - S_3)\xi_2^{(1)}, \qquad A_2'' = (T - S_3)\xi_1^{(1)} - N_1\eta_1^{(1)},$$
$$B_1'' = (T + S_3)\eta^{(1)} - N_2\xi_2^{(1)}, \qquad B_2'' = N_2\xi_1^{(1)} - (T + S_3)\eta_1^{(1)},$$
$$C_1'' = S_2\eta^{(1)} - S_1\xi_2^{(1)}, \qquad C_2'' = S_1\xi_1^{(1)} - S_2\eta_1^{(1)},$$

and six analogous formulas for P_i'', Q_i'', R_i'', with the letters in italics on the right-hand side. When we substitute these values in relations (40) and (41), we will have the equations that relate to the efforts and moments of deformation, as well as the forces and external moments, in the form that they take with the new auxiliary variables ([1]).

Obviously, one may give names to the components of effort and the moment of deformation that are analogous to the ones that we used for the deformable line. Therefore, one may call the components N_1, N_2 of the effort, the *effort of tension*. The components $T - S_3$, $T + S_3$ are the *truncated efforts* in the plane tangent to the deformed surface. The components S_1, S_2 are the *truncated efforts* normal to the deformed surface. Similarly, the components $\mathcal{N}_1$, $\mathcal{N}_2$ of the moment of deformation may be regarded as the

[1] We remark that the coefficient of S_3 in the third equation is *null*.

moments of torsion; the components $\mathcal{T} - \mathcal{S}_3$, $\mathcal{T} + \mathcal{S}_3$ have the character of the *moments of flexion;* the components $\mathcal{S}_1$, $\mathcal{S}_2$ may be called the *moments of geodesic flexion.*

36. Remarks concerning the components S_1, S_2, S_3 **and** $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$. – With regard to the expressions S_1, S_2, S_3, and their analogues $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, we clarify the following remark that we used above in order to write the transformed equations.

In a general fashion, suppose we have a segment whose projections on *Ox*, *Oy*, *Oz* are:

$$\frac{\partial y}{\partial \rho_i} C_i - \frac{\partial z}{\partial \rho_i} B_i, \qquad \frac{\partial z}{\partial \rho_i} A_i - \frac{\partial x}{\partial \rho_i} C_i, \qquad \frac{\partial x}{\partial \rho_i} B_i - \frac{\partial y}{\partial \rho_i} A_i.$$

If we think of this segment as the moment of a vector (A_i, B_i, C_i) that is applied to the point $\left(\frac{\partial x}{\partial \rho_i}, \frac{\partial y}{\partial \rho_i}, \frac{\partial z}{\partial \rho_i}\right)$ then one sees that the projections on $Mx', My', Mz',$ will be:

$$\eta_i C_i' - \varsigma_i B_i', \qquad \varsigma_i A_i' - \xi_i C_i', \qquad \xi_i B_i' - \eta_i A_i',$$

and on Mx_1', My_1', Mz_1' they will be:

$$\eta_i^{(1)} C_i'', \qquad -\xi_i^{(1)} C_i'', \qquad \xi_i^{(1)} B_i'' - \eta_i^{(1)} A_i''.$$

From this, it results that the segment whose projections on *Ox*, *Oy*, *Oz* are:

$$\sum_i \left(\frac{\partial y}{\partial \rho_i} C_i - \frac{\partial z}{\partial \rho_i} B_i\right), \qquad \sum_i \left(\frac{\partial z}{\partial \rho_i} A_i - \frac{\partial x}{\partial \rho_i} C_i\right), \qquad \sum_i \left(\frac{\partial x}{\partial \rho_i} B_i - \frac{\partial y}{\partial \rho_i} A_i\right)$$

will have:

$$\sum_i (\eta_i C_i' - \varsigma_i B_i'), \qquad \sum_i (\varsigma_i A_i' - \xi_i C_i'), \qquad \sum_i (\xi_i B_i' - \eta_i A_i')$$

for its projections on Mx', My', Mz' and:

$$\sum_i \eta_i^{(1)} C_i'' = \Delta S_1, \qquad -\sum \xi_i^{(1)} C_i'' = -\Delta S_2, \qquad \sum_i (\xi_i^{(1)} B_i'' - \eta_i^{(1)} A_i'') = 2\Delta S_3$$

for its projections on Mx_1', My_1', Mz_1'.

Naturally, there is an identical proposition for the italicized variables.

From this, one deduces that the conditions:

$$S_1 = 0, \qquad S_2 = 0, \qquad S_3 = 0$$

amount to the following:

$$\sum_i (\eta_i C_i' - \varsigma_i B_i') = 0, \quad \sum_i (\varsigma_i A_i' - \xi_i C_i') = 0, \quad \sum_i (\xi_i B_i' - \eta_i A_i') = 0,$$

and that the conditions:

$$\mathcal{S}_1 = 0, \qquad \mathcal{S}_2 = 0, \qquad \mathcal{S}_3 = 0,$$

come down to:

$$\sum_i (\eta_i R_i' - \varsigma_i Q_i') = 0, \quad \sum_i (\varsigma_i P_i' - \xi_i R_i') = 0, \quad \sum_i (\xi_i Q_i' - \eta_i P_i') = 0.$$

In these two cases, one arrives at a system of two equations that do not depend on the choice of triad $Mx_1'y_1'z_1'$.

If the conditions $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ are *conditions that result from the form of W* then W verifies the three partial differential equations:

$$\sum_i \left(\eta_i \frac{\partial W}{\partial \varsigma_i} - \varsigma_i \frac{\partial W}{\partial \eta_i} \right) = 0, \qquad \sum_i \left(\varsigma_i \frac{\partial W}{\partial \xi_i} - \varsigma_{\xi i} \frac{\partial W}{\partial \varsigma_i} \right) = 0, \qquad \sum_i \left(\xi_i \frac{\partial W}{\partial \eta_i} - \eta_i \frac{\partial W}{\partial \xi_i} \right) = 0,$$

which entails that W depends on ξ_i, η_i, ζ_i only by the intermediary of the expressions:

$$\mathcal{E} = \xi_1^2 + \eta_1^2 + \varsigma_1^2, \qquad \mathcal{F} = \xi_1\xi_2 + \eta_1\eta_2 + \varsigma_1\varsigma_2, \qquad \mathcal{G} = \xi_2^2 + \eta_2^2 + \varsigma_2^2.$$

If the conditions $\mathcal{S}_1 = 0$, $\mathcal{S}_2 = 0$, $\mathcal{S}_3 = 0$ are *conditions that result from the form of* $\mathcal{W}$ then $\mathcal{W}$ verifies the three partial differential equations:

$$\sum_i \left(\eta_i \frac{\partial W}{\partial r_i} - \varsigma_i \frac{\partial W}{\partial q_i} \right) = 0, \qquad \sum_i \left(\varsigma_i \frac{\partial W}{\partial p_i} - \varsigma_{\xi i} \frac{\partial W}{\partial r_i} \right) = 0, \qquad \sum_i \left(\xi_i \frac{\partial W}{\partial q_i} - \eta_i \frac{\partial W}{\partial p_i} \right) = 0,$$

which entails that $\mathcal{W}$ depends on p_i, q_i, r_i only by the intermediary of the three expressions:

$$p_1\xi_1 + q_1\eta_1 + r_1\zeta_1, \qquad p_1\xi_2 + q_1\eta_2 + r_1\zeta_2 + p_2\xi_1 + q_2\eta_1 + r_2\zeta_1, \qquad p_2\xi_2 + q_2\eta_2 + r_2\zeta_2,$$

expressions that reduce to the coefficients of $d\rho_1^2, d\rho_1 d\rho_2$, and $d\rho_2^2$ in the equation of the lines of curvature of (M) when $\zeta_1 = \zeta_2 = 0$.

Furthermore, observe that if one simply imposes the conditions:

$$S_1 = 0, \qquad S_2 = 0,$$

which amount to saying that the segment whose projection on Mx_1', My_1', Mz_1' has the indicated values from the preceding page is parallel to Mz_1' or that it is perpendicular to both of the vectors (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2), which gives the conditions:

$$\xi_1(\eta_2 C_2' - \varsigma_2 B_2') + \eta_1(\varsigma_2 A_2' - \xi_2 C_2') + \varsigma_1(\xi_2 B_2' - \eta_2 A_2') = 0,$$
$$\xi_2(\eta_1 C_1' - \varsigma_1 B_1') + \eta_2(\varsigma_1 A_1' - \xi_1 C_1') + \varsigma_2(\xi_1 B_1' - \eta_1 A_1') = 0,$$

which may be written:

$$(\eta_1\varsigma_2 - \varsigma_1\eta_2)A_2' + (\varsigma_1\xi_2 - \xi_1\varsigma_2)B_2' + (\xi_1\eta_2 - \eta_1\xi_2)C_2' = 0,$$
$$(\eta_1\varsigma_2 - \varsigma_1\eta_2)A_1' + (\varsigma_1\xi_2 - \xi_1\varsigma_2)B_1' + (\xi_1\eta_2 - \eta_1\xi_2)C_1' = 0,$$

and, in that form express that the vectors (A_1', B_1', C_1') and (A_2', B_2', C_2') are perpendicular to the normal Mz_1'. One thus finds *two conditions that are independent of the choice of triad* $Mx_1'y_1'z_1'$, and may be verified immediately *a posteriori* when one gives them the meaning of the truncated efforts S_1, S_2. If the conditions $S_1 = 0$, $S_2 = 0$ are *conditions that result from the form of W* then *W* verifies the two partial differential equations:

$$(\eta_1\varsigma_2 - \varsigma_1\eta_2)\frac{\partial W}{\partial \xi_1} + (\varsigma_1\xi_2 - \xi_1\varsigma_2)\frac{\partial W}{\partial \eta_1} + (\xi_1\eta_2 - \eta_1\xi_2)\frac{\partial W}{\partial \varsigma_1} = 0,$$
$$(\eta_1\varsigma_2 - \varsigma_1\eta_2)\frac{\partial W}{\partial \xi_2} + (\varsigma_1\xi_2 - \xi_1\varsigma_2)\frac{\partial W}{\partial \eta_2} + (\xi_1\eta_2 - \eta_1\xi_2)\frac{\partial W}{\partial \varsigma_2} = 0,$$

which entails that *W* is a function that depends on ξ_i, η_i, ζ_i only by the intermediary of the three expressions $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$.

The same reasoning proves that the conditions:

$$\mathcal{S}_1 = 0, \qquad \mathcal{S}_2 = 0,$$

amount to two conditions that are independent of the choice of triad $Mx_1'y_1'z_1'$, which one may ultimately write:

$$(\eta_1\varsigma_2 - \varsigma_1\eta_2)P_1' + (\varsigma_1\xi_2 - \xi_1\varsigma_2)Q_1' + (\xi_1\eta_2 - \eta_1\xi_2)R_1' = 0,$$
$$(\eta_1\varsigma_2 - \varsigma_1\eta_2)P_2' + (\varsigma_1\xi_2 - \xi_1\varsigma_2)Q_2' + (\xi_1\eta_2 - \eta_1\xi_2)R_2' = 0.$$

If the conditions $\mathcal{S}_1 = 0$, $\mathcal{S}_2 = 0$ are conditions *that result from the form of W* then *W* verifies the two partial differential equations:

$$(\eta_1\varsigma_2 - \varsigma_1\eta_2)\frac{\partial W}{\partial p_1} + (\varsigma_1\xi_2 - \xi_1\varsigma_2)\frac{\partial W}{\partial q_1} + (\xi_1\eta_2 - \eta_1\xi_2)\frac{\partial W}{\partial r_1} = 0,$$

$$(\eta_1\varsigma_2-\varsigma_1\eta_2)\frac{\partial W}{\partial p_2}+(\varsigma_1\xi_2-\xi_1\varsigma_2)\frac{\partial W}{\partial q_2}+(\xi_1\eta_2-\eta_1\xi_2)\frac{\partial W}{\partial r_2}=0,$$

which entails that W is a function that depends only on p_i, q_i, r_i only by the intermediary of the four expressions:

$$p_1\xi_1+q_1\eta_1+r_1\zeta_1,\qquad p_1\xi_2+q_1\eta_2+r_1\zeta_2,\qquad p_2\xi_1+q_2\eta_1+r_2\zeta_1,\qquad p_2\xi_2+q_2\eta_2+r_2\zeta_2.$$

Similarly, imagine the condition:

$$S_3=0\,.$$

It expresses that the segment whose projections on Mx_1', My_1', Mz_1' have the indicated values from the page (?) is perpendicular to Mz_1', which gives the condition:

$$(\eta_1\varsigma_2-\varsigma_1\eta_2)\sum_i(\eta_i C_i'-\varsigma_i B_i')+(\varsigma_1\xi_2-\xi_1\varsigma_2)\sum_i(\varsigma_i A_i'-\xi_i C_i')$$
$$+(\xi_1\eta_2-\eta_1\xi_2)\sum_i(\xi_i B_i'-\eta_i A_i')=0,$$

which does not depend on the *choice of triad* $Mx_1'y_1'z_1'$ and leads to a partial differential equation that is verified by W when the condition $S_3=0$ results from the form of W.

This equation is:

$$(\xi_2\mathcal{E}-\xi_1\mathcal{F})\frac{\partial W}{\partial \xi_1}+(\eta_2\mathcal{E}-\eta_1\mathcal{F})\frac{\partial W}{\partial \eta_1}+(\varsigma_2\mathcal{E}-\varsigma_1\mathcal{F})\frac{\partial W}{\partial \varsigma_1}$$
$$+(\xi_2\mathcal{F}-\xi_1\mathcal{G})\frac{\partial W}{\partial \xi_2}+(\eta_2\mathcal{F}-\eta_1\mathcal{G})\frac{\partial W}{\partial \eta_2}+(\varsigma_2\mathcal{F}-\varsigma_1\mathcal{G})\frac{\partial W}{\partial \varsigma_2}=0\,,$$

which is easily integrated because it admits the three particular integrals defined by $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, respectively.

The same reasoning applies to the condition:

$$\mathcal{S}_3=0,$$

which, moreover, corresponds to a *condition that is independent of the choice of the triad* $Mx_1'y_1'z_1'$ and, when it results from the form of W, leads to the partial differential equation:

$$(\xi_2\mathcal{E}-\xi_1\mathcal{F})\frac{\partial W}{\partial p_1}+(\eta_2\mathcal{E}-\eta_1\mathcal{F})\frac{\partial W}{\partial q_1}+(\varsigma_2\mathcal{E}-\varsigma_1\mathcal{F})\frac{\partial W}{\partial r_1}$$
$$+(\xi_2\mathcal{F}-\xi_1\mathcal{G})\frac{\partial W}{\partial p_2}+(\eta_2\mathcal{F}-\eta_1\mathcal{G})\frac{\partial W}{\partial q_2}+(\varsigma_2\mathcal{F}-\varsigma_1\mathcal{G})\frac{\partial W}{\partial r_2}=0\,,$$

whose integration is immediate.

37. Equations that are obtained by introducing the coordinates x, y **as independent variables in place of** ρ_1, ρ_2, **as in Poisson's example. –** We propose to form equations that are analogous to those of sec. **35**, but in which the independent variables are x, y by pursuing a certain analogy that we will also make for the deformable three-dimensional medium.

To abbreviate notation, denote the left-hand side of the transformation relations by $\mathcal{X}_0', \mathcal{Y}_0', \mathcal{Z}_0', \mathcal{L}_0', \mathcal{M}_0', \mathcal{N}_0'$; i.e., set:

$$\mathcal{X}_0' = \frac{\partial A_1}{\partial \rho_1} + \frac{\partial A_2}{\partial \rho_2} - \Delta_0 X_0, \qquad \mathcal{Y}_0' = \frac{\partial B_1}{\partial \rho_1} + \frac{\partial B_2}{\partial \rho_2} - \Delta_0 Y_0, \qquad \mathcal{Z}_0' = \frac{\partial C_1}{\partial \rho_1} + \frac{\partial C_2}{\partial \rho_2} - \Delta_0 Z_0,$$

$$\mathcal{L}_0' = \frac{\partial P_1}{\partial \rho_1} + \frac{\partial P_2}{\partial \rho_2} + C_1 \frac{\partial y}{\partial \rho_1} + C_2 \frac{\partial y}{\partial \rho_2} - B_1 \frac{\partial z}{\partial \rho_1} - B_1 \frac{\partial z}{\partial \rho_1} - \Delta_0 L_0,$$
$$\mathcal{M}_0' = \frac{\partial Q_1}{\partial \rho_1} + \frac{\partial Q_2}{\partial \rho_2} + A_1 \frac{\partial z}{\partial \rho_1} + A_2 \frac{\partial z}{\partial \rho_2} - C_1 \frac{\partial x}{\partial \rho_1} - C_1 \frac{\partial x}{\partial \rho_1} - \Delta_0 M_0,$$
$$\mathcal{M}_0' = \frac{\partial R_1}{\partial \rho_1} + \frac{\partial R_2}{\partial \rho_2} + B_1 \frac{\partial x}{\partial \rho_1} + B_2 \frac{\partial x}{\partial \rho_2} - A_1 \frac{\partial y}{\partial \rho_1} - A_1 \frac{\partial y}{\partial \rho_1} - \Delta_0 N_0.$$

We may summarize the twelve relations of sec. **35**, in which we referred the elements to fixed axes, by the following:

$$0 = \iint (\mathcal{X}_0'\lambda_1 + \mathcal{Y}_0'\lambda_2 + \mathcal{Z}_0'\lambda_3 + \mathcal{I}_0'\mu_1 + \mathcal{J}_0'\mu_2 + \mathcal{K}_0'\mu_3)\, d\rho_1 d\rho_2$$
$$+ \int \left\{ \left(F_0 - A_1 \frac{d\rho_2}{ds_0} + A_2 \frac{d\rho_1}{ds_0} \right)\lambda_1 + \left(G_0 - B_1 \frac{d\rho_2}{ds_0} + B_2 \frac{d\rho_1}{ds_0} \right)\lambda_2 \right.$$
$$+ \left(H_0 - C_1 \frac{d\rho_2}{ds_0} + C_2 \frac{d\rho_1}{ds_0} \right)\lambda_3 + \left(I_0 - P_1 \frac{d\rho_2}{ds_0} + P_2 \frac{d\rho_1}{ds_0} \right)\mu_1$$
$$\left. + \left(J_0 - Q_1 \frac{d\rho_2}{ds_0} + Q_2 \frac{d\rho_1}{ds_0} \right)\mu_2 + \left(K_0 - R_1 \frac{d\rho_2}{ds_0} + R_2 \frac{d\rho_1}{ds_0} \right)\mu_3 \right\} ds_0,$$

in which λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 are arbitrary functions, and the integrals are taken along the curve C_0 of the surface (M_0) and over the domain bounded by them.

Applying GREEN'S formula, the relation becomes the following one:

$$- \iint (X_0\lambda_1 + Y_0\lambda_2 + Z_0\lambda_3 + L_0\mu_1 + M_0\mu_2 + N_0\mu_3)\Delta_0\, d\rho_1 d\rho_2$$
$$+ \int (F_0\lambda_1 + G_0\lambda_2 + H_0\lambda_3 + I_0\mu_1 + J_0\mu_2 + K_0\mu_3)\, ds_0$$
$$- \iint \left(A_1 \frac{\partial \lambda_1}{\partial \rho_1} + A_2 \frac{\partial \lambda_1}{\partial \rho_2} + B_1 \frac{\partial \lambda_2}{\partial \rho_1} + B_2 \frac{\partial \lambda_2}{\partial \rho_2} + C_1 \frac{\partial \lambda_3}{\partial \rho_1} + C_2 \frac{\partial \lambda_3}{\partial \rho_2} \right) d\rho_1 d\rho_2$$

$$-\iint\left(P_1\frac{\partial\mu_1}{\partial\rho_1}+P_2\frac{\partial\mu_1}{\partial\rho_2}+Q_1\frac{\partial\mu_2}{\partial\rho_1}+Q_2\frac{\partial\mu_2}{\partial\rho_2}+R_1\frac{\partial\mu_3}{\partial\rho_1}+R_2\frac{\partial\mu_3}{\partial\rho_2}\right)d\rho_1 d\rho_2$$
$$+\iint\left(\frac{\partial y}{\partial\rho_1}C_1+\frac{\partial y}{\partial\rho_2}C_2-\frac{\partial z}{\partial\rho_1}B_1-\frac{\partial z}{\partial\rho_2}B_2\right)\mu_1 d\rho_1 d\rho_2$$
$$+\iint\left(\frac{\partial z}{\partial\rho_1}A_1+\frac{\partial z}{\partial\rho_2}A_2-\frac{\partial x}{\partial\rho_1}C_1-\frac{\partial x}{\partial\rho_2}C_2\right)\mu_2 d\rho_1 d\rho_2$$
$$+\iint\left(\frac{\partial x}{\partial\rho_1}B_1+\frac{\partial x}{\partial\rho_2}B_2-\frac{\partial y}{\partial\rho_1}A_1-\frac{\partial y}{\partial\rho_2}A_2\right)\mu_3 d\rho_1 d\rho_2=0.$$

We seek to transform this latter equation when one takes the functions x, y of ρ_1, ρ_2 for new variables. If one denotes an arbitrary function of ρ_1, ρ_2, which becomes a function of x, y, by j then the elementary formulas for the change of variables are:

$$\frac{\partial\varphi}{\partial\rho_1}=\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\rho_1}+\frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial\rho_1},$$
$$\frac{\partial\varphi}{\partial\rho_2}=\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\rho_2}+\frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial\rho_2}.$$

Apply these formulas to the functions λ_1, λ_2, λ_3, μ_1, μ_2, μ_3. Furthermore, if C always denotes the curve of (M) that corresponds to the curve (C_0) of (M_0) then we denote the projections of the force and external moment that is applied to the point M onto Ox, Oy, Oz by X, Y, Z, L, M, N when referred to the unit of area for the deformed surface (M), and the projections of the effort and the moment of deformation that is exerted at the point M on C onto Ox, Oy, Oz by F, G, H, I, J, K when referred to the unit of length on C. Finally, introduce twelve new auxiliary functions $A_1^{(1)}, B_1^{(1)}, C_1^{(1)}$; $A_2^{(1)}, B_2^{(1)}, C_2^{(1)}$; $P_1^{(1)}, Q_1^{(1)}, R_1^{(1)}$; $P_2^{(1)}, Q_2^{(1)}, R_2^{(1)}$ by the formulas:

$$\frac{\Delta}{\Delta_1}A_1^{(1)}=A_1\frac{\partial x}{\partial\rho_1}+A_2\frac{\partial x}{\partial\rho_2},\qquad \frac{\Delta}{\Delta_1}P_1^{(1)}=P_1\frac{\partial x}{\partial\rho_1}+P_2\frac{\partial x}{\partial\rho_2},$$
$$\frac{\Delta}{\Delta_1}A_2^{(1)}=A_1\frac{\partial y}{\partial\rho_1}+A_2\frac{\partial y}{\partial\rho_2},\qquad \frac{\Delta}{\Delta_1}P_2^{(1)}=P_1\frac{\partial y}{\partial\rho_1}+P_2\frac{\partial y}{\partial\rho_2},$$

and by the analogous formulas obtained upon replacing:

$$A_1, A_2, A_1^{(1)}, A_2^{(1)}, P_1, P_2, P_1^{(1)}, P_2^{(1)},$$

by:

$$B_1, B_2, B_1^{(1)}, B_2^{(1)}, Q_1, Q_2, Q_1^{(1)}, Q_2^{(1)},$$

and then by:

$$C_1, C_2, C_1^{(1)}, C_2^{(1)}, R_1, R_2, R_1^{(1)}, R_2^{(1)},$$

respectively.

We call the analogue of Δ, Δ_1; therefore, we set:

$$\Delta_1 = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}.$$

We obtain the transformed relation:

$$\begin{aligned}
&-\iint (X\lambda_1 + Y\lambda_2 + Z\lambda_3 + L\mu_1 + M\mu_2 + N\mu_3)\Delta_1 dxdy \\
&+\int (F\lambda_1 + G\lambda_2 + H\lambda_3 + I\mu_1 + J\mu_2 + K\mu_3) ds \\
&-\iint \left(A_1^{(1)} \frac{\partial \lambda_1}{\partial x} + A_2^{(1)} \frac{\partial \lambda_1}{\partial y} + B_1^{(1)} \frac{\partial \lambda_2}{\partial x} + B_2^{(1)} \frac{\partial \lambda_2}{\partial y} + C_1^{(1)} \frac{\partial \lambda_3}{\partial x} + C_2^{(1)} \frac{\partial \lambda_3}{\partial y} \right) dxdy \\
&-\iint \left(P_1^{(1)} \frac{\partial \mu_1}{\partial x} + P_2^{(1)} \frac{\partial \mu_1}{\partial y} + Q_1^{(1)} \frac{\partial \mu_2}{\partial x} + Q_2^{(1)} \frac{\partial \mu_2}{\partial y} + R_1^{(1)} \frac{\partial \mu_3}{\partial x} + R_2^{(1)} \frac{\partial \mu_3}{\partial y} \right) dxdy \\
&+\iint \left\{ \left(C_2^{(1)} - \frac{\partial z}{\partial x} B_1^{(1)} - \frac{\partial z}{\partial y} B_2^{(1)} \right)\mu_1 + \left(\frac{\partial z}{\partial x} A_1^{(1)} - \frac{\partial z}{\partial y} A_2^{(1)} - C_2^{(1)} \right)\mu_2 + (B_1^{(1)} - A_2^{(1)})\mu_3 \right\} dxdy = 0,
\end{aligned}$$

where the integrals are taken over the curve *C* of the surface (*M*) and the domain it bounds, and *ds* denotes the element of arc-length of *C*.

We apply GREEN'S formula to the terms that involve the derivatives of λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 with respect to *x*, *y*; since λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 are arbitrary they become:

$$\begin{aligned}
&\frac{\partial A_1^{(1)}}{\partial x} + \frac{\partial A_2^{(1)}}{\partial y} = \Delta_1 X, && F = A_1^{(1)} \frac{dy}{ds} - A_2^{(1)} \frac{dx}{ds}, \\
&\frac{\partial B_1^{(1)}}{\partial x} + \frac{\partial B_2^{(1)}}{\partial y} = \Delta_1 Y, && G = B_1^{(1)} \frac{dy}{ds} - B_2^{(1)} \frac{dx}{ds}, \\
&\frac{\partial C_1^{(1)}}{\partial x} + \frac{\partial C_2^{(1)}}{\partial y} = \Delta_1 Z, && H = C_1^{(1)} \frac{dy}{ds} - C_2^{(1)} \frac{dx}{ds}, \\
&\frac{\partial P_1^{(1)}}{\partial x} + \frac{\partial P_2^{(1)}}{\partial y} + C_2^{(1)} - \frac{\partial z}{\partial x} B_1^{(1)} - \frac{\partial z}{\partial y} B_2^{(1)} = \Delta_1 L, && I = P_1^{(1)} \frac{dy}{ds} - P_2^{(1)} \frac{dx}{ds}, \\
&\frac{\partial Q_1^{(1)}}{\partial x} + \frac{\partial Q_2^{(1)}}{\partial y} - \frac{\partial z}{\partial x} A_1^{(1)} - \frac{\partial z}{\partial y} A_2^{(1)} + C_1^{(1)} = \Delta_1 M, && J = Q_1^{(1)} \frac{dy}{ds} - Q_2^{(1)} \frac{dx}{ds}, \\
&\frac{\partial R_1^{(1)}}{\partial x} + \frac{\partial R_2^{(1)}}{\partial y} + B_1^{(1)} - A_2^{(1)} = \Delta_1 N, && K = R_1^{(1)} \frac{dy}{ds} - R_2^{(1)} \frac{dx}{ds}.
\end{aligned}$$

These formulas may be deduced *a posteriori* from the ones we previously gave. For example, take the ones on the right. We have seen (se. **35**, 1) that *F*, *G*, *H* may be obtained upon replacing the expressions $\frac{d\rho_1}{ds}, \frac{d\rho_2}{ds}$ in:

$$A_1\frac{d\rho_2}{ds}-A_2\frac{d\rho_1}{ds},\qquad B_1\frac{d\rho_2}{ds}-B_2\frac{d\rho_1}{ds},\qquad C_1\frac{d\rho_2}{ds}-C_2\frac{d\rho_1}{ds},$$

with

$$-\frac{1}{\Delta}\left(\lambda\frac{\partial x}{\partial\rho_2}+\mu\frac{\partial y}{\partial\rho_2}+\nu\frac{\partial x}{\partial\rho_2}\right),\qquad \frac{1}{\Delta}\left(\lambda\frac{\partial x}{\partial\rho_1}+\mu\frac{\partial y}{\partial\rho_1}+\nu\frac{\partial x}{\partial\rho_1}\right),$$

respectively, in which λ, μ, ν denote the direction cosines of the exterior normal to C. This gives:

$$F=\frac{1}{\Delta_1}\left[\left(\lambda+\nu\frac{\partial z}{\partial x}\right)+A_1^{(1)}\left(\mu+\nu\frac{\partial z}{\partial y}\right)A_2^{(1)}\right],$$

$$G=\frac{1}{\Delta_1}\left[\left(\lambda+\nu\frac{\partial z}{\partial x}\right)+B_1^{(1)}\left(\mu+\nu\frac{\partial z}{\partial y}\right)B_2^{(1)}\right],$$

$$H=\frac{1}{\Delta_1}\left[\left(\lambda+\nu\frac{\partial z}{\partial x}\right)+C_1^{(1)}\left(\mu+\nu\frac{\partial z}{\partial y}\right)C_2^{(1)}\right],$$

and similarly:

$$I=\frac{1}{\Delta_1}\left[\left(\lambda+\nu\frac{\partial z}{\partial x}\right)P_1^{(1)}+\left(\mu+\nu\frac{\partial z}{\partial y}\right)P_2^{(1)}\right],$$

$$J=\frac{1}{\Delta_1}\left[\left(\lambda+\nu\frac{\partial z}{\partial x}\right)Q_1^{(1)}+\left(\mu+\nu\frac{\partial z}{\partial y}\right)Q_2^{(1)}\right],$$

$$K=\frac{1}{\Delta_1}\left[\left(\lambda+\nu\frac{\partial z}{\partial x}\right)R_1^{(1)}+\left(\mu+\nu\frac{\partial z}{\partial y}\right)R_2^{(1)}\right],$$

which amounts to saying that one has:

$$\frac{dy}{ds}=\frac{\lambda+\nu\dfrac{\partial z}{\partial x}}{\Delta_1},\qquad \frac{dx}{ds}=\frac{\mu+\nu\dfrac{\partial z}{\partial y}}{\Delta_1}.$$

However, these latter relations result from the formulas:

$$\lambda\frac{dx}{ds}+\mu\frac{dy}{ds}+\nu\frac{dz}{ds}=0,\qquad dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy,$$

which entails that:

$$\frac{\dfrac{dx}{ds}}{-\left(\mu+\nu\dfrac{\partial z}{\partial y}\right)}=\frac{\dfrac{dy}{ds}}{\lambda+\nu\dfrac{\partial z}{\partial x}}=\frac{\dfrac{dz}{ds}}{\lambda\dfrac{\partial z}{\partial y}-\mu\dfrac{\partial z}{\partial x}}=\frac{1}{\sqrt{1+\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2}},$$

where the sign in the latter relation corresponds to the sense in which we traverse C, which figures in the use of GREEN'S formula.

38. Introduction of new auxiliary functions provided by considering non-tri-rectangular triads formed from Mz_1' and the tangents to the curves (ρ_1) and (ρ_2). – In sec. **35**, 3, we envisioned a tri-rectangular triad $Mx_1'y_1'z_1'$ in which the Mz_1' axis is normal to (M). The formulas that give $F_0'', G_0'', H_0'', I_0'', J_0'', K_0''$ lead us to introduce new auxiliary functions; however, we may also consider the equations to be indefinite and refer them to a triad that is no longer tri-rectangular, in general, which is formed from the Mz_1' axis and the tangents to the (ρ_1) and (ρ_2) curves. This is easily accomplished by using the calculations we already performed by the intermediary of $Mx_1'y_1'z_1'$. It suffices for us to start with the equations that are obtained with regard to the latter and show the combinations:

$$X_0''\xi_1^{(1)} + Y_0''\eta_1^{(1)}, \qquad X_0''\xi_2^{(1)} + Y_0''\eta_2^{(1)}, \qquad L_0''\xi_1^{(1)} + M_0''\eta_1^{(1)}, \qquad L_0''\xi_2^{(1)} + M_0''\eta_2^{(1)}.$$

Set:

$$\mathcal{A}_1'' = \xi_2^{(1)} A_1'' + \eta_2^{(1)} B_1'', \qquad \mathcal{A}_2'' = \xi_2^{(1)} A_2'' + \eta_2^{(1)} B_2'',$$
$$\mathcal{B}_1'' = \xi_1^{(1)} A_1'' + \eta_1^{(1)} B_1'', \qquad \mathcal{B}_2'' = \xi_1^{(1)} A_2'' + \eta_1^{(1)} B_2'',$$

as well as four analogous formulas for $\mathcal{P}_1'', \mathcal{Q}_1'', \mathcal{P}_2'', \mathcal{Q}_2''$ from them, we deduce:

$$A_1'' = \frac{\eta_2^{(1)}\mathcal{B}_1'' - \eta_1^{(1)}\mathcal{A}_1''}{\Delta} \qquad A_2'' = \frac{\eta_2^{(1)}\mathcal{B}_2'' - \eta_1^{(1)}\mathcal{A}_2''}{\Delta},$$
$$B_1'' = \frac{\xi_2^{(1)}\mathcal{A}_1'' - \xi_1^{(1)}\mathcal{B}_1''}{\Delta}, \qquad B_2'' = \frac{\xi_1^{(1)}\mathcal{A}_2'' - \xi_2^{(1)}\mathcal{B}_2''}{\Delta},$$

as well as analogous formulas for $P_1'', Q_1'', P_1'', Q_2''$. The equations may be written:

$$\frac{\partial \mathcal{A}_1''}{\partial \rho_1} + \frac{\partial \mathcal{A}_2''}{\partial \rho_2} - \Sigma_2 \mathcal{A}_1'' - \Sigma_3 \mathcal{A}_2'' - \Theta_2 \mathcal{B}_1'' - \Theta_3 \mathcal{B}_2'' - \Delta\mathcal{D}' C_1'' - \Delta\mathcal{D}'' C_2'' = \Delta_0 (\xi_2^{(1)} X_0'' + \eta_2^{(1)} Y_0''),$$
$$\frac{\partial \mathcal{B}_1''}{\partial \rho_1} + \frac{\partial \mathcal{B}_2''}{\partial \rho_2} - \Sigma_1 \mathcal{A}_1'' - \Sigma_2 \mathcal{A}_2'' - \Theta_1 \mathcal{B}_1'' - \Theta_2 \mathcal{B}_2'' - \Delta\mathcal{D} C_1'' - \Delta\mathcal{D}' C_2'' = \Delta_0 (\xi_1^{(1)} X_0'' + \eta_1^{(1)} Y_0''),$$
$$\frac{\partial C_1''}{\partial \rho_1} + \frac{\partial C_2''}{\partial \rho_2} - \frac{\mathcal{E}\mathcal{D}' - \mathcal{F}\mathcal{D}}{\Delta}\mathcal{A}_1'' - \frac{\mathcal{G}''\mathcal{D} - \mathcal{F}\mathcal{D}'}{\Delta}\mathcal{B}_1'' + \frac{\mathcal{E}\mathcal{D}'' - \mathcal{F}\mathcal{D}'}{\Delta}\mathcal{A}_2'' + \frac{\mathcal{G}\mathcal{D} - \mathcal{F}\mathcal{D}''}{\Delta}\mathcal{B}_2'' = \Delta_0 Z_0'',$$

$$\frac{\partial \mathcal{P}_1''}{\partial \rho_1} + \frac{\partial \mathcal{P}_2''}{\partial \rho_2} - \Sigma_2 \mathcal{P}_1'' - \Sigma_3 \mathcal{P}_2'' - \Theta_2 \mathcal{Q}_1'' - \Theta_3 \mathcal{Q}_2'' - \Delta\mathcal{D}' R_1'' - \Delta\mathcal{D}'' R_2'' - \Delta C_1'' = \Delta_0 (\xi_2^{(1)} L_0'' + \eta_2^{(1)} M_0'')$$

$$\frac{\partial \mathcal{Q}_1''}{\partial \rho_1} + \frac{\partial \mathcal{Q}_2''}{\partial \rho_2} - \Sigma_1 \mathcal{P}_1'' - \Sigma_2 \mathcal{P}_2'' - \Theta_1 \mathcal{Q}_1'' - \Theta_2 \mathcal{Q}_2'' - \Delta \mathcal{D} R_1'' - \Delta \mathcal{D}' R_2'' + \Delta C_2'' = \Delta_0 (\xi_1^{(1)} L_0'' + \eta_1^{(1)} M_0''),$$

$$\frac{\partial R_1''}{\partial \rho_1} + \frac{\partial R_1''}{\partial \rho_2} - \frac{\mathcal{E}\mathcal{D}' - \mathcal{F}\mathcal{D}}{\Delta} \mathcal{P}_1'' + \frac{\mathcal{G}\mathcal{D} - \mathcal{F}\mathcal{D}'}{\Delta} \mathcal{Q}_1'' + \frac{\mathcal{E}\mathcal{D}'' - \mathcal{F}\mathcal{D}'}{\Delta} \mathcal{P}_2'' + \frac{\mathcal{G}\mathcal{D}' - \mathcal{F}\mathcal{D}''}{\Delta} \mathcal{Q}_2''$$
$$+ \frac{\mathcal{E}\mathcal{A}_1'' + \mathcal{F}(\mathcal{A}_2'' - \mathcal{B}_1'') - \mathcal{G}\mathcal{B}_2''}{\Delta} = \Delta_0 N_0''.$$

In these formulas, the six CHRISTOFFEL symbols are designated by Σ_1, Σ_2, Σ_3, Θ_1, Θ_2, Θ_3:

$$\Sigma_1 = \frac{-\mathcal{E}\dfrac{\partial \mathcal{E}}{\partial \rho_2} + 2\mathcal{E}\dfrac{\partial \mathcal{F}}{\partial \rho_1} - \mathcal{F}\dfrac{\partial \mathcal{E}}{\partial \rho_1}}{2\Delta^2}, \qquad \Theta_1 = \frac{\mathcal{G}\dfrac{\partial \mathcal{E}}{\partial \rho_1} + \mathcal{F}\dfrac{\partial \mathcal{E}}{\partial \rho_1} - 2\mathcal{F}\dfrac{\partial \mathcal{F}}{\partial \rho_1}}{2\Delta^2},$$

$$\Sigma_2 = \frac{\mathcal{E}\dfrac{\partial \mathcal{G}}{\partial \rho_1} - \mathcal{F}\dfrac{\partial \mathcal{E}}{\partial \rho_2}}{2\Delta^2}, \qquad \Theta_2 = \frac{\mathcal{G}\dfrac{\partial \mathcal{E}}{\partial \rho_2} - \mathcal{F}\dfrac{\partial \mathcal{G}}{\partial \rho_1}}{2\Delta^2},$$

$$\Sigma_3 = \frac{\mathcal{E}\dfrac{\partial \mathcal{G}}{\partial \rho_2} + \mathcal{F}\dfrac{\partial \mathcal{G}}{\partial \rho_1} - 2\mathcal{F}\dfrac{\partial \mathcal{F}}{\partial \rho_2}}{2\Delta^2}, \qquad \Theta_3 = \frac{2\mathcal{G}\dfrac{\partial \mathcal{F}}{\partial \rho_2} - \mathcal{G}\dfrac{\partial \mathcal{G}}{\partial \rho_1} - \mathcal{F}\dfrac{\partial \mathcal{G}}{\partial \rho_2}}{2\Delta^2},$$

and we let $\Delta^2\mathcal{D}, \Delta^2\mathcal{D}', \Delta^2\mathcal{D}''$ denote the three determinants that are defined by the identity ([1]):

$$\Delta^2(\mathcal{D}\,d\rho_1^2 + 2\mathcal{D}'\,d\rho_1 d\rho_2 + \mathcal{D}''\,d\rho_2^2) = \begin{vmatrix} \dfrac{\partial x}{\partial \rho_1} & \dfrac{\partial x}{\partial \rho_2} & \dfrac{\partial^2 x}{\partial \rho_1^2} d\rho_1^2 + 2\dfrac{\partial^2 x}{\partial \rho_1 \partial \rho_2} d\rho_1 d\rho_2 + \dfrac{\partial^2 x}{\partial \rho_2^2} d\rho_2^2 \\ \dfrac{\partial y}{\partial \rho_1} & \dfrac{\partial y}{\partial \rho_2} & \dfrac{\partial^2 y}{\partial \rho_1^2} d\rho_1^2 + 2\dfrac{\partial^2 y}{\partial \rho_1 \partial \rho_2} d\rho_1 d\rho_2 + \dfrac{\partial^2 y}{\partial \rho_2^2} d\rho_2^2 \\ \dfrac{\partial z}{\partial \rho_1} & \dfrac{\partial z}{\partial \rho_2} & \dfrac{\partial^2 z}{\partial \rho_1^2} d\rho_1^2 + 2\dfrac{\partial^2 z}{\partial \rho_1 \partial \rho_2} d\rho_1 d\rho_2 + \dfrac{\partial^2 z}{\partial \rho_2^2} d\rho_2^2 \end{vmatrix}.$$

In the preceding calculations, we used the relations:

$$p_1^{(1)} = \xi_1^{(1)}\mathcal{D}' - \xi_2^{(1)}\mathcal{D}, \qquad q_1^{(1)} = \eta_1^{(1)}\mathcal{D}' - \eta_2^{(1)}\mathcal{D},$$
$$p_2^{(1)} = \xi_1^{(1)}\mathcal{D}'' - \xi_2^{(1)}\mathcal{D}', \qquad q_2^{(1)} = \eta_1^{(1)}\mathcal{D}'' - \eta_2^{(1)}\mathcal{D}',$$

$$\frac{\partial \xi_1}{\partial \rho_1} - \eta_1 r_1 = \Theta_1 \xi_1 + \Sigma_1 \xi_2, \qquad \frac{\partial \xi_2}{\partial \rho_1} - \eta_2 r_1 = \frac{\partial \xi_1}{\partial \rho_2} - \eta_1 r_2 = \Theta_2 \xi_1 + \Sigma_2 \xi_2,$$

$$\frac{\partial \xi_2}{\partial \rho_2} - \eta_2 r_2 = \Theta_3 \xi_1 + \Sigma_3 \xi_2,$$

[1] As we will reiterate later on, here we are letting $\Delta^2\mathcal{D}, \Delta^2\mathcal{D}', \Delta^2\mathcal{D}''$ denote the quantities that DARBOUX denoted by D, D', D''.

$$\frac{\partial \eta_1}{\partial \rho_1} + \xi_1 r_1 = \Theta_1 \xi_1 + \Sigma_1 \eta_2, \qquad \frac{\partial \eta_2}{\partial \rho_1} + \xi_2 r_1 = \frac{\partial \eta_1}{\partial \rho_2} + \xi_1 r_2 = \Theta_2 \eta_1 + \Sigma_2 \eta_2,$$

$$\frac{\partial \eta_2}{\partial \rho_2} + \xi_2 r_2 = \Theta_3 \eta_1 + \Sigma_3 \eta_2.$$

39. External virtual work; a theorem analogous to those of Varignon and Saint-Guilhem. Remarks on the auxiliary functions introduced in the preceding sections. – We give the name *external virtual work* done on the deformed surface (M) by an arbitrary virtual deformation to the expression:

$$\delta \mathcal{T}_e = -\int_{C_0} (F_0' \delta' x + G_0' \delta' y + H_0' \delta' z + I_0' \delta I' + J_0' \delta J' + K_0' \delta K') ds_0$$
$$+ \iint_{C_0} (X_0' \delta' x + Y_0' \delta' y + Z_0' \delta' z + L_0' \delta I' + M_0' \delta J' + N_0' \delta K') \Delta_0 d\rho_1 d\rho_2.$$

One may give other forms to this formula by introducing other elements. For example, suppose that one introduces the expressions X_0, Y_0, Z_0, L_0, M_0, N_0; F_0, G_0, H_0, I_0, J_0, K_0. To that effect, we let δI, δJ, δK denote the projections onto the fixed axes of the segment whose projections on Mx', My', Mz' are $\delta I', \delta J', \delta K'$, in such a way that, for example:

$$-\delta I = \alpha'' \delta \alpha' + \beta'' \delta \beta' + \gamma'' \delta \gamma' = -(\alpha' \delta \alpha'' + \beta' \delta \beta'' + \gamma' \delta \gamma''),$$

by always supposing that the axes we are considering have the same disposition. We then have:

$$\delta \mathcal{T}_e = -\int_{C_0} (F_0 \delta x + G_0 \delta y + H_0 \delta z + I_0 \delta I + J_0 \delta J + K_0 \delta K) ds_0$$
$$+ \iint_{C_0} (X_0 \delta x + Y_0 \delta y + Z_0 \delta z + L_0 \delta I + M_0 \delta J + N_0 \delta K) \Delta_0 d\rho_1 d\rho_2.$$

The force (X_0', Y_0', Z_0') or (X_0, Y_0, Z_0), the moment (L_0', M_0', N_0') or (L_0, M_0, N_0) are referred to the unit of area of the *non-deformed* surface. The effort (F_0', G_0', H_0') or (F_0, G_0, H_0), and the moment of deformation (I_0', J_0', K_0') or (I_0, J_0, K_0) are referred to the unit of length of the *non-deformed* contour C_0.

Start with the formula:

$$\iint_{C_0} \delta(W \Delta_0) d\rho_1 d\rho_2 = -\delta \mathcal{T}_e$$

taken over an arbitrary portion of the deformable surface bounded by a contour C_0.

Since $\delta(W\Delta_0)$ must be identically null, by virtue of the invariance of W and Δ_0 under the group of Euclidian displacements, when the variations δx, δy, δz are given by the formulas (9), page (?), namely:

$$\delta x = (a_1 + \omega_2 z - \omega_3 y)\delta t,$$
$$\delta y = (a_2 + \omega_3 x - \omega_1 z)\delta t,$$
$$\delta z = (a_3 + \omega_1 y - \omega_2 x)\delta t,$$

and δI, δJ, δK are given by:

$$\delta I = \omega_1 \delta t, \qquad \delta J = \omega_2 \delta t, \qquad \delta K = \omega_3 \delta t,$$

and the fact that this is true for any values of a_1, a_2, a_3, ω_1, ω_2, ω_3, we conclude, from the preceding expression of $\delta\mathcal{T}_e$, that one has:

$$\int_{C_0} F_0 ds_0 - \iint_{C_0} X_0 \Delta_0 d\rho_1 d\rho_2 = 0, \qquad \int_{C_0} G_0 ds_0 - \iint_{C_0} Y_0 \Delta_0 d\rho_1 d\rho_2 = 0,$$
$$\int_{C_0} H_0 ds_0 - \iint_{C_0} Z_0 \Delta_0 d\rho_1 d\rho_2 = 0,$$
$$\int_{C_0} (I_0 + yH_0 - zG_0) ds_0 - \iint_{C_0} (L_0 + yZ_0 - zY_0) \Delta_0 d\rho_1 d\rho_2 = 0,$$

and two analogous formulas.

These six formulas that are easily deduced from the ones that one ordinarily writes by means of the principle of solidification ([1]). *In these formulas, one may imagine that the contour C_0 is variable.*

The auxiliary functions that were introduced in the preceding sections are not the only ones that one may envision. We restrict ourselves to their consideration and simply add several obvious remarks.

By definition, we have introduced two systems of efforts and moments of deformation relative to a point M of the deformed surface. The first ones are the ones that are exerted on the curves (ρ_1) and (ρ_2). The others are the ones that are exerted on orthogonal curves that are arbitrary and to be specified, with tangents Mx_1', My_2' that have arbitrary rectangular and unspecified directions in the plane that is tangent to (M) at M.

Now suppose that one introduces the function W. The first efforts and moments of deformation have the expressions we already indicated, and one immediately deduces the expressions relative to the second from this. However, in these calculations one may explicitly describe the functions that one encounters according to the nature of the problem, and which are, for example, x, y, z, and three parameters ([2]) λ_1, λ_2, λ_3, by means of which one expresses $\alpha, \alpha', \cdots, \gamma''$.

If one introduces x, y, z, λ_1, λ_2, λ_3, and if one continues to let W denote the function that depends on ρ_1, ρ_2, the first derivatives of x, y, z with respect to ρ_1, ρ_2 on λ_1, λ_2, λ_3, and their first derivatives with respect to ρ_1, ρ_2, and, after replacing the different

[1] The passage from elements referred to the unit of area of (M_0) and the length of C_0 to elements referred to the unit of area of (M) and length of C is so immediate that it suffices to limit ourselves to the first ones, for example, as we have done.

[2] For such auxiliary functions λ_1, λ_2, λ_3, one may take, for example, the components of the rotation that makes the axes Ox, Oy, Oz parallel to Mx', My', Mz', respectively.

quantities ξ_i, η_i, ζ_i, p_i, q_i, r_i with the values they are given by means of formulas (30) and (31), we will have:

$$A_i = \Delta_0 \frac{\partial W}{\partial \dfrac{\partial x}{\partial \rho_i}}, \qquad B_i = \Delta_0 \frac{\partial W}{\partial \dfrac{\partial y}{\partial \rho_i}}, \qquad C_i = \Delta_0 \frac{\partial W}{\partial \dfrac{\partial z}{\partial \rho_i}},$$

$$\Pi_i = \Delta_0 \frac{\partial W}{\partial \dfrac{\partial \lambda_1}{\partial \rho_i}}, \qquad \Xi_i = \Delta_0 \frac{\partial W}{\partial \dfrac{\partial \lambda_2}{\partial \rho_i}}, \qquad \Sigma_i = \Delta_0 \frac{\partial W}{\partial \dfrac{\partial \lambda_3}{\partial \rho_i}}.$$

40. Notion of the energy of deformation. Natural state of a deformable surface. – Envision two states (M_0) and (M) of the deformable surface bounded by the contours C_0 and C, and consider an arbitrary sequence of states starting with (M_0) and ending with (M). To accomplish this, it suffices to consider functions $x, y, z, \alpha, \alpha', \cdots, \gamma''$ of ρ_1, ρ_2, and a variable h such that for the value 0 of h the functions reduce to x_0, y_0, z_0, x, y, z, $\alpha_0, \alpha'_0, \cdots, \gamma''_0$, respectively, and for the value h of h they reduce to the values x, y, z, $\alpha, \alpha', \cdots, \gamma''$ relative to (M).

If we make the parameter h vary in a continuous fashion then we obtain a continuous deformation that permits us to pass from the state (M_0) to the state (M). Imagine the *total work* performed by the external forces and moments that are applied to the different surface elements of the surface and the efforts and moments of deformation that are applied to the contour during this continuous deformation. To obtain this total work, it suffices to take the differential obtained by starting with one of the expressions for $\delta\mathcal{T}_e$ in the preceding section, substituting the partial differentials that correspond to increases dh in h for the variations x, y, z, $\alpha, \alpha', \cdots, \gamma''$ in that expression, and integrate it from 0 to h. Since the formula:

$$\delta\mathcal{T}_e = -\iint_{C_0} \delta(W\Delta_0)\, d\rho_1 d\rho_2$$

gives the expression $-\iint_{C_0} \dfrac{\partial(W\Delta_0)}{\partial h} dh\, d\rho_1 d\rho_2$ for the actual value of $\delta\mathcal{T}_e$, we obtain:

$$-\int_0^h \left(\iint_{C_0} \frac{\partial(W\Delta_0)}{\partial h} d\rho_1 d\rho_2 \right) dh = -\iint_{C_0} \left[(W\Delta_0)_M - (W\Delta_0)_{M_0} \right] d\rho_1 d\rho_2$$

for the total work.

The work considered is independent of the intermediary states and depends on only the extreme states considered (M_0) and (M).

This leads us to introduce the notion of the *energy of deformation*, which must be distinguished from that of action as we previously envisioned. We say that – W is the *density of the energy of deformation* referred to the unit of area of the non-deformed surface.

These considerations are only the repetition of the ones that we presented in sec. **12**; similarly, the observations relating to the *natural state* of the deformable line, which was the object of sec. **13**, may be reproduced with regard to the deformable surface.

41. Notion of hidden triad and of hidden *W*. – In the study of the deformable surface, as it is in the case of the deformable line, it is natural to direct one's attention to the particular manner in which the geometric surface is drawn by the deformable surface. This amounts to thinking in terms of x, y, z and considering $\alpha, \alpha', \cdots, \gamma''$ as simple auxiliary functions. This is what we may likewise express by imagining that one ignores the existence of the triads that determine the deformable surface and that one knows only the vertices of these triads. If we take this viewpoint in order to envision the partial differential equations that one is led to in this case then we may introduce the notion of *hidden triad,* and we are led to a resulting classification of the various circumstances that may present themselves when we eliminate $\alpha, \alpha', \cdots, \gamma''$.

The first study that presents itself is that of the reductions that are produced by the elimination of $\alpha, \alpha', \cdots, \gamma''$. In the corresponding particular case in which attention is devoted almost exclusively on the point-like surface that is drawn by the deformed surface, one may sometimes make a similar abstraction of (M_0), and, as a result, of the deformation that permits us to pass from (M_0) to (M). It is by taking the latter viewpoint that we may recover the surface called flexible and inextensible in geometry.

The triad may be employed in another fashion: we may make particular hypotheses on it and, similarly, on the surface (M). All of this amounts to envisioning particular deformations of the free deformable surface. If the relations that we impose are simple relations between ξ_i, η_i, ζ_i, p_i, q_i, r_i, as will be the case in the applications we will study, then we may account for the relations in the calculation of W and deduce more particular functions from W. The interesting question that is posed is to simply introduce these functions and consider the general function W that serves as our point of departure as *hidden,* in some sense. We thus have a *theory that will be special to the particular deformations that are suggested by the given relations* ξ_i, η_i, ζ_i, p_i, q_i, r_i.

We confirm that one may thus collect all of the particular cases and give the same origin to the equations that are the result of special problems whose solutions have only been begun up till now by means of the theory of the free deformable surface. In the latter problems, one sometimes finds oneself in the proper circumstances to avoid the consideration of deformations. In reality, they still need to be completed. This is what one may do in practical applications when we envision infinitely small deformations.

Take the case where the external force and moment refer, at the very most, to only the first derivatives of the unknowns x, y, z and λ_1, λ_2, λ_3. The second derivatives of these unknowns will be introduced into the partial differential equations only by W; however, the derivatives of x, y, z figure only in ξ_i, η_i, ζ_i, and those of λ_1, λ_2, λ_3 present themselves only in p_i, q_i, r_i. One sees that if W depends only upon ξ_i, η_i, ζ_i or only upon p_i, q_i, r_i then there will be a reduction of the orders of the derivatives that enter into the system of partial differential equations. We proceed to examine the first of these two cases.

42. Case where *W* depends only upon ρ_1, ρ_1, ξ_1, η_1, ζ_1, ξ_2, η_2, ζ_2 . The surface that leads to the membrane studied by Poisson and Lamé in the case of the infinitely small deformation. The fluid surface that refers to the surface envisioned by Lagrange, Poisson, and Duhem as a particular case. – Suppose that *W* depends only on the quantities ρ_i, ξ_i, η_i, ζ_i, and not on the p_i, q_i, r_i. The equations reduce to the following:

$$\frac{\partial}{\partial\rho_1}\frac{\partial(W\Delta_0)}{\partial\frac{\partial x}{\partial\rho_1}}+\frac{\partial}{\partial\rho_2}\frac{\partial(W\Delta_0)}{\partial\frac{\partial x}{\partial\rho_2}}=\Delta_0 X_0, \qquad \frac{\partial W}{\partial\lambda_1}+\Delta_0\,\mathrm{L}_0=0,$$

$$\frac{\partial}{\partial\rho_1}\frac{\partial(W\Delta_0)}{\partial\frac{\partial y}{\partial\rho_1}}+\frac{\partial}{\partial\rho_2}\frac{\partial(W\Delta_0)}{\partial\frac{\partial y}{\partial\rho_2}}=\Delta_0 Y_0, \qquad \frac{\partial W}{\partial\lambda_2}+\Delta_0\,\mathrm{L}_0=0,$$

$$\frac{\partial}{\partial\rho_1}\frac{\partial(W\Delta_0)}{\partial\frac{\partial z}{\partial\rho_1}}+\frac{\partial}{\partial\rho_2}\frac{\partial(W\Delta_0)}{\partial\frac{\partial z}{\partial\rho_2}}=\Delta_0 Z_0, \qquad \frac{\partial W}{\partial\lambda_3}+\Delta_0\,\mathrm{L}_0=0,$$

in which *W* depends only on ρ_1, ρ_2, $\frac{\partial x}{\partial\rho_1},\cdots,\frac{\partial z}{\partial\rho_2}$, λ_1, λ_2, λ_3 . If we take the simple case where X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are given functions ([1]) of ρ_1, ρ_2, x, y, z, $\frac{\partial x}{\partial\rho_1},\cdots,\frac{\partial z}{\partial\rho_2}$, λ_1, λ_2, λ_3 they show us that the three equations may be solved with respect to λ_1, λ_2, λ_3, and one finally obtains three partial differential equations that, under our hypotheses, refer only to ρ_1, ρ_2, x, y, z, and their first and second derivatives.

We confine ourselves to the particular case in which the given functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ are null. The same will be true for the corresponding values of the functions of any arbitrary one of the systems: (L_0, M_0, N_0), (L_0', M_0', N_0'), (L_0'', M_0'', N_0''). It then results from this that the equations:

$$\frac{\partial W}{\partial\lambda_1}=0, \qquad \frac{\partial W}{\partial\lambda_2}=0, \qquad \frac{\partial W}{\partial\lambda_3}=0$$

amount to either:

$$\frac{\partial y}{\partial\rho_1}C_1-\frac{\partial z}{\partial\rho_1}B_1+\frac{\partial y}{\partial\rho_2}C_2-\frac{\partial z}{\partial\rho_2}C_2=0,$$

$$\frac{\partial z}{\partial\rho_1}A_1-\frac{\partial x}{\partial\rho_1}C_1+\frac{\partial z}{\partial\rho_2}A_2-\frac{\partial x}{\partial\rho_2}C_2=0,$$

$$\frac{\partial x}{\partial\rho_1}B_1-\frac{\partial y}{\partial\rho_1}A_1+\frac{\partial x}{\partial\rho_2}B_2-\frac{\partial y}{\partial\rho_2}A_2=0,$$

---

[1] To simplify the discussion and indicate more easily what we will be alluding to, we suppose that X_0, Y_0, Z_0, $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$ do not refer to the derivatives of λ_1, λ_2, λ_3 .

or to:

$$S_1 = 0, \qquad S_2 = 0, \qquad S_3 = 0$$

in such a way that the effort at a point of an arbitrary curve is in the plane tangent to the deformed surface and the truncated efforts that are exerted on two rectangular directions are equal.

This said, observe that if one starts with two positions (M_0) and (M_1), which are assumed to be *given*, and one deduces the functions $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$, as in sections **34** and **35**, then, in the case where these three functions are null, one may arrive at this result by accident, i.e., for a certain set of particular deformations. However, one may arrive at this result in the case of arbitrary deformations of (M) as well, since it is a consequence of the nature of (M), i.e., of the form of W.

Envision this latter case, which is particularly interesting. W is then a simple function ([1]) of ρ_1, ρ_2, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ with the latter three quantities being defined by formula (32) of sec. **31**. The equations deduced in sec. **34** and **35** then reduce to either:

$$\sum_i \left(\frac{\partial A_i'}{\partial \rho_i} + q_i C_i' - r_i B_i'\right) = \Delta_0 X_0', \qquad F_0' = A_1' \frac{d\rho_2}{ds_0} - A_2' \frac{d\rho_1}{ds_0},$$

$$\sum_i \left(\frac{\partial B_i'}{\partial \rho_i} + r_i A_i' - p_i C_i'\right) = \Delta_0 Y_0', \qquad G_0' = B_1' \frac{d\rho_2}{ds_0} - B_2' \frac{d\rho_1}{ds_0},$$

$$\sum_i \left(\frac{\partial C_i'}{\partial \rho_i} + p_i' B_i' - q_i' A_i'\right) = \Delta_0 Z_0', \qquad H_0' = C_1' \frac{d\rho_2}{ds_0} - C_2' \frac{d\rho_1}{ds_0},$$

in which one has:

$$A_1' = \Delta_0\left(2\xi_1 \frac{\partial W}{\partial \mathcal{E}} + \xi_2 \frac{\partial W}{\partial \mathcal{F}}\right), \qquad B_1' = \Delta_0\left(2\eta_1 \frac{\partial W}{\partial \mathcal{E}} + \eta_2 \frac{\partial W}{\partial \mathcal{F}}\right),$$

$$C_1' = \Delta_0\left(2\varsigma_1 \frac{\partial W}{\partial \mathcal{E}} + \varsigma_2 \frac{\partial W}{\partial \mathcal{F}}\right)$$

$$A_2' = \Delta_0\left(\xi_1 \frac{\partial W}{\partial \mathcal{F}} + 2\xi_2 \frac{\partial W}{\partial \mathcal{G}}\right), \qquad B_2' = \Delta_0\left(\eta_1 \frac{\partial W}{\partial \mathcal{F}} + 2\eta_2 \frac{\partial W}{\partial \mathcal{G}}\right)$$

$$C_2' = \Delta_0\left(\varsigma_1 \frac{\partial W}{\partial \mathcal{F}} + 2\varsigma_2 \frac{\partial W}{\partial \mathcal{G}}\right),$$

or to:

$$\frac{\partial A_1}{\partial \rho_1} + \frac{\partial A_2}{\partial \rho_2} = \Delta_0 X_0, \qquad F_0 = A_1 \frac{d\rho_2}{ds_0} - A_2 \frac{d\rho_1}{ds_0},$$

$$\frac{\partial B_1}{\partial \rho_1} + \frac{\partial B_2}{\partial \rho_2} = \Delta_0 Y_0, \qquad G_0 = B_1 \frac{d\rho_2}{ds_0} - B_2 \frac{d\rho_1}{ds_0},$$

[1] The triad is completely hidden; we may also imagine that we have a simple pointlike surface.

$$\frac{\partial C_1}{\partial \rho_1} + \frac{\partial C_2}{\partial \rho_2} = \Delta_0 Z_0, \qquad H_0 = C_1 \frac{d\rho_2}{ds_0} - C_2 \frac{d\rho_1}{ds_0},$$

in which:

$$A_1 = \Delta_0 \left(2 \frac{\partial x}{\partial \rho_1} \frac{\partial W}{\partial \mathcal{E}} + \frac{\partial x}{\partial \rho_2} \frac{\partial W}{\partial \mathcal{F}} \right), \qquad B_1 = \Delta_0 \left(2 \frac{\partial y}{\partial \rho_1} \frac{\partial W}{\partial \mathcal{E}} + \frac{\partial y}{\partial \rho_2} \frac{\partial W}{\partial \mathcal{F}} \right),$$

$$C_1 = \Delta_0 \left(2 \frac{\partial z}{\partial \rho_1} \frac{\partial W}{\partial \mathcal{E}} + \frac{\partial z}{\partial \rho_2} \frac{\partial W}{\partial \mathcal{F}} \right),$$

$$A_2 = \Delta_0 \left(\frac{\partial x}{\partial \rho_1} \frac{\partial W}{\partial \mathcal{F}} + 2 \frac{\partial x}{\partial \rho_2} \frac{\partial W}{\partial \mathcal{G}} \right), \qquad B_2 = \Delta_0 \left(\frac{\partial y}{\partial \rho_1} \frac{\partial W}{\partial \mathcal{F}} + 2 \frac{\partial y}{\partial \rho_2} \frac{\partial W}{\partial \mathcal{G}} \right),$$

$$C_2 = \Delta_0 \left(\frac{\partial z}{\partial \rho_1} \frac{\partial W}{\partial \mathcal{F}} + 2 \frac{\partial z}{\partial \rho_2} \frac{\partial W}{\partial \mathcal{G}} \right),$$

or to:

$$\sum_i \left(\frac{\partial A_i''}{\partial \rho_i} + q_i^{(1)} C_i'' - r_i^{(1)} B_i'' \right) = \Delta_0 X_0'', \qquad F_0'' = A_1'' \frac{d\rho_2}{ds_0} - A_2'' \frac{d\rho_1}{ds_0},$$

$$\sum_i \left(\frac{\partial B_i''}{\partial \rho_i} + r_i^{(1)} A_i'' - p_i^{(1)} C_i'' \right) = \Delta_0 Y_0'', \qquad G_0'' = B_1'' \frac{d\rho_2}{ds_0} - B_2'' \frac{d\rho_1}{ds_0},$$

$$\sum_i \left(\frac{\partial C_i''}{\partial \rho_i} + p_i^{(1)} B_i'' - q_i^{(1)} A_i'' \right) = \Delta_0 Z_0'', \qquad H_0'' = C_1'' \frac{d\rho_2}{ds_0} - C_2'' \frac{d\rho_1}{ds_0},$$

in which:

$$A_1'' = \Delta_0 \left(2\xi_1^{(1)} \frac{\partial W}{\partial \mathcal{E}} + \xi_2^{(1)} \frac{\partial W}{\partial \mathcal{F}} \right), \qquad B_1'' = \Delta_0 \left(2\eta_1^{(1)} \frac{\partial W}{\partial \mathcal{E}} + \eta_2^{(1)} \frac{\partial W}{\partial \mathcal{F}} \right),$$

$$C_1'' = \Delta_0 \left(2\varsigma_1^{(1)} \frac{\partial W}{\partial \mathcal{E}} + \varsigma_2^{(1)} \frac{\partial W}{\partial \mathcal{F}} \right),$$

$$A_2'' = \Delta_0 \left(\xi_1^{(1)} \frac{\partial W}{\partial \mathcal{F}} + 2\xi_2^{(1)} \frac{\partial W}{\partial \mathcal{G}} \right), \qquad B_2'' = \Delta_0 \left(\eta_1^{(1)} \frac{\partial W}{\partial \mathcal{F}} + 2\eta_2^{(1)} \frac{\partial W}{\partial \mathcal{G}} \right),$$

$$C_2'' = \Delta_0 \left(\varsigma_1^{(1)} \frac{\partial W}{\partial \mathcal{F}} + 2\varsigma_2^{(1)} \frac{\partial W}{\partial \mathcal{G}} \right),$$

or, finally, to the equations:

$$\frac{\partial}{\partial \rho_1} \begin{vmatrix} N_1 & \xi_2^{(1)} \\ T & \eta_2^{(1)} \end{vmatrix} - r_1^{(1)} \begin{vmatrix} T & \xi_2^{(1)} \\ N_2 & \eta_2^{(1)} \end{vmatrix} + \frac{\partial}{\partial \rho_2} \begin{vmatrix} \xi_1^{(1)} & N_1 \\ \eta_1^{(1)} & T \end{vmatrix} - r_2^{(1)} \begin{vmatrix} \xi_1^{(1)} & T \\ \eta_1^{(1)} & N_2 \end{vmatrix} = \Delta_0 X_0'',$$

$$\frac{\partial}{\partial \rho_1} \begin{vmatrix} T & \xi_2^{(1)} \\ N_2 & \eta_2^{(1)} \end{vmatrix} + r_1^{(1)} \begin{vmatrix} N_1 & \xi_2^{(1)} \\ T & \eta_2^{(1)} \end{vmatrix} + \frac{\partial}{\partial \rho_2} \begin{vmatrix} \xi_1^{(1)} & T \\ \eta_1^{(1)} & N_2 \end{vmatrix} + r_2^{(1)} \begin{vmatrix} \xi_1^{(1)} & N_1 \\ \eta_1^{(1)} & T \end{vmatrix} = \Delta_0 Y_0'',$$

$$p_1^{(1)}\begin{vmatrix} T & \xi_2^{(1)} \\ N_2 & \eta_2^{(1)} \end{vmatrix} - q_1^{(1)}\begin{vmatrix} N_1 & \xi_2^{(1)} \\ T & \eta_2^{(1)} \end{vmatrix} + p_2^{(1)}\begin{vmatrix} \xi_1^{(1)} & T \\ \eta_1^{(1)} & N_2 \end{vmatrix} - q_2^{(1)}\begin{vmatrix} \xi_1^{(1)} & N_1 \\ \eta_1^{(1)} & T \end{vmatrix} = \Delta_0 Z_0'',$$

$$F_0''\frac{ds_0}{ds} = \lambda'' N_1 + \mu'' T, \qquad G_0''\frac{ds_0}{ds} = \lambda'' T + \mu'' N_2, \qquad H_0'' = 0,$$

in which:

$$N_1 = 2\frac{\Delta_0}{\Delta}\left\{(\xi_1^{(1)})^2\frac{\partial W}{\partial \mathcal{E}} + \xi_1^{(1)}\xi_2^{(1)}\frac{\partial W}{\partial \mathcal{F}} + (\xi_2^{(1)})^2\frac{\partial W}{\partial \mathcal{G}}\right\},$$

$$T = \frac{\Delta_0}{\Delta}\left\{2\xi_1^{(1)}\eta_1^{(1)}\frac{\partial W}{\partial \mathcal{E}} + (\xi_2^{(1)}\eta_1^{(1)} + \xi_1^{(1)}\eta_2^{(1)})\frac{\partial W}{\partial \mathcal{F}} + 2\xi_2^{(1)}\eta_2^{(1)}\frac{\partial W}{\partial \mathcal{G}}\right\},$$

$$N_2 = 2\frac{\Delta_0}{\Delta}\left\{(\eta_1^{(1)})^2\frac{\partial W}{\partial \mathcal{E}} + \eta_1^{(1)}\eta_2^{(1)}\frac{\partial W}{\partial \mathcal{F}} + (\eta_2^{(1)})^2\frac{\partial W}{\partial \mathcal{G}}\right\}.$$

As we said, the effort is in the plane tangent to the deformed surface. N_1 and N_2 are normal efforts, i.e., efforts of tension or compression. T is an effort that is tangent to the linear element on which it is exerted, i.e., a truncated effort.

The consideration of infinitely small deformations that are applied to the preceding surface permits us to recover the surface or membrane that was studied by POISSON and LAMÉ ([1]).

Observe that, in addition to the formula that we already used to obtain Δ, we also have the following:

$$\mathcal{E} = (\xi_1^{(1)})^2 + (\eta_1^{(1)})^2, \qquad \mathcal{F} = \xi_1^{(1)}\xi_2^{(1)} + \eta_1^{(1)}\eta_2^{(1)}, \qquad \mathcal{G} = (\xi_2^{(1)})^2 + (\eta_2^{(1)})^2,$$

by virtue of which N_1, T, N_2 may be considered as the functions that are determined by ρ_1, ρ_2 and $\xi_1^{(1)}, \xi_2^{(1)}, \eta_1^{(1)}, \eta_2^{(1)}$.

A particularly interesting case, which we call the case of the *fluid surface*, is obtained upon supposing, in regard to the three functions so defined, that one has:

$$T = 0, \qquad N_1 = N_2.$$

If one observes that one has the identities ([2]):

$$(\xi_1^{(1)})^2\mathcal{G} - 2\xi_1^{(1)}\xi_2^{(1)}\mathcal{F} + (\xi_2^{(1)})^2\mathcal{E} = \Delta^2,$$
$$\xi_1^{(1)}\eta_1^{(1)}\mathcal{G} - (\xi_1^{(1)}\eta_2^{(1)} + \xi_2^{(1)}\eta_1^{(1)})\mathcal{F} + \xi_2^{(1)}\eta_2^{(1)}\mathcal{E} = 0,$$
$$(\eta_1^{(1)})^2\mathcal{G} - 2\eta_1^{(1)}\eta_2^{(1)}\mathcal{F} + (\eta_2^{(1)})^2\mathcal{E} = \Delta^2,$$

[1] POISSON. – *Mémoire sur le mouvement des corps élastiques*, pp. 488 ff., *Mém. de l'Inst.*, T. VIII, 1829; G. LAMÉ, *Leçons sur la théorie mathématique de l'élasticité des corps solides*, 2nd edition, 1866, 9th and 10th Lessons.

[2] By virtue of the second of these identities, if $T = 0$ *for any linear element* then one is led to the conditions that follow, and, as a result, $N_1 = N_2$; one may content oneself by setting $T = 0$.

that result from the values:

$$\mathcal{E} = (\xi_1^{(1)})^2 + (\eta_1^{(1)})^2, \qquad \mathcal{F} = \xi_1^{(1)}\xi_2^{(1)} + \eta_1^{(1)}\eta_2^{(1)}, \quad \mathcal{G} = (\xi_2^{(1)})^2 + (\eta_2^{(1)})^2$$

for the expressions $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ that were defined by formula (32), one sees that the two conditions that we must set amount to the following:

$$\frac{\dfrac{\partial W}{\partial \mathcal{E}}}{\mathcal{G}} = -\frac{\dfrac{\partial W}{\partial \mathcal{F}}}{2\mathcal{F}} = \frac{\dfrac{\partial W}{\partial \mathcal{G}}}{\mathcal{E}},$$

which entails that W depends on $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ only by the intermediary of the quantity $\Delta = \sqrt{\mathcal{E}\mathcal{F} - \mathcal{G}^2}$ and is, as a result, a function of ρ_1, ρ_2, and $\mu = \dfrac{\Delta}{\Delta_0} - 1$. While continuing to denote the expression of W in terms of ρ_1, ρ_2, μ by W, one will have:

$$N_1 = N_2 = \frac{\partial W}{\partial \mu}, \qquad T = 0.$$

It is easy to obtain the particular form that the different systems of equations in question take, which are, moreover, combinations or simple consequences of each others. In particular, by virtue of the equations verified by the $\xi_i^{(1)}, \cdots, r_i^{(1)}$, and upon denoting the expression $\dfrac{\partial W}{\partial \mu}$ by N, the system on page (?) takes the following form:

$$\eta_2^{(1)} \frac{\partial N}{\partial \rho_1} - \eta_1^{(1)} \frac{\partial N}{\partial \rho_2} = \Delta_0 X_0'',$$
$$-\xi_2^{(1)} \frac{\partial N}{\partial \rho_1} + \xi_1^{(1)} \frac{\partial N}{\partial \rho_2} = \Delta_0 Y_0'',$$
$$N\left(\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2}\right) = \frac{\Delta_0}{\Delta} Z_0''$$

upon using the formula:

$$\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2} = \frac{p_2^{(1)}\xi_1^{(1)} - p_1^{(1)}\xi_2^{(1)} + q_2^{(1)}\eta_1^{(1)} - q_1^{(1)}\eta_2^{(1)}}{\xi_1^{(1)}\eta_2^{(1)} - \xi_2^{(1)}\eta_1^{(1)}},$$

in which $\mathcal{R}_1$ and $\mathcal{R}_2$, the radii of principle curvature of the deformed surface (M), figure.

If we envision the particular case in which W depends only on μ, and in which (M_0) does not figure explicitly, then we find ourselves in the presence of the surface considered by LAGRANGE ([1]), whose study has been reprised by DUHEM ([2]). Here, we must make some observations that are absolutely analogous to the ones that we presented in the context of the flexible and inextensible filament of LAGRANGE. If, as LAGRANGE and DUHEM supposed, the surface (M_0) does not figure explicitly then that surface (M_0) figures only by the quantity μ; its existence is revealed only by that quantity. If one supposes that the function W is *given*, like the quantity μ that we may introduce as an unknown auxiliary function in the usual problems, then we may substitute the unknown N. If the function W is hidden then N becomes, moreover, an unknown auxiliary function; however, knowledge of that function will give us nothing *in regard to* (M_0).

In the case where the surface (M_0) figures only by the quantity μ, one may take two other unknown variables – x, y, for example – instead of ρ_1 and ρ_2, and if W is given then one has two unknowns and three equations. If W is hidden then μ figures only in W, and one is in the same case. In the first case, the remark that was made by POISSON is repeated by DUHEM ([3]). We shall develop this remark explicitly, while putting the equations in the form that was given by LAGRANGE and, more explicitly, by POISSON and DUHEM ([4]).

If we solve the preceding equations with respect to $\dfrac{\partial N}{\partial \rho_1}$ and $\dfrac{\partial N}{\partial \rho_2}$ then we obtain:

$$\frac{\partial N}{\partial \rho_1} = +\frac{\Delta_0}{\Delta}(X_0''\xi_1^{(1)} + Y_0''\eta_1^{(1)}), \qquad \frac{\partial N}{\partial \rho_2} = +\frac{\Delta_0}{\Delta}(X_0''\xi_2^{(1)} + Y_0''\eta_2^{(1)});$$

however, upon introducing, for the moment, the direction cosines l, l', l'' of Mx_1', m, m', m'' of My_1', and n, n', n'' of Mz_1', with respect to the fixed axes, one has:

$$\xi_i^{(1)} = l\frac{\partial x}{\partial \rho_i} + l'\frac{\partial y}{\partial \rho_i} + l''\frac{\partial z}{\partial \rho_i},$$
$$\eta_i^{(1)} = m\frac{\partial x}{\partial \rho_i} + m'\frac{\partial y}{\partial \rho_i} + m''\frac{\partial z}{\partial \rho_i},$$
$$\varsigma_i^{(1)} = n\frac{\partial x}{\partial \rho_i} + n'\frac{\partial y}{\partial \rho_i} + n''\frac{\partial z}{\partial \rho_i},$$

and

$$X_0''\xi_i^{(1)} + Y_0''\eta_i^{(1)} = X_0\frac{\partial x}{\partial \rho_i} + Y_0\frac{\partial y}{\partial \rho_i} + Z_0\frac{\partial z}{\partial \rho_i}.$$

[1] LAGRANGE. – *Mécanique analytique*, 1st Part, Section V, Chap. III, sec. II, nos. 44-45, pp. 158-162, of the 4th edition.

[2] P. DUHEM. – *Hydrodynamique, Elasticité, Acoustique,* T. II, pp. 78 ff.

[3] P. DUHEM. – Ibid., T. II, pp. 92 at the top of the page.

[4] P. DUHEM. – Ibid., T. II, pp. 86 and 91.

The preceding system may be written:

$$\begin{aligned}
\frac{\partial N}{\partial \rho_1} &= \frac{\Delta_0}{\Delta}\left(X_0 \frac{\partial x}{\partial \rho_1} + Y_0 \frac{\partial y}{\partial \rho_1} + Z_0 \frac{\partial z}{\partial \rho_1} \right), \\
\frac{\partial N}{\partial \rho_2} &= \frac{\Delta_0}{\Delta}\left(X_0 \frac{\partial x}{\partial \rho_2} + Y_0 \frac{\partial y}{\partial \rho_2} + Z_0 \frac{\partial z}{\partial \rho_2} \right), \\
N\left(\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2} \right) &= \frac{\Delta_0}{\Delta}(X_0 n + Y_0 n' + Z_0 n'') ;
\end{aligned}$$

this is what one finds, up to notation, on page 86 of Tome II of the book by DUHEM that was already cited (the sense of the normal to (M) alone is changed).

Introduce the variables x, y, instead of ρ_1, ρ_2; to that effect, observe that the two relations that refer to the derivatives of N may be summarized in the following:

$$dN = \frac{\Delta_0}{\Delta}(X_0 dx + Y_0 dy + Z_0 dz),$$

which corresponds, in the particular case in which μ alone figures, to the remark made by DUHEM at the top of page 90 of Tome II of his work.

If x, y are taken for variables then we have the system:

$$\begin{aligned}
\frac{\partial N}{\partial \rho_1} &= \frac{\Delta_0}{\Delta}\left(X_0 + Z_0 \frac{\partial z}{\partial \rho_1} \right), \\
\frac{\partial N}{\partial \rho_2} &= \frac{\Delta_0}{\Delta}\left(Y_0 + Z_0 \frac{\partial z}{\partial \rho_2} \right), \\
N\left(\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2} \right) &= \frac{\Delta_0}{\Delta}(X_0 n + Y_0 n' + Z_0 n'') ;
\end{aligned}$$

which is none other, up to notations and with a suitable convention on the sense of the normal, that equations (31) and (32) of DUHEM.

If we, with POISSON and DUHEM, consider the case in which $\frac{\Delta_0}{\Delta} X_0$, $\frac{\Delta_0}{\Delta} Y_0$, $\frac{\Delta_0}{\Delta} Z_0$ are given functions of x, y, z (we may assume the same for the derivatives of z) then we have three equations that refer to the two unknowns N, z.

In the particular case in which the given functions of x, y, z, insofar as they are of issue, are such that $\frac{\Delta_0}{\Delta}(X_0 dx + Y_0 dy + Z_0 dz)$ is the total differential of a function V then the system of three equations, which may be written, as we have said:

$$dN = \frac{\Delta_0}{\Delta}(X_0 dx + Y_0 dy + Z_0 dz),$$

$$N\left(\frac{1}{\mathcal{R}_1}+\frac{1}{\mathcal{R}_2}\right)=\frac{\Delta_0}{\Delta}(X_0 n+Y_0 n'+Z_0 n'')$$

amount to the following:

$$N - V = \text{const.} = C,$$
$$N\left(\frac{1}{\mathcal{R}_1}+\frac{1}{\mathcal{R}_2}\right)=n\frac{\partial V}{\partial x}+n'\frac{\partial V}{\partial y}+n''\frac{\partial V}{\partial z}.$$

N is calculated from the formula:

$$N = V + C,$$

and the surface (M) verifies the equation (1):

$$(V+C)\left(\frac{1}{\mathcal{R}_1}+\frac{1}{\mathcal{R}_2}\right)=n\frac{\partial V}{\partial x}+n'\frac{\partial V}{\partial y}+n''\frac{\partial V}{\partial z}.$$

43. The flexible and inextensible surface of the geometers. The incompressible fluid surface. The Daniele surface. – We have considered the particular case in which W does not depend on p_i, q_i, r_i and different special cases of this case. We shall show how, by the study of particular deformations, one may approach the various surfaces that were already considered, at least in part, by the authors.

First, start with the simple case, in which the triad is hidden, i.e., the definition of a simple pointlike surface, and suppose that this is, moreover, the general case in which W is an arbitrary function of ρ_1, ρ_2, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$.

1. We may imagine that one pays attention only to the deformations of the surface for which one has:

$$\mathcal{E} = \mathcal{E}_0, \qquad \mathcal{F} = \mathcal{F}_0, \qquad \mathcal{G} = \mathcal{G}_0 .$$

In the definitions of forces, etc., it suffices to introduce these hypotheses and, if the forces, etc., are given, to introduce these three conditions. In the latter case the habitual problems, which correspond to the given of the function W, and the general case where $\mathcal{E} - \mathcal{E}_0$, $\mathcal{F} - \mathcal{F}_0$, $\mathcal{G} - \mathcal{G}_0$ are non-null may be posed only for particular givens.

If we suppose that *only* the function W_0 that is obtained by setting $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} =$

[1] Compare DUHEM, *Elasticité,* etc., T. II, pp. 92, which inspired pages 179-181 of POISSON, *Mémoire sur les surfaces élastiques,* which was written on August 1, 1814, published by extract in the May, 1815, issue of Tome III of the *Correspondence sur l'Ecole Polytechnique,* pp. 154-159, and then in the *Mémoires de l'Institut de France,* 1812, Part two, which appeared in 1816.

$\mathcal{G}_0$ in $W(\rho_1, \rho_2, \mathcal{E}, \mathcal{F}, \mathcal{G})$ is given, that one does not know the values of the derivatives of W with respect to $\mathcal{E}, \mathcal{F}, \mathcal{G}$ for $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$, and that W is *hidden* as well, then we see that N_1, T, N_2 become three auxiliary functions that one must adjoin to x, y, z in such a way that we have six partial differential equations in six unknowns in the case where the forces acting on the elements of the surface are given. One therefore has a well-defined problem only if one adds the accessory conditions. If the deformed figure is assigned *a priori* then one has three equations between the unknown functions N_1, T, N_2.

The equations that we arrive at are the ones that define the flexible and inextensible surface of geometry.

2. We may imagine that one seeks to define a surface that is deformable, *sui generis*, whose *definition includes* the conditions:

$$\mathcal{E} = \mathcal{E}_0, \qquad \mathcal{F} = \mathcal{F}_0, \qquad \mathcal{G} = \mathcal{G}_0.$$

To define the new surface while retaining the same order of ideas as in the preceding we again define $F_0', G_0', \cdots, N_0'$ by the identity:

$$\iint_{C_0} \delta(W\Delta_0)\, d\rho_1 d\rho_2 = \int_{C_0} (F_0'\delta' x + G_0'\delta' y + \cdots + K_0'\delta K')\, ds_0$$
$$- \iint_{C_0} (X_0'\delta' x + Y_0'\delta' y + \cdots + N_0'\delta K')\Delta_0\, d\rho_1 d\rho_2;$$

however, this identity no longer applies, by virtue of:

$$\mathcal{E} = \mathcal{E}_0, \qquad \mathcal{F} = \mathcal{F}_0, \qquad \mathcal{G} = \mathcal{G}_0.$$

In other words, we envision a surface for which the theory results from the *a posteriori* adjunction of the conditions $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$ to the knowledge of a function $W(\rho_1, \rho_2, \mathcal{E}, \mathcal{F}, \mathcal{G})$, as well as three auxiliary functions μ_1, μ_2, μ_3 of ρ_1, ρ_2, by means of the identity:

$$\iint_{C_0} [\delta W + \mu_1 \delta(\mathcal{E} - \mathcal{E}_0) + \mu_2 \delta(\mathcal{F} - \mathcal{F}_0) + \mu_3 \delta(\mathcal{G} - \mathcal{G}_0)]\Delta\, d\rho_1 d\rho_2$$
$$= \int_{C_0} (F_0'\delta' x + G_0'\delta' y + \cdots + K_0'\delta K')\, ds_0 - \iint_{C_0} (X_0'\delta' x + Y_0'\delta' y + \cdots + N_0'\delta K')\Delta_0\, d\rho_1 d\rho_2 .$$

This amounts to replacing W with $W_1 = W + \mu_1(\mathcal{E} - \mathcal{E}_0) + \mu_2(\mathcal{F} - \mathcal{F}_0) + \mu_3(\mathcal{G} - \mathcal{G}_0)$ in the preceding general theory rather than setting $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$.

As one sees, we return to the theory of the flexible surface that corresponds to the function W_1 of ρ_1, ρ_2, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ when one confines oneself to studying the deformations that correspond to $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$.

If we put ourselves in the case of a *hidden* W_1 then if we suppose that one knows simply the value $W_0(\rho_1, \rho_2)$ that W and W_1 take simultaneously for $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$ then we recover the classical theory of the flexible inextensible surface.

Observe that if we constitute the flexible and inextensible surface by taking the conditions $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$ on W into account *a priori* by a change of variables then we are led to replace W with $\mu_1(\mathcal{E} - \mathcal{E}_0) + \mu_2(\mathcal{F} - \mathcal{F}_0) + \mu_3(\mathcal{G} - \mathcal{G}_0)$ in the calculations relating to the general deformable surface, and we come down to formulas that once again bring us back to the study of a flexible surface when one restricts oneself to studying the deformations that correspond to $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$. If we suppose that μ_1, μ_2, μ_3 are *unknown* then these formulas bring us back to the flexible and inextensible surface of the geometers. If we take this latter viewpoint we duplicate the exposition that was given by BELTRAMI in sec. **2** of his well-known Mémoire *identically*. We may observe that in the case where X_0, Y_0, Z_0, as expressed by means of these equations, are the partial derivatives of a function φ of ρ_1, ρ_2, x, y, z with respect to x, y, z the equations in which X_0, Y_0, Z_0 figure are none other than the extremal equations of a problem of the calculus of variations that consists of determining an extremum for the integral:

$$\iint \Delta_0 \varphi \, d\rho_1 d\rho_2$$

under the conditions:

$$\mathcal{E} = \mathcal{E}_0, \qquad \mathcal{F} = \mathcal{F}_0, \qquad \mathcal{G} = \mathcal{G}_0.$$

We consider the case where the surface (M_0) disappears from the givens and does not present itself in the question. The variables ρ_1, ρ_2 appear as a system of coordinates to which the surface is referred. If these variables do not figure in the givens then one may introduce two other variables in their place at will. If we take this viewpoint, which is the one that is generally adopted, then the preceding equations, by way of particular cases, give the various known equations that were studied by the authors. We confine ourselves to giving several bibliographic indications in the following section.

Suppose that we start with a surface formed by means of a function W of ρ_1, ρ_2, Δ, or, if one prefers, of ρ_1, ρ_2, and $\mu = \dfrac{\Delta}{\Delta_0} - 1$. Imagine that one pays attention ([1]) only to the deformations of the surface for which one has:

$$\mu = 0.$$

One will then find oneself in the case of the *incompressible fluid surface.* In the definitions of forces, etc., it suffices to introduce this hypothesis, and, if the forces are given, to pose this condition. In the latter case, the *habitual* problems that correspond to the given of a function W and the general case where μ is not null demand that the givens be particular cases.

If we suppose that *only* the function W_0 that is obtained by setting $\mu = 0$ in $W(\rho_1, \rho_2,$

[1] This viewpoint appears to be the one that DUHEM assumed in his work: *Hydrodynamique,* etc.; see pp. 91 of Tome II, the last four lines, and pp. 92 at the end of sec. **5**.

μ) is given, and that one does not know the value of $\frac{\partial W}{\partial \mu}$ for $\mu = 0$, and that W is *hidden,* as well, then we see that the expression N becomes an auxiliary function that one must adjoin to x, y, z, in such a way that we have four equations in four unknowns in the case of given forces.

One may again start with a function W, which may refer to the ξ_i, η_i, ζ_i, as well as the p_i, q_i, r_i, and look for the form that it must have in order for the effort that is exerted on an arbitrary linear element to be normal and, moreover, in the plane tangent to (M). It is necessary and sufficient that W depend on ξ_i, η_i, ζ_i only by the intermediary of the expression $\Delta = \sqrt{\mathcal{E}\mathcal{F} - \mathcal{G}^2}$.

We also mention the surface that is deduced from a function $W(\rho_1, \rho_2, \mathcal{E}, \mathcal{F}, \mathcal{G})$ by the adjunction of the conditions $\mathcal{E} = \mathcal{E}_0$, $\mathcal{F} = \mathcal{F}_0$, $\mathcal{G} = \mathcal{G}_0$. In the case where W does not depend on $\mathcal{F}$ one arrives at a surface that was first studied by DANIELE ([1]). The case in which W depends on $\mathcal{F}$ agrees with that of the flexible and inextensible surface in an interesting manner. It seems to correspond – better than the latter – to what one may call *army surfaces*, or envelopes, such as those of aerostats that are formed from an elastic substance that is woven from inextensible filaments.

44. Several bibliographic indications that relate to the flexible and inextensible surface of geometry. – The flexible and inextensible surface of geometry has already given rise to a great number of works, at least from the mechanical viewpoint. It seems useful to us to assemble the following bibliographic indications here, which are attached to that surface.

LAGRANGE. – *Mécanique analytique.* 3rd edition, Part 1, Section V, Chap. III, sec. **2**, pp. 138-143; Note of J. BERTRAND, pp. 140; 4th edition, Tome XI of the *Oeuvres de* LAGRANGE, Part 1, Section V, Chap. III, sec. **2**, pp. 156-162; Note of DARBOUX, pp. 160.

POISSON. – *Mémoire sur les surfaces élastique;* written August 1, 1814; inserted in the *Mémoires de la classe des sciences mathématiques et physiques de l'Institut de France,* 1812, Part 2, pp. 167-225.

CISA DE GROSY. – *Considérations sur l'équilibre des surfaces flexible et inextensible (Memorie della R. Accademia delle scienze di Torino,* vol. XXIII, Part I, pp. 259-294, 1818).

BORDONI. – *Sull' equilibrio astratto delle volte (Memorie di Matematica e di Fisica della Società Italiana delle Scienze,* residente in Modina, **19**, pp. 155-186, 1821); *Memorie dell' I.R. Istituto Lombardo di Scienze, Lettere ed Arti,* **9**, pp. 126-142, 1863; *Sulla stabilità e l'equilibrio di un terrapieno (Memorie di Matematica et di Fisica della Società Italiana delle Scienze,* residente in Modena, **24**, pp. 75-112, 1850); *Considerazioni sulle svolte delle strade (Memorie dell' I.R. Istituto Lombardo di Scienze. Lettere ed Arti,* **9**, pp. 143-154, 1863).

MOSSOTTI. – *Lezioni di Meccanica razionale,* Firenze, 1851.

[1] E. DANIELE. – *Sull' equilibrio delle reti, Rend. del Circolo matematico di Palermo,* **13**, pp. 28-85, 1899.

BRIOSCHI. - *Intorno ad alcuni punti della teorica delle superficie (Annali di Tortolini,* **3**, pp. 293-321, 1852).

JELLETT. – *On the properties of inextensible surfaces (Transactions of the Royal Irish Academy,* **22**, pp. 343-378, 1853).

MAINARDI. – *Note che risguardano alcuni argomenti della Maccanica razionale ed applicata (Giornale dell' I.R. Istituto Lombardo di Scienze, Lettere ed Arti,* **8**, pp. 304-308, 1856).

LECORNU. – *Sur l'équilibre des surfaces flexibles et inextensible (C.R.,* **91**, pp. 809-812, 1880; *Journal de l'Ecole Polytechnique*, 48th letter, pp. 1-109, 1880).

BELTRAMI. – *Sull' equilibrio delle superficie flessibili ed inestensibili (Memorie della Academia delle Scienze dell' Istituto di Bologna,* Series 4, **3**, pp. 217-265, 1882).

KÖTTER. – *Über das Gleichgewicht biegsamer unausdehnbarer Flächen , Inaugural Dissertation,* Halle, 6 February 1883; *Anwendung der Abelschen Functionen auf ein Problem der Statik biegsamer unausdehnbarer Flächen, (Journal fhr die reine und angewandte Mathematik,* **103**, pp. 44-74, 1888).

MORERA. – *Sull' equilibrio delle superficie flessibili ed inestendibili (Atti della R. Accad. Dei Lincei, Rendiconti, Transfunti,* Series 3, **7**, pp. 268-270, 1883).

VOLTERRA. – *Sull' equilibrio delle superficie flessibili ed inflessibili,* Nota I and Nota II *(Atti della R. Acc. Dei Lincei. Transunti,* Series 3, **8**, pp. 214-217, 244-246, 1884); *Sulla deformazione delle superficie flessibili ed inestensibili (Atti della R. Accad. Dei Lincei, Rendiconti,* Series 4, **1**, pp. 274-278, 1885).

MAGGI. – *Sull' equilibrio delle superficie flessibili e inestensibili, (Rendiconti del R. Istituto Lombardo di Scienze ed Lettere,* Series 2, **17**, pp. 686-694, 1884).

PADOVA. – *Ricerche sull' equilibrio delle superficie flessibili e inestensibili,* Nota I and Nota II, *(Atti della R. Acc. Dei Lincei, Rendiconti,* Series 4, **1**, pp. 269-274, 306-309, 1885).

PENNACHIETTI. – *Sull' equilibrio delle superficie flessibili e inestensibili (Palermo Rend.,* **9**, pp. 87-95, 1895). *Sulle equazioni di equilibrio delle superficie flessibili e inestensibili (Atti Acc. Gioenia* (4), **8**, 1895). *Sulla integrazione dell' equazioni di equilibrio delle superficie flessibili e inestensibili (Atti Acc. Gionenia,* (4), **8**, 1895).

RAKHMANINOV. – *Equilibre d'une surface flexible inextensible* (in Russian). *(Recueil de la Soc. Math. de Moscou,* **19**, pp. 110-181, 1895).

LECORNU. – *Sur l'équilibre d'une envelope ellipsoïdale (Comtes rendus,* **122**, pp. 218-220, 1896; *Annales de l'Ecole normale supérieure* (3), **17**, pp. 501-539, 1900.)

DE FRANCESCO. – *Sul moto di un filo et sull' equilibrio di una superficie flessibili ed inestensibili, Napoli Rend.,* (3), **9**, pp. 227, 1903; *Napoli Atti* (2), **12**, 1905.

45. The deformable surface that is obtained by supposing that Mz' is normal to the surface (*M*). – We propose to introduce the condition that Mz' is normal to the surface (*M*). We may imagine that this is accomplished, either by starting with the previously-defined deformable surface and studying only the deformations of that surface that verify the conditions:

(42) $\zeta_1 = 0, \qquad \zeta_2 = 0,$

or by defining a new deformable surface for which one develops the theory, by analogy with the first one, but keeping conditions (42) in mind.

We take the first viewpoint and study the deformations of (M) that verify the conditions (42); suppose, in addition ([1]), in view of the study of the infinitely small deformation and in order to form a continuous sequence of surfaces that start with (M_0), that one has: $\varsigma_1^{(0)} = \varsigma_2^{(0)} = 0$.

It suffices to introduce the hypotheses (42) into the formulas of sec. **34** and following in order to obtain the expressions of the various elements that figure in the theory. Conversely, if, to fix ideas, we are given the forces and external moments then one must adjoin the two equations (42) to the six equations that result from that given, which shows that *if the function W, which serves as the point of departure, is given* then one may not give the forces and external moments arbitrarily.

However, observe that upon confining ourselves to the study of those that verify (42), we have, above all, the goal of constituting a particular surface; upon following this idea, we are therefore led to distinguish three cases: 1. the function W is hidden, and we know the function W_0 relative to the particular deformations under consideration, and constituted from the essential elements of the deformations. 2. the function W is again *hidden* (i.e., not given), and we know relations (differential, for example) that relate W_0 and the traces (here, three functions) of the function W. 3. the function W still hidden, and we know the functions that recall the existence of W, either partially or totally.

We develop these possibilities by entering into the details of the calculations. Because of conditions (42) the triad, instead of depending on the six parameters x, y, z, λ_1, λ_2, λ_3, depends on only four parameters, for example x, y, z, m where we are letting m designate the angle defined by the formula:

$$tg\ m = \frac{\eta_1}{\xi_1},$$

which represents one of the angles that the axis Mx' makes with the curve (ρ_2) in (M).

Let $\Delta^2\mathcal{D}, \Delta^2\mathcal{D}', \Delta^2\mathcal{D}''$ designate the determinants defined by the identity that we gave in sec. **38**, page (?), which depend only on the derivatives of x, y, z, and are independent of m and its derivatives. In addition, recall the formulas of the same paragraph (CHRISTOFFEL symbols):

$$\Sigma_1 = \frac{-\mathcal{E}\dfrac{\partial \mathcal{E}}{\partial \rho_2} + 2\mathcal{E}\dfrac{\partial \mathcal{F}}{\partial \rho_1} - \mathcal{F}\dfrac{\partial \mathcal{E}}{\partial \rho_1}}{2\Delta^2},$$

[1] The conditions $\varsigma_1^{(0)} = \varsigma_2^{(0)}$ may be omitted in our actual exposition and figure, in summation, only in the study of the infinitely small deformation.

$$\Sigma_2 = \frac{\mathcal{E}\dfrac{\partial \mathcal{G}}{\partial \rho_1} - \mathcal{F}\dfrac{\partial \mathcal{E}}{\partial \rho_2}}{2\Delta^2},$$

and, from the conventions we made:

$$\Delta = \xi_1 \eta_2 - \xi_2 \eta_1.$$

To determine the rotations p_1, q_1, r_1, p_2, q_2, r_2 one has the following formulas ([1]):

$$p_1 = \xi_1 \mathcal{D}' - \xi_2 \mathcal{D}, \qquad p_1 = \xi_1 \mathcal{D}' - \xi_2 \mathcal{D},$$
$$p_2 = \xi_1 \mathcal{D}'' - \xi_2 \mathcal{D}', \qquad q_2 = \eta_1 \mathcal{D}'' - \eta_2 \mathcal{D}',$$
$$r_1 = -\frac{\partial m}{\partial \rho_1} + \frac{\Sigma_1 \Delta}{\mathcal{E}}, \qquad r_2 = -\frac{\partial m}{\partial \rho_2} + \frac{\Sigma_2 \Delta}{\mathcal{E}}.$$

The translations are calculated from the prior system:

$$\frac{\eta_1}{\xi_1} = tg\ m, \qquad \xi_1^2 + \eta_1^2 = \mathcal{E}, \qquad \xi_1 \xi_2 + \eta_1 \eta_2 = \mathcal{F}, \qquad \xi_2^2 + \eta_2^2 = \mathcal{G}.$$

As one sees, the translations are expressed by means of m and the first derivatives of x, y, z. The rotations p_1, q_1, p_2, q_2 are expressed by means of m and the first and second derivatives of x, y, z. Finally, the rotations r_1, r_2 are expressed by means of the derivatives of m and the first and second derivatives of x, y, z.

If one substitutes these values in the function that is obtained by making $\zeta_1 = \zeta_2 = 0$ in W, a function that we shall denote by W^0, to avoid confusion, then we obtain the function W_0 of $\rho_1, \rho_2, m, \dfrac{\partial m}{\partial \rho_1}, \dfrac{\partial m}{\partial \rho_2}$, of x, y, z, and their first and second derivatives, which, as a result, depend on the expressions $m, \dfrac{\partial m}{\partial \rho_1}, \cdots$ by the intermediary of the nine independent expressions:

$$m, \mathcal{E}, \ \mathcal{F}, \ \mathcal{G}, r_1, r_2, \ \mathcal{D}, \mathcal{D}', \mathcal{D}'',$$

or, what amounts to the same thing, by the nine independent expressions ξ_1, η_1, ξ_2, η_2, r_1, r_2, $\mathcal{D}, \mathcal{D}'', \mathcal{D}''$.

Let W_0' designate the function of these nine latter quantities that gives W_0 upon substitution for their values; W_0' results from W_0 by the substitution for p_1, q_1, p_2, q_2.

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[1] DARBOUX, *Leçons,* T. II., pp. 363, pp. 378-379, nos. 495 and 503 give identical or equivalent formulas; we represent the quantities that DARBOUX denoted by $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ in the form $\Delta^2\mathcal{D}, \Delta^2\mathcal{D}', \Delta^2\mathcal{D}''$.

We have a function W_0' that refers to the *nine arguments* that we enumerated along with ρ_1, ρ_2, whereas W refers to the *ten arguments* ξ_1, η_1, ξ_2, η_2, p_1, q_1, r_1, p_2, q_2, r_2, along with ρ_1, ρ_2.

We must stop on an important point that results, by definition, from the consideration of one of the equations to which DARBOUX gave the name of CODAZZI, namely, $p_1\eta_2 - q_1\xi_2 - p_2\eta_1 + q_2\xi_1 = 0$, and study the equations of statics for the deformable surface in the case that we examine.

The function W_0' is deduced from W^0 by substituting the following values for p_1, q_1, p_2, q_2:

$$p_1 = \xi_1\mathcal{D}' - \xi_2\mathcal{D}, \qquad q_1 = \eta_1\mathcal{D}' - \eta_2\mathcal{D},$$
$$p_2 = \xi_1\mathcal{D}'' - \xi_2\mathcal{D}', \qquad q_2 = \eta_1\mathcal{D}'' - \eta_2\mathcal{D}',$$

it results from this that one has:

$$\frac{\partial W_0'}{\partial \xi_1} = \frac{\partial W^0}{\partial \xi_1} + \frac{\partial W^0}{\partial p_1}\mathcal{D}' + \frac{\partial W^0}{\partial p_2}\mathcal{D}'', \qquad \frac{\partial W_0'}{\partial \xi_2} = \frac{\partial W^0}{\partial \xi_2} - \frac{\partial W^0}{\partial p_1}\mathcal{D} - \frac{\partial W^0}{\partial p_2}\mathcal{D}',$$
$$\frac{\partial W_0'}{\partial \eta_1} = \frac{\partial W^0}{\partial \eta_1} + \frac{\partial W^0}{\partial q_1}\mathcal{D}' + \frac{\partial W^0}{\partial q_2}\mathcal{D}'', \qquad \frac{\partial W_0'}{\partial \eta_2} = \frac{\partial W^0}{\partial \eta_2} - \frac{\partial W^0}{\partial q_1}\mathcal{D} - \frac{\partial W^0}{\partial q_2}\mathcal{D}',$$
$$\frac{\partial W_0'}{\partial r_1} = \frac{\partial W^0}{\partial r_1}, \qquad \frac{\partial W_0'}{\partial r_2} = \frac{\partial W^0}{\partial r_2},$$
$$\frac{\partial W_0'}{\partial \mathcal{D}} = -\frac{\partial W^0}{\partial p_1}\xi_2 - \frac{\partial W^0}{\partial q_1}\eta_2, \qquad \frac{\partial W_0'}{\partial \mathcal{D}''} = \xi_1\frac{\partial W^0}{\partial p_2} + \eta_1\frac{\partial W^0}{\partial q_2},$$
$$\frac{\partial W_0'}{\partial \mathcal{D}'} = \xi_1\frac{\partial W^0}{\partial p_1} - \xi_2\frac{\partial W^0}{\partial p_2} + \eta_1\frac{\partial W^0}{\partial q_1} - \eta_2\frac{\partial W^0}{\partial q_2},$$

where we are continuing to let W^0 designate the result of substituting for p_1, q_1, p_2, q_2.

Suppose that one introduces the expressions for these variables in terms of $m, \frac{\partial m}{\partial \rho_1}, \cdots$ in these formulas, and that one takes (42) into account. Observe that the formulas:

$$C_1' = \Delta_0\frac{\partial W}{\partial \varsigma_1}, \qquad C_2' = \Delta_0\frac{\partial W}{\partial \varsigma_2},$$

do not permit us to calculate C_1', C_2', if W is hidden because we must account for (42); however, the other formulas give the other expressions $A_1', \cdots,$ in terms of the derivatives of W^0. For instance, one has:

$$\left(\frac{\partial W}{\partial p_1}\right)_{\varsigma_1=0,\varsigma_2=0} = \frac{\partial W^0}{\partial p_1}.$$

The nine formulas that we deduce are given by:

$$\Delta_0 \frac{\partial W_0}{\partial \xi_1} = A_1' + \mathcal{D}'P_1' + \mathcal{D}''P_2', \qquad \Delta_0 \frac{\partial W_0}{\partial \xi_2} = A_2' - \mathcal{D}P_1' - \mathcal{D}'P_2',$$

$$\Delta_0 \frac{\partial W_0}{\partial \eta_1} = B_1' + \mathcal{D}'Q_1' + \mathcal{D}''Q_2', \qquad \Delta_0 \frac{\partial W_0}{\partial \eta_2} = B_2' - \mathcal{D}Q_1' - \mathcal{D}'Q_2',$$

$$\Delta_0 \frac{\partial W_0}{\partial r_1} = R_1', \qquad \Delta_0 \frac{\partial W_0}{\partial r_2} = R_2',$$

$$\Delta_0 \frac{\partial W_0}{\partial \mathcal{D}} = -\xi_2 P_1' - \eta_2 Q_1', \qquad \Delta_0 \frac{\partial W_0}{\partial \mathcal{D}''} = \xi_1 P_2' + \eta_1 Q_2',$$

$$\Delta_0 \frac{\partial W_0}{\partial \mathcal{D}'} = \xi_1 P_1' + \eta_1 Q_1' - \xi_2 P_2' - \eta_2 Q_2',$$

where we write W_0 instead of W_0' in order to indicate that one must replace the arguments $\xi_1, \cdots, \mathcal{D}''$ by their values as functions of $m, \frac{\partial m}{\partial \rho_1}, \cdots$.

When only the function W_0 is known we no longer have to calculate the ten auxiliary functions $A_1', \cdots,$ besides $C_1', C_2',$ and the nine equations; by definition, *when* W_0 *alone is known*, what remains are three arbitrary functions.

In order to study the system of equations for the statics of the deformable surface we apply the formulas that relate to the triad $Mx_1'y_1'z_1'$ to the triad $Mx'y'z'$. In the former triad, we find auxiliary functions that are defined by the formulas:

$$\mathcal{A}_1' = \xi_2 A_1' + \eta_2 B_1', \qquad \mathcal{A}_2' = \xi_2 A_2' + \eta_2 B_2',$$

$$\mathcal{B}_1' = \xi_1 A_1' + \eta_1 B_1', \qquad \mathcal{B}_2' = \xi_1 A_2' + \eta_1 B_2'$$

and four analogous ones for $\mathcal{P}_1', \mathcal{Q}_1', \mathcal{P}_2', \mathcal{Q}_2'$ The nine previous formulas may be written:

$$\Delta_0 \left(\xi_2 \frac{\partial W_0}{\partial \xi_1} + \eta_2 \frac{\partial W_0}{\partial \eta_2} \right) = \mathcal{A}_1' + \mathcal{D}'\mathcal{P}_1' + \mathcal{D}''\mathcal{P}_2', \qquad \Delta_0 \left(\xi_2 \frac{\partial W_0}{\partial \xi_2} + \eta_2 \frac{\partial W_0}{\partial \eta_2} \right) = \mathcal{A}_2' - \mathcal{D}\mathcal{P}_1' - \mathcal{D}'\mathcal{P}_2',$$

$$\Delta_0 \left(\xi_1 \frac{\partial W_0}{\partial \xi_1} + \eta_1 \frac{\partial W_0}{\partial \eta_1} \right) = \mathcal{B}_1' + \mathcal{D}'\mathcal{Q}_1' + \mathcal{D}''\mathcal{Q}_2' \qquad \Delta_0 \left(\xi_1 \frac{\partial W_0}{\partial \xi_2} + \eta_1 \frac{\partial W_0}{\partial \eta_2} \right) = \mathcal{B}_2' - \mathcal{D}\mathcal{Q}_1' - \mathcal{D}'\mathcal{Q}_2',$$

$$\Delta_0 \frac{\partial W_0}{\partial r_1} = R_1', \qquad \Delta_0 \frac{\partial W_0}{\partial r_2} = R_2',$$

$$\Delta_0 \frac{\partial W_0}{\partial \mathcal{D}} = -\mathcal{P}_1', \qquad \Delta_0 \frac{\partial W_0}{\partial \mathcal{D}'} = \mathcal{Q}_1' - \mathcal{P}_2', \qquad \Delta_0 \frac{\partial W_0}{\partial \mathcal{D}''} = \mathcal{Q}_2'.$$

Consider the six equilibrium equations that were given in sec. 38; the first two of the second group give $\Delta C_1'$ and $\Delta C_2'$:

$$\Delta C_1' = \frac{\partial \mathcal{P}_1'}{\partial \rho_1} + \frac{\partial \mathcal{P}_2'}{\partial \rho_2} - \Sigma_2 \mathcal{P}_1' - \Sigma_3 \mathcal{P}_2' - \Theta_2 \mathcal{Q}_1' - \Theta_3 \mathcal{Q}_2' - \Delta \mathcal{D}' R_1' - \Delta \mathcal{D}'' R_2' - \Delta_0 (\xi_2 L_0' + \eta_2 M_0')$$

$$-\Delta C_2' = \frac{\partial \mathcal{Q}_1'}{\partial \rho_1} + \frac{\partial \mathcal{Q}_2'}{\partial \rho_2} - \Sigma_1 \mathcal{P}_1' - \Sigma_2 \mathcal{P}_2' - \Theta_1 \mathcal{Q}_1' - \Theta_2 \mathcal{Q}_2' - \Delta \mathcal{D} R_1' - \Delta \mathcal{D}' R_2' - \Delta_0 (\xi_1 L_0' + \eta_1 M_0') .$$

Substitute these values in the three equations of the first group; if we write the third equation of the second group, and we are left with the system:

$$\begin{aligned} U_2 = {} & \frac{\partial}{\partial \rho_1} (\mathcal{A}_1' + \mathcal{D}' \mathcal{P}_1' + \mathcal{D}'' \mathcal{P}_2') + \frac{\partial}{\partial \rho_2} (\mathcal{A}_2' - \mathcal{D} \mathcal{P}_1' - \mathcal{D}' \mathcal{P}_2') + \mathcal{D} \frac{\partial \mathcal{P}_1'}{\partial \rho_2} - 2\mathcal{D}' \frac{\partial \mathcal{P}_1'}{\partial \rho_2} \\ & + \mathcal{D}'' \frac{\partial \mathcal{Q}_2'}{\partial \rho_2} + \mathcal{D}'' \frac{\partial}{\partial \rho_1} (\mathcal{Q}_1' - \mathcal{P}_2') - \Sigma_2 (\mathcal{A}_1' + \mathcal{D}' \mathcal{P}_1' + \mathcal{D}'' \mathcal{P}_2') - \Sigma_3 (\mathcal{A}_2' - \mathcal{D} \mathcal{P}_1' - \mathcal{D}' \mathcal{P}_2') \\ & - \Theta_2 (\mathcal{B}_1' + \mathcal{D}' \mathcal{Q}_1' + \mathcal{D}' \mathcal{Q}_2') - \Theta_3 (\mathcal{B}_2' - \mathcal{D}' \mathcal{Q}_1' - \mathcal{D}' \mathcal{Q}_2') + 2(2\Sigma_2 \mathcal{D}' - \Sigma_1 \mathcal{D}'' - \Sigma_3 \mathcal{D}) \mathcal{P}_1' \\ & - (\Theta_3 \mathcal{D} - 2\Theta_2 \mathcal{D}_1' + \Theta_1 \mathcal{D}'')(\mathcal{Q}_1' - \mathcal{P}_2') - \Delta (\mathcal{D}\mathcal{D}'' - \mathcal{D}'^2) R_1' + \Delta_0 (\xi_2 X_0' + \eta_2 Y_0') \\ & + \Delta_0 \mathcal{D}' (\xi_2 L_0' + \eta_2 M_0') - \Delta_0 \mathcal{D}'' (\xi_1 L_0' + \eta_1 M_0') = 0 , \end{aligned}$$

$$\begin{aligned} U_1 = {} & \frac{\partial}{\partial \rho_1} (\mathcal{B}_1' + \mathcal{D}' \mathcal{Q}_1' + \mathcal{D}'' \mathcal{Q}_2'') + \frac{\partial}{\partial \rho_2} (\mathcal{B}_2' - \mathcal{D} \mathcal{Q}_1' - \mathcal{D}' \mathcal{Q}_2') + \mathcal{D} \frac{\partial_1}{\partial \rho_2} (\mathcal{Q}_1' - \mathcal{P}_2') \\ & + 2\mathcal{D}' \frac{\partial \mathcal{Q}_2'}{\partial \rho_2} - \mathcal{D}'' \frac{\partial \mathcal{Q}_2'}{\partial \rho_1} - \mathcal{D} \frac{\partial \mathcal{P}_1'}{\partial \rho_1} - \Sigma_1 (\mathcal{A}_1' + \mathcal{D}' \mathcal{Q}_1' + \mathcal{D}'' \mathcal{P}_2') - \Sigma_2 (\mathcal{A}_2' - \mathcal{D} \mathcal{P}_1' - \mathcal{D}' \mathcal{P}_2') \\ & - \Theta_1 (\mathcal{B}_1' + \mathcal{D}' \mathcal{Q}_1' + \mathcal{D}'' \mathcal{Q}_2') - \Theta_2 (\mathcal{B}_2' - \mathcal{D} \mathcal{Q}_1' - \mathcal{D}'' \mathcal{Q}_2') + 2(-2\Theta_2 \mathcal{D}' + \Theta_1 \mathcal{D}'' + \Theta_3 \mathcal{D}) \mathcal{Q}_2 \\ & + (2\Sigma_2 \mathcal{D}' - \Sigma_1 \mathcal{D}'' - \Sigma_3 \mathcal{D})(\mathcal{Q}_1' - \mathcal{P}_2') + \Delta (\mathcal{D}\mathcal{D}'' - \mathcal{D}'^2) R_2' - \Delta_0 (\xi_1 X_0' + \eta_1 Y_0') \\ & + \Delta_0 \mathcal{D} (\xi_2 L_0' + \eta_2 M_0') - \Delta_0 \mathcal{D}' (\xi_1 L_0' + \eta_1 M_0') = 0 , \end{aligned}$$

$$\begin{aligned} V = {} & \frac{1}{\Delta} \left(\frac{\partial^2 \mathcal{P}_1'^2}{\partial \rho_1^2} - \frac{\partial^2 (\mathcal{Q}_1' - \mathcal{P}_2')}{\partial \rho_1 \partial \rho_2} - \frac{\partial^2 \mathcal{Q}_2'}{\partial \rho_2^2} \right) - \frac{\Theta_1 + 2\Sigma_2}{\Delta} \frac{\partial \mathcal{P}_1'}{\partial \rho_1} + \frac{\Sigma_1}{\Delta} \frac{\partial \mathcal{P}_1'}{\partial \rho_2} + \frac{\Sigma_1}{\Delta} \frac{\partial}{\partial \rho_1} (\mathcal{Q}_1' - \mathcal{P}_1') \\ & + \frac{\Theta_1}{\Delta} \frac{\partial}{\partial \rho_2} (\mathcal{Q}_1' - \mathcal{P}_2') - \frac{\Theta_1}{\Delta} \frac{\partial \mathcal{Q}_2'}{\partial \rho_1} + \frac{2\Theta_2 + \Sigma_3}{\Delta} \frac{\partial \mathcal{Q}_2'}{\partial \rho_2} + \left[\frac{\partial}{\partial \rho_2} \frac{\Sigma_1}{\Delta} - \frac{\partial}{\partial \rho_1} \frac{\Sigma_2}{\Delta} + \frac{\mathcal{E}}{\Delta} (\mathcal{D}\mathcal{D}'' - \mathcal{D}'^2) \right] \mathcal{P}_1' \\ & + \left[\frac{\partial}{\partial \rho_2} \frac{\Theta_1}{\Delta} - \frac{\partial}{\partial \rho_1} \frac{\Theta_2}{\Delta} - \frac{\mathcal{F}}{\Delta} (\mathcal{D}\mathcal{D}'' - \mathcal{D}'^2) \right] (\mathcal{Q}_1' - \mathcal{P}_2') + \left[\frac{\partial}{\partial \rho_2} \frac{\Theta_2}{\Delta} - \frac{\partial}{\partial \rho_1} \frac{\Theta_3}{\Delta} - \frac{\mathcal{G}}{\Delta} (\mathcal{D}\mathcal{D}'' - \mathcal{D}'^2) \right] \mathcal{Q}_2' \\ & + \frac{\mathcal{E}\mathcal{D}' - \mathcal{F}\mathcal{D}}{\Delta} (\mathcal{A}_1' + \mathcal{D}' \mathcal{P}_1' + \mathcal{D}'' \mathcal{P}_2') + \frac{\mathcal{G}\mathcal{D} - \mathcal{F}\mathcal{D}'}{\Delta} (\mathcal{B}_1' + \mathcal{D}_1 \mathcal{Q}_1' + \mathcal{D}'' \mathcal{Q}_2') \\ & + \frac{\mathcal{E}\mathcal{D}'' - \mathcal{F}\mathcal{D}'}{\Delta} (\mathcal{A}_2' - \mathcal{D} \mathcal{P}_1' - \mathcal{D}' \mathcal{P}_2') + \frac{\mathcal{G}\mathcal{D}' - \mathcal{F}\mathcal{D}''}{\Delta} (\mathcal{B}_2' - \mathcal{D} \mathcal{Q}_1' - \mathcal{D}' \mathcal{Q}_2') \\ & - \frac{\partial}{\partial \rho_1} (\mathcal{D}' R_1' + \mathcal{D}'' R_2') + \frac{\partial}{\partial \rho_2} (\mathcal{D} R_1' + \mathcal{D}' R_2') - \frac{\partial}{\partial \rho_1} \left[\frac{\Delta_0}{\Delta} (\xi_2 L_0' + \eta_2 M_0') \right] \\ & + \frac{\partial}{\partial \rho_2} \left[\frac{\Delta_0}{\Delta} (\xi_1 L_0' + \eta_1 M_0') \right] - \Delta_0 Z_0' = 0 , \end{aligned}$$

$$\mathcal{W} = \frac{\partial R_1'}{\partial \rho_1} + \frac{\partial R_2'}{\partial \rho_2} + \frac{\mathcal{E}}{\Delta}(\mathcal{A}_1' + \mathcal{D}'\mathcal{P}' + \mathcal{D}''\mathcal{P}_2')$$
$$+ \frac{\mathcal{F}}{\Delta}[\mathcal{A}_2' - \mathcal{D}\mathcal{P}_1' - \mathcal{D}'\mathcal{P}_2' - (\mathcal{B}_1' + \mathcal{D}'\mathcal{Q}_1' + \mathcal{D}''\mathcal{Q}_2')] - \frac{\mathcal{G}}{\Delta}(\mathcal{B}_2' - \mathcal{D}\mathcal{Q}_1' - \mathcal{D}'\mathcal{Q}_2') - \Delta_0 N_0'' = 0,$$

upon remarking that for the formation of the first three equations the CODAZZI equations are, with our notations ([1]):

$$\mathcal{D}'\mathcal{D}'' - \mathcal{D}'^2 = \mathcal{K}$$
$$\frac{\partial \mathcal{D}}{\partial \rho_2} - \frac{\partial \mathcal{D}'}{\partial \rho_1} + \Sigma_3 \mathcal{D} - 2\Sigma_2 \mathcal{D}' + \Sigma_1 \mathcal{D}'' = 0,$$
$$\frac{\partial \mathcal{D}''}{\partial \rho_1} - \frac{\partial \mathcal{D}'}{\partial \rho_2} + \Theta_3 \mathcal{D} - 2\Theta_2 \mathcal{D}' + \Theta_1 \mathcal{D}'' = 0,$$

where $\mathcal{K}$ designates the expression that is formed uniquely from $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, and their first and second derivatives, and represents the total curvature of the surface, and we also remark that:

$$\frac{\partial \log \Delta}{\partial \rho_1} = \Theta_1 + \Sigma_2, \qquad \frac{\partial \log \Delta}{\partial \rho_2} = \Theta_2 + \Sigma_3,$$

and that, as a result, when we equate the two values of $\dfrac{\partial^2 \log \Delta}{\partial \rho_1 \partial \rho_2}$ we get:

$$\frac{\partial \Theta_2}{\partial \rho_1} - \frac{\partial \Theta_1}{\partial \rho_2} = \frac{\partial \Sigma_2}{\partial \rho_2} - \frac{\partial \Sigma_1}{\partial \rho_1},$$

or:

$$\frac{\partial}{\partial \rho_2}\frac{\Sigma_2}{\Delta} - \frac{\partial}{\partial \rho_1}\frac{\Sigma_3}{\Delta} = -\left(\frac{\partial}{\partial \rho_2}\frac{\Theta_1}{\Delta} - \frac{\partial}{\partial \rho_1}\frac{\Theta_2}{\Delta}\right) = \frac{1}{\Delta}\left(\frac{\partial \Theta_2}{\partial \rho_1} - \frac{\partial \Theta_1}{\partial \rho_2} - \Theta_2\Sigma_2 + \Theta_1\Sigma_3\right)$$
$$= \frac{1}{\Delta}\left(\frac{\partial \Sigma_2}{\partial \rho_2} - \frac{\partial \Sigma_3}{\partial \rho_1} - \Theta_2\Sigma_2 + \Theta_1\Sigma_3\right).$$

46. Reduction of the system in the preceding section to a form that is analogous to one that presents itself in the calculus of variations. – From the preceding calculations it results that the auxiliary variables $A_1', \cdots,$ or, what amounts to the same thing, the $\mathcal{A}_1'$, … are all eliminated from these equations, even though their number is

[1] These equations are immediately deduced from the ones that were given in T. III, pp. 246, 248, of *Leçons* by DARBOUX upon performing a change of notations and observing that:

$$\frac{\partial \log \Delta}{\partial \rho_1} = \Theta_1 + \Sigma_2, \qquad \frac{\partial \log \Delta}{\partial \rho_2} = \Theta_2 + \Sigma_3.$$

greater than one. This is also an *a priori* consequence of the habitual considerations that one makes in the calculus of variations when the expressions for the external forces and moments have a particular form.

We shall put the equations that result from this elimination into a form that one may deduce from the calculus of variations in the case where expressions for the external forces and moments are given in a particular form.

We begin by replacing the arguments ξ_1, η_1, ξ_2, η_2 in W_0, which are functions of the arguments m, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ by their expressions that one deduces from the formulas:

$$\frac{\eta_1}{\xi_1} = tg\ m, \qquad \xi_1^2 + \eta_1^2 = \mathcal{E}, \quad \xi_1\xi_2 + \eta_1\eta_2 = \mathcal{F}, \qquad \xi_2^2 + \eta_2^2 = \mathcal{G},$$

to which we adjoin the formula we already used:

$$\xi_1\eta_2 - \xi_2\eta_1 = \Delta,$$

which only defines the sign of ξ_2, η_2.

From this, we deduce:

$$\xi_1 = \sqrt{\mathcal{E}} \cos m, \qquad \xi_2 = \frac{\mathcal{F}}{\sqrt{\mathcal{E}}}\cos m - \frac{\Delta}{\sqrt{\mathcal{E}}}\sin m,$$

$$\eta_1 = \sqrt{\mathcal{E}} \sin m, \qquad \eta_2 = \frac{\mathcal{F}}{\sqrt{\mathcal{E}}}\sin m - \frac{\Delta}{\sqrt{\mathcal{E}}}\cos m,$$

in which $\sqrt{\mathcal{E}}$ denotes a determination of the radical.

If we let $[W_0]$ denote, for the moment, the function of ρ_1, ρ_2, and m, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, r_1, r_2, $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ so obtained then we have the relations:

$$\frac{\partial [W_0]}{\partial \mathcal{E}} = \frac{1}{2\mathcal{E}}\left(\xi_1 \frac{\partial W_0}{\partial \xi_1} + \eta_1 \frac{\partial W_0}{\partial \eta_1}\right) - \frac{1}{2\mathcal{E}}\left(\xi_2 \frac{\partial W_0}{\partial \xi_2} + \eta_2 \frac{\partial W_0}{\partial \eta_2}\right) + \frac{\mathcal{G}}{2\mathcal{E}\Delta}\left(\xi_1 \frac{\partial W_0}{\partial \eta_2} - \eta_1 \frac{\partial W_0}{\partial \xi_2}\right),$$

$$\frac{\partial [W_0]}{\partial \mathcal{F}} = \frac{1}{\Delta}\left(\eta_1 \frac{\partial W_0}{\partial \xi_2} - \xi_2 \frac{\partial W_0}{\partial \eta_2}\right),$$

$$\frac{\partial [W_0]}{\partial \mathcal{G}} = \frac{1}{\Delta}\left(\xi_1 \frac{\partial W_0}{\partial \eta_2} - \eta_1 \frac{\partial W_0}{\partial \xi_2}\right),$$

$$\frac{\partial [W_0]}{\partial m} = \xi_1 \frac{\partial W_0}{\partial \eta_1} - \eta_1 \frac{\partial W_0}{\partial \xi_1} + \xi_2 \frac{\partial W_0}{\partial \eta_2} - \eta_2 \frac{\partial W_0}{\partial \xi_2}.$$

To abbreviate the notation, we set:

$$a_1' = \mathcal{A}_1' + \mathcal{D}'\mathcal{P}_1' + \mathcal{D}''\mathcal{P}_2', \qquad a_2' = \mathcal{A}_2' - \mathcal{D}\mathcal{P}_1' - \mathcal{D}'\mathcal{P}_2',$$

$$b_1' = \mathcal{B}_1' + \mathcal{D}'\mathcal{Q}_1' + \mathcal{D}''\mathcal{Q}_2', \qquad b_2' = \mathcal{B}_2' - \mathcal{D}\mathcal{Q}_1' - \mathcal{D}'\mathcal{Q}_2'.$$

We have the relations:

$$\xi_1 \frac{\partial(W_0\Delta_0)}{\partial \xi_1} + \eta_1 \frac{\partial(W_0\Delta_0)}{\partial \eta_1} = b_1', \qquad \xi_1 \frac{\partial(W_0\Delta_0)}{\partial \xi_2} + \eta_1 \frac{\partial(W_0\Delta_0)}{\partial \eta_2} = b_2',$$
$$\xi_1 \frac{\partial(W_0\Delta_0)}{\partial \xi_1} + \eta_2 \frac{\partial(W_0\Delta_0)}{\partial \eta_1} = a_1', \qquad \xi_2 \frac{\partial(W_0\Delta_0)}{\partial \xi_2} + \eta_2 \frac{\partial(W_0\Delta_0)}{\partial \eta_2} = a_2',$$

from which we deduce the following expressions for the derivatives of ($W_0\Delta_0$):

$$\frac{\partial(W_0\Delta_0)}{\partial \xi_1} = \frac{\eta_2 b_1' - \eta_1 a_1'}{\Delta}, \qquad \frac{\partial(W_0\Delta_0)}{\partial \xi_2} = \frac{\eta_2 b_2' - \eta_1 a_2'}{\Delta},$$
$$\frac{\partial(W_0\Delta_0)}{\partial \eta_1} = \frac{\xi_2 a_1' - \xi_2 b_1'}{\Delta}, \qquad \frac{\partial(W_0\Delta_0)}{\partial \eta_2} = \frac{\xi_1 a_2' - \xi_2 b_2'}{\Delta},$$

which permits us to calculate the different combinations formed from the derivatives of ($W_0\Delta_0$) in terms a_1', b_1', a_2', b_2'. We thus obtain:

$$\Delta_0 \frac{\partial[W_0]}{\partial \mathcal{E}} = \frac{1}{2\mathcal{E}} b_1' - \frac{1}{2\mathcal{E}} a_2' + \frac{\mathcal{G}}{2\mathcal{E}\Delta}\left(-\frac{\mathcal{F}}{2\Delta} b_2' + \frac{\mathcal{E}}{\Delta} a_2'\right) = \frac{1}{2\mathcal{E}} b_1' + \frac{\mathcal{F}^2}{2\mathcal{E}\Delta^2} a_2' - \frac{\mathcal{F}\mathcal{G}}{2\mathcal{E}\Delta^2} b_2',$$
$$\Delta_0 \frac{\partial[W_0]}{\partial \mathcal{F}} = -\frac{\mathcal{F}}{\Delta^2} a_2' + \frac{\mathcal{G}}{\Delta^2} b_2',$$
$$\Delta_0 \frac{\partial[W_0]}{\partial \mathcal{E}} = \frac{\mathcal{E}}{2\Delta^2} a_2' - \frac{\mathcal{F}}{2\Delta^2} b_2',$$
$$\Delta_0 \frac{\partial[W_0]}{\partial m} = \frac{\mathcal{E}}{\Delta} a_1' - \frac{\mathcal{F}}{\Delta}(a_2' - b_1') - \frac{\mathcal{G}}{\Delta} b_1',$$

from which one deduces:

$$a_1' = \frac{\Delta}{\mathcal{E}} \Delta_0 \frac{\partial[W_0]}{\partial m} + 2\mathcal{F}\Delta_0 \frac{\partial[W_0]}{\partial \mathcal{E}} + \mathcal{G}\Delta_0 \frac{\partial[W_0]}{\partial \mathcal{F}},$$
$$b_1' = 2\mathcal{E}\Delta_0 \frac{\partial[W_0]}{\partial \mathcal{E}} + \mathcal{F}\Delta_0 \frac{\partial[W_0]}{\partial \mathcal{F}},$$
$$a_2' = \Delta\Delta_0 \left\{\mathcal{F} \frac{\partial[W_0]}{\partial \mathcal{F}} + 2\mathcal{G} \frac{\partial[W_0]}{\partial \mathcal{G}}\right\},$$
$$b_2' = \Delta\Delta_0 \left\{\mathcal{E} \frac{\partial[W_0]}{\partial \mathcal{F}} + 2\mathcal{F} \frac{\partial[W_0]}{\partial \mathcal{G}}\right\},$$

in such a way that if we denote the function $[W_0]$ by W_0 then the *ten* auxiliary functions other than C_1', C_2' are defined by the following *nine* formulas:

$$\mathcal{A}_1' + \mathcal{D}'\mathcal{P}_1' + \mathcal{D}''\mathcal{P}_2' = \frac{\Delta}{\mathcal{E}}\frac{\partial(W_0\Delta_0)}{\partial m} + 2\mathcal{F}\frac{\partial(W_0\Delta_0)}{\partial\mathcal{E}} + \mathcal{G}\frac{\partial(W_0\Delta_0)}{\partial\mathcal{F}},$$

$$\mathcal{B}_1' + \mathcal{D}'\mathcal{Q}_1' + \mathcal{D}''\mathcal{Q}_2' = 2\mathcal{E}\frac{\partial(W_0\Delta_0)}{\partial\mathcal{E}} + \mathcal{F}\frac{\partial(W_0\Delta_0)}{\partial\mathcal{F}},$$

$$\mathcal{A}_2' - \mathcal{D}\mathcal{P}_1' - \mathcal{D}'\mathcal{P}_2' = \Delta\left\{\mathcal{F}\frac{\partial(W_0\Delta_0)}{\partial\mathcal{F}} + 2\mathcal{G}\frac{\partial(W_0\Delta_0)}{\partial\mathcal{G}}\right\},$$

$$\mathcal{B}_2' - \mathcal{D}\mathcal{Q}_1' - \mathcal{D}\mathcal{Q}_2' = \Delta\left\{\mathcal{E}\frac{\partial(W_0\Delta_0)}{\partial\mathcal{F}} + 2\mathcal{F}\frac{\partial(W_0\Delta_0)}{\partial\mathcal{G}}\right\},$$

$$C_1' = \Delta_0\frac{\partial W_0}{\partial r_1}, \qquad C_2' = \Delta_0\frac{\partial W_0}{\partial r_2}.$$

Define the direction cosines $\gamma, \gamma', \gamma''$ of the normal Mz' to (M) by the formulas:

$$\gamma = \frac{1}{\Delta}\frac{\partial(y,z)}{\partial(\rho_1,\rho_2)}, \qquad \gamma' = \frac{1}{\Delta}\frac{\partial(z,x)}{\partial(\rho_1,\rho_2)}, \qquad \gamma'' = \frac{1}{\Delta}\frac{\partial(x,y)}{\partial(\rho_1,\rho_2)}.$$

First, we have the following identity, in which we introduce the notations that we just now defined in place of the derivatives of W_0:

$$\frac{\partial^2}{\partial\rho_1^2}\frac{\partial(W_0\Delta_0)}{\partial\dfrac{\partial^2 x}{\partial\rho_1^2}} + \frac{\partial^2}{\partial\rho\partial\rho_2}\frac{\partial(W_0\Delta_0)}{\partial\dfrac{\partial^2 x}{\partial\rho\partial_2}} + \frac{\partial^2}{\partial\rho_2^2}\frac{\partial(W_0\Delta_0)}{\partial\dfrac{\partial^2 x}{\partial\rho_2^2}} - \frac{\partial}{\partial\rho_1}\frac{\partial(W_0\Delta_0)}{\partial\dfrac{\partial x}{\partial\rho_1}} - \frac{\partial}{\partial\rho_2}\frac{\partial(W_0\Delta_0)}{\partial\dfrac{\partial x}{\partial\rho_2}}$$

$$+\frac{\partial(W_0\Delta_0)}{\partial x} = -\left\{\frac{\partial^2\left(\dfrac{\gamma}{\Delta}\mathcal{P}_1'\right)}{\partial\rho_1^2} - \frac{\partial^2\left[\dfrac{\gamma}{\Delta}(\mathcal{Q}_1' - \mathcal{P}_2')\right]}{\partial\rho_1\partial\rho_2} - \frac{\partial^2\left(\dfrac{\gamma}{\Delta}\mathcal{Q}_2'\right)}{\partial\rho_2^2}\right\}$$

$$+\frac{\partial}{\partial\rho_1}\left\{\frac{\partial}{\partial\rho_1}\left(\frac{\mathcal{E}\dfrac{\partial x}{\partial\rho_1} - \mathcal{F}\dfrac{\partial y}{\partial\rho_1}}{\mathcal{E}\Delta}C_1'\right) + \frac{\partial}{\partial\rho_1}\left(\frac{\mathcal{E}\dfrac{\partial x}{\partial\rho_1} - \mathcal{F}\dfrac{\partial y}{\partial\rho_1}}{\mathcal{E}\Delta}\right)C_2'\right\}$$

$$-\frac{\partial}{\partial\rho_1}\left\{\left(\frac{b_1'}{\mathcal{E}} + \frac{\mathcal{F}^2 a_2'}{\mathcal{E}\Delta^2} - \frac{\mathcal{F}\mathcal{G}b_2'}{\mathcal{E}\Delta^2}\right)\frac{\partial x}{\partial\rho_1} + \left(-\frac{\mathcal{F}^2 a_2'}{\Delta^2} - \frac{\mathcal{G}b_2'}{\Delta^2}\right)\frac{\partial x}{\partial\rho_2}\right.$$

$$+\left[\frac{\partial}{\partial\rho_1}\left(\frac{\mathcal{E}\dfrac{\partial x}{\partial\rho_2} - \mathcal{F}\dfrac{\partial x}{\partial\rho_1}}{\mathcal{E}\Delta}\right) - \mathcal{D}'\gamma\right]C_1' + \left[\frac{\partial}{\partial\rho_2}\left(\frac{\mathcal{E}\dfrac{\partial x}{\partial\rho_2} - \mathcal{F}\dfrac{\partial x}{\partial\rho_1}}{\mathcal{E}\Delta}\right) - \mathcal{D}''\gamma\right]C_2'$$

$$-\frac{\partial\mathcal{D}}{\partial\frac{\partial x}{\partial\rho_1}}\mathcal{P}_1'+\frac{\partial\mathcal{D}'}{\partial\frac{\partial x}{\partial\rho_1}}(\mathcal{Q}_1'-\mathcal{P}_2')+\frac{\partial\mathcal{D}''}{\partial\frac{\partial x}{\partial\rho_1}}\mathcal{Q}_2'\Bigg\}-\frac{\partial}{\partial\rho_2}\Bigg\{\left(-\frac{\mathcal{F}a_2'}{\Delta^2}+\frac{\mathcal{G}b_2'}{\Delta^2}\right)\frac{\partial x}{\partial\rho_1}$$

$$+\left(\frac{\mathcal{E}a_2'}{\Delta^2}-\frac{\mathcal{F}b_2'}{\Delta^2}\right)\frac{\partial x}{\partial\rho_2}+\mathcal{D}\gamma C_1'+\mathcal{D}'\gamma C_2'-\frac{\partial D}{\partial\frac{\partial x}{\partial\rho_2}}\mathcal{P}_1'$$

$$+\frac{\partial\mathcal{D}'}{\partial\frac{\partial x}{\partial\rho_2}}(\mathcal{Q}_1'-\mathcal{P}_2')+\frac{\partial\mathcal{D}''}{\partial\frac{\partial x}{\partial\rho_2}}\mathcal{Q}_2'\Bigg\}.$$

In order to obtain this identity in the form that we used we have to use the relations ([1]):

$$\frac{\partial\mathcal{E}}{\partial\rho_1}=2(\mathcal{E}\Theta_1+\mathcal{F}\Sigma_1),\qquad \frac{\partial\mathcal{E}}{\partial\rho_2}=2(\mathcal{E}\Theta_2+\mathcal{F}\Sigma_2),$$
$$2\frac{\partial\mathcal{F}}{\partial\rho_2}-\frac{\partial\mathcal{G}}{\partial\rho_1}=2(\mathcal{E}\Theta_3-\mathcal{F}\Sigma_3),\qquad 2\frac{\partial\mathcal{F}}{\partial\rho_1}-\frac{\partial\mathcal{E}}{\partial\rho_2}=2(\mathcal{F}\Theta_1+\mathcal{G}\Sigma_1),$$
$$\frac{\partial\mathcal{G}}{\partial\rho_1}=2(\mathcal{F}\Theta_2+\mathcal{G}\Sigma_2),\qquad \frac{\partial\mathcal{G}}{\partial\rho_2}=2(\mathcal{F}\Theta_3+\mathcal{G}\Sigma_3),$$

whose solution gives the values of the CHRISTOFFEL symbols Σ_1, Σ_2, Σ_3, Θ_1, Θ_2, Θ_3, or, conversely, the values of the six derivatives of $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, and this permits us to eliminate these derivatives of $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$. We have also used the relations ([2]):

$$\frac{\partial^2 x}{\partial\rho_1^2}=\Theta_1\frac{\partial x}{\partial\rho_1}+\Sigma_1\frac{\partial x}{\partial\rho_2}+\mathcal{D}\Delta\gamma,$$
$$\frac{\partial^2 x}{\partial\rho_1\partial\rho_2}=\Theta_2\frac{\partial x}{\partial\rho_1}+\Sigma_2\frac{\partial x}{\partial\rho_2}+\mathcal{D}'\Delta\gamma,$$
$$\frac{\partial^2 x}{\partial\rho_2^2}=\Theta_3\frac{\partial x}{\partial\rho_1}+\Sigma_3\frac{\partial x}{\partial\rho_2}+\mathcal{D}''\Delta\gamma,$$

which permits us to eliminate the second derivatives of x, y, z, and gives rise to two series of formulas that are analogous to the ones obtained by replacing x, γ by y, γ' and z, γ''

[1] We continue to use the relations: $\frac{\partial\log\Delta}{\partial\rho_1}=\Theta_1+\Sigma_2$, $\frac{\partial\log\Delta}{\partial\rho_2}=\Theta_2+\Sigma_3$.

[2] DARBOUX. – *Leçons*, T. III, no. 702, pp. 251.

with the direction cosines defined by formulas that are deduced from the formula for γ by circular permutation.

Consider the different expressions that are presented in the preceding calculations.

First, let:

$$\frac{\partial r_1}{\partial \frac{\partial x}{\partial \rho_1}} = \frac{\partial \frac{\Sigma_1 \Delta}{\mathcal{E}}}{\partial \frac{\partial x}{\partial \rho_1}} = \frac{\partial}{\partial \frac{\partial x}{\partial \rho_1}} \left(\frac{-\mathcal{E}\frac{\partial \mathcal{E}}{\partial \rho_2} + 2\mathcal{E}\frac{\partial \mathcal{F}}{\partial \rho_1} - \mathcal{F}\frac{\partial \mathcal{E}}{\partial \rho_1}}{2\Delta\mathcal{E}} \right)$$

$$= -\frac{2\Delta\Sigma_1}{\mathcal{E}^2}\frac{\partial x}{\partial \rho_1} - \frac{\Sigma_1}{\Delta\mathcal{E}}\left(\mathcal{G}\frac{\partial x}{\partial \rho_1} - \mathcal{F}\frac{\partial x}{\partial \rho_2}\right) + \frac{-\frac{\partial x}{\partial \rho_1}\frac{\partial \mathcal{E}}{\partial \rho_2} + 2\frac{\partial x}{\partial \rho_1}\frac{\partial \mathcal{F}}{\partial \rho_1} - \frac{1}{2}\frac{\partial x}{\partial \rho_2}\frac{\partial \mathcal{F}}{\partial \rho_1} - \mathcal{F}\frac{\partial^2 x}{\partial \rho_1^2}}{\Delta\mathcal{E}}$$

$$= -\frac{2\Delta\Sigma_1}{\mathcal{E}^2}\frac{\partial x}{\partial \rho_1} - \frac{\Sigma_1}{\Delta\mathcal{E}}\left(\mathcal{G}\frac{\partial x}{\partial \rho_1} - \mathcal{F}\frac{\partial x}{\partial \rho_2}\right) + \frac{2\frac{\partial x}{\partial \rho_1}(\mathcal{F}\Theta_1 + \mathcal{G}\Sigma_1) - \frac{\partial x}{\partial \rho_2}(\mathcal{E}\Theta_1 + \mathcal{F}\Sigma_1) - \mathcal{F}\frac{\partial^2 x}{\partial \rho_1^2}}{\Delta\mathcal{E}};$$

on the other hand, one has:

$$\frac{\partial}{\partial \rho_1}\frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}}$$

$$= \frac{\frac{\partial \mathcal{E}}{\partial \rho_1}\frac{\partial x}{\partial \rho_2} + \mathcal{E}\frac{\partial^2 x}{\partial \rho_1 \partial \rho_2} - \frac{\partial \mathcal{F}}{\partial \rho_1}\frac{\partial x}{\partial \rho_1} - \mathcal{F}\frac{\partial^2 x}{\partial \rho_1^2}}{\Delta\mathcal{E}} - \frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}^2}\left[\mathcal{E}(\Theta_1 + \Sigma_2) + \frac{\partial \mathcal{E}}{\partial \rho_1}\right]$$

$$= \frac{2\frac{\partial x}{\partial \rho_2}(\mathcal{E}\Theta_1 + \mathcal{F}\Sigma_1) + \mathcal{E}\left(\Theta_2\frac{\partial x}{\partial \rho_1} + \Sigma_2\frac{\partial x}{\partial \rho_2} + \mathcal{D}'\Delta\gamma\right) - \frac{\partial x}{\partial \rho_1}(\mathcal{F}\Theta_1 + \mathcal{G}\Sigma_1 + \mathcal{E}\Theta_2 + \mathcal{F}\Sigma_2) - F\frac{\partial^2 x}{\partial \rho_1^2}}{\Delta\mathcal{E}}$$

$$-\frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}^2}[\mathcal{E}(\Theta_1 + \Sigma_2) + 2(\mathcal{E}\Theta_1 + \mathcal{F}\Sigma_1)].$$

From this, it results that:

$$\frac{\partial \frac{\Sigma_1 \Delta}{\mathcal{E}}}{\partial \frac{\partial x}{\partial \rho_1}} - \frac{\partial}{\partial \rho_1}\frac{\mathcal{E}\frac{\partial x}{\partial \rho_2} - \mathcal{F}\frac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}} = -\mathcal{D}'\gamma$$

Similarly, one has:

$$\frac{\partial r_2}{\partial \frac{\partial x}{\partial \rho_1}} = \frac{\partial \frac{\Delta \Sigma_2}{\mathcal{E}}}{\partial \frac{\partial x}{\partial \rho_1}} = \frac{\partial}{\partial \frac{\partial x}{\partial \rho_1}} \left(\frac{\mathcal{E} \frac{\partial \mathcal{G}}{\partial \rho_1} - \mathcal{F} \frac{\partial \mathcal{E}}{\partial \rho_2}}{2\Delta \mathcal{E}} \right)$$

$$= -\frac{2\Sigma_2 \Delta}{\mathcal{E}^2} \frac{\partial x}{\partial \rho_1} - \frac{\Sigma_2}{\Delta \mathcal{E}} \left(\mathcal{G} \frac{\partial x}{\partial \rho_1} - \mathcal{F} \frac{\partial x}{\partial \rho_2} \right) + \frac{\frac{\partial x}{\partial \rho_1} \frac{\partial \mathcal{G}}{\partial \rho_1} - \frac{1}{2} \frac{\partial x}{\partial \rho_1} \frac{\partial \mathcal{E}}{\partial \rho_1} - \mathcal{F} \frac{\partial^2 x}{\partial \rho_1 \partial \rho_2}}{\Delta \mathcal{E}}$$

$$= -\frac{2\Sigma_2 \Delta}{\mathcal{E}^2} \frac{\partial x}{\partial \rho_1} - \frac{\Sigma_2}{\Delta \mathcal{E}} \left(\mathcal{G} \frac{\partial x}{\partial \rho_1} - \mathcal{F} \frac{\partial x}{\partial \rho_2} \right) + \frac{2 \frac{\partial x}{\partial \rho_1} (\mathcal{F}\Theta_2 + \mathcal{G}\Sigma_2) - \frac{\partial x}{\partial \rho_1} (\mathcal{E}\Theta_2 + \mathcal{F}\Sigma_2) - \mathcal{F} \frac{\partial^2 x}{\partial \rho_1 \partial \rho_2}}{\Delta \mathcal{E}} .$$

On the other hand:

$$\frac{\partial}{\partial \rho_2} \frac{\mathcal{E} \frac{\partial x}{\partial \rho_2} - \mathcal{F} \frac{\partial x}{\partial \rho_1}}{\Delta \mathcal{E}}$$

$$= \frac{\frac{\partial \mathcal{E}}{\partial \rho_2} \frac{\partial x}{\partial \rho_2} + \mathcal{E} \frac{\partial^2 x}{\partial \rho_2^2} - \frac{\partial \mathcal{F}}{\partial \rho_2} \frac{\partial x}{\partial \rho_1} - \mathcal{F} \frac{\partial^2 x}{\partial \rho_1 \partial \rho_2}}{\Delta \mathcal{E}} - \frac{\mathcal{E} \frac{\partial x}{\partial \rho_2} - \mathcal{F} \frac{\partial x}{\partial \rho_1}}{\Delta \mathcal{E}^2} \left[\mathcal{E}(\Theta_2 + \Sigma_3) + \frac{\partial \mathcal{E}}{\partial \rho_2} \right]$$

$$= \frac{2 \frac{\partial x}{\partial \rho_1} (\mathcal{E}\Theta_2 + \mathcal{F}\Sigma_2) + \mathcal{E}(\Theta_3 \frac{\partial x}{\partial \rho_1} + \Sigma_3 \frac{\partial x}{\partial \rho_2} + \mathcal{D}'' \Delta \gamma) - \frac{\partial x}{\partial \rho_1} (\mathcal{E}\Theta_3 + \mathcal{F}\Sigma_3 + \mathcal{F}\Theta_2 + \mathcal{G}\Sigma_2) - \mathcal{F} \frac{\partial^2 x}{\partial \rho_1 \partial \rho_2}}{\Delta \mathcal{E}}$$

$$- \frac{\mathcal{E} \frac{\partial x}{\partial \rho_2} - \mathcal{F} \frac{\partial x}{\partial \rho_1}}{\Delta \mathcal{E}^2} [\mathcal{E}(\Theta_2 + \Sigma_3) + 2(\mathcal{E}\Theta_2 + \mathcal{F}\Sigma_2)] ,$$

from which, it results that:

$$\frac{\partial \frac{\Sigma_2 \Delta}{\mathcal{E}}}{\partial \frac{\partial x}{\partial \rho_2}} - \frac{\partial}{\partial \rho_2} \frac{\mathcal{E} \frac{\partial x}{\partial \rho_2} - \mathcal{F} \frac{\partial x}{\partial \rho_1}}{\Delta \mathcal{E}} = -\mathcal{D}'' \gamma ,$$

Furthermore, one has:

$$\frac{\partial r_1}{\partial \frac{\partial x}{\partial \rho_2}} = \frac{\partial \frac{\Delta \Sigma_1}{\mathcal{E}}}{\partial \frac{\partial x}{\partial \rho_2}} = \frac{\partial}{\partial \frac{\partial x}{\partial \rho_2}} \left(\frac{-\mathcal{E} \frac{\partial \mathcal{E}}{\partial \rho_2} + 2\mathcal{E} \frac{\partial \mathcal{F}}{\partial \rho_1} - \mathcal{F} \frac{\partial \mathcal{E}}{\partial \rho_1}}{2\Delta \mathcal{E}} \right)$$

$$= -\frac{\Sigma_1}{\Delta\mathcal{E}}\left(\mathcal{E}\frac{\partial x}{\partial\rho_2} - \mathcal{F}\frac{\partial x}{\partial\rho_1}\right) + \frac{\mathcal{E}\dfrac{\partial^2 x}{\partial\rho_1^2} - \dfrac{1}{2}\dfrac{\partial x}{\partial\rho_1}\dfrac{\partial\mathcal{E}}{\partial\rho_1}}{\mathcal{E}\Delta}$$

$$= -\frac{\Sigma_1}{\Delta\mathcal{E}}\left(\mathcal{E}\frac{\partial x}{\partial\rho_2} - \mathcal{F}\frac{\partial x}{\partial\rho_1}\right) + \frac{\mathcal{E}(\Theta_1\dfrac{\partial x}{\partial\rho_1} + \Sigma_1\dfrac{\partial x}{\partial\rho_2} + \mathcal{D}\Delta\gamma) - \dfrac{\partial x}{\partial\rho_1}(\mathcal{E}\Theta_1 + \mathcal{F}\Sigma_1)}{\mathcal{E}\Delta} = \mathcal{D}\gamma,$$

$$\frac{\partial r_2}{\partial\dfrac{\partial x}{\partial\rho_2}} = \frac{\partial\dfrac{\Delta\Sigma_2}{\mathcal{E}}}{\partial\dfrac{\partial x}{\partial\rho_2}} = \frac{\partial}{\partial\dfrac{\partial x}{\partial\rho_2}}\left(\frac{\mathcal{E}\dfrac{\partial\mathcal{G}}{\partial\rho_1} - \mathcal{F}\dfrac{\partial\mathcal{E}}{\partial\rho_2}}{2\Delta\mathcal{E}}\right)$$

$$= -\frac{\Sigma_2}{\Delta\mathcal{E}}\left(\mathcal{E}\frac{\partial x}{\partial\rho_2} - \mathcal{F}\frac{\partial x}{\partial\rho_1}\right) + \frac{\mathcal{E}\dfrac{\partial^2 x}{\partial\rho_1\partial\rho_2} - \dfrac{1}{2}\dfrac{\partial x}{\partial\rho_1}\dfrac{\partial\mathcal{E}}{\partial\rho_2}}{\mathcal{E}\Delta}$$

$$= -\frac{\Sigma_2}{\Delta\mathcal{E}}\left(\mathcal{E}\frac{\partial x}{\partial\rho_2} - \mathcal{F}\frac{\partial x}{\partial\rho_1}\right) + \frac{\mathcal{E}(\Theta_2\dfrac{\partial x}{\partial\rho_1} + \Sigma_2\dfrac{\partial x}{\partial\rho_2} + \mathcal{D}'\Delta\gamma) - \dfrac{\partial x}{\partial\rho_1}(\mathcal{E}\Theta_2 + \mathcal{F}\Sigma_2)}{\Delta\mathcal{E}} = \mathcal{D}'\gamma.$$

We modify the identity that we obtained, which gives us two analogous ones upon replacing x, γ with y, γ', and then by z, γ''.

We shall develop the parentheses in such a way as to show us the left-hand sides of the equations of statics of the deformable surface with the forces abstracted. To that effect, we use the relations ([1]):

$$\frac{\partial\gamma}{\partial\rho_1} = \frac{\mathcal{F}\mathcal{D}' - \mathcal{G}\mathcal{D}}{\Delta}\frac{\partial x}{\partial\rho_1} + \frac{\mathcal{F}\mathcal{D} - \mathcal{E}\mathcal{D}'}{\Delta}\frac{\partial x}{\partial\rho_2},$$

$$\frac{\partial\gamma}{\partial\rho_2} = \frac{\mathcal{F}\mathcal{D}'' - \mathcal{G}\mathcal{D}'}{\Delta}\frac{\partial x}{\partial\rho_1} + \frac{\mathcal{F}\mathcal{D}' - \mathcal{E}\mathcal{D}''}{\Delta}\frac{\partial x}{\partial\rho_2},$$

which gives rise to two analogous systems that are obtained by replacing x, γ with y, γ', and then by z, γ'', and which entails that:

$$\frac{\partial\dfrac{\gamma}{\Delta}}{\partial\rho_1} = \frac{\mathcal{F}\mathcal{D}' - \mathcal{G}\mathcal{D}}{\Delta^2}\frac{\partial x}{\partial\rho_1} + \frac{\mathcal{F}\mathcal{D} - \mathcal{E}'\mathcal{D}'}{\Delta^2}\frac{\partial x}{\partial\rho_2} - \frac{\Theta_1 + \Sigma_2}{\Delta}\gamma$$

$$\frac{\partial\dfrac{\gamma}{\Delta}}{\partial\rho_2} = \frac{\mathcal{F}\mathcal{D}'' - \mathcal{G}\mathcal{D}'}{\Delta^2}\frac{\partial x}{\partial\rho_1} + \frac{\mathcal{F}\mathcal{D}' - \mathcal{E}\mathcal{D}''}{\Delta^2}\frac{\partial x}{\partial\rho_2} - \frac{\Theta_2 + \Sigma_3}{\Delta}\gamma,$$

i.e.:

[1] DARBOUX. – *Leçons*, T. III, no. 698, pp. 244-245.

$$\frac{\partial \dfrac{\gamma}{\Delta}}{\partial \rho_1} = -\mathcal{D}' \frac{\mathcal{E}\dfrac{\partial x}{\partial \rho_2} - \mathcal{F}\dfrac{\partial x}{\partial \rho_1}}{\Delta^2} - \mathcal{D}\frac{\mathcal{G}\dfrac{\partial x}{\partial \rho_1} - \mathcal{F}\dfrac{\partial x}{\partial \rho_2}}{\Delta^2} - \frac{\Theta_1 + \Sigma_2}{\Delta}\gamma,$$

$$\frac{\partial \dfrac{\gamma}{\Delta}}{\partial \rho_2} = -\mathcal{D}'' \frac{\mathcal{E}\dfrac{\partial x}{\partial \rho_2} - \mathcal{F}\dfrac{\partial x}{\partial \rho_1}}{\Delta^2} - \mathcal{D}'\frac{\mathcal{G}\dfrac{\partial x}{\partial \rho_1} - \mathcal{F}\dfrac{\partial x}{\partial \rho_2}}{\Delta^2} - \frac{\Theta_2 + \Sigma_3}{\Delta}\gamma.$$

We thus arrive at the statement that if one denotes the left-hand sides of the equations of statics of the deformable surface by U_2, U_1, V, $\mathcal{W}$ then they express that we are led to consider reproducing all of the terms that are independent of the external forces and moments that figure in:

$$-\gamma\mathcal{W} - \frac{\mathcal{G}\dfrac{\partial x}{\partial \rho_1} - \mathcal{F}\dfrac{\partial x}{\partial \rho_2}}{\Delta^2}U_1 - \frac{\mathcal{E}\dfrac{\partial x}{\partial \rho_2} - \mathcal{F}\dfrac{\partial x}{\partial \rho_1}}{\Delta^2}U_2 + \frac{\partial}{\partial \rho_1}\left\{\frac{\mathcal{E}\dfrac{\partial x}{\partial \rho_2} - \mathcal{F}\dfrac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}}\mathcal{W}\right\}.$$

Changing x, γ into y, γ', and then into z, γ'' gives two analogous results.

On the other hand, $\mathcal{W} = 0$, may be written:

$$\frac{\partial}{\partial \rho_1}\frac{\partial (W_0\Delta_0)}{\partial \dfrac{\partial m}{\partial \rho_1}} + \frac{\partial}{\partial \rho_2}\frac{\partial (W_0\Delta_0)}{\partial \dfrac{\partial m}{\partial \rho_2}} - \frac{\partial (W_0\Delta_0)}{\partial m} + \Delta_0 N_0' = 0.$$

One therefore sees that if one sets:

$$\Delta_0\mathcal{X}_0 = \Delta_0 X_0 + \frac{\partial}{\partial \rho_1}\left[\gamma\frac{\Delta_0}{\Delta}\left(L_0\frac{\partial x}{\partial \rho_2} + M_0\frac{\partial y}{\partial \rho_2} + N_0\frac{\partial z}{\partial \rho_2}\right) - \frac{\mathcal{E}\dfrac{\partial x}{\partial \rho_2} - \mathcal{F}\dfrac{\partial x}{\partial \rho_1}}{\Delta\mathcal{E}}\Delta_0 N_0'\right]$$
$$- \frac{\partial}{\partial \rho_2}\left[\gamma\frac{\Delta_0}{\Delta}\left(L_0\frac{\partial x}{\partial \rho_1} + M_0\frac{\partial y}{\partial \rho_1} + N_0\frac{\partial z}{\partial \rho_1}\right)\right],$$

and two analogous formulas that are obtained by replacing $\mathcal{X}_0$, X_0, x, γ with $\mathcal{Y}_0$, Y_0, y, γ', and then with $\mathcal{Z}_0$, Z_0, z, γ'', respectively (along with $L_0\dfrac{\partial x}{\partial \rho_1} + \cdots$ and $L_0\dfrac{\partial x}{\partial \rho_1} + \cdots$), the equations of statics for a deformable surface may be summarized in the following relation

(1): $$\iint \delta(W_0\Delta_0)\,d\rho_1 d\rho_2 + \iint \Delta_0(\mathcal{X}_0\delta x + \mathcal{Y}_0\delta y + \mathcal{Z}_0\delta z - N_0'\delta m)\,d\rho_1 d\rho_2 = 0\,,$$

in which one considers only the terms that are ultimately presented under the double integral sign.

The preceding result may be generalized: suppose that one expresses ξ_1, η_1, ξ_2, η_2 as a function m, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ by the formulas:

$$\xi_1 = \sqrt{\mathcal{E}}\cos(m+u)\,, \qquad \xi_2 = \frac{\mathcal{F}}{\sqrt{\mathcal{E}}}\cos(m+u) - \frac{\Delta}{\sqrt{\mathcal{E}}}\sin(m+u)$$

$$\eta_1 = \sqrt{\mathcal{E}}\sin(m+u) \qquad \eta_2 = \frac{\mathcal{F}}{\sqrt{\mathcal{E}}}\sin(m+u) + \frac{\Delta}{\sqrt{\mathcal{E}}}\cos(m+u)\,,$$

where u denotes an arbitrary function of just $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$; the equation $\mathcal{W} = 0$ may then be written:

$$\frac{\partial}{\partial\rho_1}\frac{\partial(W_0\Delta_0)}{\partial\dfrac{\partial m}{\partial\rho_1}} + \frac{\partial}{\partial\rho_2}\frac{\partial(W_0\Delta_0)}{\partial\dfrac{\partial m}{\partial\rho_2}} - \frac{\partial(W_0\Delta_0)}{\partial m} + \Delta_0 N_0' = 0.$$

Upon forming the combination:

$$-\gamma\mathcal{W} - \frac{\mathcal{G}\dfrac{\partial x}{\partial\rho_1} - \mathcal{F}\dfrac{\partial x}{\partial\rho_2}}{\Delta^2}U_1 - \frac{\mathcal{E}\dfrac{\partial x}{\partial\rho_2} - \mathcal{F}\dfrac{\partial x}{\partial\rho_1}}{\Delta^2}U_2$$
$$+\frac{\partial}{\partial\rho_1}\left\{\left[\frac{\mathcal{E}\dfrac{\partial x}{\partial\rho_2} - \mathcal{F}\dfrac{\partial x}{\partial\rho_1}}{\Delta\mathcal{E}} - \frac{\partial u}{\partial\dfrac{\partial x}{\partial\rho_1}}\right]\mathcal{W}\right\} + \frac{\partial}{\partial\rho_1}\left\{-\frac{\partial u}{\partial\dfrac{\partial x}{\partial\rho_1}}\mathcal{W}\right\}$$

and the two analogous ones that are obtained by replacing x, γ with y, γ', and then with z, γ'', one finds three equations, the first of which is:

$$\frac{\partial^2}{\partial\rho_1^2}\frac{\partial(W_0\Delta_0)}{\partial\dfrac{\partial^2 x}{\partial\rho_1^2}} + \frac{\partial^2}{\partial\rho_1\partial\rho_2}\frac{\partial(W_0\Delta_0)}{\partial\dfrac{\partial^2 x}{\partial\rho_1\partial\rho_2}} + \frac{\partial^2}{\partial\rho_2^2}\frac{\partial(W_0\Delta_0)}{\partial\dfrac{\partial^2 x}{\partial\rho_2^2}} - \frac{\partial}{\partial\rho_1}\frac{\partial(W_0\Delta_0)}{\partial\dfrac{\partial x}{\partial\rho_1}} - \frac{\partial}{\partial\rho_2}\frac{\partial(W_0\Delta_0)}{\partial\dfrac{\partial x}{\partial\rho_2}} + \Delta_0\mathcal{X}_0 = 0$$

upon setting:

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[1] This relation is analogous to the formula $\int_{t_0}^{t_1}(\delta T + U')\,dt = 0$ that TISSERAND gave for HAMILTON's principle on pp. 4 of T. I in his *Traité de Mécanique céleste.*

$$\Delta_0\mathcal{X}_0 = \Delta_0 X_0 + \frac{\partial}{\partial\rho_1}\left[\gamma\frac{\Delta_0}{\Delta}\left(L_0\frac{\partial x}{\partial\rho_2} + M_0\frac{\partial y}{\partial\rho_2} + N_0\frac{\partial z}{\partial\rho_2}\right) + \left(\frac{\partial u}{\partial\frac{\partial x}{\partial\rho_1}} - \frac{\mathcal{E}\frac{\partial x}{\partial\rho_2} - \mathcal{F}\frac{\partial x}{\partial\rho_1}}{\Delta\mathcal{E}}\right)\Delta_0 N_0'\right]$$
$$-\frac{\partial}{\partial\rho_2}\left[\gamma\frac{\Delta_0}{\Delta}\left(L_0\frac{\partial x}{\partial\rho_1} + M_0\frac{\partial y}{\partial\rho_1} + N_0\frac{\partial z}{\partial\rho_1}\right) - \frac{\partial u}{\partial\frac{\partial x}{\partial\rho_2}}\Delta_0 N_0'\right].$$

These four equations may be summarized by:

$$\iint_{C_0}\{\delta(W_0\Delta_0) + \Delta_0(\mathcal{X}_0\delta x + \mathcal{Y}_0\delta x + \mathcal{Z}_0\delta x - N_0'\delta m)\}d\rho_1 d\rho_2 = 0\,,$$

in which one considers only terms that are ultimately presented under the double integral.

The summary form that one is led to, and which will be treated according to the rules of the calculus of variations, is particularly convenient for performing changes of variables.

If we suppose that the expressions $\mathcal{X}_0$, $\mathcal{Y}_0$, $\mathcal{Z}_0$, N_0' have a particular form then we will have the extremal equations for a problem of the calculus of variations.

We consider the particular case ([1]) in which $W_0\Delta_0$ does not depend on r_1, r_2 and depends on ξ_1, ξ_2, η_1, η_2 only by the intermediary of $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$; this amounts to saying that the final expression for $W_0\Delta_0$ does not depend on $m, \frac{\partial m}{\partial\rho_1}, \frac{\partial m}{\partial\rho_2}$, and is a function of ρ_1, ρ_2, and the six functions:

$$\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{D}, \mathcal{D}', \mathcal{D}''$$

of the first and second derivatives of x, y, z.

In addition, if we suppose that $N_0' = 0$; if $\mathcal{X}_0$, $\mathcal{Y}_0$, $\mathcal{Z}_0$ do not depend on m then we ultimately have three equations that relate to only x, y, z, and which may be summarized in the formula:

$$\iint_{C_0}\delta(W_0\Delta_0)d\rho_1 d\rho_2 + \iint_{C_0}\Delta_0(\mathcal{X}_0\delta x + \mathcal{Y}_0\delta y + \mathcal{Z}_0\delta z)d\rho_1 d\rho_2 = 0\,.$$

In the particular case in which U denotes a function of ρ_1, ρ_2, and x, y, z, and

[1] What follows may also be applied to the case in which $W_0\Delta_0$ is arbitrary; the essential hypothesis is the one made for $\mathcal{L}_0$, $\mathcal{M}_0$, $\mathcal{N}_0$. One may also imagine the case in which $W_0\Delta_0$ is of degree one with respect to r_1, r_2 . The coefficients of the latter are constants or, more generally, independent of ρ_2 and ρ_1, respectively.

$\gamma = \frac{1}{\Delta_0}\frac{\partial(y,z)}{\partial(\rho_1,\rho_2)}, \gamma', \gamma''$, one has, in addition:

$$X_0 = \frac{\partial U}{\partial x}, \qquad Y_0 = \frac{\partial U}{\partial y}, \qquad Z_0 = \frac{\partial U}{\partial z},$$

$$L_0\frac{\partial x}{\partial\rho_1} + M_0\frac{\partial y}{\partial\rho_1} + N_0\frac{\partial z}{\partial\rho_1} = -\left(\frac{\partial U}{\partial\gamma}\frac{\partial\Delta}{\partial\frac{\partial x}{\partial\rho_2}} + \frac{\partial U}{\partial\gamma'}\frac{\partial\Delta}{\partial\frac{\partial y}{\partial\rho_2}} + \frac{\partial U}{\partial\gamma''}\frac{\partial\Delta}{\partial\frac{\partial z}{\partial\rho_2}}\right),$$

$$L_0\frac{\partial x}{\partial\rho_2} + M_0\frac{\partial y}{\partial\rho_2} + N_0\frac{\partial z}{\partial\rho_2} = \left(\frac{\partial U}{\partial\gamma}\frac{\partial\Delta}{\partial\frac{\partial x}{\partial\rho_1}} + \frac{\partial U}{\partial\gamma'}\frac{\partial\Delta}{\partial\frac{\partial y}{\partial\rho_1}} + \frac{\partial U}{\partial\gamma''}\frac{\partial\Delta}{\partial\frac{\partial z}{\partial\rho_1}}\right),$$

$$L_0\gamma + M_0\gamma' + N_0\gamma'' = 0,$$

one then has:

$$\Delta_0\mathcal{X}_0 = \frac{\partial(U\Delta_0)}{\partial x} - \frac{\partial}{\partial\rho_1}\frac{\partial(U\Delta_0)}{\partial\frac{\partial x}{\partial\rho_1}} - \frac{\partial}{\partial\rho_2}\frac{\partial(U\Delta_0)}{\partial\frac{\partial x}{\partial\rho_2}},$$

and two analogous formulas, and one obtains the three equations for the extremals relative to the integral:

$$\iint \Delta_0(W_0 + U)\,d\rho_1 d\rho_2.$$

The preceding formulas amount to setting:

$$L_0 = \gamma'\frac{\partial U}{\partial\gamma''} - \gamma''\frac{\partial U}{\partial\gamma'},$$

$$M_0 = \gamma''\frac{\partial U}{\partial\gamma} - \gamma\frac{\partial U}{\partial\gamma''},$$

$$N_0 = \gamma\frac{\partial U}{\partial\gamma'} - \gamma'\frac{\partial U}{\partial\gamma};$$

all of which result from the fact that the $\gamma, \gamma', \gamma''$ verify the following system, which defines a function F of $\frac{\partial x}{\partial\rho_1}, \frac{\partial y}{\partial\rho_1}, \cdots, \frac{\partial z}{\partial\rho_2}$:

$$\frac{\frac{\partial F}{\partial\frac{\partial x}{\partial\rho_1}}}{\gamma} = \frac{\frac{\partial F}{\partial\frac{\partial y}{\partial\rho_1}}}{\gamma'} = \frac{\frac{\partial F}{\partial\frac{\partial z}{\partial\rho_1}}}{\gamma''}, \qquad \frac{\frac{\partial F}{\partial\frac{\partial x}{\partial\rho_2}}}{\gamma} = \frac{\frac{\partial F}{\partial\frac{\partial y}{\partial\rho_2}}}{\gamma'} = \frac{\frac{\partial F}{\partial\frac{\partial z}{\partial\rho_2}}}{\gamma''}.$$

An interesting particular case of the preceding one is the case in which the expression $\frac{\Delta_0 W_0}{\Delta}$, when one takes x and y as the variables, depends – other than on x, y – only on the derivatives of z with respect to x, y; it is easy to find the form of W_0.

Observe that the two expressions:

$$dx^2 + dy^2 + dz^2, \qquad -(d\gamma dx + d\gamma' dy + d\gamma'' dz),$$

may be written:

$$\mathcal{E}d\rho_1^2 + 2\mathcal{F}d\rho_1 d\rho_2 + \mathcal{G}d\rho_2^2, \quad \Delta(\mathcal{D}d\rho_1^2 + 2\mathcal{D}'d\rho_1 d\rho_2 + \mathcal{D}''d\rho_2^2),$$

from which it results, by virtue of the formulas:

$$dx = \frac{\partial x}{\partial \rho_1} d\rho_1 + \frac{\partial x}{\partial \rho_1} d\rho_1,$$
$$dy = \frac{\partial y}{\partial \rho_1} d\rho_1 + \frac{\partial y}{\partial \rho_1} d\rho_1,$$

that one has the identities:

$$\mathcal{E}d\rho_1^2 + 2\mathcal{F}d\rho_1 d\rho_2 + \mathcal{G}d\rho_2^2 = (1+p^2)dx^2 + 2pq\,dx\,dy + (1+q^2)dy^2$$
$$\Delta(\mathcal{D}d\rho_1^2 + 2\mathcal{D}'d\rho_1 d\rho_2 + \mathcal{D}''d\rho_2^2 = \frac{1}{\sqrt{1+p^2+q^2}}(r\,dx^2 + 2s\,dx\,dy + t\,dy^2).$$

From the theory of the invariants of quadratic forms, one has:

$$\mathcal{E}\mathcal{F} - \mathcal{G}^2 = (1+p^2+q^2)\left[\frac{\partial(x,y)}{\partial(\rho_1,\rho_2)}\right]^2,$$
$$\Delta^2(\mathcal{D}\mathcal{D}'' - \mathcal{D}'^2) = \frac{rt-s^2}{1+p^2+q^2}\left[\frac{\partial(x,y)}{\partial(\rho_1,\rho_2)}\right]^2,$$
$$\Delta(\mathcal{G}\mathcal{D} + \mathcal{E}\mathcal{D}'' - 2\mathcal{F}\mathcal{D}') = \frac{(1+q^2)r + (1+p^2)t - 2pqs}{\sqrt{1+p^2+q^2}}\left[\frac{\partial(x,y)}{\partial(\rho_1,\rho_2)}\right]^2,$$

and, as a result, when we pass to absolute invariants, we get:

$$\mathcal{D}\mathcal{D}'' - \mathcal{D}'^2 = \frac{rt-s^2}{(1+p^2+q^2)^2},$$
$$\frac{\mathcal{G}\mathcal{D} + \mathcal{E}\mathcal{D}''' - 2\mathcal{F}\mathcal{D}'}{\Delta} = \frac{(1+q^2)r - 2pqs + (1+p^2)t}{(1+p^2+q^2)^{3/2}}.$$

We recover two well-known expressions for the total curvature and the mean curvature.

The case that we are dealing with then the one in which $\frac{\Delta_0 W_0}{\Delta}$ is a function φ of ρ_1, ρ_2, and the two expressions:

$$\frac{1}{\mathcal{R}_1\mathcal{R}_2} = \mathcal{D}\mathcal{D}'' - \mathcal{D}'^2, \qquad \frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2} = \frac{\mathcal{G}\mathcal{D} + \mathcal{E}\mathcal{D}'' - 2\mathcal{F}\mathcal{D}'}{\Delta},$$

in which $\mathcal{R}_1$ and $\mathcal{R}_2$ denote the radii of the principal curvatures.

If we take x, y for variables then the formula that summarizes the equations of statics of the deformable surface may be written:

$$\delta\iint \varphi\sqrt{1+p^2+q^2}\,dxdy + \iint \frac{\Delta_0}{\Delta}(\mathcal{X}_0\delta x + \mathcal{Y}_0\delta y + \mathcal{Z}_0\delta z)\sqrt{1+p^2+q^2}\,dxdy = 0.$$

The function under the $\iint$ in the second integral is:

$$\frac{\Delta_0}{\Delta}\left\{\left(\mathcal{X}_0\frac{\partial x}{\partial\rho_1} + \mathcal{Y}_0\frac{\partial x}{\partial\rho_1}\right)\delta\rho_1 + \left(\mathcal{X}_0\frac{\partial x}{\partial\rho_2} + \mathcal{Y}_0\frac{\partial x}{\partial\rho_2}\right)\delta\rho_2 + \mathcal{Z}_0\delta z\right\}\sqrt{1+p^2+q^2}$$

and, as a result, since φ does not refer to the derivatives of ρ_1, ρ_2 the equations of the problem become:

$$\frac{\partial^2}{\partial x^2}\frac{\partial(\varphi\sqrt{1+p^2+q^2})}{\partial r} + \cdots - \frac{\partial}{\partial x}\frac{\partial(\varphi\sqrt{1+p^2+q^2})}{\partial q} + \frac{\Delta_0}{\Delta}\sqrt{1+p^2+q^2}\,\mathcal{Z}_0 = 0,$$

$$\frac{\partial}{\partial\rho_1}(\varphi\sqrt{1+p^2+q^2}) + \left(\mathcal{X}_0\frac{\partial x}{\partial\rho_1} + \mathcal{Y}_0\frac{\partial y}{\partial\rho_1}\right)\frac{\Delta_0}{\Delta}\sqrt{1+p^2+q^2} = 0,$$

$$\frac{\partial}{\partial\rho_2}(\varphi\sqrt{1+p^2+q^2}) + \left(\mathcal{X}_0\frac{\partial x}{\partial\rho_2} + \mathcal{Y}_0\frac{\partial y}{\partial\rho_2}\right)\frac{\Delta_0}{\Delta}\sqrt{1+p^2+q^2} = 0.$$

In particular, suppose that φ does not depend on ρ_1, ρ_2, and it depends uniquely on $\frac{1}{\mathcal{R}_1+\mathcal{R}_2}$ and $\frac{1}{\mathcal{R}_1\mathcal{R}_2}$; this gives the equations:

$$\frac{\partial^2}{\partial x^2}\frac{\partial(\varphi\sqrt{1+p^2+q^2}}{\partial r} + \cdots + \frac{\Delta_0}{\Delta}\sqrt{1+p^2+q^2}\,\mathcal{Z}_0 = 0,$$

$$\mathcal{X}_0 = 0, \qquad \mathcal{Y}_0 = 0.$$

One may write:

$$\Delta_0\mathcal{X}_0 = \left\{\frac{\Delta_0}{\Delta} X_0\sqrt{1+p^2+q^2} - \frac{\partial}{\partial y}\left[\gamma\frac{\Delta_0}{\Delta}(L_0+N_0p)\right] + \frac{\partial}{\partial x}\left[\gamma\frac{\Delta_0}{\Delta}(M_0+N_0q)\right]\right\}\frac{\partial(x,y)}{\partial(\rho_1,\rho_2)},$$
$$\Delta_0\mathcal{Y}_0 = \left\{\frac{\Delta_0}{\Delta\gamma''} Y_0 - \frac{\partial}{\partial y}\left[\gamma'\frac{\Delta_0}{\Delta}(L_0+N_0p)\right] + \frac{\partial}{\partial x}\left[\gamma'\frac{\Delta_0}{\Delta}(M_0+N_0q)\right]\right\}\frac{\partial(x,y)}{\partial(\rho_1,\rho_2)},$$
$$\Delta_0\mathcal{Z}_0 = \left\{\frac{\Delta_0}{\Delta\gamma''} Z_0 - \frac{\partial}{\partial y}\left[\gamma''\frac{\Delta_0}{\Delta}(L_0+N_0p)\right] + \frac{\partial}{\partial x}\left[\gamma''\frac{\Delta_0}{\Delta}(M_0+N_0q)\right]\right\}\frac{\partial(x,y)}{\partial(\rho_1,\rho_2)}.$$

We may combine the two equations $\mathcal{X}_0 = 0$, $\mathcal{Y}_0 = 0$ with the preceding ones. For example, we may introduce the combination $\gamma\mathcal{X}_0 + \gamma'\mathcal{Y}_0 + \gamma''\mathcal{Z}_0$ upon taking:

$$\frac{\partial^2}{\partial x^2}\frac{\partial(\varphi\sqrt{1+p^2+q^2})}{\partial r} + \cdots + \frac{\Delta_0}{\Delta}\sqrt{1+p^2+q^2}\,\frac{1}{\gamma''}(\gamma\mathcal{X}_0 + \gamma'\mathcal{Y}_0 + \gamma''\mathcal{Z}_0) = 0.$$

If the givens in the equation that we write, or in other combinations, are suitable then ρ_1, ρ_2 might no longer appear and, by the preceding equation, one will thus have an equation for the surface. The equations:

$$\mathcal{X}_0 = 0, \qquad \mathcal{Y}_0 = 0,$$

serve to define ρ_1, ρ_2 as a function of x, y (or inversely), and may be left aside if one abstracts from the natural state.

Consider the particular case in which the function φ is a linear function with constant coefficients with respect to $\left(\frac{1}{\mathcal{R}_1+\mathcal{R}_2}\right)^2$ and $\frac{1}{\mathcal{R}_1\mathcal{R}_2}$; i.e., a function of the form:

$$A\left(\frac{1}{\mathcal{R}_1+\mathcal{R}_2}\right)^2 + B\frac{1}{\mathcal{R}_1\mathcal{R}_2} + C,$$

in which A, B, C are constants. The constant B disappears from the question according to a remark that was first made by POISSON in his memoir on elastic surfaces ([1]), and was then reprised and generalized by OLINDE RODRIGUES ([2]), and, in the case in which all of the external forces are null, we summarize the equation in question by:

[1] POISSON. – *Mémoire sur les surfaces élastique,* dated August 1, 1814 (Mémoires de la Classe des Sciences mathématiques et physiques, of l'Institut de France, year of 1812, second Part, pp. 167-225); an extract of this memoir first appeared in the Bulletin de la SociJté Philomatique, and then in the Correspondance sur l'Ecole Polytechnique, T. III, pp. 154-159, 1815.

[2] RODRIGUES. – *Recherches sur la théorie analytique des lignes et des rayons de courbure des surfaces et sur la transformation d'une classe d'intégrales doubles, qui ont un rapport direct aves les formulas de cette theorie.* Correspondence to l'Ecole Polytechnique, T. III, pp. 162-182, 1815; in particular, see pp. 172, et seq.

$$\delta\iint\left(\frac{1}{\mathcal{R}_1}+\frac{1}{\mathcal{R}_2}\right)^2\sqrt{1+p^2+q^2dxdy}+C\delta\iint\sqrt{1+p^2+q^2dxdy}=0\,,$$

which is the conclusion that POISSON arrived at in his own researches.

In conclusion, observe that by the consideration of infinitely small deformations the general developments of this section easily lead to the theories of THOMSON and TAIT ([1]) and LORD RAYLEIGH ([2]); we leave to the reader the burden of taking this approach and studying the case with which one is concerned in detail ([3]).

47. Dynamics of the deformable line. – The dynamics of the deformable line are attached to the preceding exposition. To see this, it suffices to regard one of the parameters – ρ_1, for example – as time *t*. One will then have an action consisting of simultaneous deformation and movement. Under the influence of the triad, the velocity of a point of the deformable line enters into *W* by way of the three arguments ξ_2, η_2, ζ_2, and one finds oneself in the presence of the notion of *anisotropic kinematics* that was already envisioned by RANKINE, and which has since been introduced into several theories of physics, such as the theories of double refraction and rotational polarization, for example.

Similarly, if *W* is independent of rotations and leads to null external moments then the argument of pure deformation $\xi_1^2+\eta_1^2+\varsigma_1^2$ and the argument $\xi_2^2+\eta_2^2+\varsigma_2^2$ are generally accompanied by the argument $\xi_1\xi_2+\eta_1\eta_2+\varsigma_1\varsigma_2$. Such a type of argument is no longer new in mechanics and appears, notably, in the theory of forces at a distance, as we shall show later on.

When *W* does not contain the mixed argument $\xi_1\xi_2+\eta_1\eta_2+\varsigma_1\varsigma_2$ it is necessary, in general, to consider the infinitesimal state of deformation and motion of the natural state in order to find oneself in the case of classical mechanics in which *the action of deformation is completely separate from the kinematical action.* One thus obtains D'ALEMBERT'S principle upon supposing that the external force and moment are null, i.e., upon expressing that the deformable line is not subject to any action from the external world, and introducing, as a result, the fundamental notion of an *isolated system*, of which we spoke at the beginning of this note.

The dynamics of the deformable surface may be established in the same manner by means of the theory of the deformable medium of three dimensions, which we shall now discuss.

[1] THOMSON and TAIT. – *Treatise*, Part II, no. 644.

[2] LORD RAYLEIGH. – *Theory of Sound*, vol. I, 2nd ed., 1894, pp. 352.

[3] It amounts to the *infinitely small* deformation of an originally *planar* surface.